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Symmetric diffusions with polynomial eigenvectors

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Abstract

We describe symmetric diffusion operators where the spectral decomposition is given through a family of orthogonal polynomials. In dimension one, this reduces to the case of Hermite, Laguerre and Jacobi polynomials. In higher dimension, some basic examples arise from compact Lie groups. We give a complete description of the bounded sets on which such operators may live. We then provide a classification of those sets when the polynomials are ordered according to their usual degrees.

MSC : 60B15, 60B20, 60G99, 43A75, 14H50.

Key words : orthogonal polynomials, diffusion operators, random matrices.

1 Introduction

Symmetric diffusion operators and their associated heat semigroups play a central rôle in the study of continuous Markov processes, and also in differential geometry and partial differential equations. The analysis of the associated heat or potential kernels have been considered from many points of view, such as short and long time asymptotics, upper and lower bounds, on the diagonal and away from it, convergence to equilibrium, e.g. All these topics had been deeply investigated along the past half century, see [3, 11, 27] for example. Unfortunately, there are very few examples where computations are explicit.

The spectral decomposition may provide one approach to the heat kernel study and the analysis of convergence to equilibrium, especially when the spectrum is discrete. Once again, there are very few models where this spectral decomposition is at hand, either for the explicit expressions of the eigenvalues or the eigenvectors. The aim of this survey is to present a family of models where this spectral decomposition is completely explicit. Namely, we shall require the eigenvectors to be a polynomials in a finite number of variables. We are then dealing with orthogonal polynomials with respect to the reversible measure of this diffusion operator. Once again, orthogonal

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polynomial have been thoroughly investigated in many aspects, going back to the early works of Legendre, Chebyshev, Markov and Stieltjes, see [12, 22, 26] e.g.

To be more precise, we shall consider some open connected set $\Omega \subset \mathbb{R}^d$, with piecewise smooth boundary $\partial\Omega$ (say at least piecewise $\mathcal{C}^1$, and may be empty), and some probability measure $\mu(dx)$ on Ω, with smooth positive density measure $\rho(x)$ with respect to the Lebesgue measure and such that polynomials are dense in $\mathcal{L}^2(\mu)$.

A diffusion operator L on Ω (but we shall also consider such objects on smooth manifolds with no further comment) is a linear second order differential operator with no 0-order terms, therefore written as

$$\mathrm{L}(f) = \sum_{ij} g^{ij}(x)\partial^2_{ij} f + \sum_i b^i(x)\partial_i f, \tag{1.1}$$

such that at every point $x \in \Omega$, the symmetric matrix $(g^{ij}(x))$ is non negative. For simplicity, we shall assume here that this matrix is always non degenerate in Ω (it may, and will in general, be degenerate at the boundary $\partial\Omega$). This is an ellipticity assumption, which will be in force throughout. We will also assume that the coefficients $g^{ij}(x)$ and $b^i(x)$ are smooth. We are mainly interested in the case where L is symmetric on $\mathcal{L}^2(\mu)$ when restricted to smooth functions compactly supported in Ω. On the generator L, this translates into

$$\mathrm{L}(f) = \frac{1}{\rho}\sum_i \partial_i\Big(\rho \sum_j g^{ij}\partial_j f\Big), \tag{1.2}$$

as is readily seen using integration by parts in Ω, see [3].

We are also interested in the case when $\mathcal{L}^2(\mu)$ admits a complete orthonormal basis $P_q(x), q \in \mathbb{N}$, of polynomials such that $\mathrm{L}(P_q) = -\lambda_q P_q$, for some real (indeed non negative) parameters λ_q. This is equivalent to the fact that there exists an increasing sequence $\mathcal{P}_n$ of finite dimensional subspaces of the set $\mathcal{P}$ of polynomials such that $\cup_n \mathcal{P}_n = \mathcal{P}$ and such that L maps $\mathcal{P}_n$ into itself.

When such happens, we have a spectral decomposition of L in this basis, and, when a function $f \in \mathcal{L}^2(\mu)$ is written as $f = \sum_q c_q P_q$, then $\mathrm{L}(f) = \sum_q -\lambda_q c_q P_q$, such that any expression depending only on the spectral decomposition may be analyzed easily.

Our aim is to describe various families of such situations, which will be referred to as polynomial models. In dimension d, many such models may be constructed with various techniques: Lie groups, root systems, Hecke algebras, etc, see [1, 2, 7, 9, 10, 12, 13, 14, 16, 17, 18, 19, 20, 21, 22], among many other possible references. Introducing weighted degrees, we shall analyse the situation where the operator maps for any $k \in \mathbb{N}$ the space $\mathcal{P}_k$ of polynomials with degree less k into itself. For bounded sets Ω with regular boundaries, this leads to an algebraic description of the admissible boundaries for such sets. In dimension 2 and for the usual degree, we give a complete description of all the admissible models (reducing to 11 families up to affine transformation). We then present some other models, with other degrees, or in larger dimension, with no claim for exhaustivity.

This short survey is organized as follows. In Section 2, we present a brief tour of the dimension 1 case, where the classical families of Hermite, Laguerre and Jacobi

polynomials appear. We provide some geometric description for the Jacobi case (which will serve as a guide in higher dimension) together with various relations between those three families. In Section 3 we describe some basic notions concerning the symmetric diffusion operators, introducing in particular the operator Γ and the integration by parts formula. We also introduce the notion of image, or projection, of operators, a key tool towards the construction of polynomial models. In Section 4, we describe the Laplace operators on spheres, $SO(n)$ and $SU(n)$, and we provide various projections arising from these examples leading to polynomials models, in particular for spectral measures in the Lie group cases. Section 5 describes the general case, with the introduction of the weighted degree models. In particular, when the set Ω is bounded, we provide the algebraic description of the sets on which such polynomial models may exist. In particular, we show that the boundaries of those admissible sets Ω must lie in some algebraic variety, satisfying algebraic restrictions. The description of the sets lead then to the description of the measures and the associated operators. Section 6 is a brief account of the complete classification for the ordinary degree of those bounded models in dimension 2. This requires a precise analysis of the algebraic nature of the boundary which is only sketched in this paper. Section 7 provides some examples of 2 dimensional models with weighted degrees, and is far from exhaustive, since no complete description is valid at the moment. Section 8 proposes some new ways (apart from tensorization) to construct higher dimensional models from low dimension ones. Finally, we give in Section 9 the various pictures corresponding to the 2 dimensional models described in Section 6.

2 Dimension 1 case : Jacobi, Laguerre and Hermite

In dimension one, given a probability measure μ for which the polynomials are dense in $\mathcal{L}^2(\mu)$, there is, up to the choice of a sign, a unique family (P_n) of polynomials with $\deg(P_n) = n$ and which is an orthonormal basis in $\mathcal{L}^2(\mu)$. It is obtained through the Gram-Schmidt orthonormalization procedure of the sequence $\{1, x, x^2, \cdots\}$. When does such a sequence consists in eigenvectors of some given diffusion operator of the form $L(f)(x) = a(x)\partial^2 f + b(x)\partial f$ (a Sturm-Liouville operator)? This had been described long ago (see e.g. [4, 23]), and reduces up to affine transformation to the classical cases of Jacobi, Hermite and Laguerre polynomials.

Those three families play a very important rôle in many branches of mathematics and ingeneering (mainly in statistics and probability for the Hermite family and in fluid mechanics for the Jacobi one), and we refer to the huge literature on them for further information. We briefly recall these models.

1. $\mathcal{I} = \mathbb{R}$. The measure μ is Gaussian centered: $\mu(dx) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx$. The associated polynomials are the Hermite polynomials (H_n). They are eigenvectors of the Ornstein–Uhlenbeck

$$\mathcal{H} = \frac{d^2}{dx^2} - x\frac{d}{dx} \quad \mathcal{H}(H_n) = -nH_n.$$

2. $\mathcal{I} = \mathbb{R}_+^*$ The measure is the gamma measure $\mu_a(dx) = C_a x^{a-1} e^{-x}\, dx$, $a > 0$. The associated polynomials are the Laguerre polynomials $L_n^{(a)}$, and the associated operator is the Laguerre operator

$$\mathcal{L}_a = x\frac{d^2}{dx^2} + (a-x)\frac{d}{dx}, \quad \mathcal{L}_a(L_n^{(a)}) = -nL_n^{(a)}.$$

3. $\mathcal{I} = (-1,1)$. The measure is the beta measure $\mu_{a,b}(dx) = C_{a,b}(1-x)^{a-1}(1+x)^{b-1}\, dx$, $a, b > 0$. The associated polynomials are the Jacobi polynomials $(J_n^{(a,b)})$ and the associated Jacobi operator is

$$\mathcal{J}_{a,b} = (1-x^2)\frac{d^2}{dx^2} - \big(a-b+(a+b)x\big)\frac{d}{dx}, \quad \mathcal{J}_{a,b}J_n^{(a,b)} = -n(n+a+b)J_n^{(a,b)}.$$

The Jacobi family contains the ultraspherical (or Gegenbauer) family (when $a = b$) with as particular cases the Legendre polynomials $a = b = 0$, the Chebyshev first and second kind ($a = b = -1/2$ and $a = b = 1/2$ respectively), which appear, after renormalization, when writing $\cos(n\theta) = P_n(\cos\theta)$ (first kind) and $\sin(n\theta) = \sin(\theta)Q_n(\cos\theta)$ (second kind). The first two families (Hermite and Laguerre) appear as limits of the Jacobi case. For example, when we chose $a = b = n/2$ and let then n go to ∞, and scale the space variable x into $x/\sqrt{n}$, the measure $\mu_{a,a}$ converges to the Gaussian measure, the Jacobi polynomials converge to the Hermite ones, and $\frac{2}{n}\mathcal{J}_{a,a}$ converges to $\mathcal{H}$.

In the same way, the Laguerre case is obtained from the Jacobi one fixing b, changing x into $\frac{2x}{a} - 1$, and letting a go to infinity. Then $\mu_{a,b}$ converges to μ_b, and $\frac{1}{a}\mathcal{J}_{a,b}$ converges to $\mathcal{L}_b$.

Also, when a is a half-integer, the Laguerre operator may be seen as the image of the Ornstein–Uhlenbeck operator in dimension d. Indeed, as the product of one dimensional Ornstein–Uhlenbeck operators, the latter has generator $H_d = \Delta - x.\nabla$. It's reversible measure is $e^{-|x|^2/2}dx/(2\pi)^{d/2}$, it's eigenvectors are the products $Q_{k_1}(x_1)\cdots Q_{k_d}(x_d)$, and the associated process $X_t = (X_t^1, \cdots, X_t^d)$, is formed of independent one dimensional Ornstein-Uhlenbeck processes. Then, if one considers $R(x) = |x|^2$, then one may observe that, for any smooth function $F : \mathbb{R}_+ \mapsto \mathbb{R}$,

$$\mathcal{H}_d\big(F(R)\big) = 2\mathcal{L}_a(F)(R),$$

where $a = d/2$. In the probabilist interpretation, this amounts to observe that if X_t is a d-dimensional Ornstein–Uhlenbeck process, then $|X_{t/2}|^2$ is a Laguerre process with parameter $a = d/2$.

In the same way, as we shall see below in Section 4, when $a = b = (d-1)/2$, $\mathcal{J}_{a,a}$ may be seen as the Laplace operator $\Delta_{S^{d-1}}$ on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$ acting on functions depending only on the first coordinate (or equivalently on functions invariant under the rotations leaving $(1, 0, \cdots, 0)$ invariant), and a similar interpretation is valid for $J_{p/2,q/2}$ for integers p and q. This interpretation comes from Zernike and Brinkman [8] and Braaksma and Meulenbeld [6] (see also Koornwinder [15]). Jacobi polynomials also play a central role in the analysis on compact Lie groups. Indeed, for (a, b) taking the various values of $(q/2, q/2)$, $((q-1)/2, 1)$, $(q-1, 2)$, $(2(q-1), 4)$

and (4, 8) the Jacobi operator $\mathcal{J}_{a,b}$ appears as the radial part of the Laplace-Beltrami (or Casimir) operator on the compact rank 1 symmetric spaces, that is spheres, real, complex and quaternionic projective spaces, and the special case of the projective Cayley plane (see Sherman [25]).

3 Basics on symmetric diffusions

Diffusion operators are associated with diffusion processes (that is continuous Markov processes) through the requirement that, if (X_t) is the Markov process with associated generator L, then for any smooth function f, $f(X_t) - \int_0^t \mathrm{L}(f)(X_s)ds$ is a local martingale, see [3]. Here, we are mainly interested in such diffusion operators which are symmetric with respect to some probability measure μ. In probabilistic terms, this amounts to require that when the law of X_0 is μ, then not only at any time the law of X_t is still μ, but also, for any time $t > 0$, the law of $(X_{t-s}, 0 \leq s \leq t)$ is the same than the law of $(X_s, 0 \leq s \leq t)$ (that is why this measure is often called the reversible measure).

For any diffusion operator as such described in equation (1.1), we may define the carré du champ operator Γ as the following bilinear application

$$\Gamma(f,g) = \frac{1}{2}\Big(\mathrm{L}(fg) - f\mathrm{L}(g) - g\mathrm{L}(f)\Big),$$

defined say for smooth functions defined on $\Omega \subset \mathbb{R}^d$. From formula (1.2), it is readily seen that

$$\Gamma(f,g) = \sum_{ij} g^{ij}\partial_i f \partial_j g,$$

such that $\Gamma(f,f) \geq 0$. As already mentioned, we restrict ourselves to the case where the matrix (g^{ij}) is everywhere positive definite. Then, the inverse matrix (g_{ij}) defines a Riemannian metric. By abuse of language, we shall refer to the matrix (g^{ij}) (or equivalently to the operator Γ) as the metric of the operator, although formally is should be called a co-metric. Then, Riemannian properties of this metric may carry important information on the models under study. Typically, most of the models described below in Section 6 have constant curvature.

The operator L satisfies the following chain rule (or change of variable formula). For any n-uple $f = (f_1, \cdots, f_k)$ and any smooth function $\Phi : \mathbb{R}^k \mapsto \mathbb{R}$,

$$\mathrm{L}\big(\Phi(f)\big) = \sum_i \partial_i \Phi(f)\mathrm{L}(f_i) + \sum_{ij} \partial^2_{ij}\Phi(f)\Gamma(f_i, f_j). \tag{3.3}$$

This allows us to compute $\mathrm{L}\big(\Phi(f)\big)$ as soon as we know $\mathrm{L}(f_i)$ and $\Gamma(f_i, f_j)$. This is in particular the case for the coordinates $x^i : \Omega \mapsto \mathbb{R}$, where $\Gamma(x^i, x^j) = g^{ij}(x)$ and $\mathrm{L}(x^i) = b^i(x)$, recovering then the form given in equation (1.1). But we may observe that the family $f = (f_i)$ in the previous formula does not need to be a coordinate system (that is a diffeomorphism from Ω into some new set Ω_1). There may be more function (f_j) then required $(k > d)$ or less $(k < d)$. This remark will play an important rôle in the sequel.

In general, when looking at such operators, one considers first the action of the operator on smooth compactly supported function on Ω. Since we want to work on polynomials, it is better to enlarge the set of functions we are working on, for example the set of $\mathcal{L}^2(\mu)$ functions which are smooth and compactly supported in a neighborhood of Ω, referred to below just as "smooth functions".

The operator is symmetric in $\mathcal{L}^2(\mu)$ when, for any pair (f, g) of smooth functions

$$\int_\Omega \mathrm{L}(f)\, g\, d\mu = \int_\Omega f\, \mathrm{L}(g)\, d\mu. \tag{3.4}$$

Usually, for this to be true, one should require one of the functions f or g to be compactly supported in Ω, or ask for some boundary conditions on f and g, such as Dirichlet or Neuman. However, in the case of polynomial models, the operator will be such that no boundary conditions will be required for equation (3.4) to hold. More precisely, at any regular point of the boundary, we shall require the unit normal vector to belong to the kernel of the matrix (g^{ij}). Under such assumption, the symmetry equation (3.4) is satisfied whenever f and g are smooth functions.

If we observe that $\mathrm{L}(\mathbf{1}) = 0$ (where $\mathbf{1}$ denotes the constant function), then applying (3.4) to $\mathbf{1}$ shows that $\int \mathrm{L}(f)\, d\mu = 0$, and therefore, applying the definition of Γ and integrating over Ω (provided $fg \in \mathcal{L}^2(\mu)$), that

$$\int \mathrm{L}(f)\, g\, d\mu = -\int \Gamma(f, g)\, d\mu = \int_\Omega f\, \mathrm{L}g\, d\mu, \tag{3.5}$$

which is called the integration by parts formula.

The fact that $\Gamma(f, f) \geq 0$ and the previous formula (3.5) shows that, for any function f, $\int_\Omega f\, \mathrm{L}(f)\, d\mu \leq 0$, and therefore all eigenvalues are non positive.

The operator is entirely determined by the knowledge of Γ and μ, as is obvious from formula (1.2), and the datum (Ω, Γ, μ) is called a Markov triple in the language of [3], to which we refer for more details about the general description of such symmetric diffusion operators

As already mentioned, we want to analyse those situations such that the set $\mathcal{P}$ of polynomials is dense in $\mathcal{L}^2(\mu)$ (and, being an algebra, all polynomials will be automatically in any $\mathcal{L}^p(\mu)$ for any $p > 1$), and such that there exists some Hilbert basis (P_n) of $\mathcal{L}^2(\mu)$ with elements in $\mathcal{P}$ such that $\mathrm{L}(P_n) = -\lambda_n P_n$. Since we also require that any polynomial is a finite linear combination of the P_n', we see that the set $\mathcal{P}_n = \{\sum_0^n \mu_k P_k\}$ is an increasing sequence of finite dimensional linear subspaces of $\mathcal{P}$ such that $\mathrm{L} : \mathcal{P}_n \mapsto \mathcal{P}_n$, and $\cup_n \mathcal{P}_n = \mathcal{P}$.

Conversely, if there exists such an increasing sequence $\mathcal{P}_n$ of finite dimensional linear subspaces of $\mathcal{P}$ such that such that $\cup_n \mathcal{P}_n = \mathcal{P}$ satisfying $\mathrm{L} : \mathcal{P}_n \mapsto \mathcal{P}_n$, then we may find a sequence (P_n) which is an orthonormal basis of $\mathcal{L}^2(\mu)$ and eigenvectors of L. Indeed, the restriction of L to the finite dimensional subspace $\mathcal{P}_n$ is symmetric when we provide it with the Euclidean structure inherited from the $\mathcal{L}^2(\mu)$ structure, and therefore may be diagonalized in some orthonormal basis. Repeating this in any space $\mathcal{P}_n$ provides the full sequence of polynomial orthogonal vectors.

It may be worth to observe that when this happens, the set of polynomials is an algebra dense in $\mathcal{L}^2(\mu)$ and stable under L and the associated heat kernel $P_t = \exp(t\mathrm{L})$. When this happens, it is automatically dense in the $\mathcal{L}^2(\mu)$ domain of

L, for the domain topology, and the set of polynomial will be therefore a core for our operator (see [3] for more details).

From now on, we shall denote by (P_n) such a sequence of eigenvectors, with $\mathrm{L}(P_n) = -\lambda_n P_n$ (and we recall that $\lambda_n \geq 0$). Since $\mathrm{L}(\mathbf{1}) = 0$, we may always chose $P_0 = \mathbf{1}$, and $\lambda_0 = 0$. In general, this eigenvalue is simple, in particular in the elliptic case. Indeed, thanks to the integration by parts formula (3.5), any function f such that $\mathrm{L}(f) = 0$ satisfies $\int \Gamma(f,f)d\mu = 0$, from which $\Gamma(f,f) = 0$. If ellipticity holds (but also under much weaker requirements), then this implies that f is constant.

As mentioned in the introduction, one is often interested in the heat kernel P_t associated with L, that is the linear operator $\exp(t\mathrm{L})$, defined through the fact that $\mathrm{L}P_n = e^{-\lambda_n t}P_n$, or equivalently by the fact $F(x,t) = P_t(f)(x)$ satisfies the heat equation $\partial_t F = \mathrm{L}_x(F)$, with $F(x,0) = f(x)$.

This heat semigroup may be represented (at least at a formal level) as

$$P_t(f)(x) = \int_\Omega f(y)p_t(x,y)d\mu(y),$$

where the heat kernel may be written

$$p_t(x,y) = \sum_n e^{-\lambda_n t}P_n(x)P_n(y), \tag{3.6}$$

provided that the series $\sum_n e^{-2\lambda_n t}$ is convergent (in practise and in all our models, this will always be satisfied). From this we see that a good knowledge on λ_n and P_n provides information of this heat kernel. However, it happens (thanks to the positivity of Γ) that P_t preserves positivity, (and of course $P_t(\mathbf{1}) = 1$), which is equivalent to the fact that $p_t(x,y)\mu(dy)$ is a probability measure for any $t > 0$ and any $x \in \Omega$, in particular $p_t(x,y) \geq 0$. This is not at all obvious from the representation (3.6). Therefore, this representation (3.6) of $p_t(x,y)$ does not carry all the information about it.

It is worth to observe the following, which will be the basic tool for the construction of our polynomial models. Start from an operator L (defined on some manifold Ω_1 or some open set in it), symmetric under some probability measure μ. Assume that we may find a set of functions $f = (f_i)_{i=1,\cdots k}$, that we consider as a function $f : \Omega \mapsto \Omega_1 \subset \mathbb{R}^k$, such that

$$L(f_i) = B^i(f), \Gamma(f_i, f_j) = G^{ij}(f).$$

Then, for any smooth function $G : \Omega_1 \mapsto \mathbb{R}$, and thanks to formula (3.3), one has

$$\mathrm{L}\big(G(f)\big) = \mathrm{L}_1(G)(f),$$

where

$$\mathrm{L}_1 G = \sum_{ij} G^{ij}\partial^2_{ij}G + \sum_i B^i\partial_i G. \tag{3.7}$$

This new diffusion operator L_1 is said to be the image of L under f. In probabilistic terms, the image $Y_t = f(X_t)$ of the diffusion process X_t associated with L under f

is still a diffusion process, and it's generator is L_1. Moreover, if L is symmetric with respect to μ, L_1 is symmetric with respect of the image measure ν of μ under f.

This could (and will) be an efficient method to determine the density ρ_1 of this image measure with respect to the Lebesgue measure, through the use of formula (1.2).

4 Examples : spheres, $SO(d)$, $SU(d)$

We now describe a few natural examples leading to polynomial models. The first basic remark is that, given a symmetric diffusion operator L described as before by a Markov triple (Ω, Γ, μ), it is enough to find a set $\{X_1, \cdots, X_k\}$ of functions such that for any $i = 1, \cdots, k$, $\mathrm{L}(X_i)$ is a degree 1 polynomial in the variables X_j, and for any pair $(i, j), i, j = 1, \cdots, k$, $\Gamma(X_i, X_j)$ is a degree 2 polynomial in those variables. Indeed, if such happens, then the image L_1 of L under $X = \{X_1, \cdots, X_k\}$ given in (3.7) is a symmetric diffusion operator with reversible measure μ_1, where μ_1 is the image of μ under X. Thanks to formula (3.3), L_1 preserves for each $n \in \mathbb{N}$ the set $\mathcal{P}_n$ of polynomials in the variables (X_i) with total degree less than n. One may then diagonalize L_1 in $\mathcal{P}_n$, and this leads to the construction of a $\mathcal{L}^2(\mu_1)$ orthonormal basis formed with polynomials.

Moreover, given polynomial models, we may consider product models, which are again polynomial models, and from them consider projections. Indeed, given two polynomial models described by triples $(\Omega_i, \Gamma_i, \mu_i)$ as in Section 3, we may introduce on $\Omega_1 \times \Omega_2$ the measure $\mu_1 \otimes \mu_2$, and the sum operator $\Gamma_1 \oplus \Gamma_2$, acting on functions $f(x, y)$ and $h(x, y)$ as

$$\big(\Gamma_1 \otimes \mathrm{Id} \oplus \mathrm{Id} \otimes \Gamma_2\big)(f, h)(x, y) = \sum_{ij} g_1^{ij} \partial_{x_i} f \partial_{x_j} h + \sum_{kl} g_2^{kl}(y) \partial_{y_k} f \partial_{y_l} h.$$

Once we have some polynomial models $(\Omega_1, \Gamma_1, \mu_1)$ and $(\Omega_2, \Gamma_2, \mu_2)$, then $(\Omega_1 \times \Omega_2, \Gamma_1 \oplus \Gamma_2, \mu_1 \otimes \mu_2)$ is again a polynomial model. At the level of processes, if (X_t^i) is associated with $(\Omega_i, \Gamma_i, \mu_i)$, and are chosen independent, then the process (X_t^1, X_t^2) is associated with this product. This kind of tensorisation procedure constructs easily new dimensional models in higher dimension once we have some in low dimension. The polynomials $R_{i,j}(x, y), x \in \Omega_1,\ y \in \Omega_2$ associated with the product are then just the tensor products $P_i(x)Q_j(y)$ of the polynomials associated with each of the components.

Moreover, one may consider quotients in these products to construct more polynomial models, as we did to pass from the one dimensional Ornstein-Uhlenbeck operator to the Laguerre operator.

The easiest example to start with is the Laplace operator $\Delta_{\mathbb{S}}$ on the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. This operator may be naively described as follows : considering some smooth function f on $\mathbb{S}^{d-1}$, we extend it in a neighborhood of $\mathbb{S}^{d-1}$ into a function which is independent of the norm, that is $\hat{f}(x) = f(\frac{x}{\|x\|})$, where $\|x\|^2 = \sum_i x_i^2$, for $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$. Then, we consider $\Delta(\hat{f})$, where Δ is the usual Laplace operator in $\mathbb{R}^d$, and restrict $\Delta(\hat{f})$ on $\mathbb{S}^d$. This is $\Delta_{\mathbb{S}}(f)$.

An easy exercise shows that, for the functions x_i which are the restriction to $\mathbb{S}^{d-1}$ of the usual coordinates in $\mathbb{R}^d$, then

$$\Delta_{\mathbb{S}}(x_i) = -(d-1)x_i,\ \Gamma_{\mathbb{S}}(x_i, x_j) = \delta_{ij} - x_i x_j.$$

The uniform measure on the sphere (that is the unique probability measure which is invariant under rotations) is the reversible measure for $\Delta_{\mathbb{S}}$. The system of functions $(x_1, \cdots, x_d)$ is not a coordinate system, since those functions are linked by the relation $\sum_i x_i^2 = 1$. For example, one sees from the previous formulae that $\Delta_{\mathbb{S}}(\|x\|^2) = \Gamma(\|x\|^2, \|x\|^2) = 0$ on $\mathbb{S}^{d-1}$ (an good way to check the right value for λ when imposing $\mathrm{L}(x_i) = -\lambda x_i$ with the same Γ operator).

But the system $(x_1, \cdots, x_{d-1})$ is a system of coordinates for say the upper half sphere. We may observe that the operator indeed projects onto $(x_1, \cdots, x_{d-1})$ into an elliptic operator in the unit ball $\mathbb{B}^{d-1} = \{\|x\| < 1\}$, with exactly the same relations for $\mathrm{L}(x_i)$ and $\Gamma(x_i, x_j)$. The image operator (that is the Laplace operator in this system of coordinates) is then

$$\mathrm{L} = \sum_{ij} (\delta_{ij} - x_i x_j)\partial_{ij}^2 - (d-1)\sum_i x_i \partial_i,$$

and from the formula (1.2), is is easy to determine the density measure (up to a multiplicative constant), which is $\rho(x) = (1 - \|x\|^2)^{-1/2}$, which happens here to be $\det(g^{ij})^{-1/2}$ (as is always the case for Laplace operators). We see then that this provides an easy way to compute the density of the uniform measure on the sphere under this map $\mathbb{S}^{d-1} \mapsto \mathbb{B}^{d-1}$, which projects the upper half sphere onto the unit ball.

One may also observe that if $M = (M_{ij})$ is a matrix (with fixed coefficients), and if $y_i = \sum_j M_{ij} x_j$, then $\Delta(y_i) = -(d-1)y_i$ and $\Gamma(y_i, y_j) = (MM^t)_{ij} - y_i y_j$. Then, when M is orthogonal, the image measure of $\Delta_{\mathbb{S}}$ under $x \mapsto Mx$ is $\Delta_{\mathbb{S}}$ itself, which tells us that the Laplace operator is invariant under orthogonal transformations.

We may also consider the projection π from $\mathbb{S}^{d-1}$ to $\mathbb{B}_p$ for $p < d-1$: π : $(x_1, \cdots, x_d) \mapsto (x_1, \cdots, x_p)$, which provides the same operator as before, except that now we are working on the unit ball $\mathbb{B}^p \subset \mathbb{R}^p$ and $L(x_i) = -(d-1)x_i$, where the parameter d is no longer correlated with the dimension p of the ball. We may as well consider the generic operator L_λ on the unit ball in $\mathbb{R}^p$ with $\Gamma(x_i, x_j) = \delta_{ij} - x_i x_j$ and $\mathrm{L}(x_i) = -\lambda x_i$, where $\lambda > p-1$. Is is readily checked that it has symmetric measure density $C_{p,d}(1 - \|x\|^2)^{\frac{\lambda-p-1}{2}}$. As a consequence, the image measure of the sphere $\mathbb{S}^{d-1}$ onto the unit ball through this projection has density $C_{d,p}(1 - \|x\|^2)^{(d-p-2)/2}$. It is worth to observe that when λ converges to $p-1$, the measure converges to the uniform measure on the boundary of $\mathbb{B}^p$, that is $\mathbb{S}^{p-1}$, and the operator converges to the operator on $\mathbb{S}^{p-1}$.

When we chose $p = 1$, we recover the symmetric Jacobi polynomial model in dimension 1 with parameters $a = b = (d-1)/2$.

For these operators, we see that, in terms of the variables (x_i), g^{ij} are polynomials with total degree 2 and $L(x_i)$ are degree 1. Therefore, in view of the chain rule formula (3.3), we see that the operator L such defined maps the space $\mathcal{P}_n$ of

polynomials with total degree less than n into itself, and this provides a first family of polynomial models.

One may also still consider the unit sphere in $\mathbb{R}^d$, and chose integers such that $p_1 + \cdots + p_k = d$. Then, setting $P_0 = 0, P_i = p_1 + \cdots + p_i$, consider the functions $X_i = \sum_{P_{i-1}+1}^{P_i} x_j^2$, for $i = 1, \cdots k-1$. The image of the sphere under this application is the simplex $\mathcal{D}_{k-1} = \{X_i \geq 0, \sum_1^{k-1} X_i \leq 1\}$. We have $\Delta_{\mathbb{S}}(X_i) = 2(p_i - dX_i)$, and $\Gamma(X_i, X_j) = 4X_i(\delta_{ij} - X_i X_j)$. The operator $\Delta_{\mathbb{S}}$ such projects on the simplex, with

$$G^{ij} = 4X_i(\delta_{ij} - X_j),\ B^i = 2(p_i - dX_i),$$

and provides again a polynomial model on it. The reversible measure is easily seen to have density $C \prod_1^{k-1} X_i^{r_i} (1 - \sum_1^{k-1} X_i)^{r_k}$, with $r_i = \frac{p_i - 2}{2}$, $ri = 1, \cdots, k$, which a Dirichlet measure on the simplex $\mathcal{D}_{k-1}$. This measure is then seen as the image of the uniform measure on the sphere through this application $X : \mathbb{S}^{d-1} \mapsto \mathcal{D}_{k-1}$. The general Dirichlet density measure $\rho_{a_1, \cdots, a_k}(X) = X_1^{a_1} \cdots X_{k-1}^{a_{k-1}} (1 - \sum_i X_i)^{a_k}$ also produces (with the same Γ operator) a new family of polynomial models. We may play a around with the Dirichlet measures for half integer parameters. For example, look at the image in a Dirichlet $\rho_{a_1, \cdots, a_k}$ measure for the image through $(X_1, X_2, \cdots, X_{k-1}) \mapsto (X_1 + X_2, X_3, \cdots, X_{k-1})$ produces a Dirichlet measure with parameters $\rho_{a_1 + a_2, a_3, \cdots, a_k}$, which is obvious from the sphere interpretation, and extends easily to the general class. The same procedure is still true at the level of the operators (or the associated stochastic processes).

Once again, when considering the case $k = 2$, we get an operator on $[0, 1]$. Changing X into $2X - 1$, we get an operator on $[-1, 1]$, which, up to the scaling factor 4, is the dissymetric Jacobi model with parameters $a = r_1/2, b = (d - r_1)/2$.

There are many other ways to produce polynomial models from spheres, (and we shall provide some later in dimension 2, see Section 7. But we want to show another very general way to construct some. Let us consider some semi-simple compact Lie group $\mathbb{G}$ (such as $SO(n), SU(n), Sp(n)$, etc). On those groups, there exist a unique invariant probability measure (that is invariant under $x \mapsto gx$ and $x \mapsto xg$, for any $g \in \mathbb{G}$). This is the Haar measure. There also exists a unique (up to a scaling constant) diffusion operator $\Delta_{\mathbb{G}}$ which is also invariant under left and right action : this means that if, for a smooth function $f : \mathbb{G} \mapsto \mathbb{R}$, one defines for any $g \in \mathbb{G}$, $L_g(f)(x) = f(gx)$, then $\Delta_{\mathbb{G}}(L_g(f)) = L_g(\Delta_{\mathbb{G}} f)$, and the same is true for the right action $R_g(f)(x) = f(xg)$. This operator is called the Laplace (or Casimir) operator on the group $\mathbb{G}$. Assume then that the Lie group $\mathbb{G}$ is represented as a group of matrices, as is the case for the natural presentation of the natural above mentioned Lie groups. The Lie algebra $\mathcal{G}$ of $\mathbb{G}$ is the tangent space at the origin, and to any element A of the Lie algebra, that is a matrix (A_{ij}), we may associate some vector field R_A on the group through $X_A(f)(g) = \partial_t f(ge^{tA})_{t=0}$. If we write $g = (g_{ij})$ and consider a function $f(g_{ij}) : \mathbb{G} \mapsto \mathbb{R}$, then

$$X_A(g) = \sum_{ijk} g_{ik} A_{kj} \partial_{g_{ij}} f,$$

and therefore X_A preserves the set $\mathcal{P}_n$ of polynomials with total degree n in the variables (g_{ij}). Now, the Casimir operator may be written as $\sum_i X_{A_i}^2$, where the

A_i form an orthonormal basis for some natural quadratic form on the Lie algebra $\mathcal{G}$ called the Killing form. This operator also preserves the set $\mathcal{P}_n$. Unfortunately, those "coordinates" g_{ij} are in general linked by algebraic relations, and may not serve as a true coordinate system on the group. However, we may then describe the operator $\Delta_{\mathbb{G}}$ through it's action on those functions $g_{ij} : \mathbb{G} \mapsto \mathbb{R}$.

Without further detail, consider the group $SO(n)$ of orthogonal matrices with determinant 1. Let m_{ij} be the entries of a matrix in $SO(n)$, considered as functions on the group. We have

$$\Delta_{SO(n)}(m_{ij}) = -(n-1)m_{ij},\ \Gamma_{SO(n)}(m_{kl}, m_{qp}) = \delta_{kq}\delta_{lp} - m_{kp}m_{ql}.$$

We now show some projections of this operator. Let $\mathcal{M}(p,q)$ be the space of $p \times q$ matrices. Select p lines and q columns in the matrix $g \in SO(n)$, (say the first ones), and consider the map $\pi : SO(n) \mapsto \mathcal{M}(p,q)$ which to $M \in SO(n)$ associates the extracted matrix $N = (m_{ij}), 1 \leq i \leq p, 1 \leq j \leq q$. From the form of $\Delta_{SO(n)}(m_{ij})$ and $\Gamma_{SO(n)}(m_{ij}, m_{kl})$, it is clear that the operator projects on $\mathcal{M}(p,q)$ through π. It may happen (whenever $p+q > n$) that the image is contained in a sub-manifold (indeed an algebraic variety) of $\mathcal{M}(p,q)$. But we have nevertheless a new diffusion operator on this image, and the associated process is known as the matrix Jacobi process. It is worth to observe that if $p = n$ and $q = 1$, this is nothing else than the spherical operator $\Delta_{\mathbb{S}}$ in $\mathbb{S}^{n-1}$. In general, whenever $p + q \leq n$, this process lives of the symmetric domain $\{NN^* \leq \mathrm{Id}\}$, and has a reversible measure with density ρ with respect to the Lebesgue measure which is $\det(\mathrm{Id} - NN^*)^{(n-p-q-1)/2}$, which is easily seen from formula (1.2). We may also now fix p and q and consider n as a parameter, and we obtain a family of polynomial processes on this symmetric domain a long as $p + q < n + 1$.

One may play another game and consider the image of the operator on the spectrum. More precisely, given the associated process $X_t \in SO(n)$, one looks at the process obtained on the eigenvalues of X_t (that is the spectral measure of X_t). This process is again a diffusion process, for which we shall compute the generator. To analyze the spectral measure (that is the eigenvalues up to permutations of them), the best is to look at the characteristic polynomial $P(X) = \det(M - X\mathrm{Id}) = \sum_{i=1}^n a_i X^i$. Then we wan to compute $\Delta_{SO(n)}(a_i)$ and $\Gamma_{SO(n)}(a_i, a_j)$.

For a generic matrix $M = (m_{ij})$, Cramer's formulae tells us that, on the set where M is invertible, $\partial_{m_{ij}} \log(\det(M)) = M^{-1}_{ji}$ and $\partial^2_{m_{ij},m_{kl}} \log(\det(M)) = -M^{-1}_{jk} M^{-1}_{li}$.

From this, and using the chain rule formula, we get that

$$\begin{aligned}\Delta_{SO(n)} \log P(X) = \quad & -(n-1)\mathrm{trace}\,(M(M - X\mathrm{Id})^{-1} \\ & -\mathrm{trace}\left((M - X\mathrm{Id})^{-1}(M^t - X\mathrm{Id})^{-1}\right) + \left(\mathrm{trace}\, M(M - X\mathrm{Id})^{-1}\right)^2.\end{aligned}$$

and

$$\begin{aligned}\Gamma\big(\log(P(X)), \log(P(Y))\big) = \quad & \mathrm{trace}\left((M - X\mathrm{Id})^{-1}(M^t - Y\mathrm{Id})^{-1}\right) \\ & -\mathrm{trace}\left(M^2(M - X\mathrm{Id})^{-1}(M - Y\mathrm{Id})^{-1}\right).\end{aligned}$$

But

$$\begin{cases} \text{trace}\,(M(M-X\mathrm{Id})^{-1} = n - X\frac{P'}{P}(X), \\ \text{trace}\,\big((M-X\mathrm{Id})^{-1}(M^t-Y\mathrm{Id})^{-1}\big) = \frac{1}{1-XY}\Big(\frac{1}{Y}\frac{P'}{P}(\frac{1}{Y}) - X\frac{P'}{P}(X)\Big) \\ \text{trace}\,\big(M^2(M-X\mathrm{Id})^{-1}(M-Y\mathrm{Id})^{-1}\big) = n + \frac{1}{X-Y}\Big(X^2\frac{P'}{P}(X) - Y^2\frac{P'}{P}(Y)\Big). \end{cases}$$

One may use the fact that $M \in SO(n)$ to see that $P(\frac{1}{X}) = (-X)^{-n}P(X)$, so that $\frac{1}{Y}\frac{P'}{P}(\frac{1}{Y}) = n - Y\frac{P'}{P}(Y)$.

In the end, we see that

$$\Delta_{SO(n)}(P) = -(d-1)XP' + X^2P'',$$

$$\Gamma_{SO(n)}\big(P(X),P(Y)\big) = \frac{XY}{1-XY}\Big(nP(X)P(Y) + \frac{(1-X^2)P(Y)P'(X) - (1-Y^2)P(X)P'(Y)}{X-Y}\Big).$$

Since $\Delta_{SO(n)}P(X) = \sum_i \Delta_{SO(n)}(a_i)X^i$ and

$$\Gamma_{SO(n)}\big(P(X),P(Y)\big) = \sum_{ij}\Gamma_{SO(n)}(a_i,a_j)X^iY^j,$$

we see form the action of $\Delta_{SO(n)}$ and $\Gamma_{SO(n)}$ on $P(X)$ that, in terms of the variables (a_k), $\Delta_{SO(n)}(a_i)$ are degree one polynomials and $\Gamma_{SO(n)}(a_i,a_j)$ are degree two. Therefore, from the same argument as before, the operator $\Delta_{SO(n)}$ projects on it's spectrum into a polynomial model in the variables a_i.

The same is true (with similar computations) for the spectra of NN^*, where N is the extracted matrix in $\mathcal{M}(p,q)$ described above (corresponding to the matrix Jacobi process), and for many more models.

Similarly, one may also look at the special unitary group $SU(n)$, where the coordinates ($z_{ij} = x_{ij} + \mathbf{i}y_{ij}$) are the entries of the matrix. Using complex coordinates, one has then to consider $\Delta_{SU(n)}(z_{ij})$ and $\Gamma(z_{ij},z_{kl})$ and $\Gamma(z_{ij},\bar{z}_{kl})$ in order to recover the various quantities corresponding to the variables x_{ij} and y_{ij} (using the linearity of Δ and the bilinearity of Γ). We have (up to some normalization)

$$\begin{cases} \Delta_{SU(n)}(z_{ij}) = -(n^2-1)z_{ij}, \\ \Gamma_{SU(n)}(z_{ij},z_{kl}) = z_{ij}z_{kl} - nz_{il}z_{kj}, \\ \Gamma_{SU(n)}(z_{ij},\bar{z}_{kl}) = n\delta_{ik}\delta_{jl} - z_{ij}\bar{z}_{kl}) \end{cases}$$

The same remark as before applies about the extracted matrices, and also, with the same method, we get for the characteristic polynomial

(4.8)

$$\begin{cases} \Delta_{SU(n)}(P) = -(n^2-1)XP' + (n+1)X^2P'', \\ \Gamma_{SU(n)}(P(X),P(Y)) = XY\Big(P'(X)P'(Y) + \frac{n}{X-Y}\big(P'(X)P(Y) - P'(Y)P(X)\big)\Big), \\ \Gamma_{SU(n)}(P(X),\bar{P}(Y)) = \frac{1}{1-XY}\Big(nP(X)\bar{P}(Y) - \bar{Y}\bar{P}'(Y)P(X) - XP'(X)\bar{P}(Y)\Big). \end{cases}$$

Once again, the Casimir operator on $SU(n)$ projects onto a polynomial model in the variables of the characteristic polynomial.

5 The general case

As we briefly showed in Section 4, there are many models for orthogonal polynomials and they are quite hard to describe in an exhaustive way. We propose in this Section a more systematic approach. Recall that we are considering probability measures μ on $\mathbb{R}^d$, on some open connected set $\Omega \subset \mathbb{R}^d$ for which the set $\mathcal{P}$ of polynomials are dense in $\mathcal{L}^2(\mu)$. Recall that it is enough for this to hold that there exists some $\epsilon > 0$ such that $\int e^{\epsilon\|x\|} d\mu < \infty$.

The first thing to do is to describe some ordering of the polynomials. For this, we chose a sequence $a = (a_1, \cdots, a_d)$ of positive integers and say that the degree of a monomial $x_1^{p_1} x_2^{p_2} \cdots x_d^{p_d}$ is $a_1 p_1 + \cdots a_d p_d$. Then, the degree $\deg_a(P)$ of a polynomial $P(x_1, \cdots, x_d)$ is the maximum value of the degree of it's monomials (such a degree is usually referred as a skew degree). Then, the space $\mathcal{P}_n$ of polynomials with $\deg_a(P) \leq n$ is finite dimensional. Moreover, $\mathcal{P}_n \subset \mathcal{P}_{n+1}$, and $\cup_n \mathcal{P}_n = \mathcal{P}$.

To chose a sequence of orthogonal polynomials (P_k) for μ, we chose at each step n some orthonormal basis in the orthogonal complement $\mathcal{H}_n$ of $\mathcal{P}_{n-1}$ in $\mathcal{P}_n$. This space in general has a large dimension (increasing with n), and there is therefore not an unique choice of such an orthonormal basis.

We are then looking for diffusion differential operators L (with associated Γ operator) on Ω, such that L admits such a sequence (P_n) as eigenvectors, with real eigenvalues . The operator L will be then automatically essentially self-adjoint on $\mathcal{P}$. The first observation is that for each n, L maps $\mathcal{P}_n$ into itself.

In particular, for each coordinate, $\mathrm{L}(x_i) \in \mathcal{P}_{a_i}$ and for each pair of coordinates (x_i, x_j), $\Gamma(x_i, x_j) \in \mathcal{P}_{a_i + a_j}$. Then, writing $\mathrm{L} = \sum_{ij} g^{ij} \partial_{ij} + \sum_i b^i \partial_i$, we see that

$$g^{ij} \in \mathcal{P}_{a_i + a_j}, \ b^i \in \mathcal{P}_{a_i} \tag{5.9}$$

Moreover, under the conditions (5.9), we see from the chain rule formula (3.3) that for each $n \in \mathbb{N}$, L maps $\mathcal{P}_n$ into itself. Provided it is symmetric on $\mathcal{P}_n$ for the Euclidean structure induced from $\mathcal{L}^2(\mu)$, we will then be able to derive an orthonormal basis formed with eigenvectors for it.

Once we have a polynomial model with a given choice of degrees $(a_1, \cdots, a_d)$, (say, in the variables $(x_i, i = 1, \cdots, d)$, and as soon as one may find polynomials $X_1, \cdots, X_k$ in the variables x_i with $\deg_a(X_i) = b_i$, as soon as $\mathrm{L}(X_i)$ and $\Gamma(X_i, X_j)$ are polynomials in those variables X_i, then we get a new model in the variables (X_i) (provided however that the ellipticity requirement is satisfied), with new degrees $b = (b_1, \cdots, b_k)$. Indeed, from the chain rule (3.3), one sees that the image operator L_1 maps the set $\mathcal{Q}_n$ of polynomials in the variables (X_i) with degree $\deg_b \leq n$ into itself.

The next task is then to describe the sets Ω on which such a choice for L and μ is possible. For that, we restrict our attention for those $\Omega \subset \mathbb{R}^d$ which are bounded with piecewise $\mathcal{C}^1$ boundaries. We call those sets admissible sets.

Then, we have the main important characterization

Theorem 5.1.

1. *If Ω is an admissible set, then $\partial\Omega$ is included into an algebraic variety, with* $\deg_a \leq 2 \sum_i a_i$.

2. *Let Q be the reduced equation of ∂_Ω. Ω is admissible if and only if there exist some $g^{ij} \in \mathcal{P}_{a_i+a_j}$ and $L_i \in \mathcal{P}_{a_i}$ such that*

$$\forall i = 1, \cdots, d, \sum_j g^{ij}\partial_j Q = L_i Q, \ (g^{ij}) \text{ with non negative in } \Omega. \tag{5.10}$$

When such happens, Q divides $\det(g)$.

3. *Let $Q = Q_1 \cdots Q_k$ the decomposition of Q into irreducible factors. If* (5.10) *is satisfied, then any function $\rho(x) = Q_1^{r_1} \cdots Q_k^{r_k}$ is an admissible density measure for the measure μ, provided $\int_\Omega \rho(x)dx < \infty$. When $Q = \det(g)$, then there are no other measures.*

4. *For any solution (g^{ij}) of* (5.10)*, and any μ as before, setting $\Gamma(f,g) = \sum_{ij} g^{ij}\partial_i\partial_j g$, then the triple (Ω, Γ, μ) is an admissible solution for the operator* L.

Remark 5.2. *Observe that equation* (5.10) *may be rewritten as $\Gamma(\log Q, x_i) = L_i$.*

Proof. — We shall not give the full details of the proof here, and just describe the main ideas.

Suppose we have a polynomial model with coefficients g^{ij}, b^i on Ω, with polynomial functions g^{ij} and b^i satisfying the above requirements on their degrees.

The first thing to observe is that if L is diagonalizable of $\mathcal{P}_n$ for each $n \in \mathbb{N}$, then for each polynomial pair (P, Q)

$$\int_\Omega \mathrm{L}(P)\, Q\, d\mu = \int_\Omega P\, \mathrm{L}(Q)\, d\mu. \tag{5.11}$$

This extends easily to any pair (f, g) of smooth functions compactly supported in Ω, so that the description (1.2) holds true. Moreover, Ω being bounded, and the coefficients g^{ij} and b^i being polynomials, formula (5.11) extends further to every pair (f, g) of smooth functions, not necessarily with support in Ω. Using Stockes formula, (and the regularity of the boundary of Ω), this imposes that, for any pair of smooth function (f, h), $\int_{\partial\Omega} \sum_{ij} f\partial_i h g^{ij} n_j dx = 0$, where (n_j) is the normal tangent vector at the boundary $\partial\Omega$. Therefore, this implies that, for any i, $\sum_j g^{ij} n_j = 0$ on the boundary, so that (n_j) is in the kernel of (g) at the boundary. This implies in turn that the boundary lies inside the algebraic set $\{\det(g) = 0\}$.

Therefore, $\partial\Omega$ is included in some algebraic variety. For any regular point $x \in \partial\Omega$ consider an irreducible polynomial Q_1 such that, in a neighborhood $\mathcal{V}$ of x, the boundary is included in $\{Q_1 = 0\}$. Then, (n_j) is parallel to $\partial_j Q_1$, so that $\sum_j g^{ij}\partial_j Q_1 = 0$ on $\mathcal{V} \cap \{Q_1 = 0\}$. From Hilbert's Nullstellensatz, $\sum_j g^{ij}\partial_j Q_1 = L_i Q_1$, for some polynomial L_i.

This being valid for any polynomial Q_1 appearing in the reduced equation of $\partial\Omega$, this is still true for the reduced equation itself (and in fact the two assertions are equivalent).

For a given polynomial Q, if equation (5.10) admits a non trivial solution (g^{ij}), then $\partial_i Q$ is in the kernel of (g^{ij}) at every regular point of $\{Q = 0\}$. Then, $\det(g)$ vanishes at that point. Q being reduced, then Q is a factor of $\det(g)$.

Now, the link between $b^i = \mathrm{L}(x_i)$ and $\sum_g^{ij} \partial_j \log\rho$ given in (1.2) shows that, in order for b^i to be a polynomial with degree less than a_i, it is enough (and in fact

equivalent) to have $\sum_j g^{ij}\partial_j \log\rho = A_i$, for some polynomial A_i with degree less than a_i. But comparing with equation (5.10) shows that it is satisfied for $Q_i^{r_i}$ for any factor Q_i of Q and any parameter r_i. Then, all the condition are satisfied and the model (Ω, Γ, μ) is a polynomial model.

∎

One sees that indeed the problem of determining polynomial models relies entirely on the study of the boundary $\partial\Omega$, at least as far as bounded sets Ω are considered. Given any algebraic curve, and a fixed choice of degrees $(a_1, \cdots, a_k)$ it is an easy task to decide if this curve is a candidate to be (or to belong to) the boundary of some set Ω on which there exist a polynomial model : equation (5.10) must have a non trivial solution. This equation is a linear system of equations in the coefficients of the polynomials g^{ij} and L_i, however in general with much more equations than variables.

Moreover, as soon as one has a model on Ω, there there exist as we already saw many other models on the same set, with the same (g^{ij}), with measures described with a finite number of parameters, depending on the number of irreducible components in the reduced equation of $\partial\Omega$.

The solutions of equation (5.10) provide a set of measures which are admissible. The admissible measures are determined through the requirement that $\sum_j g^{ij}\partial_j \log\rho = A_i$, with $\deg_a(A_i) \leq a_i$, or in other terms $\Gamma(\log\rho, x_i) = A_i$. When the reduced equation of the boundary is $\{\det(g) = 0\}$, then we have described all the measures in Theorem 5.1. But when some factor of $\det(g)$ does not appear in the reduced equation of $\partial\Omega$, it is nor excluded that those factor may provide some other admissible measure (see [5]). However, in dimension two and for the usual degree, where we are able do provide a complete description of all possible models, this situation never appears and we wonder if this may appear at all.

The fact that the boundary is included into $\{\det(g)\} = 0$ allows to restrict in general to one of the connected components of the complement of this set, so that the metric may never degenerate inside Ω. But it may happen (although we have no example for this) that there exist some solutions of this problem for which the solution (g^{ij}) is not positive inside any bounded region bounded by $\{Q = 0\}$.

But the determination of all possible admissible boundaries (that is the curves for which equation (5.10) have a non trivial solution) is a much harder problem. The possibility for an algebraic surface to have a non trivial solution in equation (5.10) is a very strong requirement, as we shall see next, and this reduces considerably the possible models.

6 Classification with usual degree in dimension 2

In this Section, we reduce to the two dimensional case, and moreover to the usual degree $a_1 = a_2 = 1$. In this situation, the problem is then invariant under affine transformation, and this allows to use classical tools of algebraic geometry to reduce the problem. The coefficients (g^{ij}) have at most degree 2 and the boundary

maximum degree 4. The main result is the following

Theorem 6.1. *In dimension* 2 *and for the usual degree* $a_1 = a_2 = 1$*, and up to affine transformations, there exist exactly 11 bounded sets* Ω *(with piecewise* $\mathcal{C}^1$ *boundary) corresponding to a polynomial model. Their boundaries are (see pictures in Section 9): the triangle, the circle, the square, two coaxial parabolas, a parabola, a tangent line and a line parallel to the axis, a parabola and two tangent lines, a cuspidal cubic and a tangent line, a cuspidal cubic and a line passing through the infinite point of the cubic, a nodal cubic, the swallow tail and the deltoid curve.*

In all the models, the only possible values for the measure are the one described in Theorem 5.1. When the boundary has maximal degree, then the metric (g^{ij}) *is unique up to scaling, and correspond to a constant curvature metric, either* 0 *or* 1 *(after appropriate scaling).*

There are models (triangle, circle) where the metric (g^{ij}) *is not unique.*

Remark 6.2. *The previous assertion is not completely exact. The family described by two axial parabolas are not reducible one to the other under affine transformations. But a simple quadratic transformation does the job.*

Proof. — It is out of scope to give the proof here, which is quite lengthy and technical. But it relies on some simple argument. The main point is to show (using appropriate change of coordinates allowed by the affine invariance of the problem) that there may be no flex point and no flat point on the boundary. That is (in complex variables), that one may no find an analytic branch locally of the form $y = x^3 + o(x^3)$ or $y = x^4 + o(x^4)$. This is done through the local study of equation (5.10). Such points correspond to singular points of the dual curve. But there is a balance between the singular points of the curve, of it's dual curve, and the genus of the curve (seen as a compact Riemann surface), known as Plucker's formulae. This allows to show that $\partial\Omega$ must indeed have many singular points, which list is easy to write since the degree is low (here 4). It remains to analyze all the possible remaining cases. See [5] for details. ∎

Observe that the triangle and the circle case where already described in Section 4 as image of the two dimensional sphere. But even in this case, equation (5.10) produces other metrics (g^{ij}) than the one already described. If one considers a single entry of a $SU(d)$ matrix, then it corresponds to a polynomial model in the unit disk which is one of these exotic metrics on the unit ball in $\mathbb{R}^2$. Typically, on the circle, one may add to the generator $a(x\partial_y - y\partial_x)^2$, which satisfies the boundary condition and corresponds to some extra random rotation in the associated stochastic process. Indeed, all these 11 models may be described, at least for some half-integer values of the parameters appearing in the description of the measure, as the image of the above mentioned models constructed on spheres, $SO(d)$ or $SU(d)$. But there are also, for other values of the measure, some constructions provided by more sophisticated geometric models, in particular root systems in Euclidean spaces. We refer to [5] for a complete description of theses models.

7 Other models with other degrees in dimension 2

When Ω is not bounded, equation (5.10) is not fully justified. If one restricts our attention to the usual degree and to those boundaries which satisfy this condition, then we obtain only the products of the various one dimensional models, and two extra models which are bounded by a cuspidal cubic or a parabola. In this situation, there are some exponential factors in the measures, as happens in the Laguerre case. When there is no boundary at all, it may be proved (although not easily) that the only admissible measures are the Gaussian ones : they correspond to the product of Ornstein-Uhlenbeck operators, but as is the case with the circle, one may add to the metric some rotational term $(x\partial_y - y\partial_x)^2$, which produces new families of orthogonal polynomials.

Beyond this, one may exhibit some examples on various bounded sets Ω with weighted degrees. There is no complete classification in general for such general models at the moment. The reason is that affine invariance is then lost (this is counterbalanced by the fact that some other polynomial change of variables are allowed), but the local analysis made above is no longer valid. To show how rich this new family of models may be, we just present here some examples.

On $SU(3)$, let Z be the trace of the matrix, considered as a function $Z : SU(3) \mapsto \mathbb{C} = \mathbb{R}^2$. Then thanks to the fact that there are 3 eigenvalues belonging to the unit circle and product 1, the characteristic polynomial $\det(M - \mathrm{Id})$ of an $SU(3)$ matrix may be written as $-X^3 + ZX^2 - \bar{Z}X + 1$ such that Z itself encodes the spectral measure. Applying formulae (4.8), and up to a change of Z into $Z/3$ and scaling, one gets

$$\begin{cases} \Gamma(Z,Z) = \bar{Z} - Z^2, \\ \Gamma(\bar{Z},Z) = 1/2(1 - Z\bar{Z}), \\ \Gamma(\bar{Z},\bar{Z}) = Z - \bar{Z}^2, \\ \mathrm{L}Z = -4Z, \mathrm{L}\bar{Z} = -4\bar{Z}. \end{cases} \tag{7.12}$$

This corresponds indeed with the deltoid model appearing in Section 6. From these formulae, one sees that functions $F(Z,\bar{Z})$ which are symmetric in $(Z,\bar{Z})$ are preserved by the image operator. Setting $S = Z + \bar{Z}$ and $P = Z\bar{Z}$, this leads to the following polynomial model with degree $\deg_S + 2\deg_P$, with

$$\begin{cases} \Gamma(S,S) = 1 + S + P - S^2, \\ \Gamma(S,P) = \frac{1}{2}S - 2P + S^2 - \frac{3}{2}SP, \\ \Gamma(P,P) = P - 3SP - 3P^2 + S^3, \\ \mathrm{L}S = -4S, \mathrm{L}(P) = 1 - 9P. \end{cases}$$

Up to some constant, the determinant of the metric is $(4P - S^2)(4S^3 - 3P^2 - 12SP - 6P + 1)$, and the boundary is the domain which is delimited by a parabola and a cuspidal cubic (a degree 5 curve), which are bi-tangent at their intersection point. This leads to a two-parameter family of measures and associated orthogonal polynomials.

One may also construct more models using discrete symmetry groups. Here are some examples.

We cut the 2-d sphere into n vertical slices along the meridians, and and look for a basis of functions invariant under the reflections around these meridians. Writing the sphere as $x_1^2 + x_2^2 + x_3^2 = 1$, we chose $X = x_3$ and writing in complex notations $x_1 + \mathbf{i}x_2 = z$,we chose $Y = \Re(z^n)$. In polynomials terms $Y = (x_1^2+x_2^2)^{n/2} P_n(\frac{x_1}{\sqrt{x_1^2+x_2^2}})$, where P_n is the n-th first kind Chebychef polynomial. For parity considerations on P_n, this is always a polynomial in the variables (x_1, x_2). We have

$$\begin{cases} \Gamma(X,X) = 1 - X^2, \\ \Gamma(X,Y) = -nXY, \\ \Gamma(Y,Y) = n^2\Big((1-X^2)^{n-1} - Y^2\Big) \\ \mathrm{L}X- = -2X,\ \mathrm{L}(Y) = -n(n+1)Y \end{cases}$$

The boundary equation is then $(1 - X^2)^n - Y^2 = 0$, which is irreducible when n is odd, leading to a one parameter family of measures, and splits into two parts when n is even, leading to a two parameters family. We may in the previous model look at functions of (X^2, Y) adding a new invariance under symmetry around the hyperplane $\{x_3 = 0\}$, or of (X, Y^2), or also of (X^2, Y^2) leading to 3 new families.

There are other ways to construct such two dimensional models. A general idea is to consider some finite sub-group of $SO(3)$, and extract from the axes of the rotation some subfamily V_i which is invariant under the group action. Then, one considers the homogeneous polynomial $P(x, y, z) = \prod_i X \cdot V_i$, where $X = (x, y, z)$ and $X \cdot V$ denotes the scalar product. It is also invariant under the group action. Let m be the degree of this polynomial. With P, one constructs a new polynomial $Q = r^{m+1/2} P(\partial_x, \partial_y, \partial_z) r^{-1/2}$, where $r = (x^2 + y^2 + z^2)^{1/2}$. Q is still homogeneous with degree m, invariant under the group action, and moreover harmonic. Then, one looks for the system $X = Q, Y = \Gamma(Q, Q)$ where Γ is the spherical Laplace operator on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. The action of the spherical operator on the pair (X, Y) may lead to a polynomial system. This is not always the case however. For example, with the symmetry group of the icosahedron, there are 3 homogeneous polynomials P_6, P_{10} and P_{15}, with degrees $6, 10$ and 15 which generate all homogeneous polynomials which are invariant under the group action. The technology that we provided works starting from P_6 but not from P_{15}. The associated formulae are too complicated to be given here. For the reader interested in explicit computations (see [24] for more examples), the explicit value of P_6 is as follows (with $c = (1 + \sqrt{5})/2$)

$$P_6(x, y, z) = (c^2x^2 - y^2)(c^2y^2 - z^2)(c^2z^2 - x^2).$$

8 Higher dimensional models

The technology which allows to describe all the bounded dimensional models for the usual degree is not available in higher dimension, mainly because of the lack of the analogues of Plucker's formulae. The many models issued from Lie group action that we produced so far provide many families, with various degrees. Beyond these

explicit constructionss, and sticking to the bounded models with the usual degree in dimension 2, it is worth to observe that one may produce new admissible sets by double cover. More explicitly, as soon as we have a model in dimension d, with reduced boundary equation $P(x) = 0$, one may look for models in dimension $d + 1$ with boundary equatio $y^2 - P(x) = 0$ (y being the extra one dimensional variable. It turns out that this produces a new model for every two dimensional model which have no cusps or tangent lines as singular points (that is, in our setting, the circle, the triangle, the square, the double parabola and the nodal cubic). The boundary has no longer maximal degree, even if the starting model has, and the metric is not unique in general. Moreover, even in the simplest cases, the curvature is not constant.

We may then pursue the construction adding new dimensions. The reason why this works (together with the obstruction about singular points) remains mysterious. Most of the questions regarding these constructions and others remain open at the moment.

9 Pictures

In this Section, we give the various pictures for the 11 bounded models in dimension 2 with natural degree. We give the reduced equation of the boundary, and we indicate when the metric is unique, up to a scaling factor. When it is unique, we indicate the cases when the curvature is constant, and what is it's sign. It is worth to observe that all the models with maximal degree (here 4) have a unique constant curvature metric.

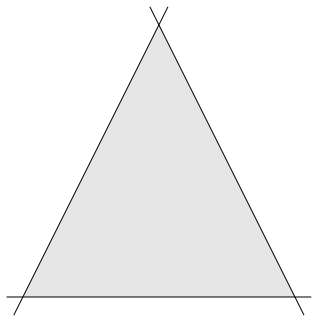

Figure 1: Triangle : $xy(1 - x - y) = 0$, metric not unique

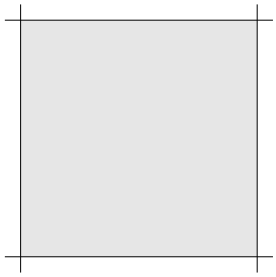

Figure 2: Square : $(1-x^2)(1-y^2)=0$, one metric , curvature 0

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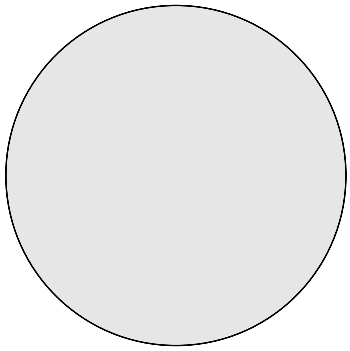

Figure 3: Circle : $x^2+y^2=1$, metric not unique

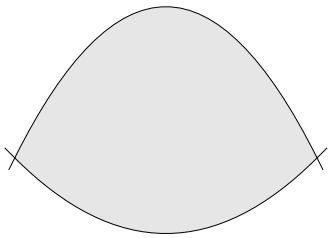

Figure 4: Double parabola : $(y+1-x^2)(y-1+ax^2)=0$, one metric, curvature 1

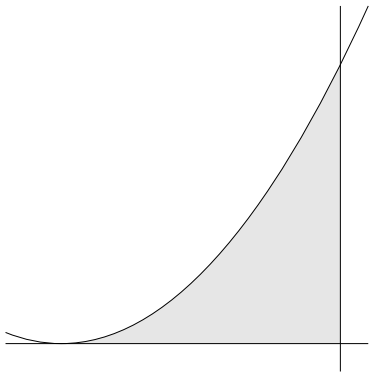

Figure 5: Parabola with 1 tangent and one secant line : $(y - x^2)y(x - 1) = 0$, one metric, curvature 1

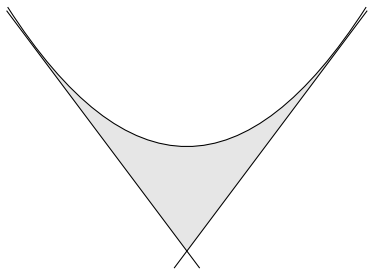

Figure 6: Parabola with 2 tangents : $(y - x^2)(y + 1 - 2x)(y + 1 + 2x) = 0$, one metric, curvature 0

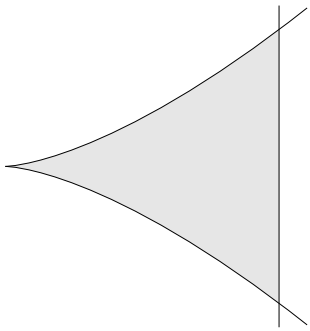

Figure 7: Cuspidal cubic with secant line : $(y^2 - x^3)(x - 1) = 0$, one metric, curvature 1

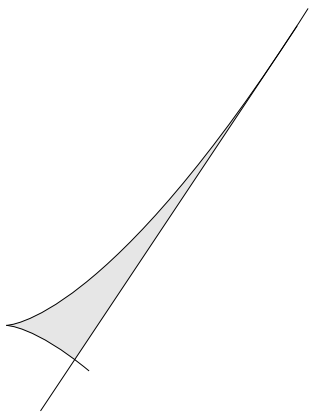

Figure 8: Cuspidal cubic with tangent line : $(y^2 - x^3)(3x - 2y - 1) = 0$, one metric, curvature 1

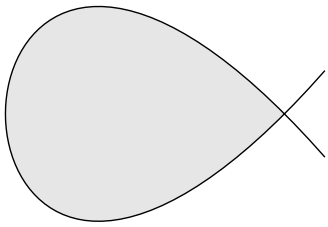

Figure 9: Nodal cubic : $y^2 - x^2(1 - x) = 0$, one metric, non constant curvature

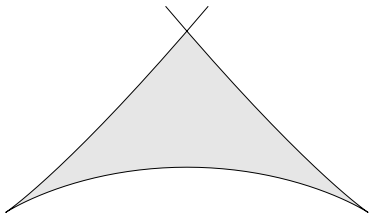

Figure 10: Swallow tail : $4x^2 - 27x^4 + 16y - 128y^2 - 144x^2y + 256y^3 = 0$, one metric, curvature 1

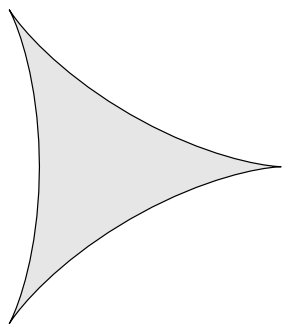

Figure 11: Deltoid : $(x^2+y^2)^2+18(x^2+y^2)-8x^3+24xy^2-27=0$, one metric, curvature 0

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