Ground state energy of the magnetic Laplacian on general three-dimensional corner domains

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GROUND STATE ENERGY OF THE MAGNETIC LAPLACIAN ON GENERAL THREE-DIMENSIONAL CORNER DOMAINS

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Abstract. The asymptotic behavior of the first eigenvalues of magnetic Laplacian operators with large magnetic fields and Neumann realization in smooth three-dimensional domains is characterized by model problems inside the domain or on its boundary. In two-dimensional polygonal domains, a new set of model problems on sectors has to be taken into account. In this paper, we consider the class of general corner domains. In dimension 3, they include as particular cases polyhedra and axisymmetric cones. We attach model problems not only to each point of the closure of the domain, but also to a hierarchy of "tangent substructures" associated with singular chains. We investigate properties of these model problems, namely continuity, semi-continuity, existence of generalized eigenfunctions satisfying exponential decay. We prove estimates for the remainders of our asymptotic formula. Lower bounds are obtained with the help of a classical IMS partition based on adequate two-scale coverings of the corner domain, whereas upper bounds are established by a novel construction of quasimodes, qualified as sitting or sliding according to spectral properties of local model problems. A part of our analysis extends to any dimension.

In a further version of the present work, we will generalize to any corner domain the improvement of upper bounds that we have proved for polyhedral domains in http://hal.archives-ouvertes.fr/hal-00864272

Contents

1. **Introduction. Main results**

The Schrödinger operator with magnetic field (also called magnetic Laplacian) in a n-dimensional space takes the form

$$(-i\nabla + \mathbf{A})^2 = \sum_{j=1}^{n} (-i\partial_{x_j} + A_j)^2,$$

where $\mathbf{A} = (A_1, \ldots, A_n)$ is a given vector field and ∂_{x_j} is the partial derivatives with respect to x_j with $\mathbf{x} = (x_1, \ldots, x_n)$ denoting the Cartesian variable in $\mathbb{R}^n$. The field $\mathbf{A}$ represents the magnetic potential and will be assumed to be regular enough.

When set on a domain Ω of $\mathbb{R}^n$ and completed by natural boundary conditions (Neumann), this operator is denoted by $H(\mathbf{A}, \Omega)$. If Ω is bounded with Lipschitz boundary[1], the form domain of $H(\mathbf{A}, \Omega)$ is the standard Sobolev space $H^1(\Omega)$ and $H(\mathbf{A}, \Omega)$ is self-adjoint, non negative, and with compact resolvent. The ground state of $H(\mathbf{A}, \Omega)$ is the eigenpair (λ, ψ)

$$\begin{cases} (-i\nabla + \mathbf{A})^2\psi = \lambda\psi & \text{in } \Omega, \\ (-i\nabla + \mathbf{A})\psi \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

associated with the lowest eigenvalue λ. Here $\mathbf{n}$ denotes the unit normal vector. If Ω is simply connected, its eigenvalues only depend on the magnetic field defined as follows (see [16, Section 1.1]): If $\omega_{\mathbf{A}}$ denotes the 1-form associated with the vector field $\mathbf{A}$

$$\omega_{\mathbf{A}} = \sum_{j=1}^{n} A_j \, \mathrm{d}x_j, \tag{1.2}$$

the corresponding 2-form $\sigma_{\mathbf{B}}$

$$\sigma_{\mathbf{B}} = \mathrm{d}\omega_{\mathbf{A}} = \sum_{j<k} B_{jk} \, \mathrm{d}x_j \wedge \mathrm{d}x_k. \tag{1.3}$$

is called the magnetic field. In dimension $n = 2$ or $n = 3$, $\sigma_{\mathbf{B}}$ can be identified in a standard way with

$$\mathbf{B} = \operatorname{curl} \mathbf{A}. \tag{1.4}$$

The eigenvectors corresponding to two different instances of $\mathbf{A}$ for the same $\mathbf{B}$ are deduced from each other by a *gauge transform*.

Introducing a (small) parameter $h > 0$ and setting

$$H_h(\mathbf{A}, \Omega) = (-ih\nabla + \mathbf{A})^2 \quad \text{with Neumann b.c. on } \partial\Omega,$$

we get the relation

$$H_h(\mathbf{A}, \Omega) = h^2 H\Big(\frac{\mathbf{A}}{h}, \Omega\Big) \tag{1.5}$$

[1] Or more generally if Ω is a finite union of bounded Lipschitz domains, cf. [28, Chapter 1] for instance.

linking the problem with large magnetic field to the semiclassical limit $h \to 0$ for the Schrödinger operator with magnetic potential. We denote by $\lambda_h = \lambda_h(\mathbf{B}, \Omega)$ the smallest eigenvalue of $H_h(\mathbf{A}, \Omega)$ and by ψ_h an associated eigenvector, so that

$$\begin{cases} (-ih\nabla + \mathbf{A})^2\psi_h = \lambda_h\psi_h & \text{in } \Omega\,, \\ (-ih\nabla + \mathbf{A})\psi_h \cdot \mathbf{n} = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1.6}$$

The behavior of $\lambda_h(\mathbf{B}, \Omega)$ as $h \to 0$ clearly provide equivalent information about the lowest eigenvalue of $H(\breve{\mathbf{A}}, \Omega)$ when $\breve{\mathbf{B}}$ is large, especially in the parametric case when $\breve{\mathbf{B}} = B\mathbf{B}$ where the real number B tends to $+\infty$ and $\mathbf{B}$ is a chosen reference magnetic field.

From now on, we consider that $\mathbf{B}$ is fixed. We assume that it is smooth enough and, unless otherwise mentioned, does not vanish[2] on $\overline{\Omega}$. The question of the semiclassical behavior of $\lambda_h(\mathbf{B}, \Omega)$ has been considered in many papers for a variety of domains, with constant or variable magnetic fields: Smooth domains [26, 20, 15, 2, 38] and polygons [4, 5, 6] in dimension $n = 2$, and mainly smooth domains [27, 21, 22, 39, 16] in dimension $n = 3$. Until now, three-dimensional non-smooth domains are only addressed in two particular configurations—rectangular cuboids [32] and lenses [34, 37], with special orientation of the (constant) magnetic field. We give more detail about the state of the art in Section 2.

1.1. **Asymptotic formulas with remainders.**

Let us briefly describe our main results in the three-dimensional setting.

Each point $\mathbf{x}$ in the closure of a 3D corner domain Ω is associated with a dilation invariant, tangent open set $\Pi_{\mathbf{x}}$, according to the following cases:

(1) If $\mathbf{x}$ is an interior point, $\Pi_{\mathbf{x}} = \mathbb{R}^3$,
(2) If $\mathbf{x}$ belongs to a *face* $\mathbf{f}$ (i.e., a connected component of the smooth part of $\partial\Omega$), $\Pi_{\mathbf{x}}$ is a half-space,
(3) If $\mathbf{x}$ belongs to an *edge* $\mathbf{e}$, $\Pi_{\mathbf{x}}$ is an infinite wedge,
(4) If $\mathbf{x}$ is a *vertex* $\mathbf{v}$, $\Pi_{\mathbf{x}}$ is an infinite cone.

Let $\mathbf{B}_{\mathbf{x}}$ be the magnetic field frozen at $\mathbf{x}$. Let $E(\mathbf{B}_{\mathbf{x}}, \Pi_{\mathbf{x}})$ be the bottom of the spectrum (ground state energy) of the tangent operator $H(\mathbf{A}_{\mathbf{x}}, \Pi_{\mathbf{x}})$ where $\mathbf{A}_{\mathbf{x}}$ is the linear approximation of $\mathbf{A}$ at $\mathbf{x}$, so that

$$\operatorname{curl} \mathbf{A}_{\mathbf{x}} = \mathbf{B}_{\mathbf{x}}\,.$$

We introduce the quantity *(lowest local energy)*

$$\mathscr{E}(\mathbf{B}, \Omega) := \inf_{\mathbf{x}\in\overline{\Omega}} E(\mathbf{B}_{\mathbf{x}}, \Pi_{\mathbf{x}}). \tag{1.7}$$

In this paper, we prove that this quantity provides the value of the limit of λ_h/h as $h \to 0$ with some control of the convergence rate as $h \to 0$, namely

$$-\,Ch^{11/10} \le \lambda_h(\mathbf{B}, \Omega) - h\mathscr{E}(\mathbf{B}, \Omega) \le Ch^{11/10}, \tag{1.8}$$

[2] Should $\mathbf{B}$ cancel, the situation would be very different, leading to another type of asymptotics [19, 14].

where the constant C is bounded by the norm of $\mathbf{A}$ in $W^{2,\infty}(\Omega)$, as proved in Theorems 5.1 and 9.1. With the point of view of large magnetic fields in the parametric case $\breve{\mathbf{B}} = B\mathbf{B}$, (1.8) yields obviously

$$-CB^{9/10} \leq \lambda(\breve{\mathbf{B}}, \Omega) - B\mathscr{E}(\mathbf{B}, \Omega) \leq CB^{9/10}, \quad \text{as } B \to +\infty. \tag{1.9}$$

Note that $B\mathscr{E}(\mathbf{B}, \Omega) = \mathscr{E}(\breve{\mathbf{B}}, \Omega)$ by homogeneity (see Lemma A.5).

If Ω is a polyhedral domain, we prove an improvement of remainders

$$-Ch^{5/4} \leq \lambda_h(\mathbf{B}, \Omega) - h\mathscr{E}(\mathbf{B}, \Omega) \leq Ch^{5/4}. \tag{1.10}$$

Finally, if $\mathbf{B}$ cancels somewhere in $\overline{\Omega}$, we obtain the upper boud in any corner domain Ω

$$\lambda_h(\mathbf{B}, \Omega) \leq Ch^{4/3}, \tag{1.11}$$

which, in view of [19, 14], is optimal.

All these results are new in this generality. Till now, there is no systematic analysis for corner domains and general results concern smooth domains. The lower bound in the polyhedral case coincides with the one obtained in the smooth case in dimensions 2 and 3 when no further assumptions are done. In the literature, improvements of the convergence rates are possible in certain particular cases when one knows more on $\mathscr{E}(\mathbf{B}, \Omega)$, namely whether the infimum is attained in some special points.

Our result does not need such extra assumptions, but our proofs have to distinguish cases whether the local ground energies $E(\mathbf{B}_{\mathbf{x}}, \Pi_{\mathbf{x}})$ are attained or not, and we have to understand the behavior of the function $\mathbf{x} \mapsto E(\mathbf{B}_{\mathbf{x}}, \Pi_{\mathbf{x}})$ when $\mathbf{x}$ spans the different regions of $\overline{\Omega}$. We have proved very general continuity and semi-continuity properties as described now.

Let $\mathfrak{F}$ be the set of faces $\mathbf{f}$, $\mathfrak{E}$ the set of edges $\mathbf{e}$ and $\mathfrak{V}$ the set of vertices of Ω. They form a partition of the closure of Ω, called stratification

$$\overline{\Omega} = \Omega \cup \Big(\bigcup_{\mathbf{f} \in \mathfrak{F}} \mathbf{f} \Big) \cup \Big(\bigcup_{\mathbf{e} \in \mathfrak{E}} \mathbf{e} \Big) \cup \Big(\bigcup_{\mathbf{v} \in \mathfrak{V}} \mathbf{v} \Big). \tag{1.12}$$

The sets Ω, $\mathbf{f}$, $\mathbf{e}$ and $\mathbf{v}$ are open sets called the strata of $\overline{\Omega}$, compare with [29] and [31, Ch. 9]. We denote them by $\mathbf{t}$ and their set by $\mathfrak{T}$. We will show the following facts

a) For each stratum $\mathbf{t} \in \mathfrak{T}$, the function $\mathbf{x} \mapsto E(\mathbf{B}_{\mathbf{x}}, \Pi_{\mathbf{x}})$ is continuous on $\mathbf{t}$.

b) The function $\mathbf{x} \mapsto E(\mathbf{B}_{\mathbf{x}}, \Pi_{\mathbf{x}})$ is lower semi-continuous on $\overline{\Omega}$.

As a consequence, the infimum determining the limit $\mathscr{E}(\mathbf{B}, \Omega)$ in (1.7) is a minimum

$$\mathscr{E}(\mathbf{B}, \Omega) = \min_{\mathbf{x} \in \overline{\Omega}} E(\mathbf{B}_{\mathbf{x}}, \Pi_{\mathbf{x}}) \,. \tag{1.13}$$

From this we can deduce that $\mathscr{E}(\mathbf{B}, \Omega) > 0$ as soon as $\mathbf{B}$ is positive and continuous on $\overline{\Omega}$.

1.2. **Contents of the paper.** In Section 2 we place our results in the framework of existing literature for dimensions 2 and 3. In Section 3 we introduce the class of corner domains defined recursively on the space dimension $n \geq 1$, alongside with their tangent cones and singular chains $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1, \ldots)$. We particularize these notions in the case of three-dimensional domains and prove weighted estimates for local maps and their derivatives. In Section 4, we introduce the tangent operators for magnetic Laplacians and establish weighted estimates of the linearization error. In Section 5 we prove the lower bound $h\mathscr{E}(\mathbf{B}, \Omega) - Ch^{11/10} \leq \lambda_h(\mathbf{B}, \Omega)$ by an IMS formula based on a two-scale partition of unity. In polyhedra, a one-scale standard partition can be used, which yields the improved lower bound $h\mathscr{E}(\mathbf{B}, \Omega) - Ch^{5/4} \leq \lambda_h(\mathbf{B}, \Omega)$. In Section 6 we classify magnetic model problems on three-dimensional tangent cones (taxonomy) and we characterize their essential spectrum. We show in Section 7 that to each point $\mathbf{x}_0 \in \overline{\Omega}$ is associated a singular chain $\mathbb{X}$ originating at $\mathbf{x}_0$ for which the tangent operator $H(\mathbf{A}_{\mathbb{X}}, \Pi_{\mathbb{X}})$ possesses an *admissible generalized eigenvector* (A.G.E.) with energy $E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0})$ and exponential decay properties in some directions. Section 8 is devoted to the properties of the local ground energy $E(\mathbf{B}_{\mathbf{x}}, \Pi_{\mathbf{x}})$. We prove the semi-continuity, the stability under perturbation and the positivity. In Section 9 we prove the upper bounds $\lambda_h(\mathbf{B}, \Omega) \leq h\mathscr{E}(\mathbf{B}, \Omega) + Ch^{\kappa}$, with $\kappa = 11/10$ or $\kappa = 5/4$ depending on whether Ω is a corner domain or a polyhedral domain, by a construction of quasimodes based on admissible generalized eigenvectors for tangent problems. Our construction critically depends on the length ν of the singular chain $\mathbb{X}$ that provides the A.G.E. When $\nu = 1$, we are in the classical situation: It suffices to concentrate the support of the quasimode around $\mathbf{x}_0$, and we qualify it as *sitting*. When $\nu = 2$, the chain has the form $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1)$: Our quasimode is decentered in the direction provided by $\mathbf{x}_1$, has a two-scale structure in general, and we qualify it as *sliding*. When $\nu = 3$, the chain has the form $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2)$ and our quasimode is *doubly sliding*. In dimension $n = 3$, considering chains of length $\nu \leq 3$ is sufficient to conclude.

In Theorems 5.1 and 9.1, the estimates depend on the magnetic potential $\mathbf{A}$. It is possible to obtain estimates depending on the magnetic field $\mathbf{B}$ and not on the magnetic potential as soon as Ω is simply connected. For this, we consider $\mathbf{B}$ as a datum and associate a potential $\mathbf{A}$ with it. Operators $\mathscr{A} : \mathbf{B} \mapsto \mathbf{A}$ lifting the curl (i.e. such that $\operatorname{curl} \circ \mathscr{A} = \mathbb{I}$) and satisfying suitable estimates do exist in the literature. We quote [10] in which it is proved that such lifting can be constructed as a pseudo-differential operator of order -1. As a consequence $\mathscr{A}$ is continuous between Hölder classes of non integer order:

$$\forall \alpha \in (0,1), \quad \exists C_\alpha > 0, \quad \|\mathscr{A}\mathbf{B}\|_{W^{2+\alpha,\infty}(\Omega)} \leq C_\alpha \|\mathbf{B}\|_{W^{1+\alpha,\infty}(\Omega)} .$$

Choosing $\mathbf{A} = \mathscr{A}\mathbf{B}$ in Theorems 5.1 and 9.1, we deduce the following result using only a Hölder norm of the magnetic field:

Proposition 1.1. *Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ be a simply connected corner domain, $\mathbf{B} \in W^{1+\alpha,\infty}(\overline{\Omega})$ be a non-vanishing Hölder continuous magnetic field of order $1 + \alpha$ with some $\alpha \in (0, 1)$.*

(1) Then there exist $C(\Omega) > 0$ and $h_0 > 0$ such that

$$\forall h \in (0, h_0), \quad |\lambda_h(\mathbf{B}, \Omega) - h\mathscr{E}(\mathbf{B}, \Omega)| \leq C(\Omega)(1 + \|\mathbf{B}\|^2_{W^{1+\alpha,\infty}(\Omega)})\, h^{11/10}\,. \tag{1.14}$$

(2) If Ω is polyhedral, the estimate is improved

$$\forall h \in (0, h_0), \quad |\lambda_h(\mathbf{B}, \Omega) - h\mathscr{E}(\mathbf{B}, \Omega)| \leq C(\Omega)(1 + \|\mathbf{B}\|^2_{W^{1+\alpha,\infty}(\Omega)})\, h^{5/4}\,. \tag{1.15}$$

If Ω is not simply connected, the first eigenvalue of the operator $H(\mathbf{A}, \Omega)$ will depend on $\mathbf{A}$, and not only on $\mathbf{B}$. A manifestation of this is the Aharonov Bohm effect, see [17] for instance. Our results (1.8), (1.10), (1.11) still hold for the first eigenvalue $\lambda_h = \lambda_h(\mathbf{A}, \Omega)$ of $H_h(\mathbf{A}, \Omega)$. Note that, in contrast, the ground state energies of tangent operators $H(\mathbf{A}_\mathbf{x}, \Pi_\mathbf{x})$ only depend on the (constant) magnetic field $\mathbf{B}_\mathbf{x}$ because the potential $\mathbf{A}_\mathbf{x}$ is linear by definition. Therefore the lowest local energy only depends on the magnetic field and can still be denoted by $\mathscr{E}(\mathbf{B}, \Omega)$ even in the non simply connected case.

1.3. **Notations.**

We denote by $\langle \cdot, \cdot \rangle_\mathcal{O}$ the L^2 Hilbert product on the open set $\mathcal{O}$ of $\mathbb{R}^n$

$$\big\langle f, g \big\rangle_\mathcal{O} = \int_\mathcal{O} f(\mathbf{x})\, \overline{g}(\mathbf{x})\, \mathrm{d}\mathbf{x}.$$

When there is no confusion, we simply write $\langle f, g \rangle$ and $\|f\| = \langle f, f \rangle^{1/2}$.

For a generic (unbounded) self-adjoint operator L we denote by $\mathrm{Dom}(L)$ its domain and $\mathfrak{S}(L)$ its spectrum. Likewise the domain of a quadratic form q is denoted by $\mathrm{Dom}(q)$.

Domains as open simply connected subsets of $\mathbb{R}^n$ are in general denoted by $\mathcal{O}$ if they are generic, Π if they are invariant by dilatation (cones) and Ω if they are bounded.

In this paper, the quadratic forms of interest are those associated with magnetic Laplacians, namely, for a positive constant h, a smooth magnetic potential $\mathbf{A}$, and a generic domain $\mathcal{O}$

$$q_h[\mathbf{A}, \mathcal{O}](f) := \big\langle (-ih\nabla + \mathbf{A})f, (-ih\nabla + \mathbf{A})f \big\rangle_\mathcal{O} = \int_\mathcal{O} (-ih\nabla + \mathbf{A})f \cdot \overline{(-ih\nabla + \mathbf{A})f}\, \mathrm{d}\mathbf{x}, \tag{1.16}$$

and its domain

$$\mathrm{Dom}(q_h[\mathbf{A}, \mathcal{O}]) = \{f \in L^2(\mathcal{O}),\ (-ih\nabla + \mathbf{A})f \in L^2(\mathcal{O})\}\,. \tag{1.17}$$

For a bounded domain Ω, $\mathrm{Dom}(q_h[\mathbf{A}, \Omega])$ coincides with $H^1(\Omega)$. For $h = 1$, we omit the index h, denoting the quadratic form by $q[\mathbf{A}, \mathcal{O}]$. In the same way we introduce the following notation for Rayleigh quotients

$$\mathscr{Q}_h[\mathbf{A}, \mathcal{O}](f) = \frac{q_h[\mathbf{A}, \mathcal{O}](f)}{\langle f, f \rangle_\mathcal{O}}, \quad f \in \mathrm{Dom}(q_h[\mathbf{A}, \mathcal{O}]),\ f \neq 0, \tag{1.18}$$

and recall that, by the min-max principle

$$\lambda_h(\mathbf{B}, \Omega) = \min_{\substack{f \in \mathrm{Dom}(q_h[\mathbf{A}, \Omega]) \\ f \neq 0}} \mathscr{Q}_h[\mathbf{A}, \Omega](f)\,. \tag{1.19}$$

In relation with changes of variables, we will also use the more general form with metric:

$$q_h[\mathbf{A},\mathcal{O},\mathrm{G}](f) = \int_{\mathcal{O}} (-ih\nabla+\mathbf{A})f \cdot \mathrm{G}\big(\overline{(-ih\nabla+\mathbf{A})f}\big)\ |\mathrm{G}|^{-1/2}\,\mathrm{d}\mathbf{x}, \tag{1.20}$$

where G is a smooth function with values in 3×3 positive symmetric matrices and $|\mathrm{G}| = \det \mathrm{G}$. Its domain is (see [22, §5] for more details)

$$\mathrm{Dom}(q_h[\mathbf{A},\mathcal{O},\mathrm{G}]) = \{f \in L^2_{\mathrm{G}}(\mathcal{O}),\ \mathrm{G}^{1/2}(-ih\nabla+\mathbf{A})f \in L^2_{\mathrm{G}}(\mathcal{O})\}\,,$$

where $L^2_{\mathrm{G}}(\mathcal{O})$ is the space of the square-integrable functions for the weight $|\mathrm{G}|^{-1/2}$ and $\mathrm{G}^{1/2}$ is the square root of the matrix G. The corresponding Raleigh quotient is denoted by $\mathscr{Q}_h[\mathbf{A},\mathcal{O},\mathrm{G}]$.

The domain of the magnetic Laplacian with Neumann boundary conditions on the set $\mathcal{O}$ is

$$\begin{aligned}\mathrm{Dom}\,(H_h(\mathbf{A},\mathcal{O})) = \big\{&f \in \mathrm{Dom}(q_h[\mathbf{A},\mathcal{O}]),\\ &(-ih\nabla+\mathbf{A})^2 f \in L^2(\mathcal{O}) \ \text{ and } \ (-ih\nabla+\mathbf{A})f\cdot\mathbf{n} = 0 \text{ on } \partial\mathcal{O}\big\}\,.\end{aligned} \tag{1.21}$$

We will also use the space of the functions which are *locally*[3] in the domain of $H_h(\mathbf{A},\mathcal{O})$:

$$\begin{aligned}\mathrm{Dom}_{\,\mathrm{loc}}\,(H_h(\mathbf{A},\mathcal{O})) := \{&f \in H^1_{\mathrm{loc}}(\overline{\mathcal{O}}),\\ &(-ih\nabla+\mathbf{A})^2 f \in H^0_{\mathrm{loc}}(\overline{\mathcal{O}}) \ \text{ and } \ (-ih\nabla+\mathbf{A})f\cdot\mathbf{n} = 0 \text{ on } \partial\mathcal{O}\}.\end{aligned} \tag{1.22}$$

When $h = 1$, we omit the index h in (1.21) and (1.22).

2. **State of the art**

Here we collect some results of the literature about the semiclassical limit for the first eigenvalue of the magnetic Laplacian depending on the geometry of the domain and the variation of the magnetic field. We briefly mention the case when the domain has no boundary, before reviewing in more detail what is known on bounded domains $\Omega \subset \mathbb{R}^n$ with Neumann boundary condition depending on the dimension $n \in \{2, 3\}$. To keep this section relatively short, we only quote results related with our problematics, i.e., the asymptotic behavior of the ground state energy with error estimates from above and from below.

2.1. **Without boundary or with Dirichlet conditions.** Here M is either a compact Riemannian manifold without boundary or $\mathbb{R}^n$, and $H_h(\mathbf{A}, M)$ is the magnetic Laplacian associated with the 1-form $\mathbf{A}$. In this very general framework, the magnetic field is the 2-form $\mathbf{B} = \mathrm{curl}\,\mathbf{A}$. Then for each $\mathbf{x} \in M$ the local energy at $\mathbf{x}$ is the intensity

$$b(\mathbf{x}) := \tfrac{1}{2}\,\mathrm{Tr}([\mathbf{B}^*(\mathbf{x})\cdot\mathbf{B}(\mathbf{x})]^{1/2})$$

[3]Here $H^m_{\mathrm{loc}}(\overline{\mathcal{O}})$ denotes for $m = 0, 1$ the space of functions which are in $H^m(\mathcal{O}\cap\mathcal{B})$ for any ball $\mathcal{B}$.

and $\mathscr{E}(\mathbf{B}, M) = b_0 := \inf_{\mathbf{x}\in M} b(\mathbf{x})$. It is proved by Helffer and Mohamed in [19] that if b_0 is positive and under a condition at infinity if $M = \mathbb{R}^n$, then

$$\exists C > 0, \quad -Ch^{5/4} \leq \lambda_h(\mathbf{B}, M) - h\mathscr{E}(\mathbf{B}, M) \leq Ch^{4/3} \ .$$

More precise results are proved in dimension 2 when b admits a unique positive non-degenerate minimum: A complete asymptotic expansion of the eigenvalues of $H_h(\mathbf{A}, M)$ in powers of $\sqrt{h}$ has been obtained by Helffer and Kordyukov in [18], and improved in by Vũ Ngọc and Raymond in [41] where it is proved that the sole integer powers of h appear in the expansion. These results imply in particular that with these assumptions there holds

$$|\lambda_h(\mathbf{B}, M) - h\mathscr{E}(\mathbf{B}, M)| \leq Ch^2.$$

The case of Dirichlet boundary condition is very close to the case without boundary.

2.2. **Neumann conditions in dimension 2.** In contrast, when Neumann boundary conditions are applied on the boundary, the local energy drops significantly as was established in [42] by Saint-James and de Gennes as early as 1963. In this review of the dimension $n = 2$, we classify the domains in two categories: those with a regular boundary and those with a polygonal boundary.

2.2.1. *Regular domains.* Let $\Omega \subset \mathbb{R}^2$ be a regular domain and B be a regular non-vanishing scalar magnetic field on $\overline{\Omega}$. To each $\mathbf{x} \in \overline{\Omega}$ is associated a tangent problem. According to whether $\mathbf{x}$ is an interior point or a boundary point, the tangent problem is the magnetic Laplacian on the plane $\mathbb{R}^2$ or the half-plane $\Pi_\mathbf{x}$ tangent to Ω at $\mathbf{x}$, with the constant magnetic field $B_\mathbf{x} \equiv B(\mathbf{x})$. The associated spectral quantities $E(B_\mathbf{x}, \mathbb{R}^2)$ and $E(B_\mathbf{x}, \Pi_\mathbf{x})$ are respectively equal to $|B_\mathbf{x}|$ and $|B_\mathbf{x}|\Theta_0$ where $\Theta_0 := E(1, \mathbb{R}^2_+)$ is a universal constant whose value is close to 0.59 (see [42]). With the quantities

$$b = \inf_{\mathbf{x}\in\overline{\Omega}} |B(\mathbf{x})| \quad \text{and} \quad b' = \inf_{\mathbf{x}\in\partial\Omega} |B(\mathbf{x})|, \tag{2.1}$$

we find

$$\mathscr{E}(B, \Omega) = \min(b, b'\Theta_0) \ .$$

In this generality, the asymptotic limit

$$\lim_{h\to 0} \frac{\lambda_h(B, \Omega)}{h} = \mathscr{E}(B, \Omega) \tag{2.2}$$

is proved by Lu and Pan in [26]. Improvements of this result depend on the geometry and the variation of the magnetic field as we describe now.

• *Constant magnetic field.* If the magnetic field is constant and normalized to 1, then $\mathscr{E}(B, \Omega) = \Theta_0$. The following estimate is proved by Helffer and Morame:

$$\exists C > 0, \quad -Ch^{3/2} \leq \lambda_h(1, \Omega) - h\Theta_0 \leq Ch^{3/2} \ ,$$

for h small enough [20, §10], while the upper bound was already given by Bernoff and Sternberg [3]. This result is improved in [20, §11] in which a two-term asymptotics is proved, showing that a remainder in $O(h^{3/2})$ is optimal. Under the additional assumption that the curvature of the boundary admits a unique and non-degenerate maximum, a complete expansion of $\lambda_h(1,\Omega)$ is provided by Fournais and Helffer [15].

• *Variable magnetic field.* Here we recall results from [20, §9] for variable magnetic fields (we use the notation (2.1))

$$\text{(2.3a)}\qquad \text{If } b < \Theta_0 b', \quad \exists C > 0, \quad |\lambda_h(B,\Omega) - hb| \le Ch^2,$$

$$\text{(2.3b)}\qquad \text{If } b > \Theta_0 b', \quad \exists C > 0, \quad -Ch^{5/4} \le \lambda_h(B,\Omega) - h\Theta_0 b' \le Ch^{3/2},$$

$$\text{(2.3c)}\qquad \text{If } b = \Theta_0 b', \quad \exists C > 0, \quad -Ch^{5/4} \le \lambda_h(B,\Omega) - hb \le Ch^2.$$

Under non-degeneracy hypotheses, the optimality of the interior estimates (2.3a) is a consequence of [18], whereas the eigenvalue asymptotics provided in [38, 40] yield that the upper bound in (2.3b) is sharp.

2.2.2. *Polygonal domains.* Let Ω be a curvilinear polygon and let $\mathfrak{V}$ be the (finite) set of its vertices. In this case, new model operators appear on infinite sectors $\Pi_{\mathbf{x}}$ tangent to Ω at vertices $\mathbf{x} \in \mathfrak{V}$. By homogeneity $E(B_{\mathbf{x}}, \Pi_{\mathbf{x}}) = |B(\mathbf{x})| E(1, \Pi_{\mathbf{x}})$ and by rotation invariance, $E(1, \Pi_{\mathbf{x}})$ only depends on the opening $\alpha(\mathbf{x})$ of the sector $\Pi_{\mathbf{x}}$. Let $\mathcal{S}_\alpha$ be a model sector of opening $\alpha \in (0, 2\pi)$. Then

$$\mathscr{E}(B,\Omega) = \min\left(b, b'\Theta_0, \min_{\mathbf{x}\in\mathfrak{V}} |B(\mathbf{x})|\, E(1, \mathcal{S}_{\alpha(\mathbf{x})})\right) .$$

In [4, §11], it is proved that

$$\exists C > 0, \quad -Ch^{5/4} \le \lambda_h(B,\Omega) - h\mathscr{E}(B,\Omega) \le Ch^{9/8}.$$

This estimate can be improved under the assumption that

$$\text{(2.4)}\qquad \mathscr{E}(B,\Omega) < \min(b, b'\Theta_0),$$

which means that at least one of the corners makes the energy lower than in the regular case: The asymptotic expansions provided in [5] then yield the sharp estimates

$$\exists C > 0, \quad |\lambda_h(B,\Omega) - h\mathscr{E}(B,\Omega)| \le Ch^{3/2} .$$

From [23, 4] follows that for all $\alpha \in (0, \frac{\pi}{2}]$ there holds

$$\text{(2.5)}\qquad E(1, \mathcal{S}_\alpha) < \Theta_0.$$

Therefore condition (2.4) holds for constant magnetic fields as soon as there is an angle opening $\alpha_{\mathbf{x}} \le \frac{\pi}{2}$. Finite element computations by Galerkin projection as presented in [6] suggest that (2.5) still holds for all $\alpha \in (0, \pi)$. Let us finally mention that if Ω has straight sides and B is constant, the convergence of $\lambda_h(B,\Omega)$ to $h\mathscr{E}(B,\Omega)$ is exponential: Their difference is bounded by $C\exp(-\beta h^{-1/2})$ for suitable positive constants C and β (see [5]).

2.3. **Neumann conditions in dimension 3.** In dimension $n = 3$ we still distinguish the regular and singular domains.

2.3.1. *Regular domains.* Here $\Omega \subset \mathbb{R}^3$ is assumed to be regular. For a continuous magnetic field $\mathbf{B}$ it is known ([27] and [21]) that (2.2) holds. In that case

$$\mathscr{E}(\mathbf{B},\Omega) = \min\big(\inf_{\mathbf{x}\in\Omega}|\mathbf{B}(\mathbf{x})|, \inf_{\mathbf{x}\in\partial\Omega}|\mathbf{B}(\mathbf{x})|\sigma(\theta(\mathbf{x}))\big),$$

where $\theta(\mathbf{x}) \in [0, \frac{\pi}{2}]$ denotes the unoriented angle between the magnetic field and the boundary at the point $\mathbf{x} \in \partial\Omega$, and the quantity $\sigma(\theta)$ is the bottom of the spectrum of a model problem, see Section 6. Let us simply mention that σ is increasing on $[0, \frac{\pi}{2}]$ and that $\sigma(0) = \Theta_0$, $\sigma(\pi/2) = 1$.

• *Constant magnetic field.* Here the magnetic field $\mathbf{B}$ is assumed to be constant and unitary. There exists a non-empty set Σ of $\partial\Omega$ on which $\mathbf{B}(\mathbf{x})$ is tangent to the boundary. In that case we have

$$\mathscr{E}(\mathbf{B},\Omega) = \Theta_0\ .$$

Theorem 1.1 of [22] states that

$$\exists C > 0, \quad |\lambda_h(\mathbf{B},\Omega) - h\Theta_0| \leq Ch^{4/3},$$

for h small enough. Under some extra assumptions on Σ, Theorem 1.2 of [22] yields a two-term asymptotics for $\lambda_h(\mathbf{B},\Omega)$ showing the optimality of the previous estimate.

• *Variable magnetic field.* Let $\mathbf{B}$ be a smooth non-vanishing magnetic field. There holds [16, Theorem 9.1.1]

$$\exists C > 0, \quad -Ch^{5/4} \leq \lambda_h(\mathbf{B},\Omega) - h\mathscr{E}(\mathbf{B},\Omega) \leq Ch^{5/4}\ .$$

The proof of this result was already sketched in [27]. In [22, Remark 6.2], the upper bound is improved to $O(h^{4/3})$.

Under the following two extra assumptions

a) The inequality $\inf_{\mathbf{x}\in\partial\Omega}|\mathbf{B}(\mathbf{x})|\,\sigma(\theta(\mathbf{x})) < \inf_{\mathbf{x}\in\Omega}|\mathbf{B}(\mathbf{x})|$ holds,

b) The function $\mathbf{x} \mapsto |\mathbf{B}(\mathbf{x})|\sigma(\theta(\mathbf{x}))$ reaches its minimum at a point $\mathbf{x}_0$ where $\mathbf{B}$ is neither normal nor tangent to the boundary,

a three-term quasimode is constructed in [39], providing the sharp upper bound:

$$\exists C > 0, \quad \lambda_h(\mathbf{B},\Omega) - h\mathscr{E}(\mathbf{B},\Omega) \leq Ch^{3/2}\ .$$

2.3.2. *Singular domains.* Until now, two examples of non-smooth domains have been addressed in the literature. In both cases, the magnetic field $\mathbf{B}$ is assumed to be constant.

• *Rectangular cuboids.* The case where Ω is a rectangular cuboid (i.e., the product of three bounded intervals) is considered by Pan [32]: The asymptotic limit (2.2) holds for such a domain and there exists a vertex $\mathbf{v} \in \mathfrak{V}$ such that $\mathscr{E}(\mathbf{B},\Omega) = E(\mathbf{B},\Pi_{\mathbf{v}})$. Moreover, in the case where the magnetic field is tangent to a face but is not tangent to any edge, there holds

$$E(\mathbf{B},\Pi_{\mathbf{v}}) < \inf_{\mathbf{x}\in\overline{\Omega}\setminus\mathfrak{V}} E(\mathbf{B},\Pi_{\mathbf{x}})$$

and eigenfunctions associated to $\lambda_h(\mathbf{B},\Omega)$ concentrate near corners as $h\to 0$.

• *Lenses.* The case where Ω has the shape of a lens is treated in [34] and [37]. The domain Ω is supposed to have two faces separated by an edge $\mathbf{e}$ that is a regular loop contained in the plane $x_3=0$. The magnetic field considered is $\mathbf{B}=(0,0,1)$.

It is proved in [34] that, if the opening angle α of the lens is constant and $\leq 0.38\pi$,

$$\inf_{\mathbf{x}\in\mathbf{e}} E(\mathbf{B},\Pi_{\mathbf{x}}) < \inf_{\mathbf{x}\in\overline{\Omega}\setminus\mathbf{e}} E(\mathbf{B},\Pi_{\mathbf{x}})$$

and that the asymptotic limit (2.2) holds with the following estimate:

$$\exists C>0,\quad |\lambda_h(\mathbf{B},\Omega)-h\mathscr{E}(\mathbf{B},\Omega)|\leq Ch^{5/4}\,.$$

When the opening angle of the lens is variable and under some non-degeneracy hypotheses, a complete eigenvalue asymptotics is obtained in [37] resulting into the optimal estimate

$$\exists C>0,\quad |\lambda_h(\mathbf{B},\Omega)-h\mathscr{E}(\mathbf{B},\Omega)|\leq Ch^{3/2}\,.$$

3. Domains with corners and their singular chains

For the sake of completeness and for ease of further discussion, in the same spirit as in [12, Section 2], we introduce here a recursive definition of two intertwining classes of domains

a) $\mathfrak{P}_n$, a class of infinite open cones in $\mathbb{R}^n$.

b) $\mathfrak{D}(M)$, a class of bounded connected open subsets of a smooth manifold without boundary—actually, $M=\mathbb{R}^n$ or $M=\mathbb{S}^n$, with $\mathbb{S}^n$ the unit sphere of $\mathbb{R}^{n+1}$,

3.1. Tangent cones and corner domains.

We call a *cone* any open subset Π of $\mathbb{R}^n$ satisfying

$$\forall\rho>0 \;\text{ and }\; \mathbf{x}\in\Pi,\quad \rho\mathbf{x}\in\Pi,$$

and the *section* of the cone Π is its subset $\Pi\cap\mathbb{S}^{n-1}$. Note that $\mathbb{S}^0=\{-1,1\}$.

Definition 3.1 (Tangent cone)**.** Let Ω be an open subset of $M=\mathbb{R}^n$ or $M=\mathbb{S}^n$. Let $\mathbf{x}_0\in\overline{\Omega}$. The cone $\Pi_{\mathbf{x}_0}$ is said to be *tangent to* Ω at $\mathbf{x}_0$ if there exists a local $\mathscr{C}^\infty$ diffeomorphism $\mathrm{U}^{\mathbf{x}_0}$ which maps a neighborhood $\mathcal{U}_{\mathbf{x}_0}$ of $\mathbf{x}_0$ in M onto a neighborhood $\mathcal{V}_{\mathbf{x}_0}$ of $\mathbf{0}$ in $\mathbb{R}^n$ and such that

$$\mathrm{U}^{\mathbf{x}_0}(\mathbf{x}_0)=\mathbf{0},\quad \mathrm{U}^{\mathbf{x}_0}(\mathcal{U}_{\mathbf{x}_0}\cap\Omega)=\mathcal{V}_{\mathbf{x}_0}\cap\Pi_{\mathbf{x}_0}\quad\text{and}\quad \mathrm{U}^{\mathbf{x}_0}(\mathcal{U}_{\mathbf{x}_0}\cap\partial\Omega)=\mathcal{V}_{\mathbf{x}_0}\cap\partial\Pi_{\mathbf{x}_0}. \tag{3.1}$$

We denote by $\mathrm{J}^{\mathbf{x}_0}$ the Jacobian of the inverse of $\mathrm{U}^{\mathbf{x}_0}$, that is

$$\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}):=\mathrm{d}_{\mathbf{v}}(\mathrm{U}^{\mathbf{x}_0})^{-1}(\mathbf{v}),\qquad \forall\mathbf{v}\in\mathcal{V}_{\mathbf{x}_0}\,. \tag{3.2}$$

We assume without restriction that the Jacobian at $\mathbf{0}$ is the identity matrix: $\mathrm{J}^{\mathbf{x}_0}(\mathbf{0})=\mathbb{I}_n$. The open set $\mathcal{U}_{\mathbf{x}_0}$ is called a *map-neighborhood* and $(\mathcal{U}_{\mathbf{x}_0},\mathrm{U}^{\mathbf{x}_0})$ a *local map*.

The metric associated with the local map $(\mathcal{U}_{\mathbf{x}_0},\mathrm{U}^{\mathbf{x}_0})$ is denoted by $\mathrm{G}^{\mathbf{x}_0}$ and defined as

$$\mathrm{G}^{\mathbf{x}_0}=(\mathrm{J}^{\mathbf{x}_0})^{-1}((\mathrm{J}^{\mathbf{x}_0})^{-1})^\top. \tag{3.3}$$

The metric $\mathsf{G}^{\mathbf{x}_0}$ at $\mathbf{0}$ is the identity matrix. ■

The tangent cone $\Pi_{\mathbf{x}_0}$ does not depend on the choice of the map-neighborhood $\mathcal{U}_{\mathbf{x}_0}$ or the local map $(\mathcal{U}_{\mathbf{x}_0}, \mathsf{U}^{\mathbf{x}_0})$ because of the constraint $\mathsf{J}^{\mathbf{x}_0}(\mathbf{0}) = \mathbb{I}_n$. Therefore when there exists a tangent cone to Ω at $\mathbf{x}_0$, it is unique.

Definition 3.2 (Class of corner domains)**.** The classes of corner domains $\mathfrak{D}(M)$ ($M = \mathbb{R}^n$ or $M = \mathbb{S}^n$) and tangent cones $\mathfrak{P}_n$ are defined as follow:

Initialization: $\mathfrak{P}_0$ has one element, $\{0\}$. $\mathfrak{D}(\mathbb{S}^0)$ is formed by all subsets of $\mathbb{S}^0$.

Recurrence: For $n \geq 1$,

(1) $\Pi \in \mathfrak{P}_n$ if and only if the section of Π belongs to $\mathfrak{D}(\mathbb{S}^{n-1})$,
(2) $\Omega \in \mathfrak{D}(M)$ if and only if for any $\mathbf{x}_0 \in \overline{\Omega}$, there exists a tangent cone $\Pi_{\mathbf{x}_0} \in \mathfrak{P}_n$ to Ω at $\mathbf{x}_0$. ■

Polyhedral domains and polyhedral cones form important subclasses of $\mathfrak{D}(M)$ and $\mathfrak{P}_n$.

Definition 3.3 (Class of polyhedral cones and domains)**.** The classes of polyhedral domains $\overline{\mathfrak{D}}(M)$ ($M = \mathbb{R}^n$ or $M = \mathbb{S}^n$) and polyhedral cones $\overline{\mathfrak{P}}_n$ are defined as follow:

(1) The cone $\Pi \in \mathfrak{P}_n$ is a polyhedral cone if its boundary is contained in a finite union of subspaces of codimension 1. We write $\Pi \in \overline{\mathfrak{P}}_n$.
(2) The domain $\Omega \in \mathfrak{D}(M)$ is a polyhedral domain if all its tangent cones $\Pi_{\mathbf{x}}$ are polyhedral. We write $\Omega \in \overline{\mathfrak{D}}(M)$. ■

Here is a rapid description of corner domains in lower dimensions.

Example 3.4. In dimensions $n = 1, 2, 3$ we have:

- The elements of $\mathfrak{P}_1$ are $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_-$.
- The elements of $\mathfrak{D}(\mathbb{S}^1)$ are $\mathbb{S}^1$ and all open intervals $\mathcal{I} \subset \mathbb{S}^1$ such that $\overline{\mathcal{I}} \neq \mathbb{S}^1$.
- The elements of $\mathfrak{P}_2$ are $\mathbb{R}^2$ and all sectors with opening $\alpha \in (0, 2\pi)$, including half-spaces ($\alpha = \pi$).
- The elements of $\mathfrak{D}(\mathbb{R}^2)$ are curvilinear polygons with piecewise non-tangent smooth sides. Note that corner angles do not take the values 0 or 2π, and that $\mathfrak{D}(\mathbb{R}^2)$ includes smooth domains.
- The elements of $\mathfrak{D}(\mathbb{S}^2)$ are $\mathbb{S}^2$ and all curvilinear polygons with piecewise non-tangent smooth sides in the sphere $\mathbb{S}^2$.
- The elements of $\mathfrak{P}_3$ are all cones with section in $\mathfrak{D}(\mathbb{S}^2)$. This includes $\mathbb{R}^3$, half-spaces, dihedra and many different cones like octants or axisymmetric cones.
- The elements of $\mathfrak{D}(\mathbb{R}^3)$ are tangent in each point $\mathbf{x}_0$ to a cone $\Pi_{\mathbf{x}_0} \in \mathfrak{P}_3$. Note that the nature of the section of the tangent cone determines whether the 3D domain has a vertex, an edge, or is regular near $\mathbf{x}_0$.

We will give later on a more exhaustive description of the class $\mathfrak{D}(\mathbb{R}^3)$ of 3D corner domains, see §3.5.

Remark 3.5. In dimension 2, the cones are sectors. So their sides are contained in one-dimensional subspaces, and they are "polyhedral". We deduce that

$$\mathfrak{P}_2 = \overline{\mathfrak{P}}_2 \quad \text{and} \quad \mathfrak{D}(M) = \overline{\mathfrak{D}}(M) \quad \text{for} \quad M = \mathbb{R}^2 \text{ or } \mathbb{S}^2. \tag{3.4}$$

In dimension 3, a non-degenerate axisymmetric cone (i.e. different from $\mathbb{R}^3$ or a half-space) is not polyhedral, whereas an octant is.

3.2. **Admissible atlantes.** We are going to introduce the notion of admissible atlas for a corner domain, so that the associated diffeomorphisms satisfy some uniformity properties. We need some definition and preliminary result first.

Notation 3.6. For $\mathbf{v} \in \mathbb{R}^n$, we denote by $\langle \mathbf{v} \rangle$ the vector space generated by $\mathbf{v}$. For $r > 0$, we denote by $\mathsf{N}_r(\mathbf{v}) := r^{-1}\mathbf{v}$ the scaling of ratio r^{-1} which will serve as normalization. Note that $\mathsf{N}_{r^{-1}} = \mathsf{N}_r^{-1}$.

The following lemma emphasizes the robustness of Definition 3.1 as soon as a tangent cone belongs to $\mathfrak{P}_n$:

Lemma 3.7. *Let Ω be an open subset of M and $\mathbf{x}_0 \in \overline{\Omega}$ such that there exists a tangent cone $\Pi_{\mathbf{x}_0} \in \mathfrak{P}_n$ to Ω at $\mathbf{x}_0$ with map-neighborhood $\mathcal{U}_{\mathbf{x}_0}$. Then for all $\mathbf{u}_0 \in \mathcal{U}_{\mathbf{x}_0} \cap \overline{\Omega}$ there exists a tangent cone $\Pi_{\mathbf{u}_0} \in \mathfrak{P}_n$ to Ω at $\mathbf{u}_0$.*

Proof. Let $\mathbf{u}_0 \in \mathcal{U}_{\mathbf{x}_0} \cap \overline{\Omega}$. We have to prove that there exists a tangent cone $\Pi_{\mathbf{u}_0}$ at $\mathbf{u}_0$ in the sense of Definition 3.1 and that $\Pi_{\mathbf{u}_0} \in \mathfrak{P}_n$. Let $\widehat{\Omega}_{\mathbf{x}_0} = \Pi_{\mathbf{x}_0} \cap \mathbb{S}^{n-1}$ be the section of $\Pi_{\mathbf{x}_0}$. Let $(\mathcal{U}_{\mathbf{x}_0}, \mathsf{U}^{\mathbf{x}_0})$ be a local map and $\mathbf{v}_0 = \mathsf{U}^{\mathbf{x}_0}(\mathbf{u}_0) \in \overline{\Pi}_{\mathbf{x}_0}$. We denote by $(r(\mathbf{v}_0), \theta(\mathbf{v}_0)) \in (0, +\infty) \times \overline{\widehat{\Omega}}_{\mathbf{x}_0}$ its polar coordinates:

$$r(\mathbf{v}_0) := \|\mathbf{v}_0\| \qquad \text{and} \qquad \theta(\mathbf{v}_0) := \frac{\mathbf{v}_0}{\|\mathbf{v}_0\|}\,. \tag{3.5}$$

By the recursive definition there exists a tangent cone $\Pi_{\theta(\mathbf{v}_0)} \in \mathfrak{P}_{n-1}$ to $\widehat{\Omega}_{\mathbf{x}_0}$ at $\theta(\mathbf{v}_0)$. Let $\mathsf{U}^{\theta(\mathbf{v}_0)}$ be an associated diffeomorphism which sends a map-neighborhood $\mathcal{U}_{\theta(\mathbf{v}_0)}$ of $\theta(\mathbf{v}_0)$ onto a neighborhood $\mathcal{V}_{\theta(\mathbf{v}_0)}$ of $\mathbf{0} \in \mathbb{R}^{n-1}$. We may assume without restriction that there exists a n-dimensional ball with center $\theta(\mathbf{v}_0)$ and radius $\rho_1 \in (0,1)$ such that

$$\mathcal{U}_{\theta(\mathbf{v}_0)} = \mathcal{B}(\theta(\mathbf{v}_0), \rho_1) \cap \mathbb{S}^{n-1}. \tag{3.6}$$

Then we set[4] $\mathcal{U}_{(1,\theta(\mathbf{v}_0))} = \mathcal{B}(\theta(\mathbf{v}_0), \rho_1)$ and define on $\mathcal{U}_{(1,\theta(\mathbf{v}_0))}$ the diffeomorphism—using polar coordinates $(r(\mathbf{v}), \theta(\mathbf{v}))$:

$$\mathsf{U}^{(1,\theta(\mathbf{v}_0))} : \mathbf{v} \mapsto \left(r(\mathbf{v}) - 1, \mathsf{U}^{\theta(\mathbf{v}_0)}(\theta(\mathbf{v}))\right). \tag{3.7}$$

[4]We distinguish between the point $\theta(\mathbf{v}_0) \in \widehat{\Omega}_{\mathbf{x}_0}$ and its polar coordinates $(1, \theta(\mathbf{v}_0))$.

There holds $\mathrm{d}_{(1,\theta(\mathbf{v}_0))}\mathrm{U}^{(1,\theta(\mathbf{v}_0))} = \mathbb{I}_n$. Define

$$\Pi_{\mathbf{v}_0} := \langle \mathbf{v}_0 \rangle \times \Pi_{\theta(\mathbf{v}_0)}\,. \tag{3.8}$$

Notice that $\Pi_{\mathbf{v}_0} \in \mathfrak{P}_n$. It is the tangent cone to $\Pi_{\mathbf{x}_0}$ at the point $(1, \theta(\mathbf{v}_0))$ and $\mathrm{U}^{(1,\theta(\mathbf{v}_0))}$ maps $\mathcal{U}_{(1,\theta(\mathbf{v}_0))}$ on a neighborhood of $\mathbf{0} \in \mathbb{R}^n$. Let

$$\mathrm{U}^{\mathbf{v}_0} := \mathrm{N}^{-1}_{r(\mathbf{v}_0)} \circ \mathrm{U}^{(1,\theta(\mathbf{v}_0))} \circ \mathrm{N}_{r(\mathbf{v}_0)}\,. \tag{3.9}$$

Then $\mathrm{U}^{\mathbf{v}_0}$ is a diffeomorphism defined on

$$\mathcal{U}_{\mathbf{v}_0} := \|\mathbf{v}_0\|\, \mathcal{U}_{(1,\theta(\mathbf{v}_0))} = \mathcal{B}(\mathbf{v}_0, \rho_1\|\mathbf{v}_0\|) \tag{3.10}$$

with values in $\Pi_{\mathbf{v}_0}$. Let us define

$$\mathcal{U}_{\mathbf{u}_0} := (\mathrm{U}^{\mathbf{x}_0})^{-1}(\mathcal{U}_{\mathbf{v}_0})\,. \tag{3.11}$$

It is a neighborhood of $\mathbf{u}_0$. Let

$$\mathrm{U}^{\mathbf{u}_0}(\mathbf{u}) := \mathrm{J}^{\mathbf{x}_0}(\mathbf{v}_0)\ (\mathrm{U}^{\mathbf{v}_0} \circ \mathrm{U}^{\mathbf{x}_0}(\mathbf{u})) \tag{3.12}$$

be defined for $\mathbf{u} \in \mathcal{U}_{\mathbf{u}_0}$. Note that the differential of $\mathrm{U}^{\mathbf{u}_0}$ at the point $\mathbf{u}_0$ is the identity matrix $\mathbb{I}_n$. Let us set finally

$$\Pi_{\mathbf{u}_0} := \mathrm{J}^{\mathbf{x}_0}(\mathbf{v}_0)(\Pi_{\mathbf{v}_0})\,. \tag{3.13}$$

Then the map-neighborhood $\mathcal{U}_{\mathbf{u}_0}$, the diffeomorphism $\mathrm{U}^{\mathbf{u}_0}$ and the cone $\Pi_{\mathbf{u}_0}$ satisfy the requirements of Definition 3.1 and $\Pi_{\mathbf{u}_0}$ is the tangent cone to Ω at $\mathbf{u}_0$. Since $\Pi_{\mathbf{v}_0} \in \mathfrak{P}_n$, there holds $\Pi_{\mathbf{u}_0} \in \mathfrak{P}_n$. □

Remark 3.8. If the tangent cone $\Pi_{\mathbf{x}_0}$ is *polyhedral*, the procedure for constructing $\mathrm{U}^{\mathbf{u}_0}$ can be simplified as follows: We define $\mathbf{v}_0$ and its polar coordinates $(r(\mathbf{v}_0), \theta(\mathbf{v}_0))$ as before. Since $\Pi_{\mathbf{x}_0}$ is polyhedral, the ball $\mathcal{B}(\theta(\mathbf{v}_0), \rho_1)$ (3.6) is such that the set $\widetilde{\mathcal{U}} := \overline{\mathcal{B}(\theta(\mathbf{v}_0), \rho_1)} \cap \Pi_{\mathbf{x}_0}$ is homogeneous with respect to $\theta(\mathbf{v}_0)$, that is

$$\mathbf{v} \in \widetilde{\mathcal{U}} \ \text{ and } \ \rho \in \Big[0, \frac{\rho_1}{\|\mathbf{v} - \theta(\mathbf{v}_0)\|}\Big] \quad \Longrightarrow \quad \rho\mathbf{v} + (1-\rho)\theta(\mathbf{v}_0) \in \widetilde{\mathcal{U}}.$$

The set $\widetilde{\mathcal{V}} := \{\mathbf{v} \in \mathbb{R}^n |\ \mathbf{v} + \theta(\mathbf{v}_0) \in \widetilde{\mathcal{U}}\}$ defines a polyhedral cone $\widetilde{\Pi}$ in a natural way by $\{\mathbf{v} \in \mathbb{R}^n |\ \exists \rho > 0\ \rho\mathbf{v} \in \widetilde{\mathcal{V}}\}$. Defining $\mathrm{U}^{\mathbf{v}_0}$ as the translation $\mathrm{T}_{\mathbf{v}_0} : \mathbf{v} \mapsto \mathbf{v} - \mathbf{v}_0$, we find that $\widetilde{\Pi} = \Pi_{\mathbf{v}_0}$. Then, with this simple definition of $\mathrm{U}^{\mathbf{v}_0}$ we still define $\mathrm{U}^{\mathbf{u}_0}$ by (3.12). On the other hand, by uniqueness of tangent cones, the new definition of $\Pi_{\mathbf{v}_0}$ coincides with the old one (3.8). Finally, $\Pi_{\mathbf{u}_0}$ is still defined by (3.13).

Lemma 3.9. *Let* $(\mathcal{U}_{\mathbf{x}_0}, \mathrm{U}^{\mathbf{x}_0})$ *be a local map with image a neighborhood* $\mathcal{V}_{\mathbf{x}_0}$ *of* $\mathbf{0}$, *and such that* $\mathrm{J}^{\mathbf{x}_0}(\mathbf{0}) = \mathbb{I}_n$. *There exists* $r_0 > 0$ *such that*

$$\|\mathbf{u}' - \mathbf{u} - (\mathbf{v}' - \mathbf{v})\| \le \tfrac{1}{2}\,\|\mathbf{v}' - \mathbf{v}\|, \qquad \textit{with} \quad \mathbf{v} = \mathrm{U}^{\mathbf{x}_0}(\mathbf{u}), \quad \mathbf{v}' = \mathrm{U}^{\mathbf{x}_0}(\mathbf{u}'), \tag{3.14}$$

for any $\mathbf{v}, \mathbf{v}' \in \mathcal{B}(\mathbf{0}, r_0) \subset \mathcal{V}_{\mathbf{x}_0}$.

Proof. Let r_1 be such that $\mathbf{v}, \mathbf{v}' \in \mathcal{B}(\mathbf{0}, r_1) \subset \mathcal{V}_{\mathbf{x}_0}$. A Taylor expansion of $(\mathrm{U}^{\mathbf{x}_0})^{-1}(\mathbf{v}')$ around $\mathbf{v}$ gives

$$\|(\mathrm{U}^{\mathbf{x}_0})^{-1}(\mathbf{v}') - (\mathrm{U}^{\mathbf{x}_0})^{-1}(\mathbf{v}) - \mathrm{J}^{\mathbf{x}_0}(\mathbf{v})(\mathbf{v}' - \mathbf{v})\| \leq \tfrac{1}{2}\,\|\mathrm{dJ}^{\mathbf{x}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},r_1))}\|\mathbf{v}' - \mathbf{v}\|^2.$$

Another Taylor expansion of $\mathrm{J}^{\mathbf{x}_0}(\mathbf{v})$ around $\mathbf{0}$ gives

$$\|\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}) - \mathrm{J}^{\mathbf{x}_0}(\mathbf{0})\| \leq \|\mathrm{dJ}^{\mathbf{x}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},r_1))}\|\mathbf{v}\| \;.$$

Since $(\mathrm{U}^{\mathbf{x}_0})^{-1}(\mathbf{v}) = \mathbf{u}$, $(\mathrm{U}^{\mathbf{x}_0})^{-1}(\mathbf{v}') = \mathbf{u}'$ and $\mathrm{J}^{\mathbf{x}_0}(\mathbf{0}) = \mathbb{I}_n$, we deduce

$$\|(\mathbf{u}' - \mathbf{u}) - (\mathbf{v}' - \mathbf{v})\| \leq \Big(\|\mathrm{dJ}^{\mathbf{x}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},r_1))}\|\mathbf{v}\| \; + \tfrac{1}{2}\,\|\mathrm{dJ}^{\mathbf{x}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},r_1))}\|\mathbf{v}' - \mathbf{v}\|\Big)\;\|\mathbf{v}' - \mathbf{v}\|.$$

If we choose $r_0 \leq \min\big\{r_1,\, 1/(4\|\mathrm{dJ}^{\mathbf{x}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},r_1))})\big\}$, we have

$$\|\mathrm{dJ}^{\mathbf{x}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},r_1))}\|\mathbf{v}\| \; + \tfrac{1}{2}\,\|\mathrm{dJ}^{\mathbf{x}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},r_1))}\|\mathbf{v}' - \mathbf{v}\| \leq \tfrac{1}{2}, \qquad \forall \mathbf{v}, \mathbf{v}' \in \mathcal{B}(\mathbf{0}, r_0),$$

which ends the proof. □

Proposition 3.10. *(i) The domain Ω belongs to $\mathfrak{D}(\mathbb{R}^n)$ if and only if there exist a finite set $\mathfrak{X} \subset \overline{\Omega}$ and, for each $\mathbf{x}_0 \in \mathfrak{X}$, a cone $\Pi_{\mathbf{x}_0} \in \mathfrak{P}_n$ and a local map $(\mathcal{U}_{\mathbf{x}_0}, \mathrm{U}^{\mathbf{x}_0})$ such that (3.1) holds, with the condition that, moreover, $\cup_{\mathbf{x}_0 \in \mathfrak{X}} \mathcal{U}_{\mathbf{x}_0} \supset \overline{\Omega}$.*

(ii) The equivalence (i) still holds if one requires moreover that for all $\mathbf{x}_0 \in \mathfrak{X}$ and all $\mathbf{u}, \mathbf{u}' \in \mathcal{U}_{\mathbf{x}_0}$, (3.14) holds.

Proof. (i) The "if" direction is a consequence of the definition of $\mathfrak{D}(\mathbb{R}^n)$ and, in particular, the fact that $\overline{\Omega}$ is compact and can be covered by a finite number of map-neighborhoods. The "only if" direction is a consequence of Lemma 3.7.

(ii) is then a consequence of Lemma 3.9 (and of the compactness of $\overline{\Omega}$, of course). □

Definition 3.11. Let $\Omega \in \mathfrak{D}(M)$. An atlas $(\mathcal{U}_{\mathbf{x}}, \mathrm{U}^{\mathbf{x}})_{\mathbf{x} \in \overline{\Omega}}$ is called *admissible* if it comes from the following recursive procedure:

1. Take a finite set $\mathfrak{X} \subset \overline{\Omega}$ as in Proposition 3.10 together with the associated map-neighborhoods and diffeomorphisms $(\mathcal{U}_{\mathbf{x}_0}, \mathrm{U}^{\mathbf{x}_0})$ for $\mathbf{x}_0 \in \mathfrak{X}$, satisfying moreover (3.14).
2. We assume moreover that for each $\mathbf{x}_0 \in \mathfrak{X}$ the map-neighborhood $\mathcal{U}_{\mathbf{x}_0}$ contains a ball $\mathcal{B}(\mathbf{x}_0, 2R_{\mathbf{x}_0})$ for some $R_{\mathbf{x}_0} > 0$ and that the balls with half-radius $\mathcal{B}(\mathbf{x}_0, R_{\mathbf{x}_0})$ cover $\overline{\Omega}$.
3. All the other map-neighborhoods and diffeomorphisms $(\mathcal{U}_{\mathbf{x}}, \mathrm{U}^{\mathbf{x}})$ with $\mathbf{x} \in \overline{\Omega} \setminus \mathfrak{X}$ are constructed by the recursive procedure (3.5)–(3.12), based on admissible atlantes for the sections $\widehat{\Omega}_{\mathbf{x}_0}$ associated with the set of reference points $\mathbf{x}_0 \in \mathfrak{X}$. In the polyhedral case, the straightforward construction described in Remark 3.8 is preferred. ■

As a direct consequence of Lemmas 3.7, 3.9, and Proposition 3.10, we obtain the existence of admissible atlantes.

Theorem 3.12. *Let Ω be a corner domain in $\mathfrak{D}(M)$. Then Ω admits an admissible atlas.*

For an admissible atlas, we can express the derivative of the diffeomorphism as follows: Let $\mathbf{x}_0 \in \mathfrak{X}$, $\mathbf{u}_0 \in \mathcal{U}_{\mathbf{x}_0}$ and $\mathbf{v}_0 := \mathrm{U}^{\mathbf{x}_0}(\mathbf{u}_0)$. Differentiating (3.12), we get

$$\forall \mathbf{v} \in \mathcal{V}_{\mathbf{u}_0}, \quad \mathrm{J}^{\mathbf{u}_0}(\mathbf{v}) = \mathrm{J}^{\mathbf{x}_0}(\mathbf{v})\ \mathrm{J}^{\mathbf{v}_0}(\mathrm{U}^{\mathbf{v}_0}(\mathbf{v}))\ (\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}_0))^{-1}\,, \tag{3.15}$$

and (3.9) provides:

$$\mathrm{J}^{\mathbf{v}_0}(\mathrm{U}^{\mathbf{v}_0}(\mathbf{v})) = \mathrm{J}^{(1,\theta(\mathbf{v}_0))}\left(\mathrm{U}^{(1,\theta(\mathbf{v}_0))}(\tfrac{\mathbf{v}}{\|\mathbf{v}_0\|})\right). \tag{3.16}$$

3.3. **Estimates for local Jacobian matrices.** We give in Proposition 3.13 several estimates for the Jacobians $\mathrm{J}^{\mathbf{x}_0}$ (3.2) and the metric $\mathrm{G}^{\mathbf{x}_0}$ (3.3) of all the diffeomorphisms contained in an admissible atlas of a corner domain Ω. All estimates are consequence of local bounds in L^∞ norm on the derivative of Jacobian functions. We denote for any $\mathbf{x}_0 \in \overline{\Omega}$

$$\mathrm{K}^{\mathbf{x}_0}(\mathbf{v}) = \mathrm{d}_{\mathbf{v}}\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}), \quad \mathbf{v} \in \mathcal{V}_{\mathbf{x}_0}\,. \tag{3.17}$$

After considering the case of reference points $\mathbf{x}_0 \in \mathfrak{X}$, we deal with points $\mathbf{u}_0 \in \overline{\Omega}$ close to a reference point $\mathbf{x}_0$ such that $\Pi_{\mathbf{x}_0} \in \overline{\mathfrak{P}}_n$: in that case the quantities $\mathrm{K}^{\mathbf{u}_0}$ for $\mathbf{u}_0 \in \mathcal{U}_{\mathbf{x}_0}$ remain bounded uniformly in $\mathcal{U}_{\mathbf{x}_0}$. The next estimate is a global version of the first one when assuming that $\Omega \in \overline{\mathfrak{D}}(M)$. The last estimate deals with points $\mathbf{u}_0$ close to a reference point $\mathbf{x}_0$ such that the section $\widehat{\Omega}_{\mathbf{x}_0}$ of $\Pi_{\mathbf{x}_0}$ is polyhedral[5]: in that case we show that for $\mathbf{u}_0 \in \mathcal{U}_{\mathbf{x}_0}$, the quantity $\mathrm{K}^{\mathbf{u}_0}$ is controlled by $\|\mathbf{u}_0 - \mathbf{x}_0\|^{-1}$. These estimates will be useful when using change of variables on quadratic form defined on corner domains in dimension 3. An important feature of these estimates is a recursive control of their domain of validity: In each case we exhibit such domains as balls with explicit centers and implicit radii. The principle is to start from the finite number of reference points $\mathbf{x}_0 \in \mathfrak{X}$ provided by an admissible atlas and proceed with points $\mathbf{u}_0$ which are not in this set using Lemma 3.7 and Remark 3.8. The outcome is that estimates are valid in a ball around $\mathbf{u}_0$ with radius $\rho(\mathbf{u}_0)$ proportional to the distance $\mathrm{dist}(\mathbf{u}_0, \mathfrak{X})$ of $\mathbf{u}_0$ to the set of reference points, the proportion ratio $\rho(\widehat{\mathbf{u}}_1)$ being a similar radius associated with the section $\widehat{\Omega}_{\mathbf{x}_0} \in \mathfrak{D}(\mathbb{S}^{n-1})$.

Proposition 3.13. *Let $\Omega \in \mathfrak{D}(M)$ and $(\mathcal{U}_{\mathbf{x}}, \mathrm{U}^{\mathbf{x}})_{\mathbf{x}\in\overline{\Omega}}$ be an admissible atlas with set of reference points $\mathfrak{X} \subset \overline{\Omega}$. Then we have the following assertions:*

(i) Let $\mathbf{x}_0 \in \mathfrak{X}$. With $R_{\mathbf{x}_0}$ introduced in Definition 3.11, there exists $c(\mathbf{x}_0)$ such that

$$\begin{aligned} &\|\mathrm{K}^{\mathbf{x}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},R_{\mathbf{x}_0}))} \le c(\mathbf{x}_0), \\ &\|\mathrm{J}^{\mathbf{x}_0} - \mathbb{I}_n\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} + \|\mathrm{G}^{\mathbf{x}_0} - \mathbb{I}_n\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} \le r\, c(\mathbf{x}_0) \quad \textit{for all } r \le R_{\mathbf{x}_0}\,. \end{aligned} \tag{3.18}$$

(ii) Let $\mathbf{x}_0 \in \mathfrak{X}$ such that $\Pi_{\mathbf{x}_0} \in \overline{\mathfrak{P}}_n$. Then there exists a constant $c(\mathbf{x}_0)$ such that for all $\mathbf{u}_0 \in \overline{\Omega} \cap \mathcal{B}(\mathbf{x}_0, R_{\mathbf{x}_0})$, $\mathbf{u}_0 \neq \mathbf{x}_0$, there holds, denoting $\widehat{\mathbf{u}}_1 := \mathrm{U}^{\mathbf{x}_0}\mathbf{u}_0/\|\mathrm{U}^{\mathbf{x}_0}\mathbf{u}_0\| \in \widehat{\Omega}_{\mathbf{x}_0}$

$$\begin{aligned} &\|\mathrm{K}^{\mathbf{u}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},\rho(\mathbf{u}_0)))} \le c(\mathbf{x}_0) \quad \textit{with} \quad \rho(\mathbf{u}_0) = \tfrac{1}{3}\rho(\widehat{\mathbf{u}}_1)\,\|\mathbf{u}_0 - \mathbf{x}_0\|, \\ &\|\mathrm{J}^{\mathbf{u}_0} - \mathbb{I}_n\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} + \|\mathrm{G}^{\mathbf{u}_0} - \mathbb{I}_n\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} \le r\, c(\mathbf{x}_0) \quad \textit{for all } r \le \rho(\mathbf{u}_0)\,. \end{aligned} \tag{3.19}$$

[5]But this does not imply that the tangent cone $\Pi_{\mathbf{x}_0}$ is polyhedral.

(iii) Let $\Omega \in \overline{\mathfrak{D}}(\mathbb{R}^n)$, then there exists $c(\Omega)$ such that for all $\mathbf{u}_0 \in \overline{\Omega}$, there holds, with $\widehat{\mathbf{u}}_1$ as above,

$$\begin{aligned} &\|\mathrm{K}^{\mathbf{u}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},\rho(\mathbf{u}_0)))} \le c(\Omega) \quad \textit{with} \quad \rho(\mathbf{u}_0) = \tfrac{1}{3}\rho(\widehat{\mathbf{u}}_1)\,\mathrm{dist}(\mathbf{u}_0,\mathfrak{X}), \\ &\|\mathrm{J}^{\mathbf{u}_0} - \mathbb{I}_n\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} + \|\mathrm{G}^{\mathbf{u}_0} - \mathbb{I}_n\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} \le r\,c(\Omega) \quad \textit{for all } r \le \rho(\mathbf{u}_0)\,. \end{aligned} \tag{3.20}$$

(iv) Let $\mathbf{x}_0 \in \mathfrak{X}$ be such that the section $\widehat{\Omega}_{\mathbf{x}_0} = \Pi_{\mathbf{x}_0} \cap \mathbb{S}^{n-1}$ belongs to $\overline{\mathfrak{D}}(\mathbb{S}^{n-1})$. Then there exists $c(\mathbf{x}_0)$ such that for all $\mathbf{u}_0 \in \overline{\Omega} \cap \mathcal{B}(\mathbf{x}_0, R_{\mathbf{x}_0})$, $\mathbf{u}_0 \neq \mathbf{x}_0$, there holds,

$$\begin{aligned} &\|\mathrm{K}^{\mathbf{u}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},\rho(\mathbf{u}_0)))} \le \frac{1}{\|\mathbf{u}_0 - \mathbf{x}_0\|}\, c(\mathbf{x}_0) \quad \textit{with} \quad \rho(\mathbf{u}_0) = \tfrac{1}{3}\rho(\widehat{\mathbf{u}}_1)\,\|\mathbf{u}_0 - \mathbf{x}_0\|, \\ &\|\mathrm{J}^{\mathbf{u}_0} - \mathbb{I}_n\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} + \|\mathrm{G}^{\mathbf{u}_0} - \mathbb{I}_n\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} \le \frac{r}{\|\mathbf{u}_0 - \mathbf{x}_0\|}\, c(\mathbf{x}_0) \quad \textit{for all } r \le \rho(\mathbf{u}_0)\,. \end{aligned} \tag{3.21}$$

Proof. *(i)* The estimate for $\mathrm{K}^{\mathbf{x}_0}$ in (3.18) comes from the definition of a map-neighborhood. The bound in (3.18) on $\mathrm{J}^{\mathbf{x}_0} - \mathbb{I}_n$ follows immediately because of the Taylor estimate

$$\|\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}) - \mathbb{I}_n\| \le \|\mathbf{v}\|\,\|\mathrm{K}^{\mathbf{x}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},\|\mathbf{v}\|))}, \quad \mathbf{v} \in \mathcal{V}_{\mathbf{x}_0}\,. \tag{3.22}$$

Concerning the bound (3.18) on $\mathrm{G}^{\mathbf{x}_0} - \mathbb{I}_n$, we rely on the Taylor estimate

$$\|\mathrm{G}^{\mathbf{x}_0}(\mathbf{v}) - \mathbb{I}_n\| \le \|\mathbf{v}\|\,\|\mathrm{K}^{\mathbf{x}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},\|\mathbf{v}\|))}\,\|(\mathrm{J}^{\mathbf{x}_0})^{-1}\|^3_{L^\infty(\mathcal{B}(\mathbf{0},\|\mathbf{v}\|))}\,. \tag{3.23}$$

(ii) Since $\Pi_{\mathbf{x}_0}$ is polyhedral, we can take advantage of Remark 3.8: For $\mathbf{u}_0$ in the ball $\mathcal{B}(\mathbf{x}_0, R_{\mathbf{x}_0})$, the local map $(\mathcal{U}_{\mathbf{u}_0}, \mathrm{U}^{\mathbf{u}_0})$ is defined by (3.10)–(3.12) where, for some $\rho_1 < 1$,

$$\mathbf{v}_0 = \mathrm{U}^{\mathbf{x}_0}(\mathbf{u}_0), \quad \mathcal{U}_{\mathbf{v}_0} = \mathcal{B}(\mathbf{v}_0, \rho_1\|\mathbf{v}_0\|), \quad \text{and} \quad \mathrm{U}^{\mathbf{v}_0}(\mathbf{v}) = \mathbf{v} - \mathbf{v}_0\,.$$

Note that the radius ρ_1 is the radius $\rho(\widehat{\mathbf{u}}_1)$ of a map neighborhood of $\widehat{\mathbf{u}}_1 := \mathbf{v}_0/\|\mathbf{v}_0\|$, which plays the same role as $\rho(\mathbf{u}_0)$ in one dimension less.

We recall that our admissible atlas satisfies Condition (1) of Definition 3.11. Applying (3.14) with the couples $\{(\mathbf{u},\mathbf{u}_0),(\mathbf{v},\mathbf{v}_0)\}$ and $\{(\mathbf{u}_0,\mathbf{x}_0),(\mathbf{v}_0,\mathbf{0})\}$, we deduce that $\mathcal{U}_{\mathbf{u}_0}$ contains the ball $\mathcal{B}(\mathbf{u}_0, \frac{1}{3}\rho_1\|\mathbf{u}_0 - \mathbf{x}_0\|)$. On the other hand, in this case (3.15) reduces to

$$\forall \mathbf{v} \in \mathcal{V}_{\mathbf{u}_0}, \quad \mathrm{J}^{\mathbf{u}_0}(\mathbf{v}) = \mathrm{J}^{\mathbf{x}_0}(\mathbf{v})\;(\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}_0))^{-1}. \tag{3.24}$$

Thus, we deduce from the above formula that

$$\|\mathrm{K}^{\mathbf{u}_0}\|_{L^\infty(\mathcal{V}_{\mathbf{u}_0})} \le \|\mathrm{K}^{\mathbf{x}_0}\|_{L^\infty(\mathcal{V}_{\mathbf{x}_0})}\|(\mathrm{J}^{\mathbf{x}_0})^{-1}\|_{L^\infty(\mathcal{U}_{\mathbf{x}_0})}\,. \tag{3.25}$$

All of this proves estimate for $\mathrm{K}^{\mathbf{u}_0}$ in (3.19).

The bound in (3.19) on $\mathrm{J}^{\mathbf{u}_0} - \mathbb{I}_n$ follows immediately because of the Taylor estimate (3.22) where $\mathbf{x}_0$ is replaced by $\mathbf{u}_0$. Concerning the bound on $\mathrm{G}^{\mathbf{u}_0} - \mathbb{I}_n$, we start from the Taylor estimate (3.23) where we replace $\mathbf{x}_0$ by $\mathbf{u}_0$. It remains to bound $\|(\mathrm{J}^{\mathbf{u}_0})^{-1}\|$. We note that we have, thanks to (3.24)

$$\mathrm{J}^{\mathbf{u}_0}(\mathbf{v})^{-1} = (\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}_0))\,(\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}))^{-1}\,.$$

Whence the bound (3.19) on $\mathrm{G}^{\mathbf{u}_0} - \mathbb{I}_n$.

(iii) Applying Proposition 3.10 to $\Omega\in\overline{\mathfrak{D}}(M)$, we deduce from (3.25):

$$\sup_{x\in\overline{\Omega}}\|\mathrm{K}^{\mathbf{x}}\|_{L^\infty(\mathcal{V}_{\mathbf{x}})}\le\max_{\mathbf{x}_0\in\mathfrak{X}}\left(\|\mathrm{K}^{\mathbf{x}_0}\|_{L^\infty(\mathcal{V}_{\mathbf{x}})}\|(\mathrm{J}^{\mathbf{x}_0})^{-1}\|_{L^\infty(\mathcal{U}_{\mathbf{x}})}\right)<+\infty. \tag{3.26}$$

(iv) Differentiating (3.15) with respect to $\mathbf{v}$ yields

$$\mathrm{K}^{\mathbf{u}_0}(\mathbf{v})=\mathrm{K}^{\mathbf{x}_0}(\mathbf{v})\ \mathrm{J}^{\mathbf{v}_0}(\mathrm{U}^{\mathbf{v}_0}(\mathbf{v}))\ (\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}_0))^{-1}+\mathrm{J}^{\mathbf{x}_0}(\mathbf{v})\ \mathrm{d}_{\mathbf{v}}\mathrm{J}^{\mathbf{v}_0}(\mathrm{U}^{\mathbf{v}_0}(\mathbf{v}))\ \ (\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}_0))^{-1}. \tag{3.27}$$

Using in turn (3.16) we calculate

$$\begin{aligned}\mathrm{d}_{\mathbf{v}}\mathrm{J}^{\mathbf{v}_0}(\mathrm{U}^{\mathbf{v}_0}(\mathbf{v}))&=\mathrm{d}_{\mathbf{v}}\Big\{\mathrm{J}^{(1,\theta(\mathbf{v}_0))}\left(\mathrm{U}^{(1,\theta(\mathbf{v}_0))}(\tfrac{\mathbf{v}}{\|\mathbf{v}_0\|})\right)\Big\}\\&=\tfrac{1}{\|\mathbf{v}_0\|}\ \mathrm{K}^{(1,\theta(\mathbf{v}_0))}\left(\mathrm{U}^{(1,\theta(\mathbf{v}_0))}(\tfrac{\mathbf{v}}{\|\mathbf{v}_0\|})\right)\ \left(\mathrm{J}^{(1,\theta(\mathbf{v}_0))}\left(\mathrm{U}^{(1,\theta(\mathbf{v}_0)}(\tfrac{\mathbf{v}}{\|\mathbf{v}_0\|})\right)\right)^{-1}.\end{aligned} \tag{3.28}$$

Recall that $\mathrm{U}^{(1,\theta)}$ is deduced from U^{θ} by formula (3.7) on the domain $\mathcal{U}_{(1,\theta(\mathbf{v}_0))}=\mathcal{B}(\theta(\mathbf{v}_0),\rho_0)$, cf. (3.6). Therefore there exists a constant $c(\rho_0)\ge1$ such that

$$\|\mathrm{J}^{(1,\theta)}\|_{L^\infty(\mathcal{V}_{(1,\theta)})}\le c(\rho_0)\|\mathrm{J}^{\theta}\|_{L^\infty(\mathcal{V}_{\theta})}\quad\text{and}\quad\|\mathrm{K}^{(1,\theta)}\|_{L^\infty(\mathcal{V}_{(1,\theta)})}\le c(\rho_0)\|\mathrm{K}^{\theta}\|_{L^\infty(\mathcal{V}_{\theta})}.$$

We deduce

$$\|\mathrm{K}^{\mathbf{u}_0}\|\le c'(\rho_0)\left(\|\mathrm{K}^{\mathbf{x}_0}\|\,\|\mathrm{J}^{\theta(\mathbf{v}_0)}\|\,\|(\mathrm{J}^{\mathbf{x}_0})^{-1}\|+\frac{\|(\mathrm{J}^{\theta(\mathbf{v}_0)})^{-1}\|}{\|\mathbf{v}_0\|}\,\|\mathrm{J}^{\mathbf{x}_0}\|\,\|\mathrm{K}^{\theta(\mathbf{v}_0)}\|\,\|(\mathrm{J}^{\mathbf{x}_0})^{-1}\|\right) \tag{3.29}$$

where we have omitted the mention of the L^∞ norms. Since the section $\widehat{\Omega}_{\mathbf{x}_0}$ belongs to $\overline{\mathfrak{D}}(\mathbb{S}^{n-1})$, we deduce from *(iii)* and (3.26) applied to the section $\widehat{\Omega}_{\mathbf{x}_0}$ that

$$\sup_{\theta\in\overline{\widehat{\Omega}}_{\mathbf{x}_0}}\|\mathrm{J}^{\theta}\|_{L^\infty(\mathcal{V}_{\theta})}<+\infty\quad\text{and}\quad\sup_{\theta\in\overline{\widehat{\Omega}}_{\mathbf{x}_0}}\|\mathrm{K}^{\theta}\|_{L^\infty(\mathcal{V}_{\theta})}<+\infty.$$

Therefore the r.h.s. of (3.29) is controlled by $c(\mathbf{x}_0)/\|\mathbf{v}_0\|$. Using (3.14) we obtain that $\|\mathbf{v}_0\|\simeq\|\mathbf{u}_0-\mathbf{x}_0\|$, whence the bound (3.21) on $\mathrm{K}^{\mathbf{u}_0}$. The bound (3.21) for $\mathrm{J}^{\mathbf{u}_0}-\mathbb{I}_n$ follows immediately as in point *(i)*. Finally, to prove the bound on $\mathrm{G}^{\mathbf{u}_0}-\mathbb{I}_n$, we combine the Taylor estimate (3.23) (at $\mathbf{u}_0$) with the estimate of $\mathrm{K}^{\mathbf{u}_0}$ in (3.21) and the formula for $(\mathrm{J}^{\mathbf{u}_0})^{-1}$

$$(\mathrm{J}^{\mathbf{u}_0}(\mathbf{v}))^{-1}=(\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}_0))\ (\mathrm{J}^{\mathbf{v}_0}(\mathrm{U}^{\mathbf{v}_0}(\mathbf{v})))^{-1}\ (\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}))^{-1},$$

deduced from (3.15). It remains to use (3.16) to bound $(\mathrm{J}^{\mathbf{v}_0}(\mathrm{U}^{\mathbf{v}_0}(\mathbf{v})))^{-1}$, which ends the proof. □

Remark 3.14. In dimension $n=2$, domains $\Omega\in\mathfrak{D}(\mathbb{R}^2)$ are always in case *(ii)* or *(iii)* of Proposition 3.13 since $\mathfrak{D}(\mathbb{R}^2)=\overline{\mathfrak{D}}(\mathbb{R}^2)$, cf. (3.4). In dimension $n=3$, Proposition 3.13 still covers all possibilities: Indeed, since $\mathfrak{D}(\mathbb{S}^2)=\overline{\mathfrak{D}}(\mathbb{S}^2)$, one is at least in case *(iv)*. In higher dimensions $n\ge4$, Proposition 3.13 does not provide estimates for all possible singular points. General estimates would involve distance to non-discrete sets of points, see (3.36) later on. However Proposition 3.13 is sufficient for the core of our investigation, which, for independent reasons, is limited to dimension $n\le3$.

Remark 3.15. We can use the computation of $\mathsf{K}^{\mathbf{u}_0}$ in the proof of Proposition 3.13 to obtain estimates for its differential $\mathrm{d}\mathsf{K}^{\mathbf{u}_0}$. Note that in (3.29), the worst term is $1/\|\mathbf{v}_0\|$. By differentiating (3.27), we obtain an upper bound in $1/\|\mathbf{v}_0\|^2$. Thus we have the following improvements in Proposition 3.13:

(1) In cases *(i)*, *(ii)* and *(iii)*, the estimates for $\mathsf{K}^{\mathbf{x}_0}$ and $\mathsf{K}^{\mathbf{u}_0}$ are still valid for their differentials $\mathrm{d}\mathsf{K}^{\mathbf{x}_0}$ and $\mathrm{d}\mathsf{K}^{\mathbf{u}_0}$, respectively.

(2) Let $\mathbf{x}_0 \in \mathfrak{X}$ such that $\widehat{\Omega}_{\mathbf{x}_0} = \Pi_{\mathbf{x}_0} \cap \mathbb{S}^{n-1}$ belongs to $\overline{\mathfrak{D}}(\mathbb{S}^{n-1})$. Then there exists $c(\mathbf{x}_0)$ such that for all $\mathbf{u}_0 \in \overline{\Omega} \cap \mathcal{B}(\mathbf{x}_0, R_{\mathbf{x}_0})$, $\mathbf{u}_0 \neq \mathbf{x}_0$, there holds, with $\widehat{\mathbf{u}}_1 := \mathsf{U}^{\mathbf{x}_0}\mathbf{u}_0/\|\mathsf{U}^{\mathbf{x}_0}\mathbf{u}_0\|$

$$\|\mathrm{d}\mathsf{K}^{\mathbf{u}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},\rho(\mathbf{u}_0)))} \leq \frac{1}{\|\mathbf{u}_0 - \mathbf{x}_0\|^2}\, c(\mathbf{x}_0) \quad \text{with} \quad \rho(\mathbf{u}_0) = \tfrac{1}{3}\rho(\widehat{\mathbf{u}}_1)\,\|\mathbf{u}_0 - \mathbf{x}_0\|. \tag{3.30}$$

3.4. **Strata and singular chains.** In this section, we exhibit a canonical structure of tangent cones and corner domains.

Definition 3.16. Let $\mathfrak{O}_n$ denote the group of orthogonal linear transformations of $\mathbb{R}^n$.

a) We say that a cone Π is *equivalent* to another cone Π' and denote

$$\Pi \equiv \Pi'$$

if there exists $\underline{\mathsf{U}} \in \mathfrak{O}_n$ such that $\underline{\mathsf{U}}\Pi = \Pi'$.

b) Let $\Pi \in \mathfrak{P}_n$. If Π is equivalent to $\mathbb{R}^{n-d} \times \Gamma$ with $\Gamma \in \mathfrak{P}_d$ and d is minimal for such an equivalence, Γ is said to be *a minimal reduced cone* associated with Π and we denote by $d(\Pi) := d$ the *reduced dimension* of the cone Π.

c) Let $\mathbf{x} \in \overline{\Omega}$ and let $\Pi_{\mathbf{x}}$ be its tangent cone. We denote by $d_0(\mathbf{x})$ the dimension of the minimal reduced cone associated with $\Pi_{\mathbf{x}}$. ■

Remark 3.17. If there exists a linear isomorphism between Π and Π' then $d(\Pi) = d(\Pi')$.

3.4.1. *Recursive definition of the singular chains.* A singular chain $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_\nu) \in \mathfrak{C}(\Omega)$ (with ν an integer) is a finite collection of points according to the following recursive definition.

Initialization: $\mathbf{x}_0 \in \overline{\Omega}$,

- Let $C_{\mathbf{x}_0}$ be the tangent cone to Ω at $\mathbf{x}_0$ (here $C_{\mathbf{x}_0} = \Pi_{\mathbf{x}_0}$).
- Let $\Gamma_{\mathbf{x}_0} \in \mathfrak{P}_{d_0}$ be its minimal reduced cone: $C_{\mathbf{x}_0} = \underline{\mathsf{U}}^0(\mathbb{R}^{n-d_0} \times \Gamma_{\mathbf{x}_0})$.
- Alternative:
  - If $\nu = 0$, stop here.
  - If $\nu > 0$, then[6] $d_0 > 0$ and let $\Omega_{\mathbf{x}_0} \in \mathfrak{D}(\mathbb{S}^{d_0-1})$ be the section of $\Gamma_{\mathbf{x}_0}$

Recurrence: $\mathbf{x}_j \in \overline{\Omega}_{\mathbf{x}_0,\ldots,\mathbf{x}_{j-1}} \in \mathfrak{D}(\mathbb{S}^{d_{j-1}-1})$. If $d_{j-1} = 1$, stop here ($\nu = j$). If not:

- Let $C_{\mathbf{x}_0,\ldots,\mathbf{x}_j}$ be the tangent cone to $\Omega_{\mathbf{x}_0,\ldots,\mathbf{x}_{j-1}}$ at $\mathbf{x}_j$,

[6] If $d_0 = 0$, we have necessarily $\nu = 0$.

- Let $\Gamma_{\mathbf{x}_0,\ldots,\mathbf{x}_j} \in \mathfrak{P}_{d_j}$ be its minimal reduced cone: $C_{\mathbf{x}_0,\ldots,\mathbf{x}_j} = \underline{\mathrm{U}}^j(\mathbb{R}^{d_{j-1}-1-d_j} \times \Gamma_{\mathbf{x}_0,\ldots,\mathbf{x}_j})$.
- Alternative:
  - If $\nu = j$, stop here.
  - If $\nu > j$, then $d_j > 0$ and let $\Omega_{\mathbf{x}_0,\ldots,\mathbf{x}_j} \in \mathfrak{D}(\mathbb{S}^{d_j-1})$ be the section of $\Gamma_{\mathbf{x}_0,\ldots,\mathbf{x}_j}$.

Note that $n \geq d_0 > d_1 > \ldots > d_\nu$. Hence $\nu \leq n$. Note also that for $\nu = 0$, we obtain the trivial one element chain $(\mathbf{x}_0)$ for any $\mathbf{x}_0 \in \overline{\Omega}$.

Definition 3.18. For any $\mathbf{x} \in \overline{\Omega}$, we denote by $\mathfrak{C}_{\mathbf{x}}(\Omega)$ the subset of chains $\mathbb{X} \in \mathfrak{C}(\Omega)$ originating at $\mathbf{x}$, i.e., the set of chains $\mathbb{X} = (\mathbf{x}_0, \ldots, \mathbf{x}_\nu)$ with $\mathbf{x}_0 = \mathbf{x}$. Note that the one element chain $(\mathbf{x})$ belongs to $\mathfrak{C}_{\mathbf{x}}(\Omega)$. We also set

$$\mathfrak{C}^*_{\mathbf{x}}(\Omega) = \{\mathbb{X} \in \mathfrak{C}_{\mathbf{x}}(\Omega),\ \nu > 0\} = \mathfrak{C}_{\mathbf{x}}(\Omega) \setminus \{(\mathbf{x})\}. \tag{3.31}$$

We set finally, with the notation $\langle \mathbf{y} \rangle$ for the vector space generated by $\mathbf{y}$,

$$\Pi_{\mathbb{X}} = \begin{cases} C_{\mathbf{x}_0} = \Pi_{\mathbf{x}_0} & \text{if } \ \nu = 0, \\ \underline{\mathrm{U}}^0\big(\mathbb{R}^{n-d_0} \times \langle \mathbf{x}_1 \rangle \times C_{\mathbf{x}_0,\mathbf{x}_1}\big) & \text{if } \ \nu = 1, \\ \underline{\mathrm{U}}^0\Big(\mathbb{R}^{n-d_0} \times \langle \mathbf{x}_1 \rangle \times \ldots \times \underline{\mathrm{U}}^{\nu-1}\big(\mathbb{R}^{d_{\nu-2}-1-d_{\nu-1}} \times \langle \mathbf{x}_\nu \rangle \times C_{\mathbf{x}_0,\ldots,\mathbf{x}_\nu}\big) \ldots\Big) & \text{if } \ \nu \geq 2. \end{cases} \tag{3.32}$$

Note that if $d_\nu = 0$, the cone $C_{\mathbf{x}_0,\ldots,\mathbf{x}_\nu}$ coincides with $\mathbb{R}^{d_{\nu-1}-1}$, leading to $\Pi_{\mathbb{X}} = \mathbb{R}^n$.

Definition 3.19. Let $\mathbb{X} = (\mathbf{x}_0, \ldots, \mathbf{x}_\nu)$ be a chain in $\mathfrak{C}(\Omega)$.

(i) The cone $\Pi_{\mathbb{X}}$ defined in (3.32) is called a *tangent substructure* of $\Pi_{\mathbf{x}_0}$.

(ii) Let $\mathbb{X}' = (\mathbf{x}'_0, \ldots, \mathbf{x}'_{\nu'})$ be another chain in $\mathfrak{C}(\Omega)$. We say that $\mathbb{X}'$ is equivalent to $\mathbb{X}$ if $\mathbf{x}'_0 = \mathbf{x}_0$ and $\Pi_{\mathbb{X}'} = \Pi_{\mathbb{X}}$. ■

This notion of equivalence is well suited to the class of operators that we consider in this paper, see also Remark 4.3 later on.

3.4.2. *Strata of a corner domain.* For $d \in \{0, \ldots, n\}$, let

$$\mathfrak{A}_d(\Omega) = \{\mathbf{x} \in \overline{\Omega}, \quad d_0(\mathbf{x}) = d\}. \tag{3.33}$$

The strata of $\overline{\Omega}$ are the connected components of $\mathfrak{A}_d(\Omega)$, for $d \in \{0, \ldots, n\}$. They are denoted by $\mathbf{t}$ and their set by $\mathfrak{T}$.

Examples:

- $\mathfrak{A}_0(\Omega)$ coincides with Ω.
- $\mathfrak{A}_1(\Omega)$ is the subset of $\partial\Omega$ of the regular points of the boundary (the corresponding strata being the faces in dimension $n = 3$ and the sides in dimension $n = 2$).
- If $n = 2$, $\mathfrak{A}_2(\Omega)$ is the set of corners.
- If $n = 3$, $\mathfrak{A}_2(\Omega)$ is the set of edge points.
- If $n = 3$, $\mathfrak{A}_3(\Omega)$ is the set of corners.

Proposition 3.20. *Let* $\mathbf{t} \in \mathfrak{A}_d(\Omega)$ *be a stratum. Then* $\mathbf{t}$ *is a smooth submanifold*[7] *of codimension* d*. In particular* $\mathfrak{A}_n(\Omega)$ *is a finite subset of* $\partial\Omega$*.*

Proof. Let $\mathbf{x}_0 \in \mathbf{t}$ and $(\mathcal{U}_{\mathbf{x}_0}, \mathrm{U}^{\mathbf{x}_0})$ be an associated local map. The tangent cone at $\mathbf{x}_0$ writes $\Pi_{\mathbf{x}_0} = \underline{\mathrm{U}}\left(\mathbb{R}^{n-d} \times \Gamma_{\mathbf{x}_0}\right)$, with $\Gamma_{\mathbf{x}_0} \in \mathfrak{P}_d$. For simplicity, we may assume that $\underline{\mathrm{U}} = \mathbb{I}_n$. Denote by π the orthogonal projection on $\mathbb{R}^{n-d}$ and set $\pi^\perp := \mathbb{I}_n - \pi$. Let $\mathbf{u} \in \mathcal{U}_{\mathbf{x}_0}$ and $\mathbf{v} = \mathrm{U}^{\mathbf{x}_0}(\mathbf{u})$. According as $\pi^\perp(\mathbf{v})$ is 0 or not, the tangent cone $\Pi_\mathbf{v}$ at $\mathbf{v}$ to $\Pi_{\mathbf{x}_0}$ has distinct expressions.

(1) If $\pi^\perp(\mathbf{v}) = 0$, then $\mathrm{U}^\mathbf{v}$ is the translation by $\mathbf{v}$ and $\Pi_\mathbf{v} = \Pi_{\mathbf{x}_0}$.

(2) If $\pi^\perp(\mathbf{v}) \neq 0$, we introduce the cylindrical coordinates $(r(\mathbf{v}), \theta(\mathbf{v}), \pi(\mathbf{v}))$ of $\mathbf{v}$ with:

$$r(\mathbf{v}) = \|\pi^\perp(\mathbf{v})\|, \quad \theta(\mathbf{v}) = \frac{\pi^\perp(\mathbf{v})}{\|\pi^\perp(\mathbf{v})\|} \in \overline{\Omega}_{\mathbf{x}_0} \quad \text{with} \quad \Omega_{\mathbf{x}_0} = \Gamma_{\mathbf{x}_0} \cap \mathbb{S}^{d-1}\,. \tag{3.34}$$

Let $\Pi_{\theta(\mathbf{v})} \in \mathfrak{P}_{d-1}$ be the tangent cone to $\Omega_{\mathbf{x}_0}$ at $\theta(\mathbf{v})$. We have, cf. proof of Lemma 3.7,

$$\Pi_\mathbf{v} := \mathbb{R}^{n-d} \times \langle \pi^\perp(\mathbf{v}) \rangle \times \Pi_{\theta(\mathbf{v})}\,. \tag{3.35}$$

In any case, the tangent cone $\Pi_\mathbf{u}$ is linked to $\Pi_\mathbf{v}$ by the formula $\Pi_\mathbf{u} = \mathrm{J}^{\mathbf{x}_0}(\mathbf{v})(\Pi_\mathbf{v})$. We deduce:

(1) If $\pi^\perp(\mathbf{v}) = 0$, then $d(\Pi_\mathbf{u}) = d(\Pi_{\mathbf{x}_0})$ (cf. Remark 3.17), therefore $d(\mathbf{u}) = d(\mathbf{x}_0) = d$ and $\mathbf{u} \in \mathfrak{A}_d(\Omega)$.

(2) If $\pi^\perp(\mathbf{v}) \neq 0$, then $d(\Pi_\mathbf{u}) = d(\Pi_\mathbf{v})$ and we have $d(\mathbf{u}) \geq d + 1 > d(\mathbf{x}_0) = d$.

Therefore $\mathbf{u} \in \mathfrak{A}_d(\Omega)$ if and only if $\pi^\perp(\mathbf{v}) = 0$. We conclude that

$$\mathfrak{A}_d(\Omega) \cap \mathcal{U}_{\mathbf{x}_0} = (\mathrm{U}^{\mathbf{x}_0})^{-1}(\pi(\mathcal{V}_{\mathbf{x}_0})).$$

Hence the stratum $\mathbf{t}$ is a smooth submanifold of codimension d. □

Remark 3.21. Let Ω be a corner domain and $\mathfrak{X}$ be the set of reference points of an admissible atlas, cf. Definition 3.11. Let $\mathbf{x}_0 \in \mathfrak{X}$. As a consequence of the above proof we find that for any $\mathbf{u}_0 \in \mathcal{B}(\mathbf{x}_0, R_{\mathbf{x}_0})$, $d(\mathbf{u}_0) \geq d(\mathbf{x}_0)$. Thus, in particular, the set of corners $\mathfrak{A}_n(\Omega)$ is contained in $\mathfrak{X}$.

3.4.3. *Topology on singular chains.* Here we introduce a distance on equivalence classes of the set of chains $\mathfrak{C}(\Omega)$, for the equivalence already introduced in Definition 3.19. This will allow to introduce natural notions of continuity and lower semi-continuity on chains.

Let us denote by $\mathrm{BGL}(n)$ the ring of linear isomorphisms L with norm $\|L\| \leq 1$, where

$$\|L\| = \max_{\mathbf{x} \in \mathbb{R}^n \setminus \{0\}} \frac{\|L\mathbf{x}\|}{\|\mathbf{x}\|}\,.$$

[7] This means that for each $\mathbf{x}_0 \in \mathbf{t}$ there exists a neighborhood $\mathcal{U} \subset \mathbf{t}$ of $\mathbf{x}_0$ and an associate local diffeomorphism from $\mathcal{U}$ onto an open set in $\mathbb{R}^{n-d}$.

Definition 3.22. Let $\mathbb{X} = (\mathbf{x}_0, \dots, \mathbf{x}_\nu)$ and $\mathbb{X}' = (\mathbf{x}'_0, \dots, \mathbf{x}'_{\nu'})$ be two singular chains in $\mathfrak{C}(\Omega)$. We define the distance $\mathbb{D}(\mathbb{X}, \mathbb{X}') \in \mathbb{R}_+ \cup \{+\infty\}$ as

$$\mathbb{D}(\mathbb{X}, \mathbb{X}') = \|\mathbf{x}_0 - \mathbf{x}'_0\| + \frac{1}{2}\Big\{ \min_{\substack{L\in \mathrm{BGL}(n) \\ L\Pi_{\mathbb{X}} = \Pi_{\mathbb{X}'}}} \|L - \mathbb{I}_n\| + \min_{\substack{L\in \mathrm{BGL}(n) \\ L\Pi_{\mathbb{X}'} = \Pi_{\mathbb{X}}}} \|L - \mathbb{I}_n\| \Big\},$$

where the second term is set to $+\infty$ if $\Pi_{\mathbb{X}}$ and $\Pi_{\mathbb{X}'}$ do not belong to the same orbit for the action of $\mathrm{BGL}(n)$ on $\mathfrak{P}_n$.

Remark 3.23. (i) The distance $\mathbb{D}(\mathbb{X}, \mathbb{X}')$ is zero if and only if the chains $\mathbb{X}$ and $\mathbb{X}'$ are equivalent.

(ii) As a consequence of the proof of Proposition 3.20, the strata of $\overline{\Omega}$ are contained in orbits of the natural action of $\mathrm{BGL}(n)$ on chains.

(iii) For strata of polyhedral domains, the distance $\mathbb{D}$ between chains of length 1 is equivalent to the standard distance in $\mathbb{R}^n$. This is no longer true for strata containing conical points in their closure.

We define a partial order on chains.

Definition 3.24. Let $\mathbb{X} = (\mathbf{x}_0, \dots, \mathbf{x}_\nu)$ and $\mathbb{X}' = (\mathbf{x}'_0, \dots, \mathbf{x}'_{\nu'})$ be two singular chains in $\mathfrak{C}(\Omega)$. We say that $\mathbb{X} \le \mathbb{X}'$ if $\nu \le \nu'$ and $\mathbf{x}_j = \mathbf{x}'_j$ for all $0 \le j \le \nu$.

Theorem 3.25. *Let Ω be a corner domain in $\mathfrak{D}(M)$ with $M = \mathbb{R}^n$ or $\mathbb{S}^n$, and $F : \mathfrak{C}(\Omega) \to \mathbb{R}$ be a function such that*

(i) F is continuous on $\mathfrak{C}(\Omega)$ for the distance $\mathbb{D}$

(ii) F is order-preserving on $\mathfrak{C}(\Omega)$ (i.e. $\mathbb{X} \le \mathbb{X}'$ implies $F(\mathbb{X}) \le F(\mathbb{X}')$).

Then for all chain $\mathbb{X} = (\mathbf{x}_0, \dots, \mathbf{x}_\nu) \cup \{\emptyset\}$, the function (with the convention that $\Omega_\emptyset = \Omega$)

$$\overline{\Omega}_{\mathbf{x}_0,\dots,\mathbf{x}_\nu} \ni \mathbf{x} \longmapsto F((\mathbf{x}_0, \dots, \mathbf{x}_\nu, \mathbf{x}))$$

is lower semi-continuous. In particular $\overline{\Omega} \ni \mathbf{x} \mapsto F((\mathbf{x}))$ is lower semi-continuous.

Proof. The proof is recursive over the dimension n.

Initialization. $n = 1$. Let Ω belong to $\mathfrak{D}(M)$ with $M = \mathbb{R}$ or $\mathbb{S}^1$. Then Ω is an open interval $(\mathbf{c}, \mathbf{c}')$. The chains in $\mathfrak{C}(\Omega)$ are

- $\mathbb{X} = (\mathbf{x}_0)$ for $\mathbf{x}_0 \in (\mathbf{c}, \mathbf{c}')$ with $\Pi_{\mathbb{X}} = \mathbb{R}$,
- $\mathbb{X} = (\mathbf{x}_0)$ for $\mathbf{x}_0 = \mathbf{c}$ and $\mathbf{x}_0 = \mathbf{c}'$, with $\Pi_{\mathbb{X}} = \mathbb{R}_+$ and $\mathbb{R}_-$, respectively,
- $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1)$ for $\mathbf{x}_0 = \mathbf{c}$ or $\mathbf{x}_0 = \mathbf{c}'$, and $\mathbf{x}_1 = 1$, with $\Pi_{\mathbb{X}} = \mathbb{R}$.

The function F is continuous on $\mathfrak{C}(\Omega)$. By definition of the distance $\mathbb{D}$ there holds

$$\mathbb{D}\big((\mathbf{x}), (\mathbf{c}, 1)\big) = \|\mathbf{x} - \mathbf{c}\| \quad \text{and} \quad \mathbb{D}\big((\mathbf{x}), (\mathbf{c}', 1)\big) = \|\mathbf{x} - \mathbf{c}'\|, \qquad \forall \mathbf{x} \in (\mathbf{c}, \mathbf{c}')\,.$$

Therefore, as $\mathbf{x} \to \mathbf{c}$, with $\mathbf{x} \neq \mathbf{c}$, $F((\mathbf{x}))$ tends to $F((\mathbf{c}, 1))$. By assumption $F((\mathbf{c}, 1)) \ge F((\mathbf{c}))$, and the same at the other end $\mathbf{c}'$. This proves that F is lower semi-continuous on $\overline{\Omega} = [\mathbf{c}, \mathbf{c}']$.

Recurrence. We assume that Theorem 3.25 holds for any dimension $n^\star < n$. Let us prove it for the dimension n.

a) Let $\mathbb{X}_0$ be a non-empty chain in $\mathfrak{C}(\Omega)$. Then $\Omega_{\mathbb{X}_0}$ belongs to $\mathfrak{D}(\mathbb{S}^{n^\star})$ for an $n^\star < n$. The chains $\mathbb{Y} \in \mathfrak{C}(\Omega_{\mathbb{X}_0})$ correspond to the chains $(\mathbb{X}_0, \mathbb{Y})$ in $\mathfrak{C}(\Omega)$ and the corresponding tangent substructures $\Pi_{\mathbb{Y}} \in \mathfrak{P}_{n^\star}$ and $\Pi_{\mathbb{X}_0,\mathbb{Y}} \in \mathfrak{P}_n$ are linked by a relation of the type, cf. (3.32)

$$\Pi_{\mathbb{X}_0,\mathbb{Y}} = \underline{\mathrm{U}}^0\big(\mathbb{R}^{n-d_0} \times \langle \mathbf{x}_1 \rangle \times \ldots \times \Pi_{\mathbb{Y}}\big)$$

Hence the distances $\mathbb{D}\big((\mathbb{X}_0, \mathbb{Y}), (\mathbb{X}_0, \mathbb{Y}')\big)$ and $\mathbb{D}\big(\mathbb{Y}, \mathbb{Y}'\big)$ can be compared:

$$\begin{aligned}\mathbb{D}\big((\mathbb{X}_0, \mathbb{Y}), (\mathbb{X}_0, \mathbb{Y}')\big) &= \frac{1}{2}\Big\{ \min_{\substack{L\in \mathrm{BGL}(n)\\ L\Pi_{\mathbb{X}_0,\mathbb{Y}}=\Pi_{\mathbb{X}_0,\mathbb{Y}'}}} \|L-\mathbb{I}_n\| + \min_{\substack{L\in \mathrm{BGL}(n)\\ L\Pi_{\mathbb{X}_0,\mathbb{Y}'}=\Pi_{\mathbb{X}_0,\mathbb{Y}}}} \|L-\mathbb{I}_n\| \Big\} \\ &\le \frac{1}{2}\Big\{ \min_{\substack{L^\star\in \mathrm{BGL}(n^\star)\\ L^\star\Pi_{\mathbb{Y}}=\Pi_{\mathbb{Y}'}}} \|L^\star-\mathbb{I}_{n^\star}\| + \min_{\substack{L^\star\in \mathrm{BGL}(n^\star)\\ L^\star\Pi_{\mathbb{Y}'}=\Pi_{\mathbb{Y}}}} \|L^\star-\mathbb{I}_{n^\star}\| \Big\} \\ &\le \ \mathbb{D}\big(\mathbb{Y}, \mathbb{Y}'\big).\end{aligned}$$

Let us define the function $F^\star$ on $\mathfrak{C}(\Omega_{\mathbb{X}_0})$ by the partial application

$$F^\star(\mathbb{Y}) = F((\mathbb{X}_0, \mathbb{Y})), \quad \mathbb{Y} \in \mathfrak{C}(\Omega_{\mathbb{X}_0}).$$

Since F is continuous on $\mathfrak{C}(\Omega)$, the above inequality between distances prove that $F^\star$ is continuous on $\mathfrak{C}(\Omega_{\mathbb{X}_0})$. Likewise the monotonicity property is obviously transported from F to $F^\star$. Therefore the recurrence assumption provides the lower semi-continuity of $F^\star$ on $\overline{\Omega}_{\mathbb{X}_0}$, hence of $\mathbf{x} \mapsto F((\mathbb{X}_0, \mathbf{x}))$ on the same set.

b) It remains to prove that $\mathbf{x} \mapsto F((\mathbf{x}))$ is lower semi-continuous on $\overline{\Omega}$. Let $\mathbf{x}_0 \in \overline{\Omega}$. At this point we follow the proof of Proposition 3.20. For any $\mathbf{u} \in \mathcal{U}_{\mathbf{x}_0}$, we define π, $\pi^\perp$ and $\mathbf{v}$ like there and encounter the same two cases:

(1) If $\pi^\perp(\mathbf{v}) = 0$, then $\Pi_{\mathbf{v}} = \Pi_{\mathbf{x}_0}$. Hence $\Pi_{\mathbf{u}} = \mathrm{J}^{\mathbf{x}_0}(\mathbf{v})(\Pi_{\mathbf{x}_0})$. Since $\mathrm{J}^{\mathbf{x}_0}(\mathbf{v})$ tends to $\mathbb{I}_n$ as $\mathbf{v} \to \mathbf{0}$, the distance $\mathbb{D}((\mathbf{x}_0), (\mathbf{u}))$ tends to 0 as $\mathbf{u}$ tends to $\mathbf{x}_0$. By the continuity assumption, $F((\mathbf{u}))$ tends to $F((\mathbf{x}_0))$.

(2) If $\pi^\perp(\mathbf{v}) \neq 0$, let $\mathbf{x}_1$ be the element of $\overline{\Omega}_{\mathbf{x}_0}$ defined by $\mathbf{x}_1 = \pi^\perp(\mathbf{v})\,\|\pi^\perp(\mathbf{v})\|^{-1}$. Let $\Pi_{\mathbf{x}_1} \in \mathfrak{P}_{d-1}$ be the tangent cone to $\Omega_{\mathbf{x}_0}$ at $\mathbf{x}_1$. We find

$$\Pi_{\mathbf{v}} = \mathbb{R}^{n-d} \times \langle \pi^\perp(\mathbf{v}) \rangle \times \Pi_{\mathbf{x}_1} = \Pi_{\mathbf{x}_0,\mathbf{x}_1}\,.$$

Hence $\Pi_{\mathbf{u}} = \mathrm{J}^{\mathbf{x}_0}(\mathbf{v})(\Pi_{\mathbf{x}_0,\mathbf{x}_1})$. Like before, we deduce that the distance $\mathbb{D}((\mathbf{x}_0, \mathbf{x}_1), (\mathbf{u}))$ tends to 0 as $\mathbf{u}$ tends to $\mathbf{x}_0$. By the continuity assumption, $F((\mathbf{u}))$ tends to $F((\mathbf{x}_0, \mathbf{x}_1))$, which by the monotonicity assumption, is larger than $F((\mathbf{x}_0))$.

This ends the proof of the theorem. □

3.4.4. *Singular chains and admissible atlantes.* The aim of this section is to provide an overview of map-neighborhoods and Jacobian estimates in the framework of singular chains. In their generality, these facts are not needed for our study of magnetic Laplacians, which is restricted to dimension $n \leq 3$ for distinct reasons that we will explain later on. Nevertheless, full generality sheds some light on the recursive structure present in the very definition of admissible atlantes and in the domain of validity of estimates in Proposition 3.13.

• *Chains of atlantes.* Denote by $\mathfrak{X}(\Omega)$ the set of reference points of an admissible atlas for a corner domain Ω. The chain of atlantes of a corner domain Ω is defined as follows:

(0) Start from the set $\mathfrak{X}(\Omega)$ of reference points $\mathbf{x}_0 \in \overline{\Omega}$, as in Definition 3.11.
(1) For each $\mathbf{x}_0 \in \mathfrak{X}(\Omega)$, choose an admissible atlas of the section $\Omega_{\mathbf{x}_0} \in \mathfrak{D}(\mathbb{S}^{d_0-1})$, with set $\mathfrak{X}(\Omega_{\mathbf{x}_0})$ of reference points $\mathbf{x}_1 \in \overline{\Omega}_{\mathbf{x}_0}$.
(2) For each $\mathbf{x}_1 \in \mathfrak{X}(\Omega_{\mathbf{x}_0})$, choose an admissible atlas of the section $\Omega_{\mathbf{x}_0,\mathbf{x}_1} \in \mathfrak{D}(\mathbb{S}^{d_1-1})$, with set $\mathfrak{X}(\Omega_{\mathbf{x}_0,\mathbf{x}_1})$ of reference points $\mathbf{x}_2 \in \overline{\Omega}_{\mathbf{x}_0,\mathbf{x}_1}$. And so on...

• *Cylindrical coordinates.* The natural coordinates associated with chains of atlantes are recursively defined cylindrical coordinates. Let $\mathbf{u}_0 \in \overline{\Omega}$.

(1) If $\mathbf{u}_0 \notin \mathfrak{X}(\Omega)$, pick $\mathbf{x}_0 \in \mathfrak{X}(\Omega)$ such that $\mathbf{u}_0 \in \mathcal{B}^n(\mathbf{x}_0, R_{\mathbf{x}_0})$ (n-dimensional ball). Then define $\mathbf{v}_0 = \mathrm{U}^{\mathbf{x}_0}\mathbf{u}_0$ and, if $d_0 > 0$, its cylindrical coordinates
$$\pi_0(\mathbf{v}_0) \in \mathbb{R}^{n-d_0}, \quad r(\mathbf{v}_0) = \|\mathbf{v}_0 - \pi_0(\mathbf{v}_0)\|, \quad \text{and} \quad \mathbf{u}_1 = \frac{\mathbf{v}_0 - \pi_0(\mathbf{v}_0)}{r(\mathbf{v}_0)} \in \overline{\Omega}_{\mathbf{x}_0}\,.$$
If $d_0 = 0$, $\pi_0 = \mathbb{I}_n$, then stop.
(2) If $\mathbf{u}_1 \notin \mathfrak{X}(\Omega_{\mathbf{x}_0})$, pick $\mathbf{x}_1 \in \mathfrak{X}(\Omega_{\mathbf{x}_0})$ such that $\mathbf{u}_1 \in \mathcal{B}^{d_0}(\mathbf{x}_1, R_{\mathbf{x}_1}) \cap \mathbb{S}^{d_0-1}$. Then define $\mathbf{v}_1 = \mathrm{U}^{\mathbf{x}_0,\mathbf{x}_1}\mathbf{u}_1$ and, if $d_1 > 0$, its cylindrical coordinates
$$\pi_1(\mathbf{v}_1) \in \mathbb{R}^{d_0-1-d_1}, \quad r(\mathbf{v}_1) = \|\mathbf{v}_1 - \pi_1(\mathbf{v}_1)\|, \quad \text{and} \quad \mathbf{u}_2 = \frac{\mathbf{v}_1 - \pi_1(\mathbf{v}_1)}{r(\mathbf{v}_1)} \in \overline{\Omega}_{\mathbf{x}_0,\mathbf{x}_1}\,.$$
If $d_1 = 0$, $\pi_1 = \mathbb{I}_n$, then stop. And so on...

Let $\mathbf{v}_{\nu_*}$ be the last element of the sequence $\mathbf{v}_0, \mathbf{v}_1, \ldots$. In any case $\nu_* \leq n$.

• *Local maps.* The local maps are recursively constructed using the natural coordinates associated with chains.

(0) If $\mathbf{u}_0 = \mathbf{x}_0 \in \mathfrak{X}(\Omega)$, use the local map $(\mathcal{U}_{\mathbf{x}_0}, \mathrm{U}^{\mathbf{x}_0})$ and stop.
(1) If $\mathbf{u}_0 \notin \mathfrak{X}(\Omega)$, a local map $(\mathcal{U}_{\mathbf{u}_0}, \mathrm{U}^{\mathbf{u}_0})$ is defined by the formulas hereafter. The map neighborhood $\mathcal{U}_{\mathbf{u}_0}$ can be chosen as $(\mathrm{U}^{\mathbf{x}_0})^{-1}(\mathcal{U}_{\mathbf{v}_0})$ with
$$\mathcal{U}_{\mathbf{v}_0} = \mathcal{B}^{n-d_0}(\pi_0(\mathbf{v}_0), R_{\mathbf{x}_0}) \times r(\mathbf{v}_0)\, \mathcal{U}_{(1,\mathbf{u}_1)}, \quad \mathcal{U}_{(1,\mathbf{u}_1)} = \mathcal{B}^{d_0}(\mathbf{u}_1, \rho_1), \quad \mathcal{U}_{\mathbf{u}_1} = \mathcal{U}_{(1,\mathbf{u}_1)} \cap \mathbb{S}^{d_0-1}.$$
The diffeomorphism $\mathrm{U}^{\mathbf{u}_0}$ is defined by $\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}_0)\ (\mathrm{U}^{\mathbf{v}_0} \circ \mathrm{U}^{\mathbf{x}_0})$ with
$$\mathrm{U}^{\mathbf{v}_0} = \big(\mathrm{T}_{\pi_0(\mathbf{v}_0)}\,,\ \mathrm{N}^{-1}_{r(\mathbf{v}_0)} \circ \mathrm{U}^{(1,\mathbf{u}_1)} \circ \mathrm{N}_{r(\mathbf{v}_0)}\big) \quad \text{and} \quad \mathrm{U}^{(1,\mathbf{u}_1)} = (\mathrm{T}_1\,,\ \mathrm{U}^{\mathbf{u}_1})\,,$$
where $\mathrm{T}_{\pi_0(\mathbf{v}_0)}$ is the translation $\mathbf{v} \mapsto \mathbf{v} - \pi_0(\mathbf{v}_0)$ in $\mathbb{R}^{n-d_0}$, and T_1 is the translation by 1 for the radius in polar coordinates. If $\mathbf{u}_1 = \mathbf{x}_1 \in \mathfrak{X}(\Omega_{\mathbf{x}_0})$, stop.

(2) If $\mathbf{u}_1 \notin \mathfrak{X}(\Omega_{\mathbf{x}_0})$, a local map $(\mathcal{U}_{\mathbf{u}_1}, \mathsf{U}^{\mathbf{u}_1})$ is defined like in step (1), replacing $\mathbf{x}_0$ by $\mathbf{x}_1$, $\mathbf{v}_0$ by $\mathbf{v}_1$, $\mathcal{B}^{n-d_0}$ by $\mathcal{B}^{d_0-1-d_1}$, $\pi_0(\mathbf{v}_0)$ by $\pi_1(\mathbf{v}_1)$, $\mathcal{B}^{d_0}$ by $\mathcal{B}^{d_1}$, and finally $\mathbf{u}_1$ by $\mathbf{u}_2$...

• *Estimates on Jacobian matrices.* Let $\mathbf{u}_0 \in \overline{\Omega}$. As explained in Remark 3.8, as soon as a polyhedral cone $\Gamma_{\mathbf{x}_0,\ldots,\mathbf{x}_\nu}$ is reached in the construction, the corresponding diffeomorphism $\mathsf{U}^{(1,\mathbf{u}_{\nu+1})}$ is chosen as a translation, so it is the same for $\mathsf{U}^{\mathbf{u}_{\nu+1}}$, and the norm of its differential is bounded. By recursion, this implies the estimate for the differential $\mathsf{K}^{\mathbf{u}_0}$ of $\mathsf{J}^{\mathbf{u}_0}$

$$\|\mathsf{K}^{\mathbf{u}_0}\| \leq \frac{c(\Omega)}{r(\mathbf{v}_0)\,\cdots\, r(\mathbf{v}_{\nu-1})} \tag{3.36}$$

with the convention that if $\nu - 1 < 0$, the denominator is 1.The same estimate is valid if $\mathbf{u}_\nu \in \mathfrak{X}(\Omega_{\mathbf{x}_0,\ldots,\mathbf{x}_{\nu-1}})$ with the convention that $\Omega_{\mathbf{x}_0,\ldots,\mathbf{x}_{\nu-1}} = \Omega$ if $\nu - 1 < 0$. Note that $\nu = 0$ for any $\mathbf{u}_0$ if the domain Ω is polyhedral. In turn, the domain of validity of estimates (3.36) is (at least) a ball centered at $\mathbf{u}_0$ of radius

$$\rho(\mathbf{u}_0) = r(\Omega)\, r(\mathbf{v}_0)\,\cdots\, r(\mathbf{v}_{\nu_*})\,. \tag{3.37}$$

3.5. **3D domains.** In this section we refine our analysis for the particular case of 3D domains. In each case we provide an exhaustive description of the possible singular chains. We also give the consequences of Proposition 3.13.

3.5.1. *Faces, edges and corners.*

Definition 3.26. Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$. We denote by $\mathfrak{F}$ the set of the connected components of $\mathfrak{A}_1(\Omega)$ (faces), $\mathfrak{E}$ those of $\mathfrak{A}_2(\Omega)$ (edges) and $\mathfrak{V}$ the finite set $\mathfrak{A}_3(\Omega)$ (corners).
Let $\mathbf{x}_0 \in \mathfrak{A}_d(\Omega)$ with $d < 3$, then $\Pi_{\mathbf{x}_0} \in \overline{\mathfrak{P}}_3$. Let $\mathbf{x}_0 \in \mathfrak{V}$, we distinguish between two cases:

(1) If $\Pi_{\mathbf{x}_0} \in \overline{\mathfrak{P}}_3$, then $\mathbf{x}_0$ is a polyhedral corner.

(2) If $\Pi_{\mathbf{x}_0} \notin \overline{\mathfrak{P}}_3$, then $\mathbf{x}_0$ is a conical corner. We denote by $\mathfrak{V}^\circ$ the set of conical corners. ■

Combining Proposition 3.13 and Remark 3.5, we obtain local estimates for the Jacobian matrix and the metric issued from changes of variables pertaining to an admissible atlas:

Corollary 3.27. *Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ and $(\mathcal{U}_{\mathbf{x}}, \mathsf{U}^{\mathbf{x}})_{\mathbf{x}\in\overline{\Omega}}$ be an admissible atlas. Note that the set of its reference points $\mathfrak{X}$ contains $\mathfrak{V}$ (cf. Remark 3.21), thus in particular the set of conical corners $\mathfrak{V}^\circ$. There exists $c(\Omega)$ such that*

(i) for all $\mathbf{x}_0 \in \mathfrak{X}$, there holds

$$\|\mathsf{J}^{\mathbf{x}_0} - \mathbb{I}_3\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} + \|\mathsf{G}^{\mathbf{x}_0} - \mathbb{I}_3\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} \leq r\, c(\Omega), \quad \textit{for all } r \leq R_{\mathbf{x}_0}\,, \tag{3.38}$$

(ii) for all $\mathbf{u}_0 \in \overline{\Omega} \setminus \mathfrak{X}$, there holds

$$\|\mathsf{J}^{\mathbf{u}_0} - \mathbb{I}_n\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} + \|\mathsf{G}^{\mathbf{u}_0} - \mathbb{I}_n\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} \leq \frac{r}{d_{\mathfrak{V}^\circ}(\mathbf{u}_0)}\, c(\Omega), \quad \textit{for all } r \leq \rho(\mathbf{u}_0)\,, \tag{3.39}$$

with $\rho(\mathbf{u}_0)$ *as in Proposition 3.13 and*

$$d_{\mathfrak{V}^\circ}(\mathbf{u}_0) = \begin{cases} 1 & \textit{if} \quad \mathfrak{V}^\circ = \emptyset, \\ \operatorname{dist}(\mathbf{u}_0, \mathfrak{V}^\circ) & \textit{else}. \end{cases} \tag{3.40}$$

Remark 3.28. Note that estimate (3.39) blows up when we get closer to a conical point without reaching it, while at any conical point $\mathbf{x}_0 \in \mathfrak{V}^\circ$, we have the good estimate (3.38). These two estimates can be written in the synthetic way

$$\|\mathrm{J}^{\mathbf{u}_0} - \mathbb{I}_n\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} + \|\mathrm{G}^{\mathbf{u}_0} - \mathbb{I}_n\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} \leq \frac{r}{d^*_{\mathfrak{V}^\circ}(\mathbf{u}_0)}\, c(\Omega), \quad \text{for all } r \leq \rho(\mathbf{u}_0)\,,$$

if we define $\rho(\mathbf{u}_0)$ as $R_{\mathbf{u}_0}$ if $\mathbf{u}_0 \in \mathfrak{X}$ and given in Proposition 3.13 if $\mathbf{u}_0 \in \overline{\Omega}\setminus\mathfrak{X}$, and introduce

$$d^*_{\mathfrak{V}^\circ}(\mathbf{u}_0) = \begin{cases} 1 & \text{if} \quad \mathfrak{V}^\circ = \emptyset, \\ 1 & \text{if} \quad \mathbf{u}_0 \in \mathfrak{V}^\circ, \\ \operatorname{dist}(\mathbf{u}_0, \mathfrak{V}^\circ) & \text{else}. \end{cases}$$

Note that the function $d^*_{\mathfrak{V}^\circ}$ is discontinuous when $\mathfrak{V}^\circ \neq \emptyset$.

In view of the spectral analysis of magnetic Laplacians, the latter corollary will be used in several localization processes around points $\mathbf{x}_0 \in \overline{\Omega}$, leading to distinct analyses depending on how far $\mathbf{x}_0$ is from $\mathfrak{V}^\circ$.

3.5.2. *Singular chains of 3D corner domains.*

Proposition 3.29. *Let* $\Omega \in \mathfrak{D}(\mathbb{R}^3)$*. Then chains of length* ≤ 3 *are sufficient to describe all equivalence classes of the set of chains* $\mathfrak{C}(\Omega)$*. If moreover* $\Omega \in \overline{\mathfrak{D}}(\mathbb{R}^3)$*, chains of length* 2 *are sufficient.*

Proof. Let $\mathbf{x}_0 \in \overline{\Omega}$. In Description 3.30 and Figure 1, we enumerate all the chains starting from $\mathbf{x}_0$ and the associated tangent substructures according as $\mathbf{x}_0$ is an interior point, a face point, an edge point, or a vertex.

Description 3.30.

(1) Interior point $\mathbf{x}_0 \in \Omega$. Only one chain in $\mathfrak{C}_{\mathbf{x}_0}(\Omega)$: $\mathbb{X} = (\mathbf{x}_0)$. $\Pi_{\mathbb{X}} \equiv \mathbb{R}^3$.

(2) Let $\mathbf{x}_0$ belong to a face. There are two chains in $\mathfrak{C}_{\mathbf{x}_0}(\Omega)$:
   (a) $\mathbb{X} = (\mathbf{x}_0)$ with $\Pi_{\mathbb{X}} = \Pi_{\mathbf{x}_0}$, the tangent half-space. $\Pi_{\mathbb{X}} \equiv \mathbb{R}^2 \times \mathbb{R}_+$.
   (b) $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1)$ where $\mathbf{x}_1 = 1$ is the only element in $\mathbb{R}_+ \cap \mathbb{S}^0$. Thus $\Pi_{\mathbb{X}} = \mathbb{R}^3$.

(3) Let $\mathbf{x}_0$ belong to an edge. There are three possible lengths for chains in $\mathfrak{C}_{\mathbf{x}_0}(\Omega)$:
   (a) $\mathbb{X} = (\mathbf{x}_0)$ with $\Pi_{\mathbb{X}} = \Pi_{\mathbf{x}_0}$, the tangent wedge (which is not a half-plane). The reduced cone of $\Pi_{\mathbf{x}_0}$ is a sector $\Gamma_{\mathbf{x}_0}$ the section of which is an interval $\mathcal{I}_{\mathbf{x}_0} \subset \mathbb{S}^1$.
   (b) $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1)$ where $\mathbf{x}_1 \in \overline{\mathcal{I}}_{\mathbf{x}_0}$.
      (i) If $\mathbf{x}_1$ is interior to $\mathcal{I}_{\mathbf{x}_0}$, $\Pi_{\mathbb{X}} = \mathbb{R}^3$. No further chain.

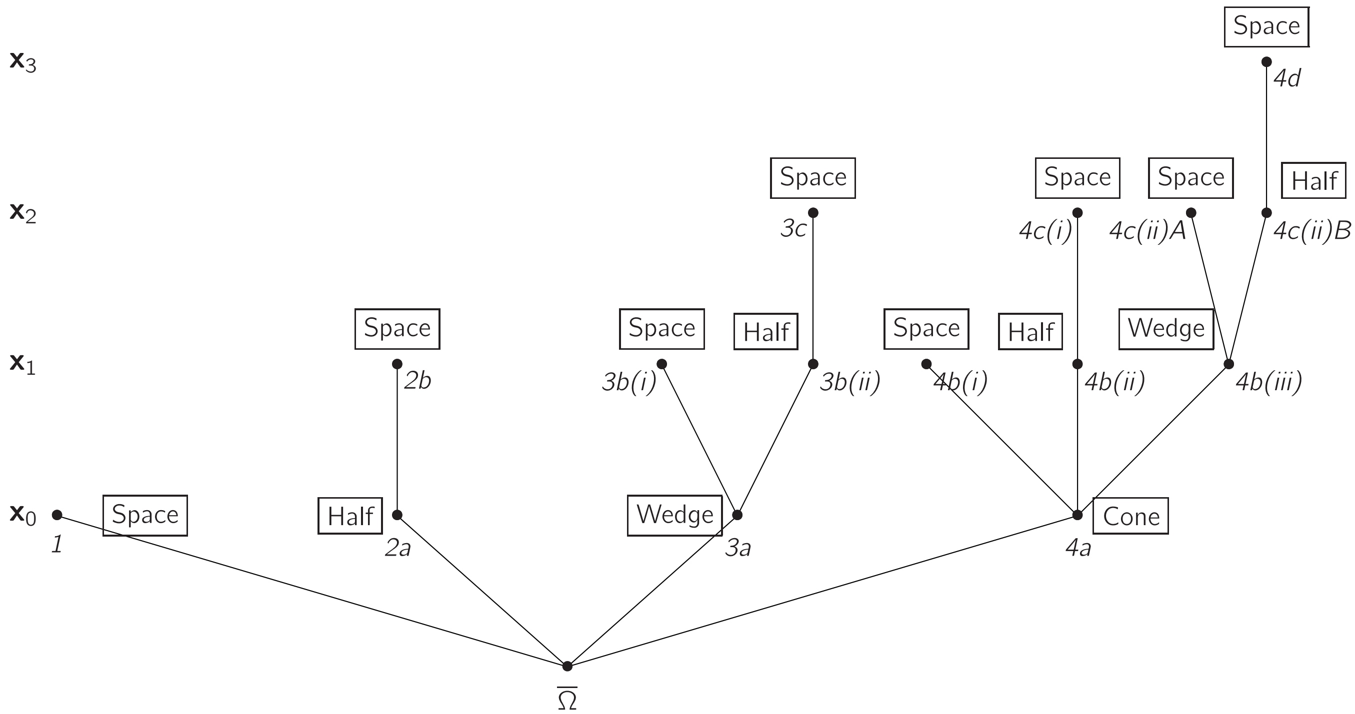

Figure 1. The tree of singular chains with numbering according to Description 3.30 (Half is for half-space)

(ii) If $\mathbf{x}_1$ is a boundary point of $\mathcal{I}_{\mathbf{x}_0}$, $\Pi_{\mathbb{X}}$ is a half-space, containing one of the two faces $\partial^{\pm}\Pi_{\mathbf{x}_0}$ of the wedge $\Pi_{\mathbf{x}_0}$.

(c) $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2)$ where $\mathbf{x}_1 \in \partial\mathcal{I}_{\mathbf{x}_0}$, $\mathbf{x}_2 = 1$ and $\Pi_{\mathbb{X}} = \mathbb{R}^3$.

(4) Let $\mathbf{x}_0$ be a corner. There are four possible lengths for chains in $\mathfrak{C}_{\mathbf{x}_0}(\Omega)$:

(a) $\mathbb{X} = (\mathbf{x}_0)$ with $\Pi_{\mathbb{X}} = \Pi_{\mathbf{x}_0}$, the tangent cone (which is not a wedge). It coincides with its reduced cone. Its section $\Omega_{\mathbf{x}_0}$ is a polygonal domain in $\mathbb{S}^2$.

(b) $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1)$ where $\mathbf{x}_1 \in \overline{\Omega}_{\mathbf{x}_0}$.

(i) If $\mathbf{x}_1$ is interior to $\Omega_{\mathbf{x}_0}$, $\Pi_{\mathbb{X}} = \mathbb{R}^3$. No further chain.

(ii) If $\mathbf{x}_1$ is in a side of $\Omega_{\mathbf{x}_0}$, $\Pi_{\mathbb{X}}$ is a half-space.

(iii) If $\mathbf{x}_1$ is a corner of $\Omega_{\mathbf{x}_0}$, $\Pi_{\mathbb{X}}$ is a wedge. Its edge contains one of the edges of $\Pi_{\mathbf{x}_0}$.

(c) $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2)$ where $\mathbf{x}_1 \in \partial\Omega_{\mathbf{x}_0}$

(i) If $\mathbf{x}_1$ is in a side of $\Omega_{\mathbf{x}_0}$, $\mathbf{x}_2 = 1$, $\Pi_{\mathbb{X}} = \mathbb{R}^3$. No further chain.

(ii) If $\mathbf{x}_1$ is a corner of $\Omega_{\mathbf{x}_0}$, $C_{\mathbf{x}_0,\mathbf{x}_1}$ is plane sector, and $\mathbf{x}_2 \in \overline{\mathcal{I}}_{\mathbf{x}_0,\mathbf{x}_1}$ where the interval $\mathcal{I}_{\mathbf{x}_0,\mathbf{x}_1}$ is its section.

(A) If $\mathbf{x}_2$ is an interior point of $\mathcal{I}_{\mathbf{x}_0,\mathbf{x}_1}$, then $\Pi_{\mathbb{X}} = \mathbb{R}^3$.

(B) If $\mathbf{x}_2$ is a boundary point of $\mathcal{I}_{\mathbf{x}_0,\mathbf{x}_1}$, then $\Pi_{\mathbb{X}}$ is a half-space.

(d) $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ where $\mathbf{x}_1$ is a corner of $\Omega_{\mathbf{x}_0}$, $\mathbf{x}_2 \in \partial\mathcal{I}_{\mathbf{x}_0,\mathbf{x}_1}$ and $\mathbf{x}_3 = 1$. Then $\Pi_{\mathbb{X}} = \mathbb{R}^3$.

To sum up, there are 4 equivalence classes in $\mathfrak{C}_{\mathbf{x}_0}(\Omega)$ in the case of an edge point $\mathbf{x}_0$:

- $\mathbb{X} = (\mathbf{x}_0)$
- $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1^{\pm})$ with $\{\mathbf{x}_1^-, \mathbf{x}_1^+\} = \partial\mathcal{I}_{\mathbf{x}_0}$
- $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1^{\circ})$ with $\mathbf{x}_1^{\circ}$ any chosen point in $\mathcal{I}_{\mathbf{x}_0}$.

If $\mathbf{x}_0$ is a polyhedral corner, the set of the equivalence classes of $\mathfrak{C}_{\mathbf{x}_0}(\Omega)$ is finite according to the following description. Let $\mathbf{x}_1^j$, $1 \le j \le N$, be the corners of $\Omega_{\mathbf{x}_0}$, and $\mathbf{f}_1^j$, $1 \le j \le N$, be its sides (notice that there are as many corners as sides). There are $2N+2$ equivalence classes in $\mathfrak{C}_{\mathbf{x}_0}(\Omega)$:

- $\mathbb{X} = (\mathbf{x}_0)$ (vertex)
- $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1^j)$ with $1 \le j \le N$ (edge-point limit)
- $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1^{\circ,j})$ with $\mathbf{x}_1^{\circ,j}$ any chosen point inside $\mathbf{f}_1^j$ (face-point limit)
- $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1^{\circ})$ with $\mathbf{x}_1^{\circ}$ any chosen point in $\Omega_{\mathbf{x}_0}$ (interior point limit).

Finally, if $\mathbf{x}_0$ belongs to $\mathfrak{V}^{\circ}$, the set of chains which are face-point limits is infinite. Moreover, chains $(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2)$ obtained by the general above procedure (4)-(c)-(ii)-(B) can be irreducible: Such chains represent the limit of a conical face close to an edge. □

4. Magnetic Laplacians and their tangent operators

Let $\mathbf{A}$ be a magnetic potential associated with the magnetic field $\mathbf{B}$ on a corner domain $\Omega \in \mathfrak{D}(\mathbb{R}^3)$. We recall that the corresponding magnetic Laplacian is

$$H_h(\mathbf{A}, \Omega) = (-ih\nabla + \mathbf{A})^2.$$

At each point $\mathbf{x}_0 \in \overline{\Omega}$ is associated a local map $(\mathcal{U}_{\mathbf{x}_0}, \mathrm{U}^{\mathbf{x}_0})$ and a tangent cone $\Pi_{\mathbf{x}_0}$, cf. (3.1). We will associate a tangent magnetic potential to $\Pi_{\mathbf{x}_0}$ and provide formulas and estimates for the operator transformed from the magnetic Laplacian $H_h(\mathbf{A}, \Omega)$ by the local map $(\mathcal{U}_{\mathbf{x}_0}, \mathrm{U}^{\mathbf{x}_0})$.

4.1. **Change of variables.** Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$. We consider a magnetic potential $\mathbf{A} \in \mathscr{C}^1(\overline{\Omega})$. Let $\mathbf{x}_0 \in \overline{\Omega}$. Let us recall from Section 3.1 that the local smooth diffeomorphism $\mathrm{U}^{\mathbf{x}_0}$ maps a neighborhood $\mathcal{U}_{\mathbf{x}_0}$ of $\mathbf{x}_0$ onto a neighborhood $\mathcal{V}_{\mathbf{x}_0}$ of $\mathbf{0}$ so that

$$\mathrm{U}^{\mathbf{x}_0}(\mathcal{U}_{\mathbf{x}_0} \cap \Omega) = \mathcal{V}_{\mathbf{x}_0} \cap \Pi_{\mathbf{x}_0} \quad \text{and} \quad \mathrm{U}^{\mathbf{x}_0}(\mathcal{U}_{\mathbf{x}_0} \cap \partial\Omega) = \mathcal{V}_{\mathbf{x}_0} \cap \partial\Pi_{\mathbf{x}_0}.$$

The differential of $\mathrm{U}^{\mathbf{x}_0}$ at the point $\mathbf{x}_0$ is the identity matrix $\mathbb{I}_3$. We recall that

$$\mathrm{J}^{\mathbf{x}_0} = \mathrm{d}(\mathrm{U}^{\mathbf{x}_0})^{-1} \quad \text{and} \quad \mathrm{G}^{\mathbf{x}_0} = (\mathrm{J}^{\mathbf{x}_0})^{-1}((\mathrm{J}^{\mathbf{x}_0})^{-1})^{\top}$$

denote the jacobian matrix of the inverse of $\mathrm{U}^{\mathbf{x}_0}$ and the associated metric. According to formulas (A.4)–(A.5), we introduce the magnetic potential $\mathbf{A}^{\mathbf{x}_0}$ and magnetic field $\mathbf{B}^{\mathbf{x}_0} = \operatorname{curl} \mathbf{A}^{\mathbf{x}_0}$ transformed by $\mathrm{U}^{\mathbf{x}_0}$ in $\mathcal{V}_{\mathbf{x}_0} \cap \Pi_{\mathbf{x}_0}$

$$\mathbf{A}^{\mathbf{x}_0} := (\mathrm{J}^{\mathbf{x}_0})^{\top}\big((\mathbf{A} - \mathbf{A}(\mathbf{x}_0)) \circ (\mathrm{U}^{\mathbf{x}_0})^{-1}\big) \quad \text{and} \quad \mathbf{B}^{\mathbf{x}_0} := |\det \mathrm{J}^{\mathbf{x}_0}|\,(\mathrm{J}^{\mathbf{x}_0})^{-1}\big(\mathbf{B} \circ (\mathrm{U}^{\mathbf{x}_0})^{-1}\big)\,. \tag{4.1}$$

We also introduce the phase shift

$$\zeta_h^{\mathbf{x}_0}(\mathbf{x}) = e^{i\langle \mathbf{A}(\mathbf{x}_0),\mathbf{x}\rangle/h}, \quad \mathbf{x} \in \Omega, \tag{4.2}$$

so that there holds for any f in $H^1(\Omega)$

$$q_h[\mathbf{A},\Omega](f) = q_h[\mathbf{A} - \mathbf{A}(\mathbf{x}_0),\Omega](\zeta_h^{\mathbf{x}_0} f). \tag{4.3}$$

To $f \in H^1(\Omega)$ with support in $\mathcal{U}_{\mathbf{x}_0}$ we associate the function ψ

$$\psi := (\zeta_h^{\mathbf{x}_0} f) \circ (\mathrm{U}^{\mathbf{x}_0})^{-1}, \tag{4.4}$$

defined in $\Pi_{\mathbf{x}_0}$, with support in $\mathcal{V}_{\mathbf{x}_0}$. For any $h > 0$ Lemma A.4 provides the identities

$$q_h[\mathbf{A},\Omega](f) = q_h[\mathbf{A}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0},\mathrm{G}^{\mathbf{x}_0}](\psi) \quad \text{and} \quad \|f\|_{L^2(\Omega)} = \|\psi\|_{L^2_{\mathrm{G}}(\Pi_{\mathbf{x}_0})}, \tag{4.5}$$

where the quadratic forms $q_h[\mathbf{A},\Omega]$ and $q_h[\mathbf{A}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0},\mathrm{G}^{\mathbf{x}_0}]$ are defined in (1.16) and (1.20), respectively. Using the Rayleigh quotient, we immediately deduce

$$\mathcal{Q}_h[\mathbf{A},\Omega](f) = \mathcal{Q}_h[\mathbf{A}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0},\mathrm{G}^{\mathbf{x}_0}](\psi). \tag{4.6}$$

4.2. **Model and tangent operators.**

Definition 4.1. We call *model operator* any magnetic Laplacian $H(\mathbf{A},\Pi)$ where $\Pi \in \mathfrak{P}_3$ and $\mathbf{A}$ is a linear potential associated with the constant magnetic field $\mathbf{B}$. We denote by $E(\mathbf{B},\Pi)$ the bottom of the spectrum (ground state energy) of $H(\mathbf{A},\Pi)$ and by $\lambda_{\mathrm{ess}}(\mathbf{B},\Pi)$ the bottom of its essential spectrum.

Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ and $\mathbf{A} \in \mathscr{C}^1(\overline{\Omega})$. For each $\mathbf{x}_0 \in \overline{\Omega}$ we set

$$\mathbf{B}_{\mathbf{x}_0} = \mathbf{B}(\mathbf{x}_0) \quad \text{and} \quad \mathbf{A}_{\mathbf{x}_0}(\mathbf{v}) = \nabla\mathbf{A}(\mathbf{x}_0)\cdot\mathbf{v}, \quad \mathbf{v} \in \Pi_{\mathbf{x}_0}, \tag{4.7}$$

so that $\mathbf{B}_{\mathbf{x}_0}$ is the magnetic field frozen at $\mathbf{x}_0$ and $\mathbf{A}_{\mathbf{x}_0}$ the linear part[8] of the potential at $\mathbf{x}_0$. By extension, for each singular chain $\mathbb{X} = (\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_\nu) \in \mathfrak{C}(\Omega)$ we set

$$\mathbf{B}_{\mathbb{X}} = \mathbf{B}(\mathbf{x}_0) \quad \text{and} \quad \mathbf{A}_{\mathbb{X}}(\mathbf{x}) = \nabla\mathbf{A}(\mathbf{x}_0)\cdot\mathbf{x}, \quad \mathbf{x} \in \Pi_{\mathbb{X}}. \tag{4.8}$$

We have obviously

$$\operatorname{curl}\mathbf{A}_{\mathbb{X}} = \mathbf{B}_{\mathbb{X}}\,.$$

Definition 4.2. Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ and $\mathbf{A} \in \mathscr{C}^1(\overline{\Omega})$. Let $\mathbb{X} \in \mathfrak{C}(\Omega)$ be a singular chain of Ω. The model operator $H(\mathbf{A}_{\mathbb{X}},\Pi_{\mathbb{X}})$ is called a *tangent operator*.

Remark 4.3. The notion of equivalence classes between singular chains as introduced in Definition 3.19 is sufficient for the analysis of operators $H_h(\mathbf{A},\Omega)$ in the case of magnetic fields $\mathbf{B}$ smooth in Cartesian variables. Should $\mathbf{B}$ be smooth in polar variables only, the whole hierarchy of singular chains would be needed.

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[8] In (4.7), $\nabla\mathbf{A}$ is the 3×3 matrix with entries $\partial_k A_j$, $1 \le j,k \le 3$, and $\cdot\,\mathbf{v}$ denotes the multiplication by the column vector $\mathbf{v} = (v_1,v_2,v_3)^\top$.

The potential $\mathbf{A}_{\mathbf{x}_0}$ and the field $\mathbf{B}_{\mathbf{x}_0}$ are connected to the potential $\mathbf{A}^{\mathbf{x}_0}$ and field $\mathbf{B}^{\mathbf{x}_0}$ (4.1) obtained through the local map: Since $\mathrm{dU}^{\mathbf{x}_0}(\mathbf{x}_0) = \mathbb{I}_3$ by definition, there holds

$$\mathbf{B}^{\mathbf{x}_0}(\mathbf{0}) = \mathbf{B}(\mathbf{x}_0)\,. \tag{4.9}$$

Likewise, let $\mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}}$ be the linear part of $\mathbf{A}^{\mathbf{x}_0}$ at the vertex $\mathbf{0}$ of Π_{x_0}. Then, there holds

$$\mathbf{A}^{\mathbf{x}_0}(\mathbf{0}) = 0 \quad \text{and} \quad \mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}} = \mathbf{A}_{\mathbf{x}_0}\,. \tag{4.10}$$

Local and minimum energies are introduced as follows.

Definition 4.4. Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ and $\mathbf{B} \in \mathscr{C}^0(\overline{\Omega})$. The application $\mathbf{x} \mapsto E(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x})$ is called *local ground energy* (with $E(\mathbf{B}, \Pi)$ introduced in Definition 4.1). We define the *lowest local energy* of $\mathbf{B}$ on $\overline{\Omega}$ by

$$\mathscr{E}(\mathbf{B}, \Omega) := \inf_{\mathbf{x}\in\overline{\Omega}} E(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x}). \tag{4.11}$$

■

The relations with singular chains and the question whether $\mathscr{E}(\mathbf{B}, \Omega)$ is a minimum are addressed later on Section 8.

4.3. **Linearization.** Starting from the identity (4.5) $q_h[\mathbf{A}, \Omega](f) = q_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}, \mathrm{G}^{\mathbf{x}_0}](\psi)$, we want to compare $q_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}, \mathrm{G}^{\mathbf{x}_0}](\psi)$ with the term $q_h[\mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}}, \Pi_{\mathbf{x}_0}](\psi) = q_h[\mathbf{A}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\psi)$ obtained by linearizing the potential and the metric.

4.3.1. *Change of metric.* Here we compare L^2 norm and quadratic forms associated with the metric $\mathrm{G}^{\mathbf{x}_0}$, with the corresponding quantities associated with the trivial metric $\mathbb{I}_3$. Like in Proposition 3.13 and Corollary 3.27, and for the same reasons, we have essentially two distinct cases, resulting into a uniform approximation in a polyhedral domain, and a controlled blow up close to conical points when they are present.

Lemma 4.5. *Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ and $(\mathcal{U}_\mathbf{x}, \mathrm{U}^\mathbf{x})_{\mathbf{x}\in\overline{\Omega}}$ be an admissible atlas. We recall that the set of reference points $\mathfrak{X}$ contains the set of conical vertices $\mathfrak{V}^\circ$. Let $\mathbf{A} \in W^{1,\infty}(\Omega)$ be a magnetic potential and, for $\mathbf{x}_0 \in \overline{\Omega}$, let $\mathbf{A}^{\mathbf{x}_0}$ be the potential (4.1) produced by the local map $\mathrm{U}^{\mathbf{x}_0}$. There exists $c(\Omega)$ such that*

(i) for all $\mathbf{x}_0 \in \mathfrak{X}$ and $r \in (0, R_{\mathbf{x}_0})$, for all $\psi \in H^1(\Pi_{\mathbf{x}_0})$ satisfying $\operatorname{supp}(\psi) \subset \mathcal{B}(\mathbf{0}, r)$, *there holds*

$$\begin{aligned} &\big|q_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}, \mathrm{G}^{\mathbf{x}_0}](\psi) - q_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\psi)\big| \le c(\Omega)\, r\, q_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}, \mathrm{G}^{\mathbf{x}_0}](\psi),\\ &\big|\|\psi\|_{L^2_\mathrm{G}(\Pi_{\mathbf{x}_0})} - \|\psi\|_{L^2(\Pi_{\mathbf{x}_0})}\big| \le c(\Omega)\, r\, \|\psi\|_{L^2_\mathrm{G}(\Pi_{\mathbf{x}_0})}\,. \end{aligned} \tag{4.12}$$

(ii) for all $\mathbf{u}_0 \in \overline{\Omega} \setminus \mathfrak{X}$ and $r \in (0, \rho(\mathbf{u}_0))$ (with $\rho(\mathbf{u}_0)$ given by Proposition 3.13), for all $\psi \in H^1(\Pi_{\mathbf{u}_0})$ satisfying $\operatorname{supp}(\psi) \subset \mathcal{B}(\mathbf{0}, r)$, *there holds*

$$\begin{aligned} &\big|q_h[\mathbf{A}^{\mathbf{u}_0}, \Pi_{\mathbf{u}_0}, \mathrm{G}^{\mathbf{u}_0}](\psi) - q_h[\mathbf{A}^{\mathbf{u}_0}, \Pi_{\mathbf{u}_0}](\psi)\big| \le c(\Omega)\, \frac{r}{d_{\mathfrak{V}^\circ}(\mathbf{u}_0)}\, q_h[\mathbf{A}^{\mathbf{u}_0}, \Pi_{\mathbf{u}_0}, \mathrm{G}^{\mathbf{u}_0}](\psi),\\ &\big|\|\psi\|_{L^2_\mathrm{G}(\Pi_{\mathbf{u}_0})} - \|\psi\|_{L^2(\Pi_{\mathbf{u}_0})}\big| \le c(\Omega)\, \frac{r}{d_{\mathfrak{V}^\circ}(\mathbf{u}_0)}\, \|\psi\|_{L^2_\mathrm{G}(\Pi_{\mathbf{u}_0})}\,, \end{aligned} \tag{4.13}$$

with $d_{\mathfrak{Y}^\circ}$ *defined in* (3.40).

Proof. The lemma is a direct consequence of Corollary 3.27 providing estimates for the L^∞ norm of the difference $\mathrm{G}^{\mathbf{x}_0} - \mathbb{I}_3$. Indeed, an estimate on $\mathrm{G}^{\mathbf{x}_0} - \mathbb{I}_3$ implies a similar estimate for $\max\{\|\tau_i - 1\|_{L^\infty}, 1 \leq i \leq 3\}$, with $\tau_i = \tau_i(\mathbf{x})$ the eigenvalues of $\mathrm{G}^{\mathbf{x}_0}(\mathbf{x})$, which allows to compare the quadratic forms associated with $\mathrm{G}^{\mathbf{x}_0}$ and with $\mathbb{I}_3$. □

Combining the identities (4.5) with Lemma 4.5, we see that it is equivalent to deal with $q_h[\mathbf{A}, \Omega](f)$ or $q_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\psi)$ modulo a well-controlled error. This will be useful later on when we will estimate the corresponding Rayleigh quotients (see Sections 5 and 9).

4.3.2. *Linearization of the potential.* We estimate the remainders due to the linearization $\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}$ at the vertex $\mathbf{0}$ of the tangent cone $\Pi_{\mathbf{x}_0}$ of the potential $\mathbf{A}^{\mathbf{x}_0}$ resulting from a local map. For this, we first use a Taylor expansion around $\mathbf{0}$ in $\Pi_{\mathbf{x}_0}$.

Lemma 4.6. *Let* $\mathbf{x}_0 \in \overline{\Omega}$*. For any* $r > 0$ *such that* $\mathcal{V}_{\mathbf{x}_0} \supset \mathcal{B}(\mathbf{0}, r)$

$$\forall \mathbf{v} \in \mathcal{B}(\mathbf{0}, r) \cap \Pi_{\mathbf{x}_0}, \quad |\mathbf{A}^{\mathbf{x}_0}(\mathbf{v}) - \mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}(\mathbf{v})| \leq \tfrac{1}{2}\|\mathbf{A}^{\mathbf{x}_0}\|_{W^{2,\infty}(\mathcal{B}(\mathbf{0},r)\cap\Pi_{\mathbf{x}_0})} |\mathbf{v}|^2 . \tag{4.14}$$

So we have to estimate the second derivatives of the mapped potentials $\mathbf{A}^{\mathbf{x}_0}$.

Lemma 4.7. *Let* $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ *with an associated admissible atlas with set of reference points* $\mathfrak{X}$*. Let* $\mathbf{A} \in W^{2,\infty}(\Omega)$ *be a magnetic potential. For* $\mathbf{x}_0 \in \overline{\Omega}$*, let* $\mathbf{A}^{\mathbf{x}_0}$ *be the potential* (4.1)*. There exists* $c(\Omega)$ *such that*

(i) for all $\mathbf{x}_0 \in \mathfrak{X}$*,*

$$\|\mathrm{d}^2\mathbf{A}^{\mathbf{x}_0}(\mathbf{v})\| \leq c(\Omega)\|\mathbf{A}\|_{W^{2,\infty}(\Omega)}, \qquad \forall \mathbf{v} \in \mathcal{B}(\mathbf{0}, R_{\mathbf{x}_0}). \tag{4.15}$$

(ii) for all $\mathbf{u}_0 \in \overline{\Omega} \setminus \mathfrak{X}$*, with* $\rho(\mathbf{u}_0)$ *given in Proposition* 3.13 *and* $d_{\mathfrak{Y}^\circ}$ *defined in* (3.40)*,*

$$\|\mathrm{d}^2\mathbf{A}^{\mathbf{u}_0}(\mathbf{v})\| \leq c(\Omega)\left(\frac{\|\mathbf{A}\|_{W^{1,\infty}(\Omega)}}{d_{\mathfrak{Y}^\circ}(\mathbf{u}_0)} + \|\mathbf{A}\|_{W^{2,\infty}(\Omega)}\right), \qquad \forall \mathbf{v} \in \mathcal{B}(\mathbf{0}, \rho(\mathbf{u}_0)). \tag{4.16}$$

Proof. Let $\mathbf{u}_0 \in \overline{\Omega}$. Differentiating twice (4.1), we obtain, for $\mathbf{u} \in \mathcal{U}_{\mathbf{x}_0}$ and $\mathbf{v} = \mathrm{U}^{\mathbf{u}_0}(\mathbf{u})$,

$$\|\mathrm{d}^2\mathbf{A}^{\mathbf{u}_0}(\mathbf{v})\| \lesssim \|\mathrm{dK}^{\mathbf{u}_0}(\mathbf{v})\|\ |\mathbf{A}(\mathbf{u}) - \mathbf{A}(\mathbf{u}_0)| + \|\mathrm{K}^{\mathbf{u}_0}(\mathbf{v})\|\ \|\mathrm{J}^{\mathbf{u}_0}(\mathbf{v})\|\ \|\mathrm{d}\mathbf{A}(\mathbf{u})\| + \|\mathrm{J}^{\mathbf{u}_0}(\mathbf{v})\|^3\ \|d^2\mathbf{A}(\mathbf{u})\|.$$

(i) When $\mathbf{u}_0 = \mathbf{x}_0 \in \mathfrak{X}$, (4.15) is a consequence of Proposition 3.13 and Remark 3.15 (1).

(ii) Let $\mathbf{u}_0 \in \overline{\Omega} \setminus \mathfrak{X}$ and $\mathbf{x}_0 \in \mathfrak{X}$ such that $\mathbf{u}_0 \in \mathcal{U}_{\mathbf{x}_0}$. The above inequality, Proposition 3.13 and Remark 3.15 (2) yield for $\mathbf{v} \in \mathcal{B}(\mathbf{0}, \rho(\mathbf{u}_0))$,

$$\begin{aligned}\|\mathrm{d}^2\mathbf{A}^{\mathbf{u}_0}(\mathbf{v})\| &\lesssim \frac{|\mathbf{u} - \mathbf{u}_0|}{|\mathbf{u}_0 - \mathbf{x}_0|^2}\|\mathbf{A}\|_{W^{1,\infty}} + \frac{1}{|\mathbf{u}_0 - \mathbf{x}_0|}\|\mathbf{A}\|_{W^{1,\infty}} + \|\mathbf{A}\|_{W^{2,\infty}} \\ &\lesssim \frac{1}{|\mathbf{u}_0 - \mathbf{x}_0|}\|\mathbf{A}\|_{W^{1,\infty}} + \|\mathbf{A}\|_{W^{2,\infty}}.\end{aligned}$$

Here we have used the inequality $|\mathbf{u} - \mathbf{u}_0| \leq |\mathbf{u}_0 - \mathbf{x}_0|$ which holds by construction of the admissible atlas. □

Estimates between $\mathbf{A}^{\mathbf{x}_0}$ and $\mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}}$ deduced from the combination of Lemmas 4.6 and 4.7 allow to compare $q_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\psi)$ and $q_h[\mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}}, \Pi_{\mathbf{x}_0}](\psi)$ via identity (A.7) which writes

$$q_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\psi) = q_h[\mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}}, \Pi_{\mathbf{x}_0}](\psi) + 2\,\mathrm{Re}\left\langle(-ih\nabla + \mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}})\psi, (\mathbf{A}^{\mathbf{x}_0} - \mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}})\psi\right\rangle + \|(\mathbf{A}^{\mathbf{x}_0} - \mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}})\psi\|^2.$$

This will be extensively used in Sections 5 and 9.

4.4. **A general rough upper bound.** As a first consequence of a weaker form of Lemmas 4.5 and 4.7, we are going to prove a very general rough upper bound for the Rayleigh quotients of $q_h[\mathbf{A}, \Omega]$ as $h \to 0$. In fact this reasoning holds in a natural way for n-dimensional corner domains. In the n-dimensional case, the magnetic field is a 2-form and associated magnetic potentials are 1-forms that we write by using their representation as vector fields in a canonical basis of $\mathbb{R}^n$, see (1.2)–(1.3). In dimension n, $E(\mathbf{B}, \Pi)$ and $\mathscr{E}(\mathbf{B}, \Omega)$ are defined as in Definition 4.4.

In this context we prove a rough upper bound on the first eigenvalue of $H_h(\mathbf{A}, \Omega)$ by using only elementary arguments. We need the following Lemma, that will also be useful later:

Lemma 4.8. *Let $\Omega \in \mathfrak{D}(\mathbb{R}^n)$ and let $\mathbf{A} \in W^{2,\infty}(\overline{\Omega})$ be a twice differentiable magnetic potential associated with the magnetic field $\mathbf{B}$. Let $\mathbf{x}_0 \in \overline{\Omega}$ be a chosen point and let $\varepsilon > 0$. Then there exists $h_0 > 0$ such that for all $h \in (0, h_0)$ there exists a function f_h supported near $\mathbf{x}_0$ satisfying*

$$\frac{1}{h}\frac{q_h[\mathbf{A}, \Omega](f_h)}{\|f_h\|^2} \leq E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}) + \varepsilon\,,$$

where $E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0})$ is the ground state energy of $H(\mathbf{A}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0})$.

Proof. Let $(\mathcal{U}_{\mathbf{x}_0}, \mathrm{U}^{\mathbf{x}_0})$ be a local map with $\mathrm{U}^{\mathbf{x}_0} : \mathcal{U}_{\mathbf{x}_0} \mapsto \mathcal{V}_{\mathbf{x}_0} \subset \Pi_{\mathbf{x}_0}$, cf. (3.1). This change of variables transforms the magnetic potential into $\mathbf{A}^{\mathbf{x}_0}$ given by (4.1):

$$\mathbf{A}^{\mathbf{x}_0} = (\mathrm{J}^{\mathbf{x}_0})^\top \big((\mathbf{A} - \mathbf{A}(\mathbf{x}_0)) \circ (\mathrm{U}^{\mathbf{x}_0})^{-1}\big)\,.$$

Denote by $\mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}}$ its linear part. Recall that $\operatorname{curl} \mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}} = \mathbf{B}_{\mathbf{x}_0}$. By definition of $E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0})$ there exists $\psi \in \mathrm{Dom}(q[\mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}}, \Pi_{\mathbf{x}_0}])$ a L^2-normalized function such that

$$q[\mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}}, \Pi_{\mathbf{x}_0}](\psi) \leq E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}) + \tfrac{\varepsilon}{4}\,.$$

Let us consider a smooth cut-off function χ with support in $\mathcal{B}(\mathbf{0}, 1)$ and equal to 1 on $\mathcal{B}(\mathbf{0}, \frac{1}{2})$. Then the functions with compact support

$$\mathbf{x} \longmapsto \chi(\tfrac{\mathbf{x}}{R})\,\psi(\mathbf{x})$$

converge to ψ in $\mathrm{Dom}(q[\mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}}, \Pi_{\mathbf{x}_0}])$ as $R \to \infty$. Therefore there exists $R = R(\varepsilon, \mathbf{x}_0) > 0$ and a new function $\psi \in \mathrm{Dom}(q[\mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}}, \Pi_{\mathbf{x}_0}])$ with support in $\mathcal{B}(\mathbf{0}, R)$ which satisfies

$$q[\mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}}, \Pi_{\mathbf{x}_0}](\psi) \leq E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}) + \tfrac{\varepsilon}{2}\,.$$

For all $h > 0$ we define the L^2-normalized function

$$\psi_h(\mathbf{x}) := h^{-n/4}\psi(h^{-1/2}\mathbf{x})$$

so that (see Lemma A.5)

$$q_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\psi_h) \leq h(E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}) + \tfrac{\varepsilon}{2}) .$$

We have $\operatorname{supp}(\psi_h) \subset \mathcal{B}(\mathbf{0}, h^{1/2}R)$ and therefore

$$\exists h_\varepsilon > 0, \ \forall h \in (0, h_\varepsilon), \quad \operatorname{supp}(\psi_h) \subset \mathcal{V}_{\mathbf{x}_0} .$$

Combining (A.7) with a Cauchy-Schwarz inequality we find

$$\begin{aligned}(4.17)\quad q_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\psi_h) \leq{}& q_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\psi_h) \\ &+ 2\sqrt{q_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\psi_h)}\ \|(\mathbf{A}^{\mathbf{x}_0} - \mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0})\psi_h\| + \|(\mathbf{A}^{\mathbf{x}_0} - \mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0})\psi_h\|^2.\end{aligned}$$

Notice now that the estimates *(i)* of Proposition 3.13 are still valid for any chosen $\mathbf{x}_0$ in $\overline{\Omega}$ with constants $c(\mathbf{x}_0)$ and radius $R_{\mathbf{x}_0}$ depending on $\mathbf{x}_0$. Hence estimates *(i)* of Lemma 4.7 holds at $\mathbf{x}_0$ with a constant $c(\mathbf{x}_0)$ replacing the uniform constant $c(\Omega)$. Therefore applying Lemma 4.6 with $r = h^{1/2}R$ we get $c = c(\varepsilon, \mathbf{x}_0) > 0$ such that

$$(4.18)\qquad \|(\mathbf{A}^{\mathbf{x}_0} - \mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0})\psi_h\| \leq cR^2h\|\psi_h\|, \quad \forall h \in (0, h_\varepsilon).$$

Let $\mathrm{G}^{\mathbf{x}_0}$ be the metric associated with the change of variables (see Section 4.1). Again *(i)* of Lemma 4.5 is valid for all $\mathbf{x}_0 \in \overline{\Omega}$ with $c(\mathbf{x}_0)$ instead of $c(\Omega)$. Applying this with $r = h^{1/2}R$ provides another constant $c = c(\varepsilon, \mathbf{x}_0) > 0$ such that

$$(4.19)\qquad \left| q_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}, \mathrm{G}^{\mathbf{x}_0}](\psi_h) - q_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\psi_h) \right| \leq c\, Rh^{1/2}\, q_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\psi_h),$$

$$(4.20)\qquad \left| \|\psi_h\|_{L^2_{\mathrm{G}}(\Pi_{\mathbf{x}_0})} - \|\psi_h\|_{L^2(\Pi_{\mathbf{x}_0})} \right| \leq c\, Rh^{1/2}\, \|\psi_h\|_{L^2_{\mathrm{G}}(\Pi_{\mathbf{x}_0})} .$$

According to Section 4.1 (4.1)–(4.5), we define for $h \in (0, h_\varepsilon)$:

$$f_h := (\zeta_h^{\mathbf{x}_0})^{-1}\, \psi_h \circ \mathrm{U}^{\mathbf{x}_0} \qquad \text{with} \qquad \zeta_h^{\mathbf{x}_0}(\mathbf{x}) = \mathrm{e}^{i\langle \mathbf{A}(\mathbf{x}_0), \mathbf{x}\rangle/h}, \quad \mathbf{x} \in \mathcal{U}_{\mathbf{x}_0} \cap \overline{\Omega}$$

and we have

$$q_h[\mathbf{A}, \Omega](f_h) = q_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}, \mathrm{G}^{\mathbf{x}_0}](\psi_h) \quad \text{and} \quad \|f_h\|_{L^2(\Omega)} = \|\psi_h\|_{L^2_{\mathrm{G}}(\Pi_{\mathbf{x}_0})}.$$

Thus, combining with (4.17)–(4.20) we deduce

$$\begin{aligned}\frac{q_h[\mathbf{A}, \Omega](f_h)}{\|f_h\|^2} &\leq (1 + cRh^{1/2})\left(\frac{q_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\psi_h)}{\|\psi_h\|^2} + c\left(R^2h^{3/2} + R^4h^2\right)\right) \\ &\leq (1 + cRh^{1/2})\Big(h\big(E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}) + \tfrac{\varepsilon}{2}\big) + c\left(R^2h^{3/2} + R^4h^2\right)\Big).\end{aligned}$$

We can write this in the form

$$\frac{1}{h}\frac{q_h[\mathbf{A}, \Omega](f_h)}{\|f_h\|^2} \leq E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}) + \tfrac{\varepsilon}{2} + h^{1/2}M_\varepsilon(h),$$

where $M_\varepsilon(h)$ is a bounded function for $h \in [0, h_\varepsilon]$ that depends on $\varepsilon > 0$. We deduce the lemma by choosing h so small that $h^{1/2}M_\varepsilon(h) \leq \frac{\varepsilon}{2}$. ☐

Proposition 4.9. *Let $\Omega \in \mathfrak{D}(\mathbb{R}^n)$ and let $\mathbf{A} \in W^{2,\infty}(\overline{\Omega})$ be a magnetic potential associated with the magnetic field $\mathbf{B}$. Then the first eigenvalue $\lambda_h(\mathbf{B}, \Omega)$ of $H(\mathbf{A}, \Omega)$ satisfies*

$$\limsup_{h\to 0} \frac{\lambda_h(\mathbf{B}, \Omega)}{h} \leq \mathscr{E}(\mathbf{B}, \Omega)\,,$$

where $\mathscr{E}(\mathbf{B}, \Omega)$ is the lowest local energy as introduced in Definition 4.4.

Proof. Let $\varepsilon > 0$. Let $\mathbf{x}_0 \in \overline{\Omega}$ such that $E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}) < \mathscr{E}(\mathbf{B}, \Omega) + \varepsilon$. The min-max principle and Lemma 4.8 provide h_ε such that

$$\forall h \in (0, h_\varepsilon), \qquad \frac{\lambda_h(\mathbf{B}, \Omega)}{h} \leq \mathscr{E}(\mathbf{B}, \Omega) + 2\varepsilon$$

that gives the proposition. □

5. Lower bound for ground state energy in corner domains

In this section we establish a lower bound for the first eigenvalue $\lambda_h(\mathbf{B}, \Omega)$ of the magnetic Laplacian $H_h(\mathbf{A}, \Omega)$ with Neumann boundary conditions.

Theorem 5.1. *Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ be a general 3D corner domain, and let $\mathbf{A} \in W^{2,\infty}(\overline{\Omega})$ be a twice differentiable magnetic potential.*

(i) Then there exist $C(\Omega) > 0$ and $h_0 > 0$ such that

$$\forall h \in (0, h_0), \quad \lambda_h(\mathbf{B}, \Omega) \geq h\mathscr{E}(\mathbf{B}, \Omega) - C(\Omega)\big(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)h^{11/10}\,. \tag{5.1}$$

(ii) If Ω is a polyhedral domain, this lower bound is improved:

$$\forall h \in (0, h_0), \quad \lambda_h(\mathbf{B}, \Omega) \geq h\mathscr{E}(\mathbf{B}, \Omega) - C(\Omega)\big(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)h^{5/4}\,. \tag{5.2}$$

We recall that the quantity $\mathscr{E}(\mathbf{B}, \Omega)$ is the lowest local energy defined in (4.11).

Remark 5.2. If the magnetic field $\mathbf{B}$ vanishes, then $\mathscr{E}(\mathbf{B}, \Omega) = 0$ and Theorem 5.1 is obvious. In contrast, if $\mathbf{B}$ does not vanish on $\overline{\Omega}$, we will see in Corollary 8.6 that $\mathscr{E}(\mathbf{B}, \Omega) > 0$.

• *Structure of the proof.* The proof proceeds in a classical way from an IMS partition argument coupled with the analysis of remainders due to the cut-off effects, the local maps and the linearization of the potential. The less classical piece of the analysis is our special construction of cut-off functions in regions close to conical points $\mathbf{x}_0 \in \mathfrak{V}^\circ$, where a second, smaller, scale is introduced.

We choose first an admissible atlas on $\overline{\Omega}$ according to Definition 3.11 and we recall that the conical points are part of the set $\mathfrak{X}$ of its reference points.

• *Splitting off the conical points.* We start with a (smooth) macro partition of unity on $\overline{\Omega}$, independent of h, $(\Xi_0, (\Xi_{\mathbf{x}})_{\mathbf{x}\in\mathfrak{V}^\circ})$ aimed at separating the conical points, i.e. such that

- $\overline{\operatorname{supp}\Xi_0} \cap \mathfrak{V}^\circ = \emptyset$,
- for any $\mathbf{x}_0 \in \mathfrak{V}^\circ$, $\operatorname{supp}\Xi_{\mathbf{x}_0} \subset \mathcal{B}(\mathbf{x}_0, R_{\mathbf{x}_0})$.

Here $R_{\mathbf{x}_0}$ is the radius associated with the reference point $\mathbf{x}_0$ in the admissible atlas. In the polyhedral case, i.e., when $\mathfrak{V}^\circ = \emptyset$, we simply set $\Xi_0 \equiv 1$.

For any $f \in H^1(\Omega)$ IMS formula (see Lemma A.7) gives

$$q_h[\mathbf{A},\Omega](f) = q_h[\mathbf{A},\Omega](\Xi_0 f) + \sum_{\mathbf{x}\in\mathfrak{V}^\circ} q_h[\mathbf{A},\Omega](\Xi_{\mathbf{x}} f) - h^2\Big(\|(\nabla\Xi_0)f\|^2 + \sum_{\mathbf{x}\in\mathfrak{V}^\circ}\|(\nabla\Xi_{\mathbf{x}})f\|^2\Big)$$

$$\geq q_h[\mathbf{A},\Omega](\Xi_0 f) + \sum_{\mathbf{x}_0\in\mathfrak{V}^\circ} q_h[\mathbf{A},\Omega](\Xi_{\mathbf{x}} f) - Ch^2\|f\|^2. \tag{5.3}$$

In Section 5.1, we give a lower bound of $q_h[\mathbf{A},\Omega](\Xi_0 f)$. In the polyhedral case, this will finish the proof. Section 5.2 is devoted to conical points and estimates of $q_h[\mathbf{A},\Omega](\Xi_{\mathbf{x}} f)$.

5.1. **Estimates outside conical points.**

Here we prove a lower bound for $q_h[\mathbf{A},\Omega](\Xi_0 f)$.

• *IMS localization.* Let $\delta \in (0, \frac{1}{2})$ be an exponent which will be determined later on. Now, we make a h-dependent partition of $\operatorname{supp}\Xi_0 \cap \overline{\Omega}$ with size h^δ. Relying on Lemma B.2, we can choose for $0 < h \leq h_0$ (h_0 small enough) a finite set $\mathscr{C}(h)$ of points $\mathbf{c} \in \overline{\Omega}$ together with radii $\rho_{\mathbf{c}}$ equivalent to h^δ (with uniformity as $h \to 0$) such that

(1) The union of balls $\mathcal{B}(\mathbf{c}, \rho_{\mathbf{c}})$ covers $\operatorname{supp}\Xi_0 \cap \overline{\Omega}$
(2) Each ball $\mathcal{B}(\mathbf{c}, 2\rho_{\mathbf{c}})$ is contained in a map-neighborhood of the admissible atlas
(3) The finite covering condition holds

Relying on Lemma B.7, we choose an associate partition of unity $\big(\xi_{\mathbf{c}}\big)_{\mathbf{c}\in\mathscr{C}(h)}$ such that

$$\xi_{\mathbf{c}} \in \mathscr{C}_0^\infty(\mathcal{B}(\mathbf{c}, \rho_{\mathbf{c}})), \quad \forall \mathbf{c} \in \mathscr{C}(h) \qquad \text{and} \qquad \Xi_0 \sum_{\mathbf{c}\in\mathscr{C}(h)} \xi_{\mathbf{c}}^2 = \Xi_0 \quad \text{on} \quad \overline{\Omega},$$

and satisfying the uniform estimate of gradients

$$\exists C > 0, \quad \forall h \in (0, h_0), \quad \forall \mathbf{c} \in \mathscr{C}(h), \quad \|\nabla\xi_{\mathbf{c}}\|_{L^\infty(\Omega)} \leq Ch^{-\delta}\,. \tag{5.4}$$

The IMS formula (see Lemma A.7) provides for all $f \in H^1(\Omega)$

$$q_h[\mathbf{A},\Omega](\Xi_0 f) = \sum_{\mathbf{c}\in\mathscr{C}(h)} q_h[\mathbf{A},\Omega](\xi_{\mathbf{c}}\,\Xi_0 f) - h^2 \sum_{\mathbf{c}\in\mathscr{C}(h)} \|\nabla\xi_{\mathbf{c}}\;\Xi_0 f\|^2_{L^2(\Omega)}$$

and using (5.4) we get $C = C(\Omega) > 0$ such that

$$q_h[\mathbf{A},\Omega](\Xi_0 f) \geq \sum_{\mathbf{c}\in\mathscr{C}(h)} q_h[\mathbf{A},\Omega](\xi_{\mathbf{c}}\Xi_0 f) - C\,h^{2-2\delta}\|\Xi_0 f\|^2_{L^2(\Omega)}\,. \tag{5.5}$$

• *Local control of the energy.* For each center $\mathbf{c} \in \mathscr{C}(h)$, we are going to bound from below the term $q_h[\mathbf{A}, \Omega](\xi_\mathbf{c}\, \Xi_0 f)$ appearing in (5.5). By construction $\operatorname{supp}(\xi_\mathbf{c}\, \Xi_0 f)$ is contained in the map-neighborhood $\mathcal{U}_\mathbf{c}$. Using (4.2) and (4.4), we set

$$\psi_\mathbf{c} := (\zeta_h^\mathbf{c} \xi_\mathbf{c}\, \Xi_0 f) \circ (\mathrm{U}^\mathbf{c})^{-1}, \qquad \text{with} \qquad \zeta_h^\mathbf{c}(\mathbf{x}) = \mathrm{e}^{i\langle \mathbf{A}(\mathbf{c}), \mathbf{x}\rangle / h}. \tag{5.6}$$

According to (4.5) with $\mathbf{x}_0$ replaced by $\mathbf{c}$, we have

$$q_h[\mathbf{A}, \Omega](\xi_\mathbf{c} \Xi_0 f) = q_h[\mathbf{A}^\mathbf{c}, \Pi_\mathbf{c}, \mathrm{G}^\mathbf{c}](\psi_\mathbf{c}) \quad \text{and} \quad \|\xi_\mathbf{c}\, \Xi_0 f\|_{L^2(\Omega)} = \|\psi_\mathbf{c}\|_{L^2_{\mathrm{G}^\mathbf{c}}(\Pi_\mathbf{c})} . \tag{5.7}$$

In order to replace the metric $\mathrm{G}^\mathbf{c}$ by the identity, we apply Lemma 4.5 with $r \simeq h^\delta$. Using that the distance $d_{\mathfrak{V}^\circ}$ to conical points is bounded from below by a positive number on $\operatorname{supp} \Xi_0$, we obtain the existence of a constant $c(\Omega) > 0$ such that for all centers $\mathbf{c} \in \mathscr{C}(h)$

$$\frac{q_h[\mathbf{A}^\mathbf{c}, \Pi_\mathbf{c}, \mathrm{G}^\mathbf{c}](\psi_\mathbf{c})}{\|\psi_\mathbf{c}\|^2_{L^2_{\mathrm{G}}(\Pi_\mathbf{c})}} \geq (1 - c(\Omega) h^\delta) \frac{q_h[\mathbf{A}^\mathbf{c}, \Pi_\mathbf{c}](\psi_\mathbf{c})}{\|\psi_\mathbf{c}\|^2}. \tag{5.8}$$

We now want to replace $\mathbf{A}^\mathbf{c}$ in the above Rayleigh quotient by its linear part $\mathbf{A}^\mathbf{c}_\mathbf{0}$ at $\mathbf{0}$. For this we use identity (A.7) with $\psi = \psi_\mathbf{c}$ and $\mathcal{O} = \Pi_\mathbf{c}$:

$$\begin{aligned} q_h[\mathbf{A}^\mathbf{c}, \Pi_\mathbf{c}](\psi_\mathbf{c}) &= q_h[\mathbf{A}^\mathbf{c}_\mathbf{0}, \Pi_\mathbf{c}](\psi_\mathbf{c}) \\ &\quad + 2 \operatorname{Re} \big\langle (-ih\nabla + \mathbf{A}^\mathbf{c}_\mathbf{0})\psi_\mathbf{c}, (\mathbf{A}^\mathbf{c} - \mathbf{A}^\mathbf{c}_\mathbf{0})\psi_\mathbf{c} \big\rangle + \|(\mathbf{A}^\mathbf{c} - \mathbf{A}^\mathbf{c}_\mathbf{0})\psi_\mathbf{c}\|^2. \end{aligned} \tag{5.9}$$

The Cauchy-Schwarz inequality then yields:

$$q_h[\mathbf{A}^\mathbf{c}, \Pi_\mathbf{c}](\psi_\mathbf{c}) \geq q_h[\mathbf{A}^\mathbf{c}_\mathbf{0}, \Pi_\mathbf{c}](\psi_\mathbf{c}) - 2 \left(q_h[\mathbf{A}^\mathbf{c}_\mathbf{0}, \Pi_\mathbf{c}](\psi_\mathbf{c}) \right)^{1/2} \|(\mathbf{A}^\mathbf{c} - \mathbf{A}^\mathbf{c}_\mathbf{0})\psi_\mathbf{c}\| ,$$

leading to the parametric estimate

$$\forall \eta > 0, \quad q_h[\mathbf{A}^\mathbf{c}, \Pi_\mathbf{c}](\psi_\mathbf{c}) \geq (1 - \eta) q_h[\mathbf{A}^\mathbf{c}_\mathbf{0}, \Pi_\mathbf{c}](\psi_\mathbf{c}) - \eta^{-1} \|(\mathbf{A}^\mathbf{c} - \mathbf{A}^\mathbf{c}_\mathbf{0})\psi_\mathbf{c}\|^2 , \tag{5.10}$$

based on the simple inequality $2ab \leq \eta a^2 + \eta^{-1} b^2$. Since $\operatorname{curl} \mathbf{A}^\mathbf{c}_\mathbf{0} = \mathbf{B}_\mathbf{c}$, we have the lower bound by the minimum local energy at $\mathbf{c}$:

$$q_h[\mathbf{A}^\mathbf{c}_\mathbf{0}, \Pi_\mathbf{c}](\psi_\mathbf{c}) \geq h E(\mathbf{B}_\mathbf{c}, \Pi_\mathbf{c}) \|\psi_\mathbf{c}\|^2 \tag{5.11}$$

$$\geq h \mathscr{E}(\mathbf{B}, \Omega) \|\psi_\mathbf{c}\|^2 . \tag{5.12}$$

According to Lemmas 4.6 and 4.7 (note that $d_{\mathfrak{V}^\circ} \geq r_0 > 0$ on $\operatorname{supp} \Xi_0$), we have

$$\|(\mathbf{A}^\mathbf{c} - \mathbf{A}^\mathbf{c}_\mathbf{0})\psi_\mathbf{c}\| \leq c(\Omega) \|\mathbf{A}\|_{W^{2,\infty}(\Omega)} h^{2\delta} \|\psi_\mathbf{c}\| . \tag{5.13}$$

Combining (5.10)–(5.13) we deduce for all $\eta > 0$:

$$q_h[\mathbf{A}^\mathbf{c}, \Pi_\mathbf{c}](\psi_\mathbf{c}) \geq (1 - \eta) h \mathscr{E}(\mathbf{B}, \Omega) \|\psi_\mathbf{c}\|^2 - \eta^{-1} h^{4\delta} c(\Omega)^2 \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)} \|\psi_\mathbf{c}\|^2.$$

Choosing $\eta = h^{2\delta - \frac{1}{2}}$ to equilibrate ηh and $\eta^{-1} h^{4\delta}$ we get for another constant $C = C(\Omega)$

$$q_h[\mathbf{A}^\mathbf{c}, \Pi_\mathbf{c}](\psi_\mathbf{c}) \geq \Big(h \mathscr{E}(\mathbf{B}, \Omega) - C(\Omega) \big(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big) h^{2\delta + \frac{1}{2}} \Big) \|\psi_\mathbf{c}\|^2, \quad \forall \mathbf{c} \in \mathscr{C}(h). \tag{5.14}$$

• *Conclusion.* Combining the previous localized estimate (5.14) with (5.8) we deduce:

$$q_h[\mathbf{A},\Omega](\xi_{\mathbf{c}}\,\Xi_0 f) \geq \Big(h\mathscr{E}(\mathbf{B},\Omega) - C(\Omega)(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})(h^{2\delta+\frac12}+h^{1+\delta})\Big)\,\|\xi_{\mathbf{c}}\,\Xi_0 f\|^2. \tag{5.15}$$

Summing up in $\mathbf{c}\in\mathscr{C}(h)$, we obtain

$$\frac{\sum_{\mathbf{c}\in\mathscr{C}(h)} q_h[\mathbf{A},\Omega](\xi_{\mathbf{c}}\,\Xi_0 f)}{\|\Xi_0 f\|^2_{L^2(\Omega)}} \geq h\mathscr{E}(\mathbf{B},\Omega) - C(\Omega)(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})(h^{2\delta+\frac12}+h^{1+\delta}). \tag{5.16}$$

Using (5.5), we get another constant $C(\Omega)>0$ such that

$$\forall f\in H^1(\Omega),\quad \frac{q_h[\mathbf{A},\Omega](\Xi_0 f)}{\|\Xi_0 f\|^2_{L^2(\Omega)}} \geq h\mathscr{E}(\mathbf{B},\Omega) - C(\Omega)(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})\left(h^{2\delta+\frac12}+h^{1+\delta}+h^{2-2\delta}\right). \tag{5.17}$$

In the polyhedral case, $\Xi_0\equiv 1$. So we can optimize the remainders by taking $\delta=\frac38$ in (5.17), thus deducing point (ii) of Theorem 5.1 from the min-max principle.

5.2. **Estimates near conical points.** Let $\mathbf{x}_0\in\mathfrak{V}^\circ$. We are going to estimate $q_h[\mathbf{A},\Omega](\Xi_{\mathbf{x}_0}f)$ from below.

• *IMS partition.* For $h>0$ small enough we construct a special covering of the support of $\Xi_{\mathbf{x}_0}$. We recall that this support is included in the ball $\mathcal{B}(\mathbf{x}_0,R_{\mathbf{x}_0})$. We cover $\mathcal{B}(\mathbf{x}_0,R_{\mathbf{x}_0})\cap\overline{\Omega}$ by a finite collection of h-dependent balls $\mathcal{B}(\mathbf{c},\rho_{\mathbf{c}})$:

- The first ball is centered at $\mathbf{x}_0$ itself and its radius is $2h^{\delta_0}$: $\mathcal{B}(\mathbf{c},\rho_{\mathbf{c}})=\mathcal{B}(\mathbf{x}_0,2h^{\delta_0})$. Here the exponent $\delta_0\in(0,\frac12)$ will be chosen later on.
- The other balls $\mathcal{B}(\mathbf{c},\rho_{\mathbf{c}})$ cover the annular region $h^{\delta_0}\leq|\mathbf{x}-\mathbf{x}_0|<R_{\mathbf{x}_0}$ and their radii are $\simeq h^{\delta_0+\delta_1}$ where the new exponent $\delta_1>0$ is such that $\delta_0+\delta_1<\frac12$ and will be also chosen later on. Thanks to Lemma B.2 the set $\mathscr{C}(h,\mathbf{x}_0)$ of the centers and the corresponding radii can be taken so that the conditions of this lemma are satisfied (inclusion in map-neighborhoods, finite covering), see previous case §5.1.

So this covering contains a "large" ball centered at the corner and a whole bunch of smaller ones covering the remaining part.

Relying on Lemma B.7, we choose an associate partition of unity $(\xi_{\mathbf{c}})_{\mathbf{c}\in\{\mathbf{x}_0\}\cup\mathscr{C}(h,\mathbf{x}_0)}$ such that

$$\xi_{\mathbf{c}}\in\mathscr{C}^\infty_0(\mathcal{B}(\mathbf{c},\rho_{\mathbf{c}})),\quad \forall\mathbf{c}\in\{\mathbf{x}_0\}\cup\mathscr{C}(h,\mathbf{x}_0),\qquad\text{and}\qquad \Xi_{\mathbf{x}_0}\sum_{\mathbf{c}\in\{\mathbf{x}_0\}\cup\mathscr{C}(h,\mathbf{x}_0)}\xi^2_{\mathbf{c}}=\Xi_{\mathbf{x}_0}\quad\text{on}\quad\overline{\Omega},$$

and satisfying the following uniform estimate of gradients for all $h\in(0,h_0)$:

$$\text{for}\quad \mathbf{c}=\mathbf{x}_0,\quad \|\nabla\xi_{\mathbf{c}}\|_{L^\infty(\Omega)}\leq Ch^{-\delta_0}\quad\text{and}\quad\forall\mathbf{c}\in\mathscr{C}(h,\mathbf{x}_0),\quad \|\nabla\xi_{\mathbf{c}}\|_{L^\infty(\Omega)}\leq Ch^{-\delta_0-\delta_1}. \tag{5.18}$$

Using the IMS formula (see Lemma A.7), we have like previously in (5.5)

$$q_h[\mathbf{A},\Omega](\Xi_{\mathbf{x}_0}f)\geq q_h[\mathbf{A},\Omega](\xi_{\mathbf{x}_0}\,\Xi_{\mathbf{x}_0}f)+\sum_{\mathbf{c}\in\mathscr{C}(h,\mathbf{x}_0)}q_h[\mathbf{A},\Omega](\xi_{\mathbf{c}}\,\Xi_{\mathbf{x}_0}f)-Ch^{2-2(\delta_0+\delta_1)}\|\Xi_{\mathbf{x}_0}f\|^2. \tag{5.19}$$

• *Local control of the energy.* When $\mathbf{c} = \mathbf{x}_0$, we can proceed in the same way as in the polyhedral case due to the "good" estimates stated in Lemma 4.5 (i) and Lemma 4.7 (i). So we obtain a similar estimate as in (5.15): There exists a constant $C = C(\Omega)$ such that for any function $f \in H^1(\Omega)$

$$q_h[\mathbf{A},\Omega](\xi_{\mathbf{x}_0}\Xi_{\mathbf{x}_0}f) \geq \Big(h\mathscr{E}(\mathbf{B},\Omega) - C(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})(h^{2\delta_0+\frac12} + h^{1+\delta_0})\Big)\,\|\xi_{\mathbf{x}_0}\Xi_{\mathbf{x}_0}f\|^2. \tag{5.20}$$

When $\mathbf{c} \in \mathscr{C}(h,\mathbf{x}_0)$, we have to revisit the arguments leading from (5.6) to the final individual estimate (5.15). First we define $\psi_{\mathbf{c}}$ like in (5.6), replacing the cut-off Ξ_0 by $\Xi_{\mathbf{x}_0}$. Then we have (5.7) *mutatis mutandis*. Next we have to use Lemma 4.5 (ii) with $\mathbf{u}_0 = \mathbf{c}$ to flatten the metric. Here we have to take the distance $d_{\mathfrak{Y}^\circ}(\mathbf{c})$ to conical points into account. By construction $d_{\mathfrak{Y}^\circ}(\mathbf{c})$ coincides with $|\mathbf{c}-\mathbf{x}_0|$, so is larger than h^{δ_0}, while the quantity r equals $\rho_{\mathbf{c}}$, thus is $\lesssim h^{\delta_0+\delta_1}$: In short

$$\frac{r}{d_{\mathfrak{Y}^\circ}(\mathbf{c})} = \frac{\rho_{\mathbf{c}}}{|\mathbf{c}-\mathbf{x}_0|} \lesssim h^{\delta_1}.$$

Hence, we obtain in place of (5.8):

$$\frac{q_h[\mathbf{A}^{\mathbf{c}},\Pi_{\mathbf{c}},\mathrm{G}^{\mathbf{c}}](\psi_{\mathbf{c}})}{\|\psi_{\mathbf{c}}\|^2_{L^2_{\mathrm{G}}(\Pi_{\mathbf{c}})}} \geq (1-c(\Omega)h^{\delta_1})\frac{q_h[\mathbf{A}^{\mathbf{c}},\Pi_{\mathbf{c}}](\psi_{\mathbf{c}})}{\|\psi_{\mathbf{c}}\|^2}. \tag{5.21}$$

For the linearization of the potential $\mathbf{A}^{\mathbf{c}}$, the expressions (5.9)–(5.12) are still valid, leading to the parametric estimate

$$\forall\eta>0,\quad q_h[\mathbf{A}^{\mathbf{c}},\Pi_{\mathbf{c}}](\psi_{\mathbf{c}}) \geq (1-\eta)h\mathscr{E}(\mathbf{B},\Omega)\|\psi_{\mathbf{c}}\|^2 - \eta^{-1}\|(\mathbf{A}^{\mathbf{c}}-\mathbf{A}^{\mathbf{c}}_{\mathbf{0}})\psi_{\mathbf{c}}\|^2\,. \tag{5.22}$$

Here we use Lemmas 4.6 and 4.7 (ii) and obtain, since $\rho_{\mathbf{c}} \lesssim h^{\delta_0+\delta_1}$ and $d_{\mathfrak{Y}^\circ}(\mathbf{c}) \geq h^{\delta_0}$

$$\|(\mathbf{A}^{\mathbf{c}}-\mathbf{A}^{\mathbf{c}}_{\mathbf{0}})\psi_{\mathbf{c}}\| \leq c(\Omega)\frac{\rho^2_{\mathbf{c}}}{d_{\mathfrak{Y}^\circ}(\mathbf{c})}\|\mathbf{A}\|_{W^{2,\infty}(\Omega)}\|\psi_{\mathbf{c}}\| \leq c(\Omega)h^{\delta_0+2\delta_1}\|\mathbf{A}\|_{W^{2,\infty}(\Omega)}\|\psi_{\mathbf{c}}\|\,. \tag{5.23}$$

Combining (5.22) with (5.23) and taking $\eta = h^{\delta_0+2\delta_1-\frac12}$ we deduce

$$q_h[\mathbf{A}^{\mathbf{c}},\Pi_{\mathbf{c}}](\psi_{\mathbf{c}}) \geq \Big(h\mathscr{E}(\mathbf{B},\Omega) - C(\Omega)\big(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)h^{\delta_0+2\delta_1+\frac12}\Big)\,\|\psi_{\mathbf{c}}\|^2,\quad \forall\mathbf{c}\in\mathscr{C}(h,\mathbf{x}_0), \tag{5.24}$$

and then with (5.21) (and (5.7) with $\Xi_{\mathbf{x}_0}$)

$$q_h[\mathbf{A},\Omega](\xi_{\mathbf{c}}\Xi_{\mathbf{x}_0}f) \geq \Big(h\mathscr{E}(\mathbf{B},\Omega) - C(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})(h^{\delta_0+2\delta_1+\frac12} + h^{1+\delta_1})\Big)\,\|\xi_{\mathbf{c}}\Xi_{\mathbf{x}_0}f\|^2. \tag{5.25}$$

Summing up (5.20) and (5.25) for $\mathbf{c} \in \mathscr{C}(h,\mathbf{x}_0)$, and combining with the IMS formula, we deduce

$$\frac{q_h[\mathbf{A},\Omega](\Xi_{\mathbf{x}_0}f)}{\|\Xi_{\mathbf{x}_0}f\|^2_{L^2(\mathcal{U}_{\mathbf{x}_0})}} \geq h\mathscr{E}(\mathbf{B},\Omega) - C(h^{2\delta_0+\frac12} + h^{1+\delta_0} + h^{\frac12+\delta_0+2\delta_1} + h^{1+\delta_1} - h^{2-2(\delta_0+\delta_1)}), \tag{5.26}$$

with $C = c(\Omega)(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})$.

• *Conclusion.* Combining (5.3), (5.17) and (5.26), we deduce

$$\frac{q_h[\mathbf{A},\Omega](f)}{\|f\|^2} \geq h\mathscr{E}(\mathbf{B},\Omega) - Ch^2 - C\left(h^{2\delta+\frac{1}{2}} + h^{1+\delta} + h^{2-2\delta}\right) - C\left(h^{2\delta_0+\frac{1}{2}} + h^{1+\delta_0} + h^{\frac{1}{2}+\delta_0+2\delta_1} + h^{1+\delta_1} - h^{2-2(\delta_0+\delta_1)}\right), \tag{5.27}$$

with $C = c(\Omega)(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})$.

Remind that the error with power δ_0 and δ_1 only appears when Ω has conical points. To optimize the remainder, we first choose $\delta = 3/8$. We have now to optimize parameters δ_0, δ_1 under the constraints $0 < \delta_0 + \delta_1 < \frac{1}{2}$, $\delta_0 > 0$, $\delta_1 > 0$. We have

$$\min(1+\delta_0, \tfrac{1}{2}+2\delta_0) = \tfrac{1}{2}+2\delta_0,$$

and

$$\min(1+\delta_1, \tfrac{1}{2}+\delta_0+2\delta_1) = \tfrac{1}{2}+\delta_0+2\delta_1.$$

We are reduced to solve

$$\begin{cases} \frac{1}{2}+2\delta_0 = \frac{1}{2}+\delta_0+2\delta_1 \\ \frac{1}{2}+2\delta_0 = 2-2\delta_0-2\delta_1 \end{cases} \iff \begin{cases} 2\delta_1 = \delta_0 \\ \frac{3}{2} = 4\delta_0+2\delta_1 \end{cases} \iff \delta_0 = \frac{3}{10} \text{ and } \delta_1 = \frac{3}{20}.$$

Then we get $C(\Omega) > 0$ such that

$$\forall f \in H^1(\Omega), \quad \frac{q_h[\mathbf{A},\Omega](f)}{\|f\|^2_{L^2(\Omega)}} \geq h\mathscr{E}(\mathbf{B},\Omega) - C(\Omega)\big(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)h^{1+\frac{1}{10}}. \tag{5.28}$$

For further use we extract the following corollary of the previous proof:

Corollary 5.3. *Let* $\mathbf{x}_0 \in \overline{\Omega}$ *and* $K := \mathcal{B}(\mathbf{x}_0, \delta)$ *with* $\delta > 0$. *We define*

$$\mathscr{E}_K(\mathbf{B},\Omega) := \inf_{\mathbf{x}\in\overline{\Omega}\cap\overline{K}} E(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x}).$$

Then there exists $C > 0$ *and* $h_0 > 0$ *such that for all* $h \in (0, h_0)$ *and for all* $f \in \mathrm{Dom}(q_h[\mathbf{A},\Omega])$ *with support* $\operatorname{supp} f \subset\subset K$, *there holds*

$$\frac{1}{h}\frac{q_h[\mathbf{A},\Omega](f)}{\|f\|^2} \geq \mathscr{E}_K(\mathbf{B},\Omega) - Ch^{1/10}.$$

Proof. The corollary is obtained by slight modifications in the above proof. First we make a covering of $\overline{\Omega}\cap K$ instead in $\overline{\Omega}$. Therefore in the lower bound (5.11), we only have to consider $\mathbf{c} \in K$, and the energy is bounded below by $\mathscr{E}_K(\mathbf{B},\Omega)$ in (5.12). We finally reached (5.28) and deduce the Corollary. □

5.3. **Comments on possible generalizations.** As we can see from the proofs above, we used very few knowledge on the magnetic Laplacians—essentially the change of gauge, the change of variables, and the perturbation identity (A.7). The refined analysis is related to the corner structure. With the same approach and relying on the general estimates presented in Section 3.4.4, we are able to establish lower bounds for the ground state energy of magnetic Laplacians in n-dimensional corner domains.

Let $\Omega \in \mathfrak{D}(\mathbb{R}^n)$, and let ν be the maximal integer such that there exists a singular chain $(\mathbf{x}_0, \ldots, \mathbf{x}_{\nu-1})$ with a non-polyhedral reduced cone $\Gamma_{\mathbf{x}_0,\ldots,\mathbf{x}_{\nu-1}}$. We make the convention that $\nu = 0$ if all tangent cones are polyhedral.

Using an IMS partition on a hierarchy of balls of size h^{δ_0}, $h^{\delta_0+\delta_1}$, ..., $h^{\delta_0+\delta_1+\ldots+\delta_\nu}$ according to the position of their centers, and taking advantage of estimates (3.36), we arrive to the following collection of errors

$$
\begin{aligned}
&h^{1+\delta_0},\ h^{1+\delta_1},\ \ldots,\ h^{1+\delta_\nu}\\
&h^{\frac12+2\delta_0},\ h^{\frac12+\delta_0+2\delta_1},\ \ldots,\ h^{\frac12+\delta_0+\ldots+\delta_{\nu-1}+2\delta_\nu}\\
&h^{2-2(\delta_0+\delta_1+\ldots+\delta_\nu)},
\end{aligned}
$$

which is optimized choosing

$$\delta_k = 2^{\nu-k}\delta_\nu,\ k = 0, \ldots, \nu, \quad \text{with} \quad \delta_\nu = \frac{3}{3\cdot 2^{\nu+2}-4}\,.$$

The outcome is the following lower bound

$$\lambda_h(\mathbf{B}, \Omega) \geq h\mathscr{E}(\mathbf{B}, \Omega) - C(\Omega)\big(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)h^{1+1/(3\cdot 2^{\nu+1}-2)}\,.$$

Here $\mathscr{E}(\mathbf{B}, \Omega)$ is the natural generalization of (4.11) to n-dimensional domains. The results of Theorem 5.1 correspond to the values $\nu = 1$ and $\nu = 0$. Note that the remainder $\mathcal{O}(h^{5/4})$ is valid in a polyhedral domain in any dimension ($\nu = 0$).

6. **Taxonomy of model problems**

Refined estimates for an *upper bound* of the ground state energy $\lambda_h(\mathbf{B}, \Omega)$ will be obtained with the help of quasimode constructions. This relies on a better knowledge of tangent model problems $H(\mathbf{A}_\mathbb{X}, \Pi_\mathbb{X})$ for any singular chain $\mathbb{X}$ of Ω. In this section, we review and, when required, complete, essential facts concerning three-dimensional model problems, that is magnetic Laplacians $H(\mathbf{A}, \Pi)$ where Π is a cone in $\mathfrak{P}_3$ and $\mathbf{A}$ is a linear potential.

With the aim of constructing quasimodes for our original problem on Ω, we need generalized eigenvectors for its tangent problems. For this we make use of the localized domain $\mathrm{Dom}_{\mathrm{loc}}(H(\mathbf{A}, \Pi))$ of the model magnetic Laplacian $H(\mathbf{A}, \Pi)$ as introduced in (1.22).

Definition 6.1 (Generalized eigenvector)**.** Let $\Pi \in \mathfrak{P}_3$ be a cone and $\mathbf{A}$ a linear magnetic potential. We call *generalized eigenvector* for $H(\mathbf{A}, \Pi)$ a nonzero function $\Psi \in$

$\mathrm{Dom}_{\mathrm{loc}}(H(\mathbf{A},\Pi))$ associated with a real number Λ, so that

$$\begin{cases}(-i\nabla+\mathbf{A})^2\Psi=\Lambda\Psi & \text{in } \Pi,\\ (-i\nabla+\mathbf{A})\Psi\cdot\mathbf{n}=0 & \text{on } \partial\Pi.\end{cases} \tag{6.1}$$

■

Let $\Pi\in\mathfrak{P}_3$ be a 3D cone and let $\mathbf{B}$ be a constant magnetic field associated with a linear potential $\mathbf{A}$. Let $\Gamma\in\mathfrak{P}_d$ be a minimal reduced cone associated with Π. We recall that this means that $\Pi\equiv\mathbb{R}^{3-d}\times\Gamma$ and that the dimension d is minimal for such an equivalence. By analogy with Definition 3.18, $\mathfrak{C}_{\mathbf{0}}(\Pi)$ denotes the set of singular chains of Π originating at its vertex $\mathbf{0}$ and $\mathfrak{C}^*_{\mathbf{0}}(\Pi)$ is the subset of chains of length ≥ 2. Note that $\mathfrak{C}^*_{\mathbf{0}}(\Pi)$ is empty if and only if $\Pi=\mathbb{R}^3$, i.e., if $d=0$. We introduce the energy along higher chains:

Definition 6.2 (Energy along higher chains)**.** We define the quantity

$$\mathscr{E}^*(\mathbf{B},\Pi):=\begin{cases}\inf_{\mathbb{X}\in\mathfrak{C}^*_0(\Pi)}E(\mathbf{B},\Pi_{\mathbb{X}}) & \text{if } d>0,\\ +\infty & \text{if } d=0,\end{cases} \tag{6.2}$$

which is the infimum of the ground state energy of the magnetic Laplacian over all the singular chains of length ≥ 2. ■

We will see that this quantity will play a key role in the existence of generalized eigenvector that have some good decay properties (see Section 7).

In each of Sections 6.1–6.4 we consider one value of d, from 0 to 3 and give in each case relations between the ground stat energy $E(\mathbf{B},\Pi)$ and the energy along higher chains $\mathscr{E}^*(\mathbf{B},\Pi)$, and we provide generalized eigenvectors Ψ if they exist.

On one hand, thanks to Lemma A.5, we may reduce to deal with unitary magnetic field $|\mathbf{B}|=1$. On the other hand, quantities $E(\mathbf{B},\Pi)$ and $\mathscr{E}^*(\mathbf{B},\Pi)$ are independent of a choice of Cartesian coordinates. Thus, for each value of d, ranging from 0 to 3, once Π and a constant unitary magnetic field $\mathbf{B}$ are chosen, we exhibit a system of Cartesian coordinates $\mathbf{x}=(x_1,x_2,x_3)$ that allows the simplest possible description of the configuration $(\mathbf{B},\Pi)$. In these coordinates, the magnetic field can be viewed as a reference field, and for convenience, we denote it by $\underline{\mathbf{B}}=(b_0,b_1,b_2)$. We also choose a corresponding reference linear potential $\underline{\mathbf{A}}$, since we have gauge independence by virtue of Lemma A.1.

6.1. **Full space.** $d=0$. Π is the full space. We take coordinates $\mathbf{x}=(x_1,x_2,x_3)$ so that

$$\Pi=\mathbb{R}^3 \quad\text{and}\quad \underline{\mathbf{B}}=(1,0,0),$$

and choose as reference potential

$$\underline{\mathbf{A}}=(0,-\tfrac{x_3}{2},\tfrac{x_2}{2}).$$

Hence

$$H(\underline{\mathbf{A}},\Pi)=H(\underline{\mathbf{A}},\mathbb{R}^3)=\mathrm{D}_1^2+(\mathrm{D}_2-\tfrac{x_3}{2})^2+(\mathrm{D}_3+\tfrac{x_2}{2})^2 \quad\text{with}\quad \mathrm{D}_j=-i\partial_{x_j}.$$

Due to translation invariance, we have $\lambda_{\mathrm{ess}}(\underline{\mathbf{B}}, \mathbb{R}^3) = E(\underline{\mathbf{B}}, \mathbb{R}^3)$. It is classical (see [25]) that the spectrum of $H(\underline{\mathbf{A}}, \mathbb{R}^3)$ is $[1, +\infty)$. Therefore

$$E(\underline{\mathbf{B}}, \mathbb{R}^3) = 1\ . \tag{6.3}$$

A generalized eigenfunction associated with the ground state energy is

$$\Psi(\mathbf{x}) = \mathrm{e}^{-(x_2^2+x_3^2)/4} \quad \text{with} \quad \Lambda = 1. \tag{6.4}$$

6.2. **Half space.** $d = 1$. Π is a half-space. We take coordinates $\mathbf{x} = (x_1, x_2, x_3)$ so that

$$\Pi = \mathbb{R}^3_+ := \{(x_1, x_2, x_3) \in \mathbb{R}^3,\ x_3 > 0\} \quad \text{and} \quad \underline{\mathbf{B}} = (0, b_1, b_2) \text{ with } b_1^2 + b_2^2 = 1\ ,$$

and choose as reference potential

$$\underline{\mathbf{A}} = (b_1 x_3 - b_2 x_2, 0, 0)\ .$$

Hence

$$H(\underline{\mathbf{A}}, \Pi) = H(\underline{\mathbf{A}}, \mathbb{R}^3_+) = (\mathrm{D}_1 + b_1 x_3 - b_2 x_2)^2 + \mathrm{D}_2^2 + \mathrm{D}_3^2.$$

Due to translation invariance, we have $\lambda_{\mathrm{ess}}(\underline{\mathbf{B}}, \mathbb{R}^3_+) = E(\underline{\mathbf{B}}, \mathbb{R}^3_+)$. We note that

$$\mathscr{E}^*(\underline{\mathbf{B}}, \mathbb{R}^3_+) = E(\underline{\mathbf{B}}, \mathbb{R}^3) = 1. \tag{6.5}$$

There exists $\theta \in [0, 2\pi)$ such that $b_1 = \cos\theta$ and $b_2 = \sin\theta$, so that θ is the angle between the magnetic field and the boundary of $\mathbb{R}^3_+$. Due to symmetries we can reduce to $\theta \in [0, \frac{\pi}{2}]$. Denote by $\mathcal{F}_1$ the Fourier transform in x_1-variable, by τ the Fourier variable associated with x_1, and introduce the operator

$$\widehat{H}_\tau(\underline{\mathbf{A}}, \mathbb{R}^3_+) := (\tau + b_1 x_3 - b_2 x_2)^2 + \mathrm{D}_2^2 + \mathrm{D}_3^2,$$

acting on $L^2(\mathbb{R} \times \mathbb{R}_+)$ with natural boundary conditions. There holds

$$\mathcal{F}_1\ H(\underline{\mathbf{A}}, \mathbb{R}^3_+)\ \mathcal{F}_1^* = \int_{\tau \in \mathbb{R}}^{\oplus} \widehat{H}_\tau(\underline{\mathbf{A}}, \mathbb{R}^3_+)\, \mathrm{d}\tau.$$

We discriminate three cases:

• *Tangent field.* $\theta = 0$, then $\widehat{H}_\tau(\underline{\mathbf{A}}, \mathbb{R}^3_+) := \mathrm{D}_2^2 + \mathrm{D}_3^2 + (\tau + x_3)^2$, let ξ be the partial Fourier variable associated with x_2 and define the new operators

$$\widehat{H}_{\xi,\tau}(\underline{\mathbf{A}}, \mathbb{R}^3_+) := \xi^2 + D_3^2 + (\tau + x_3)^2, \quad \mathcal{H}(\tau) = \mathrm{D}_3^2 + (\tau + x_3)^2\ ,$$

where $\mathcal{H}(\tau)$ (sometimes called the de Gennes operator) acts on $L^2(\mathbb{R}_+)$ with Neumann boundary condition. Its first eigenvalue is denoted by $\mu(\tau)$. There holds

$$\inf \mathfrak{S}(\widehat{H}_{\tau,\xi}(\underline{\mathbf{A}}, \mathbb{R}^3_+)) = \mu(\tau) + \xi^2.$$

The behavior of the first eigenvalue $\mu(\tau)$ of $\mathcal{H}(\tau)$ is well-known (see [13]): The function μ admits a unique minimum denoted by $\Theta_0 \simeq 0.59$ for the value $\tau = -\sqrt{\Theta_0}$. Hence

$$E(\underline{\mathbf{B}}, \mathbb{R}^3_+) = \Theta_0 < \mathscr{E}^*(\underline{\mathbf{B}}, \mathbb{R}^3_+).$$

If Φ denotes an eigenvector of $\mathcal{H}(\tau)$ associated with Θ_0 (function of $x_3 \in \mathbb{R}_+$), a corresponding generalized eigenvector is

$$\Psi(\mathbf{x}) = \mathrm{e}^{-i\sqrt{\Theta_0}\,x_1}\,\Phi(x_3) \quad \text{with} \quad \Lambda = \Theta_0. \tag{6.6}$$

• *Normal field.* $\theta = \frac{\pi}{2}$, then $\widehat{H}_\tau(\underline{\mathbf{A}}, \mathbb{R}^3_+) := \mathrm{D}_2^2 + \mathrm{D}_3^2 + (\tau - x_2)^2$. There holds for all $\tau \in \mathbb{R}$, $\inf \mathfrak{S}(\widehat{H}_\tau(\underline{\mathbf{A}}, \mathbb{R}^3_+)) = 1$ (see [27, Theorem 3.1]), hence

$$E(\underline{\mathbf{B}}, \mathbb{R}^3_+) = 1 = \mathscr{E}^*(\underline{\mathbf{B}}, \mathbb{R}^3_+).$$

• *Neither tangent nor normal.* $\theta \in (0, \frac{\pi}{2})$. Then for any $\tau \in \mathbb{R}$, $\widehat{H}_\tau(\underline{\mathbf{A}}, \mathbb{R}^3_+)$ is isospectral to $\widehat{H}_0(\underline{\mathbf{A}}, \mathbb{R}^3_+)$ the ground state energy of which is an eigenvalue $\sigma(\theta) < 1$ (see [21]). We deduce

$$E(\underline{\mathbf{B}}, \mathbb{R}^3_+) = \sigma(\theta) \quad \text{with} \quad \sigma(\theta) < 1.$$

The first eigenvalue $\sigma(\theta)$ of $\widehat{H}_0(\underline{\mathbf{A}}, \mathbb{R}^3_+)$ is associated with an exponentially decreasing eigenvector Φ that is a function of $(x_2, x_3) \in \mathbb{R} \times \mathbb{R}_+$. The corresponding generalized eigenvector for $H(\underline{\mathbf{A}}, \mathbb{R}^3_+)$ is given by

$$\Psi(\mathbf{x}) = \Phi(x_2, x_3) \quad \text{with} \quad \Lambda = \sigma(\theta). \tag{6.7}$$

• *Conclusion.* Collecting the previous three cases we get

$$E(\underline{\mathbf{B}}, \mathbb{R}^3_+) \leq \mathscr{E}^*(\underline{\mathbf{B}}, \mathbb{R}^3_+)\,, \tag{6.8}$$

with equality if and only if the magnetic field is normal to the boundary.

We recall from the literature:

Lemma 6.3. *The function $\theta \mapsto \sigma(\theta)$ is continuous and increasing on $(0, \frac{\pi}{2})$ ([21, 27]). Set $\sigma(0) = \Theta_0$ and $\sigma(\frac{\pi}{2}) = 1$. Then the function $\theta \mapsto \sigma(\theta)$ is of class $\mathscr{C}^1$ on $[0, \frac{\pi}{2}]$ ([7]).*

6.3. **Wedges.** $d = 2$. Π is a wedge and let $\alpha \in (0, \pi) \cup (\pi, 2\pi)$ denote its opening. Let us introduce the model sector $\mathcal{S}_\alpha$ and the model wedge $\mathcal{W}_\alpha$

$$\mathcal{S}_\alpha = \begin{cases} \{x = (x_2, x_3),\ x_2 \tan\frac{\alpha}{2} > |x_3|\} & \text{if } \alpha \in (0, \pi) \\ \{x = (x_2, x_3),\ x_2 \tan\frac{\alpha}{2} > -|x_3|\} & \text{if } \alpha \in (\pi, 2\pi) \end{cases} \quad \text{and} \quad \mathcal{W}_\alpha = \mathbb{R} \times \mathcal{S}_\alpha\,. \tag{6.9}$$

We take coordinates $\mathbf{x} = (x_1, x_2, x_3)$ so that

$$\Pi = \mathcal{W}_\alpha \quad \text{and} \quad \underline{\mathbf{B}} = (b_0, b_1, b_2) \text{ with } b_0^2 + b_1^2 + b_2^2 = 1\,,$$

and choose as reference potential

$$\underline{\mathbf{A}} = (b_1 x_3 - b_2 x_2, 0, b_0 x_2)\,.$$

Hence

$$H(\underline{\mathbf{A}}, \Pi) = H(\underline{\mathbf{A}}, \mathcal{W}_\alpha) = (\mathrm{D}_1 + b_1 x_3 - b_2 x_2)^2 + \mathrm{D}_2^2 + (\mathrm{D}_3 + b_0 x_2)^2.$$

Due to translation invariance, we have $\lambda_{\rm ess}(\underline{\mathbf{B}}, \mathcal{W}_\alpha) = E(\underline{\mathbf{B}}, \mathcal{W}_\alpha)$. Denote by τ the Fourier variable associated with x_1, and

$$\widehat{H}_\tau(\underline{\mathbf{A}}, \mathcal{W}_\alpha) := (\tau + b_1 x_3 - b_2 x_2)^2 + \mathrm{D}_2^2 + (\mathrm{D}_3 + b_0 x_2)^2 \tag{6.10}$$

acting on $L^2(\mathcal{S}_\alpha)$ with natural Neumann boundary condition. We introduce the notation:

$$s(\underline{\mathbf{B}}, \mathcal{S}_\alpha; \tau) := \inf \mathfrak{S}(\widehat{H}_\tau(\underline{\mathbf{A}}, \mathcal{W}_\alpha)),$$

so that we have the direct Fourier integral decomposition

$$\mathcal{F}_1 \ H(\underline{\mathbf{A}}, \mathcal{W}_\alpha) \ \mathcal{F}_1^* = \int_{\tau\in\mathbb{R}}^{\oplus} \widehat{H}_\tau(\underline{\mathbf{A}}, \mathcal{W}_\alpha) \, \mathrm{d}\tau$$

and the relation

$$E(\underline{\mathbf{B}}, \mathcal{W}_\alpha) = \inf_{\tau\in\mathbb{R}} s(\underline{\mathbf{B}}, \mathcal{S}_\alpha; \tau) \, . \tag{6.11}$$

The singular chains of $\mathfrak{C}_{\mathbf{0}}^*(\mathcal{W}_\alpha)$ have three equivalence classes, cf. Definition 3.19 and Description 3.30 (3): The full space $\mathbb{R}^3$ and the two half-spaces $\Pi_\alpha^\pm$ corresponding to the two faces $\partial^\pm\mathcal{W}_\alpha$ of $\mathcal{W}_\alpha$. Thus

$$\mathscr{E}^*(\underline{\mathbf{B}}, \mathcal{W}_\alpha) = \min\{E(\underline{\mathbf{B}}, \mathbb{R}^3), \ E(\underline{\mathbf{B}}, \Pi_\alpha^+), \ E(\underline{\mathbf{B}}, \Pi_\alpha^-)\}.$$

Let $\theta^\pm \in [0, \frac{\pi}{2}]$ be the angle between $\underline{\mathbf{B}}$ and the face $\partial\Pi_\alpha^\pm$. We have, cf. Lemma 6.3,

$$\mathscr{E}^*(\underline{\mathbf{B}}, \mathcal{W}_\alpha) = \min\{1, \sigma(\theta^+), \sigma(\theta^-)\} = \sigma(\min\{\theta^+, \ \theta^-\}). \tag{6.12}$$

When $\Pi = \mathcal{W}_\alpha$, we quote the following result from [36, Theorem 3.5]:

Lemma 6.4. *Let $\alpha \in (0, \pi) \cup (\pi, 2\pi)$. There holds the inequality*

$$E(\underline{\mathbf{B}}, \mathcal{W}_\alpha) \le \mathscr{E}^*(\underline{\mathbf{B}}, \mathcal{W}_\alpha). \tag{6.13}$$

Moreover, if $E(\underline{\mathbf{B}}, \mathcal{W}_\alpha) < \mathscr{E}^*(\underline{\mathbf{B}}, \mathcal{W}_\alpha)$, then the function $\tau \mapsto s(\underline{\mathbf{B}}, \mathcal{S}_\alpha; \tau)$ reaches its infimum. Let τ^* be a minimizer. Then $E(\underline{\mathbf{B}}, \mathcal{W}_\alpha)$ is the first eigenvalue of the operator $\widehat{H}_{\tau^*}(\underline{\mathbf{A}}, \mathcal{W}_\alpha)$ and any associated eigenfunction Φ has exponential decay. The function

$$\Psi(\mathbf{x}) = \mathrm{e}^{i\tau^* x_1} \Phi(x_2, x_3) \tag{6.14}$$

is a generalized eigenvector for the operator $H(\underline{\mathbf{A}}, \mathcal{W}_\alpha)$ associated with $\Lambda = E(\underline{\mathbf{B}}, \mathcal{W}_\alpha)$.

Finally, let us quote now the continuity result on dihedra from [36, Theorem 4.5]:

Lemma 6.5. *The function $(\mathbf{B}, \alpha) \mapsto E(\mathbf{B}, \mathcal{W}_\alpha)$ is continuous on $\mathbb{S}^2 \times ((0, \pi) \cup (\pi, 2\pi))$.*

6.4. **3D cones.** $d=3$ The aim of this paragraph is the characterization of the bottom of the essential spectrum $\lambda_{\rm ess}(\mathbf{B},\Pi)$ of $H(\mathbf{A},\Pi)$:

Theorem 6.6. *Let $\Pi\in\mathfrak{P}_3$ be a cone with $d=3$, which means that Π is not a wedge, nor a half-space, nor the full space. Let $\mathbf{B}$ be a constant magnetic field. With the quantity $\mathscr{E}^*(\mathbf{B},\Pi)$ introduced in (6.2), there holds*

$$\lambda_{\rm ess}(\mathbf{B},\Pi)=\mathscr{E}^*(\mathbf{B},\Pi)\,.$$

We recall the Persson Lemma that gives a characterization of the bottom of the essential spectrum (see [33]):

Lemma 6.7. *Let $\Pi\in\mathfrak{P}_3$ and let $\mathbf{A}$ be a linear magnetic potential associated with $\mathbf{B}$. For $R>0$, we define $\mathrm{Dom}_0^R(q[\mathbf{A},\Pi])$ as the subspace of functions Ψ in $\mathrm{Dom}(q[\mathbf{A},\Pi])$ with compact support, and $\operatorname{supp}\Psi\cap\mathcal{B}(\mathbf{0},R)=\emptyset$. Then we have*

$$\lambda_{\rm ess}(\mathbf{B},\Pi)=\lim_{R\to+\infty}\left(\inf_{\substack{\Psi\in\mathrm{Dom}_0^R(q[\mathbf{A},\Pi])\\ \Psi\neq0}}\frac{q[\mathbf{A},\Pi](\Psi)}{\|\Psi\|^2}\right)\,.$$

Before proving Theorem 6.6, we show

Lemma 6.8. *Let $\Pi\in\mathfrak{P}_3$ be a cone with $d=3$, let $\Omega_{\mathbf{0}}=\Pi\cap\mathbb{S}^2$ be its section. Then $\mathscr{E}^*(\mathbf{B},\Pi)$ coincides with the infimum over singular chains of length 2:*

$$\mathscr{E}^*(\mathbf{B},\Pi)=\inf_{\mathbf{x}_1\in\overline{\Omega}_{\mathbf{0}}}E(\mathbf{B},\Pi_{\mathbf{0},\mathbf{x}_1})\,. \tag{6.15}$$

Proof. For all singular chains $\mathbb{X}$ and $\mathbb{X}'$ in $\mathfrak{C}^*(\Pi)$ such that $\mathbb{X}\leq\mathbb{X}'$, we have $E(\Pi_{\mathbb{X}},\mathbf{B})\leq E(\Pi_{\mathbb{X}'},\mathbf{B})$ as a consequence of (6.8) and (6.13). Hence (6.15). ☐

Proof of Theorem 6.6. Combining Lemmas 6.7 and A.5, we get that

$$\lambda_{\rm ess}(\mathbf{B},\Pi)=\lim_{h\to0}\left(h^{-1}\inf_{\substack{\Psi\in\mathrm{Dom}_0^1(q_h[\mathbf{A},\Pi])\\ \Psi\neq0}}\frac{q_h[\mathbf{A},\Pi](\Psi)}{\|\Psi\|^2}\right)\,. \tag{6.16}$$

• *Upper bound for $\lambda_{\rm ess}(\mathbf{B},\Pi)$.* Let $\varepsilon>0$. By Lemma 6.8 there exist $\mathbf{x}\in\overline{\Omega}_{\mathbf{0}}$ and an associated chain $\mathbb{X}=(\mathbf{0},\mathbf{x})$ of length 2 such that

$$E(\mathbf{B},\Pi_{\mathbb{X}})<\mathscr{E}^*(\mathbf{B},\Pi)+\varepsilon\,. \tag{6.17}$$

Let $\mathbf{x}':=2\mathbf{x}$. Notice that the tangent cone to Π at $\mathbf{x}'$ is $\Pi_{\mathbf{x}'}=\Pi_{\mathbb{X}}$ and therefore $E(\mathbf{B},\Pi_{\mathbf{x}'})=E(\mathbf{B},\Pi_{\mathbb{X}})$. We use Lemma 4.8 (that clearly applies even though Π is unbounded): So there exists $h_0>0$ such that for all $h\in(0,h_0)$ we can find f_h normalized and supported near $\mathbf{x}'$ satisfying $h^{-1}q_h[\mathbf{A},\Pi](f_h)\leq E(\mathbf{B},\Pi_{\mathbb{X}})+\varepsilon$. Since $|\mathbf{x}'|=2$, we may assume without restriction that $\operatorname{supp}(f_h)\cap\mathcal{B}(0,1)=\emptyset$. Combining this with (6.17) we get

$$\frac{1}{h}q_h[\mathbf{A},\Pi](f_h)\leq\mathscr{E}^*(\mathbf{B},\Pi)+2\varepsilon\,,$$

and therefore deduce from (6.16) the upper bound of $\lambda_{\mathrm{ess}}(\mathbf{B},\Pi)$ by $\mathscr{E}^*(\mathbf{B},\Pi)$.

• *Lower bound for* $\lambda_{\mathrm{ess}}(\mathbf{B},\Pi)$. Notice that for all $\mathbf{x}\in\overline{\Pi}\setminus\mathcal{B}(0,1)$, we have $\Pi_{\mathbf{x}}=\Pi_{\mathbb{X}}$ where $\mathbb{X}=(\mathbf{0},\mathbf{x}/|\mathbf{x}|)$. Therefore (see (6.15)):

$$\inf_{\mathbf{x}\in\overline{\Pi}\setminus\mathcal{B}(0,1)} E(\mathbf{B},\Pi_{\mathbf{x}})=\mathscr{E}^*(\mathbf{B},\Pi)\,.$$

Then we easily deduce the lower bound from Corollary 5.3 and (6.16). □

7. **Substructures for model problems**

Whereas the previous section provides an exhaustive description of model problems in 3 dimensions, this section goes deeper into the substructure of the magnetic Laplacian on a model cone Π with constant magnetic field $\mathbf{B}$. The main result of this section is Theorem 7.2, which compares the ground state energy of the magnetic Laplacian on the tangent substructures of Π, and states that we can always find a canonical generalized eigenvector (called *admissible*) living on a tangent substructure and associated with the ground state energy $E(\mathbf{B},\Pi)$.

7.1. **Admissible Generalized Eigenvectors and dichotomy theorem.**

Definition 7.1 (Admissible Generalized Eigenvector)**.** Let $\Pi\in\mathfrak{P}_3$ be a cone. Recall that $d(\Pi)\in[0,3]$ is the dimension of its minimal reduced cone. Let $\mathbf{A}$ be a linear magnetic potential. A generalized eigenvector Ψ for $H(\mathbf{A},\Pi)$ (cf. Definition 6.1) is said to be *admissible* if there exist an integer $k\geq d(\Pi)$ and a rotation $\underline{\mathrm{U}}:\mathbf{x}\mapsto(\mathbf{y},\mathbf{z})$ that maps Π onto the product $\mathbb{R}^{3-k}\times\Upsilon$ with Υ a cone in $\mathfrak{P}_k$, and such that

$$\Psi\circ\underline{\mathrm{U}}^{-1}(\mathbf{y},\mathbf{z})=\mathrm{e}^{i\vartheta(\mathbf{y},\mathbf{z})}\,\Phi(\mathbf{z})\qquad\forall\mathbf{y}\in\mathbb{R}^{3-k},\quad\forall\mathbf{z}\in\Upsilon,\tag{7.1}$$

with some real polynomial function ϑ of degree ≤ 2 and some exponentially decreasing function Φ, namely there exist positive constants c_Ψ and C_Ψ such that

$$\|\mathrm{e}^{c_\Psi|\mathbf{z}|}\Phi\|_{L^2(\Upsilon)}\leq C_\Psi\|\Phi\|_{L^2(\Upsilon)}\,.\tag{7.2}$$

"Admissible Generalized Eigenvector" will be shortened as A.G.E. ■

The main result that we prove in this section is a dichotomy statement, as follows.

Theorem 7.2 (Dichotomy Theorem)**.** *Let $\Pi\in\mathfrak{P}_3$ be a cone and $\mathbf{B}$ be a constant nonzero magnetic field. Let $\mathbf{A}$ be any associated linear magnetic potential. Recall that $E(\mathbf{B},\Pi)$ is the ground state energy of $H(\mathbf{A},\Pi)$ and $\mathscr{E}^*(\mathbf{B},\Pi)$ is the energy along higher chains introduced in Definition 6.2. Then,*

$$E(\mathbf{B},\Pi)\leq\mathscr{E}^*(\mathbf{B},\Pi)\tag{7.3}$$

and we have the dichotomy:

(i) *If* $E(\mathbf{B},\Pi) < \mathscr{E}^*(\mathbf{B},\Pi)$*, then* $H(\mathbf{A},\Pi)$ *admits an Admissible Generalized Eigenvector associated with the value* $E(\mathbf{B},\Pi)$*.*

(ii) *If* $E(\mathbf{B},\Pi) = \mathscr{E}^*(\mathbf{B},\Pi)$*, then there exists a singular chain* $\mathbb{X} \in \mathfrak{C}_{\mathbf{0}}^*(\Pi)$ *such that*

$$E(\mathbf{B},\Pi_{\mathbb{X}}) = E(\mathbf{B},\Pi) \quad \textit{and} \quad E(\mathbf{B},\Pi_{\mathbb{X}}) < \mathscr{E}^*(\mathbf{B},\Pi_{\mathbb{X}}).$$

Remark 7.3. In the case (ii), we note that by statement (i) applied to the cone $\Pi_{\mathbb{X}}$, $H(\mathbf{A},\Pi_{\mathbb{X}})$ admits an A.G.E. associated with the value $E(\mathbf{B},\Pi)$.

Remark 7.4. If $\mathbf{B}=0$, there is no magnetic field and $E(\Pi,\mathbf{B})=0$. An associated A.G.E. is the constant function $\Psi\equiv 1$.

Proof of Theorem 7.2. The proof relies on an exhaustion of cases based on the analysis provided in Section 6 combined with a hierarchical classification of model problems on tangent substructures of a cone Π.

• *Geometrical invariance.* As already mentioned at the beginning of Section 6 we may assume that $\mathbf{B}$ is unitary and choose any suitable Cartesian coordinates. Moreover the notion of A.G.E. is gauge invariant: If $\mathbf{A}$ and $\mathbf{A}'$ are two linear magnetic potentials satisfying $\operatorname{curl}\mathbf{A} = \operatorname{curl}\mathbf{A}'$, then, as a consequence of Lemma A.1, there exists an A.G.E. for $H(\mathbf{A},\Pi)$ if and only if there is one for $H(\mathbf{A}',\Pi)$. Hence, to prove the theorem, we may reduce to the reference configurations investigated in Sections 6.1–6.3.

• *Algorithm of the proof.* We first establish the theorem when $d=0$, then we apply the following analysis for increasing values of $d=d(\Pi)$ from 1 to 3:

(1) Check inequality (7.3).
(2) Check assertion (i).
(3) Prove that there exists a singular chain $\mathbb{X}\in\mathfrak{C}_{\mathbf{0}}^*(\Pi)$ such that $\mathscr{E}^*(\mathbf{B},\Pi) = E(\mathbf{B},\Pi_{\mathbb{X}})$. Since $d(\Pi_{\mathbb{X}}) < d$, assertion (ii) will be a consequence of the analysis made for lower dimensions.

This procedure applied to reference problems described in Section 6 will provide the theorem.

• *d=0.* Here $\Pi=\mathbb{R}^3$ (see Section 6.1). We have $E(\mathbf{B},\mathbb{R}^3)=1$ and $\mathscr{E}^*(\mathbf{B},\mathbb{R}^3)=+\infty$, moreover there always exists an admissible generalized eigenvector associated with 1, see (6.4). Theorem 7.2 is proved for $d=0$.

• *d=1.* The model cone is $\mathbb{R}^3_+$, see Section 6.2. Inequality (7.3) has already been proved, see (6.8). We know that $E(\mathbf{B},\mathbb{R}^3_+) < \mathscr{E}^*(\mathbf{B},\mathbb{R}^3_+)$ if and only if $\mathbf{B}$ is not normal to the boundary. In this case, A.G.E. have already been written and their form depends on whether the magnetic field is tangent to the boundary or not, see (6.6) and (6.7) respectively. Therefore (i) of Theorem 7.2 holds in the non-normal case. When $\mathbf{B}$ is normal to the boundary, we have $E(\mathbf{B},\mathbb{R}^3_+) = \mathscr{E}^*(\mathbf{B},\mathbb{R}^3_+)$. The only possible chain is $\mathbb{R}^3$ and we have $\mathscr{E}^*(\mathbf{B},\mathbb{R}^3_+) = E(\mathbf{B},\mathbb{R}^3) < \mathscr{E}^*(\mathbf{B},\mathbb{R}^3)$ (see the above paragraph $d=0$). Therefore Theorem 7.2 is proved for $d=1$.

• *d=2.* The model cone is the wedge $\mathcal{W}_\alpha$, see Section 6.3. Inequality (7.3) and assertion (i) come from Lemma 6.4. To deal with case (ii), denote by $\Pi_\alpha^\pm$ the two half-spaces associated with the two faces of the wedge. Denote by $\theta^\pm \in [0, \frac{\pi}{2})$ the angles between $\mathbf{B}$ and the boundary of $\Pi_\alpha^\pm$. Let $\circ \in \{-,+\}$ satisfying $\theta^\circ = \min(\theta^-, \theta^+)$ and Π_α° be the corresponding face. Due to Lemma 6.3, we have $\min(1, \sigma(\theta^+), \sigma(\theta^-)) = \sigma(\theta^\circ)$ and $\mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha) = \sigma(\theta^\circ) = E(\mathbf{B}, \Pi_\alpha^\circ)$. Therefore in case (ii) we are led to the analysis done in the previous paragraph ($d = 1$) and Theorem 7.2 is proved for $d = 2$.

• *d=3.* Due to Theorem 6.6, we have $\mathscr{E}^*(\mathbf{B}, \Pi) = \lambda_{\text{ess}}(\mathbf{B}, \Pi)$ and therefore (7.3). Moreover if $E(\mathbf{B}, \Pi) < \mathscr{E}^*(\mathbf{B}, \Pi)$, then there exists an eigenvector for $H(\mathbf{A}, \Pi)$, which by standard arguments based on Agmon estimates is exponentially decreasing, see [1]. Therefore (i) is proved.

It remains to find $\mathbb{X} \in \mathfrak{C}_0^*(\Pi)$ such that $\mathscr{E}^*(\mathbf{B}, \Pi) = E(\mathbf{B}, \Pi_\mathbb{X})$. Define on $\mathfrak{C}_\mathbf{0}^*(\Pi)$ the function $F(\mathbb{X}) = E(\mathbf{B}, \Pi_\mathbb{X})$. Let $\Omega_\mathbf{0}$ denotes the section of Π, define the function $F^\star$ on $\mathfrak{C}(\Omega_\mathbf{0})$ by the partial application

$$F^\star(\mathbb{Y}) = F((\mathbf{0}, \mathbb{Y})), \quad \mathbb{Y} \in \mathfrak{C}(\Omega_\mathbf{0}).$$

Since (7.3) has already been proved for $d \leq 2$, we have for all $\mathbb{Y}$ and $\mathbb{Y}'$ in $\mathfrak{C}(\Omega_\mathbf{0})$:

$$\mathbb{Y} \leq \mathbb{Y}' \implies F^\star(\mathbb{Y}) \leq F^\star(\mathbb{Y}') \,. \tag{7.4}$$

Let us show that $F^\star$ is continuous with respect to the distance $\mathbb{D}$ introduced in Definition 3.22. Since $\Omega_\mathbf{0}$ has a finite number of vertices, the chains $\mathbb{Y} \in \mathfrak{C}(\Omega_\mathbf{0})$ such that $\Pi_\mathbb{Y}$ is a sector (and $\Pi_\mathbb{X} = \Pi_{(\mathbf{0},\mathbb{Y})}$ is a wedge) are isolated for the topology associated with the distance $\mathbb{D}$. If $\mathbb{Y}$ is such that $\Pi_{(\mathbf{0},\mathbb{Y})} = \mathbb{R}^3$, then $F^\star(\mathbb{Y}) = 1$ (see (6.3)). Therefore it remains to treat the case where the tangent substructures $\Pi_{(\mathbf{0},\mathbb{Y})}$ are half-spaces. Let $\mathbb{Y}$ and $\mathbb{Y}'$ be such chains. Denote by θ (resp. θ') the unoriented angle in $[0, \frac{\pi}{2})$ between $\mathbf{B}$ and $\Pi_\mathbb{X}$ (resp. between $\mathbf{B}$ and $\Pi_{\mathbb{X}'}$). We have $|\theta - \theta'| \to 0$ as $\mathbb{D}(\mathbb{Y}, \mathbb{Y}') \to 0$. Moreover

$$F^\star(\mathbb{Y}) - F^\star(\mathbb{Y}') = E(\mathbf{B}, \Pi_\mathbb{X}) - E(\mathbf{B}, \Pi_{\mathbb{X}'}) = \sigma(\theta) - \sigma(\theta') \,.$$

As a consequence of the continuity of the function σ, see Lemma 6.3, we get that $F^\star(\mathbb{Y}) - F^\star(\mathbb{Y}')$ goes to 0 as $\mathbb{D}(\mathbb{Y}, \mathbb{Y}')$ goes to 0. This shows that $F^\star$ is continuous on $\mathfrak{C}(\Omega_\mathbf{0})$. Thanks to (7.4), we can apply Theorem 3.25: the function $\overline{\Omega_\mathbf{0}} \ni \mathbf{x} \mapsto F^\star((\mathbf{x})) = E(\mathbf{B}, \Pi_{\mathbf{0},\mathbf{x}})$ is lower semi-continuous on $\overline{\Omega_\mathbf{0}}$. Since $\overline{\Omega_\mathbf{0}}$ is compact, it reaches its infimum. Combining this with Lemma 6.8, we get:

$$\exists \mathbf{x}_1 \in \overline{\Omega_\mathbf{0}}, \quad \mathscr{E}^*(\mathbf{B}, \Pi) = E(\mathbf{B}, \Pi_{\mathbf{0},\mathbf{x}_1}) \,.$$

Therefore (ii) follows from the analysis made for lower dimensions and Theorem 7.2 is proved for $d = 3$. □

7.2. **Structure of the Admissible Generalized Eigenvectors.** In this section we list the model configurations $(\mathbf{B}, \Pi)$ owning A.G.E. and give a comprehensive overview of their structure in a table.

Let $\mathbf{B}$ be a constant magnetic field and Π a cone in $\mathfrak{P}_3$. Let us assume that $E(\mathbf{B},\Pi) < \mathscr{E}^*(\mathbf{B},\Pi)$. Therefore by Theorem 7.2 there exist A.G.E. Ψ that have the form (7.1). We recall the discriminant parameter $k \in \{1,2,3\}$ which is the number of directions in which the generalized eigenvector has an exponential decay. For further use we call (G1), (G2), and (G3) the situation where $k = 1$, 2, and 3, respectively.

(k,d)	Model $(\mathbf{B},\Pi)$	Potential $\underline{\mathbf{A}}$	Υ	Explicit Ψ	Φ eigenvector of
(3,3)	———	———	$\Pi = \Gamma$	$\Phi(\mathbf{z})$	$H(\mathbf{A},\Pi)$
(2,2)	(b_0,b_1,b_2) $\Pi = \mathbb{R}\times\mathcal{S}_\alpha$	$(b_1z_2 - b_2z_1, 0, b_0z_1)$	$\mathcal{S}_\alpha = \Gamma$	$e^{i\tau^* y}\Phi(\mathbf{z})$	$\widehat{H}_\tau(\underline{\mathbf{A}},\mathcal{W}_\alpha)$, cf. (6.10)
(2,1)	$(0,b_1,b_2)$, $b_2\neq 0$ $\Pi = \mathbb{R}^2\times\mathbb{R}_+$	$(b_1z_2 - b_2z_1, 0, 0)$	$\mathbb{R}\times\mathbb{R}_+$	$\Phi(\mathbf{z})$	$-\Delta_{\mathbf{z}} + (b_1z_2 - b_2z_1)^2$
(2,0)	$(1,0,0)$ $\Pi = \mathbb{R}^3$	$(0, -\frac{1}{2}z_2, \frac{1}{2}z_1)$	$\mathbb{R}^2$	$e^{-\lvert\mathbf{z}\rvert^2/4}$	$-\Delta_{\mathbf{z}} + i\mathbf{z}\times\nabla_{\mathbf{z}} + \frac{\lvert\mathbf{z}\rvert^2}{4}$
(1,1)	$(0,1,0)$ $\Pi = \mathbb{R}^2\times\mathbb{R}_+$	$(z,0,0)$	$\mathbb{R}_+ = \Gamma$	$e^{-iy_1\sqrt{\Theta_0}}\Phi(z)$	$-\partial_z^2 + (z-\sqrt{\Theta_0})^2$

Table 1. A.G.E. of $H(\mathbf{A},\Pi)$ depending on $(\mathbf{B},\Pi)$ when $E(\mathbf{B},\Pi) < \mathscr{E}^*(\mathbf{B},\Pi)$, written in variables $(\mathbf{y},\mathbf{z}) \in \mathbb{R}^{3-k}\times\Upsilon$.

In Table 1 we gather all possible situations for (k,d) where d is the dimension $d(\Pi)$ of the reduced cone of Π. We assume that the magnetic field $\mathbf{B}$ is unitary, similar formulas can be found using Lemma A.5 for any non-zero constant magnetic field. We provide the explicit form of an admissible generalized eigenfunction Ψ of $H(\underline{\mathbf{A}},\Pi)$ in variables $(\mathbf{y},\mathbf{z}) \in \mathbb{R}^{3-k}\times\Upsilon$ where $\underline{\mathbf{A}}$ is a model linear potential associated with $\mathbf{B}$ in these variables. We also give the cone Υ on which the generalized eigenfunction has exponential decay (note that Υ does not always coincide with the reduced cone Γ of Π).

Remark 7.5. In the case where Π is a half-space and $\mathbf{B}$ is normal to $\partial\Pi$, we have $E(\mathbf{B},\Pi) = \mathscr{E}^*(\mathbf{B},\Pi) = 1$ and we are in case (ii) of Theorem 7.2, therefore there exists an admissible generalized eigenvector for the strict singular chain $\mathbb{R}^3$ of Π. However there also exists an admissible generalized eigenfunction for the operator on the half-plane Π. Let $\mathbf{z} = (z_1,z_2)$ be coordinates of $\partial\Pi$ and y coordinate normal to $\partial\Pi$. Let $\underline{\mathbf{A}}(y,z_1,z_2) := (0, -\frac{z_2}{2}, \frac{z_1}{2})$. As described in [27, Lemma 4.3], the function $\Psi : (y,z_1,z_2) \mapsto e^{-(z_1^2+z_2^2)/4}$ is an admissible generalized eigenvector for $H(\underline{\mathbf{A}},\Pi)$ associated with 1, indeed it satisfies the Neumann boundary condition at the boundary $y = 0$ since it is constant in the y direction and it is a solution of the eigenvalue equation $H(\underline{\mathbf{A}},\Pi)\Psi = \Psi$, see Section 6.1.

7.3. **Examples.** In the case $d = 1$, it is known whether we are in situation (i) or (ii) of Theorem 7.2 depending on the geometry. This is not the case in general for model cones

Π with $d \geq 2$, and only in few cases it is known whether (7.3) is strict or not. We provide below some examples of wedges and 3d cones where $E(\mathbf{B}, \Pi)$ has been studied.

Example 7.6 (Wedges)**.** Let $\mathbf{B} \in \mathbb{S}^2$ be a constant magnetic field. Let α be chosen in $(0, \pi) \cup (\pi, 2\pi)$.

(a) For α small enough there holds $E(\mathbf{B}, \mathcal{W}_\alpha) < \mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha)$ (see [36] when the magnetic field is not tangent to the plane of symmetry of the wedge and [34, Ch. 7] otherwise) and we are in case (i) of Theorem 7.2.

(b) Let $\mathbf{B} = (0, 0, 1)$ be tangent to the edge. Then $\mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha) = \Theta_0$ and $E(\mathbf{B}, \mathcal{W}_\alpha) = E(1, \mathcal{S}_\alpha)$, cf. Section 2.2.2. According to whether the ground state energy $E(1, \mathcal{S}_\alpha)$ of the plane sector $\mathcal{S}_\alpha$ is less than Θ_0 or equal to Θ_0, we are in case (i) or (ii) of the dichotomy.

(c) Let $\mathbf{B}$ be tangent to a face of the wedge and normal to the edge. Then $\mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha) = \Theta_0$. It is proved in [35] that $E(\mathbf{B}, \mathcal{W}_\alpha) = \Theta_0$ for $\alpha \in [\frac{\pi}{2}, \pi)$ (case (ii)).

Example 7.7 (Octant)**.** Let $\Pi = (\mathbb{R}_+)^3$ be the model octant and $\mathbf{B}$ be a constant magnetic field with $|\mathbf{B}| = 1$. We quote from [32, §8]:

(a) If the magnetic field $\mathbf{B}$ is tangent to a face but not to an edge of Π, there exists an edge $\mathbf{e}$ such that $\mathscr{E}^*(\mathbf{B}, \Pi) = E(\mathbf{B}, \Pi_\mathbf{e})$ and there holds $E(\mathbf{B}, \Pi) < E(\mathbf{B}, \Pi_\mathbf{e})$. We are in case (i).

(b) If the magnetic field $\mathbf{B}$ is tangent to an edge $\mathbf{e}$ of Π, $\mathscr{E}^*(\mathbf{B}, \Pi) = E(\mathbf{B}, \Pi_\mathbf{e}) = E(\mathbf{B}, \Pi)$. Moreover by [32, §4], $E(\mathbf{B}, \Pi_\mathbf{e}) = E(1, \mathcal{S}_{\pi/2}) < \Theta_0 = \mathscr{E}^*(\mathbf{B}, \Pi_\mathbf{e})$. We are in case (ii).

Example 7.8 (Circular cone)**.** Let $\mathcal{C}_\alpha$ be the right circular cone of angular opening $\alpha \in (0, \pi)$:

$$\mathcal{C}_\alpha = \{\mathbf{x} \in \mathbb{R}^3,\ x_3 > 0,\ x_1^2 + x_2^2 < x_3^2 \tan^2 \alpha\}.$$

We consider a constant magnetic field $\mathbf{B}$ with $|\mathbf{B}| = 1$. It is proved in [8, 9] that

(a) For α small enough, $E(\mathbf{B}, \mathcal{C}_\alpha) < \mathscr{E}^*(\mathbf{B}, \mathcal{C}_\alpha)$.

(b) If $\mathbf{B} = (0, 0, 1)$, then $\mathscr{E}^*(\mathbf{B}, \mathcal{C}_\alpha) = \sigma(\alpha/2)$.

7.4. **Scaling and truncating Admissible Generalized Eigenvectors.** A.G.E.'s are the corner stone(s) for our construction of quasimodes. In this section, as a preparatory step towards final construction, we show a couple of useful properties when suitable scalings and cut-off are performed.

Let $H(\mathbf{A}, \Pi)$ be a model operator that has an A.G.E. Ψ associated with the value Λ. Then for any positive h, the scaled function

$$\Psi_h(\mathbf{x}) := \Psi\Big(\frac{\mathbf{x}}{\sqrt{h}}\Big), \quad \text{for} \quad \mathbf{x} \in \Pi, \tag{7.5}$$

defines an A.G.E. for the operator $H_h(\mathbf{A}, \Pi)$ associated with $h\Lambda$:

$$\begin{cases} (-ih\nabla + \mathbf{A})^2 \Psi_h = h\Lambda \Psi_h & \text{in } \Pi, \\ (-ih\nabla + \mathbf{A})\Psi_h \cdot \mathbf{n} = 0 & \text{on } \partial\Pi. \end{cases} \tag{7.6}$$

We will need to localize Ψ_h. For doing this, let us choose, once for all, a model cut-off function $\underline{\chi} \in \mathscr{C}^\infty(\mathbb{R}^+)$ such that

$$\underline{\chi}(r) = \begin{cases} 1 \text{ if } r \leq 1, \\ 0 \text{ if } r \geq 2. \end{cases} \tag{7.7}$$

For any $R > 0$, let $\underline{\chi}_R$ be the cut-off function defined by

$$\underline{\chi}_R(r) = \underline{\chi}\left(\frac{r}{R}\right), \tag{7.8}$$

and, finally

$$\chi_h(\mathbf{x}) = \underline{\chi}_R\left(\frac{|\mathbf{x}|}{h^\delta}\right) = \underline{\chi}\left(\frac{|\mathbf{x}|}{Rh^\delta}\right) \quad \text{with} \quad 0 \leq \delta \leq \frac{1}{2}. \tag{7.9}$$

Here the exponent δ is the decay rate of the cut-off. It will be tuned later to optimize remainders.

Since Ψ_h belongs to $\mathrm{Dom}_{\mathrm{loc}}(H_h(\mathbf{A}, \Pi))$, we can rely on Lemma A.8 to obtain the following identity for the Rayleigh quotient of $\chi_h \Psi_h$:

$$\frac{q_h[\mathbf{A}, \Pi](\chi_h \Psi_h)}{\|\chi_h \Psi_h\|^2} = h\Lambda + h^2 \rho_h \qquad \text{with} \qquad \rho_h = \frac{\|\,|\nabla \chi_h|\,\Psi_h\|^2}{\|\chi_h \Psi_h\|^2}. \tag{7.10}$$

The following lemma estimates the remainder ρ_h:

Lemma 7.9. *Let Ψ be an admissible generalized eigenvector for the model operator $H(\mathbf{A}, \Pi)$. Let k be the number of independent decaying directions of Ψ, cf. (7.1)–(7.2). Let Ψ_h be the rescaled function given by (7.5) and let χ_h be the cut-off function defined by (7.7)–(7.9) involving parameters $R > 0$ and $\delta \in [0, \frac{1}{2}]$. Then there exist constants $C_0 > 0$ and $c_0 > 0$ depending only on $h_0 > 0$, $R_0 > 0$ and Ψ such that*

$$\rho_h = \frac{\|\,|\nabla \chi_h|\,\Psi_h\|^2}{\|\chi_h \Psi_h\|^2} \leq \begin{cases} C_0\, h^{-2\delta} & \text{if } k < 3, \\ C_0\, h^{-2\delta}\, \mathrm{e}^{-c_0 h^{\delta - 1/2}} & \text{if } k = 3, \end{cases} \qquad \forall R \geq R_0,\ \forall h \leq h_0,\ \forall \delta \in [0, \tfrac{1}{2}].$$

Proof. By assumption $\Psi(\mathbf{x}) = \mathrm{e}^{i\vartheta(\mathbf{y},\mathbf{z})}\,\Phi(\mathbf{z})$ for $\underline{\mathrm{U}}\mathbf{x} = (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^{3-k} \times \Upsilon$, where $\underline{\mathrm{U}}$ is a suitable rotation, and there exist positive constants c_Ψ, C_Ψ controlling the exponential decay of Φ in the cone $\Upsilon \in \mathfrak{P}_k$

$$\int_\Upsilon \mathrm{e}^{2c_\Psi|\mathbf{z}|} |\Phi(\mathbf{z})|^2 \mathrm{d}\mathbf{z} \leq C_\Psi \|\Phi\|^2_{L^2(\Upsilon)}. \tag{7.11}$$

Let us set $T = Rh^\delta$, so that $\chi_h(\mathbf{x}) = \underline{\chi}(|\mathbf{x}|/T)$.

Let us first give an upper bound for $\|\,|\nabla \chi_h|\,\Psi_h\|$:
If $k < 3$, then

$$\begin{aligned} \|\,|\nabla \chi_h|\,\Psi_h\|^2 &\leq CT^{-2} \int_{|\mathbf{y}| \leq 2T} \mathrm{d}\mathbf{y} \int_{\Upsilon \cap \{|\mathbf{z}| \leq 2T\}} \left|\Phi\left(\frac{\mathbf{z}}{\sqrt{h}}\right)\right|^2 \mathrm{d}\mathbf{z} \\ &\leq CT^{-2}\, T^{3-k}\, h^{k/2} \|\Phi\|^2_{L^2(\Upsilon)}, \end{aligned}$$

else, if $k = 3$

$$\begin{aligned}
\| \, |\nabla \chi_h| \, \Psi_h \|^2 &\leq CT^{-2} \int_{\Upsilon \cap \{T \leq |\mathbf{z}| \leq 2T\}} \left| \Phi\Big(\frac{\mathbf{z}}{\sqrt{h}}\Big) \right|^2 \mathrm{d}\mathbf{z} \\
&\leq CT^{-2} \, h^{k/2} \int_{\Upsilon \cap \left\{ Th^{-\frac12} \leq |\mathbf{z}| \leq 2Th^{-\frac12} \right\}} |\Phi(\mathbf{z})|^2 \, \mathrm{d}\mathbf{z} \\
&\leq CT^{-2} \, h^{k/2} \, \mathrm{e}^{-2c_\Psi T/\sqrt{h}} \int_{\Upsilon \cap \left\{ Th^{-\frac12} \leq |\mathbf{z}| \leq 2Th^{-\frac12} \right\}} \mathrm{e}^{2c|\mathbf{z}|} \, |\Phi(\mathbf{z})|^2 \, \mathrm{d}\mathbf{z} \\
&\leq CT^{-2} \, h^{k/2} \, \mathrm{e}^{-2c_\Psi T/\sqrt{h}} \, \|\Phi\|^2_{L^2(\Upsilon)}.
\end{aligned}$$

Let us now consider $\|\chi_h \Psi_h\|$ (we use that $2|\mathbf{y}| < R$ and $2|\mathbf{z}| < R$ implies $|\mathbf{x}| < R$):

$$\|\chi_h \Psi_h\|^2 \geq \int_{2|\mathbf{y}| \leq T} \mathrm{d}\mathbf{y} \int_{\Upsilon \cap \{2|\mathbf{z}| \leq T\}} \left| \Phi\Big(\frac{\mathbf{z}}{\sqrt{h}}\Big) \right|^2 \mathrm{d}\mathbf{z}$$

$$\geq CT^{3-k} h^{k/2} \int_{\Upsilon \cap \left\{ 2|\mathbf{z}| \leq Th^{-\frac12} \right\}} |\Phi(\mathbf{z})|^2 \, \mathrm{d}\mathbf{z} \tag{7.12}$$

$$\geq CT^{3-k} h^{k/2} \, \mathcal{I}(Th^{-\frac12}) \, \|\Phi\|^2_{L^2(\Upsilon)} \tag{7.13}$$

where we have set for any $S \geq 0$

$$\mathcal{I}(S) := \Big(\int_{\Upsilon \cap \{2|\mathbf{z}| \leq S\}} |\Phi(\mathbf{z})|^2 \, \mathrm{d}\mathbf{z} \Big) \Big(\int_{\Upsilon} |\Phi(\mathbf{z})|^2 \, \mathrm{d}\mathbf{z} \Big)^{-1}.$$

The function $S \mapsto \mathcal{I}(S)$ is continuous, non-negative and non-decreasing on $[0, +\infty)$. It is moreover *increasing and positive* on $(0, \infty)$ since Φ, as a solution of an elliptic equation with polynomial coefficients and null right hand side, is analytic inside Υ. Consequently, $\mathcal{I}(Th^{-\frac12}) = \mathcal{I}(Rh^{\delta - \frac12})$ is uniformly bounded from below for $R \geq R_0$, $h \in (0, h_0)$, $\delta \in [0, \frac12]$ and thus

$$\rho_h \leq \begin{cases} CT^{-2} \left\{ \mathcal{I}(Th^{-\frac12}) \right\}^{-1} \leq C_0 h^{-2\delta} & \text{if } k < 3, \\ CT^{-2} \, \mathrm{e}^{-2c_\Psi T/\sqrt{h}} \left\{ \mathcal{I}(Th^{-\frac12}) \right\}^{-1} \leq C_0 h^{-2\delta} \mathrm{e}^{-c_0 h^{\delta - 1/2}} & \text{if } k = 3, \end{cases}$$

where the constants C_0 and c_0 in the above estimation depend only on the lower bound R_0 on R, the upper bound h_0 on h, and on the model problem associated with $\mathbf{x}_0$, provided $\delta \in [0, \frac12]$. Lemma 7.9 is proved. □

Remark 7.10. The estimate of ρ_h provided by Lemma 7.9 is still true when $k = 0$, i.e. when Ψ has no decay direction (but is of modulus 1 everywhere).

If the exponent δ is bounded from above by a number $\delta_0 < \frac12$, we obtain the following improvement of the previous lemma.

Lemma 7.11. *Under the conditions of Lemma 7.9, let $\delta_0 < \frac12$ be a positive number. Let $R_0 > 0$. Then there exist constants $h_0 > 0$, $C_1 > 0$ and $c_1 > 0$ depending only on R_0, δ_0*

and on the constants c_Ψ, C_ψ in (7.11) such that

$$\rho_h = \frac{\| \, |\nabla\chi_h| \, \Psi_h\|^2}{\|\chi_h\Psi_h\|^2} \leq \begin{cases} C_1\, h^{-2\delta} & \text{if } k<3, \\ C_1\, \mathrm{e}^{-c_1 h^{\delta-1/2}} & \text{if } k=3, \end{cases} \qquad \forall R \geq R_0, \ \forall h \leq h_0, \ \forall \delta \in [0,\delta_0]\,.$$

Proof. We obtain an upper bound of $\| \, |\nabla\chi_h| \, \Psi_h\|^2$ as in the proof of Lemma 7.9. Let us now deal with the lower-bound of $\|\chi_h\Psi_h\|^2$. With $T = Rh^\delta$, we have

$$\begin{aligned} \|\chi_h\Psi_h\|^2 \;&\geq\; CT^{3-k}h^{k/2}\int_{\Upsilon\cap\left\{2|\mathbf{z}|\leq Th^{-\frac12}\right\}} |\Phi(\mathbf{z})|^2\,\mathrm{d}\mathbf{z} \\ &\geq\; CT^{3-k}h^{k/2}\left(1 - C_\Psi \mathrm{e}^{-c_\Psi R h^{\delta-1/2}}\right)\|\Phi\|^2_{L^2(\Upsilon)}. \end{aligned} \tag{7.14}$$

Since $0 \leq \delta \leq \delta_0 < \frac12$, there holds $C_\Psi \mathrm{e}^{-c_\Psi R h^{\delta-1/2}} < \frac12$ for h small enough or R large enough. Thus we deduce the lemma. □

8. Properties of the local ground energy

In this section we describe the regularity properties of the local ground energy. The main result of this section is that the function $\mathbf{x} \mapsto E(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x})$ is lower semi-continuous on a corner domain and therefore it reaches its infimum. We also prove some stability properties of the generalized eigenvectors and the associated energy under perturbation of the magnetic field $\mathbf{B}$. Finally we provide strict diamagnetic inequality for the ground state energy.

8.1. Lower semi-continuity.

Theorem 8.1. *Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ and let $\mathbf{B} \in \mathscr{C}^0(\overline{\Omega})$ be a continuous magnetic field. Then the function $\Lambda_{\overline{\Omega}} : \mathbf{x} \mapsto E(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x})$ is lower semi-continuous on $\overline{\Omega}$.*

Proof. For $\mathbb{X} = (\mathbf{x}_0, \ldots) \in \mathfrak{C}(\Omega)$, define the function $F(\mathbb{X}) := E(\mathbf{B}_{\mathbf{x}_0}, \Pi_\mathbb{X})$, which coincides on the chains of length 1 with the function $\Lambda_{\overline{\Omega}}$: $F((\mathbf{x}_0)) = \Lambda_{\overline{\Omega}}(\mathbf{x}_0)$. Recall that we have introduced a partial order on $\mathfrak{C}(\Omega)$, see Definition 3.24. Then due to (7.3) applied to $\Pi_\mathbb{X}$ for any chain $\mathbb{X}$, the function $F : \mathfrak{C}(\Omega) \mapsto \mathbb{R}_+$ is clearly order preserving.

Let us show that it is continuous with respect to the distance $\mathbb{D}$ (see Definition 3.22). Let $\mathbb{X} \in \mathfrak{C}(\Omega)$ and $\mathbb{X}'$ tending to $\mathbb{X}$. This means that $\mathbf{x}'_0$ tends to $\mathbf{x}_0$ in $\mathbb{R}^3$ and that there exists $J \in \mathrm{BGL}(3)$ tending to the identity $\mathbb{I}_3$ such that $J(\Pi_\mathbb{X}) = \Pi_{\mathbb{X}'}$. In particular for $\mathbb{X}'$ close enough to $\mathbb{X}$, the reduced dimensions of the cones $\Pi_\mathbb{X}$ and $\Pi_{\mathbb{X}'}$ are equal: $d(\Pi_{\mathbb{X}'}) = d(\Pi_\mathbb{X})$.

(1) If $\Pi_\mathbb{X} = \mathbb{R}^3$, then $F(\mathbb{X}) = |\mathbf{B}_{\mathbf{x}_0}|$ and $F(\mathbb{X}') = |\mathbf{B}_{\mathbf{x}'_0}|$, and since $\mathbf{B}$ is continuous, $F(\mathbb{X}')$ converges toward $F(\mathbb{X})$ when $\mathbb{D}(\mathbb{X}', \mathbb{X}) \to 0$.

(2) When $\Pi_\mathbb{X}$ is a half-space, we denote by $\theta(\mathbb{X})$ the angle between $\Pi_\mathbb{X}$ and $\mathbf{B}_{\mathbf{x}_0}$. We have $\theta(\mathbb{X}') \to \theta(\mathbb{X})$ when $\mathbb{D}(\mathbb{X}', \mathbb{X}) \to 0$. Moreover

$$F(\mathbb{X}') - F(\mathbb{X}) = |\mathbf{B}_{\mathbf{x}'_0}|\sigma(\theta(\mathbb{X}')) - |\mathbf{B}_{\mathbf{x}_0}|\sigma(\theta(\mathbb{X})),$$

therefore $F(\mathbb{X}')$ tends to $F(\mathbb{X})$ due to Lemma 6.3 and the continuity of $\mathbf{B}$.

(3) When $\Pi_{\mathbb{X}}$ is a wedge, there exists $(\underline{\mathrm{U}}, \underline{\mathrm{U}}')$ in $\mathfrak{O}_3$ and (α, α') in $(0,\pi) \cup (\pi, 2\pi)$ such that $\underline{\mathrm{U}}(\Pi_{\mathbb{X}}) = \mathcal{W}_\alpha$ and $\underline{\mathrm{U}}'(\Pi_{\mathbb{X}'}) = \mathcal{W}_{\alpha'}$. Therefore

$$F(\mathbb{X}') - F(\mathbb{X}) = E(\underline{\mathrm{U}}(\mathbf{B}_{\mathbf{x}_0}), \mathcal{W}_\alpha) - E(\underline{\mathrm{U}}'(\mathbf{B}_{\mathbf{x}'_0}), \mathcal{W}_{\alpha'}),$$

with $\alpha' \to \alpha$ and $\underline{\mathrm{U}}' \to \underline{\mathrm{U}}$ when $\mathbb{D}(\mathbb{X}', \mathbb{X}) \to 0$. Lemma 6.5 and the continuity of $\mathbf{B}$ ensure that $F(\mathbb{X}')$ tends to $F(\mathbb{X})$.

(4) Finally chains $\mathbb{X}$ such that $\Pi_{\mathbb{X}}$ is a 3D cone are of length 1 and are isolated in $\mathfrak{C}(\Omega)$ for the topology associated with $\mathbb{D}$ (see Proposition 3.20).

Therefore F is continuous on $\mathfrak{C}(\Omega)$. We apply Theorem 3.25: So the function $\mathbf{x} \mapsto F((\mathbf{x})) = \Lambda_{\overline{\Omega}}(\mathbf{x})$ is lower semi-continuous on $\overline{\Omega}$. ☐

Remark 8.2. Recall that any stratum $\mathbf{t} \in \mathfrak{T}$ has a smooth submanifold structure (see Proposition 3.20). Denote by $\Lambda_{\mathbf{t}}$ the restriction of the local ground energy to $\mathbf{t}$. Then it follows from above that $\Lambda_{\mathbf{t}}$ is continuous. Moreover if $\Omega \in \overline{\mathfrak{D}}(\mathbb{R}^3)$, one can prove that $\Lambda_{\mathbf{t}}$ admits a continuous extension to $\overline{\mathbf{t}}$. But this is not true anymore if $\overline{\mathbf{t}}$ contains a conical point.

As a consequence of the above theorem, the function $\mathbf{x} \mapsto \Lambda_{\overline{\Omega}}(\mathbf{x})$ reaches its infimum over $\overline{\Omega}$. This fact will be one of the key ingredients to prove an upper bound with remainder for $\lambda_h(\mathbf{B}, \Omega)$ in the semiclassical limit.

8.2. **Stability under perturbation.** Here we describe stability of the ground state energy and of the associated A.G.E. under perturbations .

Assume that we are in case (i) of the dichotomy (Theorem 7.2). We recall that the notations (G1), (G2) and (G3) refer to the number $k = 1, 2, 3$, of independent decaying directions for the A.G.E, see Section 7.2. A perturbation of the magnetic field has distinct effects according to the situation. The only geometrical situation leading to (G1) is clearly not stable. In case (G3), the ground state energy is a discrete eigenvalue and one might expect the ground state energy to be Lipschitz under perturbations.[9] The case of (G2) is more intriguing and we prove in the following lemma the stability together with local uniform estimates for exponential decay.

Lemma 8.3. *Let $\mathbf{B}_0$ be a non zero constant magnetic field and Π be a cone in $\mathfrak{P}_3$ with $d < 3$. Assume that $E(\mathbf{B}_0, \Pi) < \mathscr{E}^*(\mathbf{B}_0, \Pi)$.*

(a) *In a ball $\mathcal{B}(\mathbf{B}_0, \varepsilon)$, the function $\mathbf{B} \mapsto E(\mathbf{B}, \Pi)$ is Lipschitz-continuous and*

$$E(\mathbf{B}, \Pi) < \mathscr{E}^*(\mathbf{B}, \Pi).$$

(b) *We suppose moreover that $(\mathbf{B}_0, \Pi)$ is in situation (G2). For $\mathbf{B} \in \mathcal{B}(\mathbf{B}_0, \varepsilon)$, we denote by $\Psi^{\mathbf{B}}$ an admissible generalized eigenfunction given by Theorem 7.2. For ε small enough, $(\mathbf{B}, \Pi)$ is still in situation (G2) and $\Psi^{\mathbf{B}}$ has the form*

$$\Psi^{\mathbf{B}}(\mathbf{x}) = e^{i\varphi^{\mathbf{B}}(\mathbf{y},\mathbf{z})}\Phi^{\mathbf{B}}(\mathbf{z}) \quad \textit{for} \quad \underline{\mathrm{U}}^{\mathbf{B}}\mathbf{x} = (\mathbf{y}, \mathbf{z}) \in \mathbb{R} \times \Upsilon,$$

[9] but this is of no use in our analysis since the points $\mathbf{x}$ in $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ such that $d(\Pi_{\mathbf{x}}) = 3$ are isolated.

with $\underline{\mathrm{U}}^{\mathbf{B}}$ a suitable rotation, and there exist constants $c > 0$ and $C > 0$ such that

$$\forall \mathbf{B} \in \mathcal{B}(\mathbf{B}_0, \varepsilon), \quad \|\Phi^{\mathbf{B}} e^{c|\mathbf{z}|}\|_{L^2(\Upsilon)} \leq C\|\Phi^{\mathbf{B}}\|_{L^2(\Upsilon)}\,. \tag{8.1}$$

Proof. Let us distinguish the three possible situations according to the value of d:

$d = 0$: When $\Pi = \mathbb{R}^3$, we have $E(\mathbf{B}, \Pi) = |\mathbf{B}|$ and $\mathscr{E}^*(\mathbf{B}, \Pi) = +\infty$. The admissible generalized eigenvector $\Psi^{\mathbf{B}}$ is explicit as explained above. Thus (a) and (b) are established in this case.

$d = 1$: When Π is a half-space, we denote by $\theta(\mathbf{B})$ the unoriented angle in $[0, \frac{\pi}{2}]$ between $\mathbf{B}$ and the boundary. The function $\mathbf{B} \mapsto \theta(\mathbf{B})$ is Lipschitz. Moreover the function σ is $\mathscr{C}^1$ on $[0, \pi/2]$ (see Lemma 6.3). We deduce that the function $\mathbf{B} \mapsto \sigma(\theta(\mathbf{B}))$ is Lipschitz outside any neighborhood of $\mathbf{B} = 0$. Thus point (a) is proved. Assuming furthermore that $(\Pi, \mathbf{B}_0)$ is in situation (G2), we have $\theta(\mathbf{B}_0) \in (0, \frac{\pi}{2})$ and there exist $\varepsilon > 0$, $\theta_{\min}$ and $\theta_{\max}$ such that

$$\forall \mathbf{B} \in \mathcal{B}(\mathbf{B}_0, \varepsilon), \quad \theta(\mathbf{B}) \in [\theta_{\min}, \theta_{\max}] \subset (0, \tfrac{\pi}{2})\,.$$

The admissible generalized eigenvector is constructed above. The uniform exponential estimate is proved in [7, §2].

$d = 2$: When Π is a wedge, point (a) comes from [36, Proposition 4.6]. Due to the continuity of $\mathbf{B} \mapsto E(\mathbf{B}, \Pi)$ there exist $c > 0$ and $\varepsilon > 0$ such that

$$\forall \mathbf{B} \in \mathcal{B}(\mathbf{B}_0, \varepsilon), \quad \mathscr{E}^*(\mathbf{B}, \Pi) - E(\mathbf{B}, \Pi) > c.$$

Point (b) is then a direct consequence of [36, Proposition 4.2].

The proof of Lemma 8.3 is complete. □

Remark 8.4. When $d = 2$, it is proved in [36] that the ground state energy is also Lipschitz with respect to the aperture angle of the wedge in case (i) of the dichotomy in Theorem 7.2, whereas it is only $\frac{1}{3}$-Hölder under perturbations in the general case. By arguments similar to those of [36, Section 4] one could prove a similar Hölder regularity for $\mathbf{B} \mapsto E(\mathbf{B}, \Pi)$ when $d(\Pi) = 3$.

8.3. **Positivity of the ground state energy.** The classical diamagnetic inequality (see [24, 43] for example) implies that the ground state energy is in general larger than the one without magnetic field, that is 0 in our case due to Neumann boundary condition. Usually it is harder to show that this inequality is strict. A strict diamagnetic inequality has been proved for the Neumann magnetic Laplacian in a bounded regular domain, in [16, Section 2.2]. For our unbounded domains Π with constant magnetic field, we have:

Proposition 8.5. *Let $\Pi \in \mathfrak{P}_3$ and $\mathbf{B}$ be a non zero constant magnetic field. Then we have $E(\mathbf{B}, \Pi) > 0$.*

Proof. It is enough to make the proof for unitary magnetic field, see Lemma A.5. Let $d \in [0, 3]$ be the reduced dimension of the cone Π. If $d = 0$, then $E(\mathbf{B}, \Pi) = 1$ (see (6.3)).

If $d = 1$, then $E(\mathbf{B}, \Pi)$ is expressed with the function σ that satisfies $\sigma(\theta) \geq \Theta_0 > 0$ for all $\theta \in [0, \frac{\pi}{2}]$, see Lemma 6.3. When $d = 2$, the strict positivity has been shown in [36, Corollary 3.9].

Assume now that $d = 3$. If we are in case (i) of Theorem 7.2, then there exists an eigenfunction $\Psi \in L^2(\Pi)$ for $H(\mathbf{A}, \Pi)$ associated with $E(\mathbf{B}, \Pi)$. Assume that $E(\mathbf{B}, \Pi) = 0$, then due to the standard diamagnetic inequality (see [24, Lemma A]), we have

$$0 \leq \int_\Pi \big|\nabla|\Psi|\big|^2 \leq \int_\Pi |(-i\nabla - \mathbf{A})\Psi|^2 = 0,$$

that leads to $\Psi = 0$, which is a contradiction. If we are in case (ii) of Theorem 7.2, then there exists a tangent substructure $\Pi_{\mathbb{X}}$ of Π with $d(\Pi_{\mathbb{X}}) < 3$ such that $E(\mathbf{B}, \Pi) = E(\mathbf{B}, \Pi_{\mathbb{X}})$ that is strictly positive due to the analysis of the cases $d \leq 2$, see above. □

Combining the above proposition with Theorem 8.1, we get:

Corollary 8.6. *Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ and let $\mathbf{B} \in \mathscr{C}^0(\overline{\Omega})$ be a non-vanishing magnetic field. Then we have $\mathscr{E}(\mathbf{B}, \Omega) > 0$.*

9. Upper bound for ground state energy in corner domains

In this section, we prove an upper bound involving error estimates that contains the same powers of h than the lower bound in Theorem 5.1.

Theorem 9.1. *Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ be a general 3D corner domain, and let $\mathbf{A} \in W^{2,\infty}(\overline{\Omega})$ be a twice differentiable magnetic potential.*

(i) Then there exist $C(\Omega) > 0$ and $h_0 > 0$ such that

$$\forall h \in (0, h_0), \quad \lambda_h(\mathbf{B}, \Omega) \leq h\mathscr{E}(\mathbf{B}, \Omega) + C(\Omega)\big(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)h^{11/10} \,. \tag{9.1}$$

(ii) If Ω is a polyhedral domain, this upper bound is improved:

$$\forall h \in (0, h_0), \quad \lambda_h(\mathbf{B}, \Omega) \leq h\mathscr{E}(\mathbf{B}, \Omega) + C(\Omega)\big(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)h^{5/4} \,. \tag{9.2}$$

(iii) If there exists a point $\mathbf{x}_0 \in \overline{\Omega}$ such that $\mathbf{B}(\mathbf{x}_0) = 0$, then $\mathscr{E}(\mathbf{B}, \Omega) = 0$ and we have the optimal upper bound

$$\forall h \in (0, h_0), \quad \lambda_h(\mathbf{B}, \Omega) \leq C(\Omega)\big(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)h^{4/3} \,. \tag{9.3}$$

We recall the notation (1.18) for Rayleigh quotients

$$\mathscr{Q}_h[\mathbf{A}, \Omega](\varphi) = \frac{q_h[\mathbf{A}, \Omega](\varphi)}{\|\varphi\|^2}, \quad \varphi \in H^1(\Omega), \ \varphi \neq 0,$$

and that, by the min-max principle

$$\lambda_h(\mathbf{B}, \Omega) = \min_{\substack{\varphi \in H^1(\Omega) \\ \varphi \neq 0}} \mathscr{Q}_h[\mathbf{A}, \Omega](\varphi) \,.$$

9.1. **Principles of construction for quasimodes.** By lower semi-continuity (see Theorem 8.1), the energy $\mathbf{x} \mapsto E(\mathbf{B}_{\mathbf{x}}, \Pi_{\mathbf{x}})$ reaches its infimum over $\overline{\Omega}$. Let $\mathbf{x}_0 \in \overline{\Omega}$ be a point such that

$$E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}) = \mathscr{E}(\mathbf{B}, \Omega).$$

By the dichotomy result (Theorem 7.2) there exists a singular chain $\mathbb{X}$ starting at $\mathbf{x}_0$ such that (see also notation (4.8)):

$$E(\mathbf{B}_{\mathbb{X}}, \Pi_{\mathbb{X}}) = E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}) \quad \text{and} \quad E(\mathbf{B}_{\mathbb{X}}, \Pi_{\mathbb{X}}) < \mathscr{E}^*(\mathbf{B}_{\mathbb{X}}, \Pi_{\mathbb{X}}).$$

For shortness, we denote $\Lambda_{\mathbb{X}} = E(\mathbf{B}_{\mathbb{X}}, \Pi_{\mathbb{X}})$. Still by Theorem 7.2, there exists an A.G.E. for the tangent model operator $H(\mathbf{A}_{\mathbb{X}}, \Pi_{\mathbb{X}})$ denoted by $\Psi^{\mathbb{X}}$ and associated with $\Lambda_{\mathbb{X}}$

$$\begin{cases} (-i\nabla + \mathbf{A}_{\mathbb{X}})^2 \Psi^{\mathbb{X}} = \Lambda_{\mathbb{X}} \Psi^{\mathbb{X}} & \text{in } \Pi_{\mathbb{X}}, \\ (-i\nabla + \mathbf{A}_{\mathbb{X}})\Psi^{\mathbb{X}} \cdot \mathbf{n} = 0 & \text{on } \partial\Pi_{\mathbb{X}}. \end{cases} \tag{9.4}$$

For $h > 0$, we define $\Psi_h^{\mathbb{X}}$ by using the canonical scaling (7.5). This gives an A.G.E. for the operator $H_h(\mathbf{A}_{\mathbb{X}}, \Pi_{\mathbb{X}})$ associated with the value $h\Lambda_{\mathbb{X}}$:

$$\begin{cases} (-ih\nabla + \mathbf{A}_{\mathbb{X}})^2 \Psi_h^{\mathbb{X}} = h\Lambda_{\mathbb{X}} \Psi_h^{\mathbb{X}} & \text{in } \Pi_{\mathbb{X}}, \\ (-ih\nabla + \mathbf{A}_{\mathbb{X}})\Psi_h^{\mathbb{X}} \cdot \mathbf{n} = 0 & \text{on } \partial\Pi_{\mathbb{X}}. \end{cases} \tag{9.5}$$

Let χ_h be the cut-off function defined by (7.7)–(7.9) involving the parameter $R > 0$ and the exponent $\delta \in (0, \frac{1}{2})$. Then the function

$$(\chi_h \Psi_h^{\mathbb{X}})(\mathbf{x}) = \underline{\chi}\left(\frac{|\mathbf{x}|}{Rh^\delta}\right) \Psi^{\mathbb{X}}\Big(\frac{\mathbf{x}}{\sqrt{h}}\Big), \quad \text{for} \quad \mathbf{x} \in \Pi_{\mathbb{X}}\,, \tag{9.6}$$

is a canonical quasimode on the substructure $\Pi_{\mathbb{X}}$ for the model operator $H_h(\mathbf{A}_{\mathbb{X}}, \Pi_{\mathbb{X}})$: Indeed the identity (7.10) and Lemma 7.9 yield

$$\mathscr{Q}_h[\mathbf{A}_{\mathbb{X}}, \Pi_{\mathbb{X}}](\chi_h \Psi_h^{\mathbb{X}}) = h\Lambda_{\mathbb{X}} + \mathcal{O}(h^{2-2\delta}). \tag{9.7}$$

Let us recall that the fact that $\chi_h \Psi_h^{\mathbb{X}}$ belongs to $\mathrm{Dom}_{\,\mathrm{loc}}(H_h(\mathbf{A}_{\mathbb{X}}, \Pi_{\mathbb{X}}))$ is essential for the validity of the identity above.

In order to prove Theorem 9.1, we are going to construct a family of quasimodes $\varphi_h^{[0]} \in H^1(\Omega)$ satisfying the estimate for $h > 0$ small enough and the suitable power κ

$$\mathscr{Q}_h[\mathbf{A}, \Omega](\varphi_h^{[0]}) \le h\Lambda_{\mathbb{X}} + C(\Omega)(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})h^\kappa. \tag{9.8}$$

The rationale of this construction is to build a link between the canonical quasimode $\chi_h \Psi_h^{\mathbb{X}}$ on the substructure $\Pi_{\mathbb{X}}$ with our original operator $H_h(\mathbf{A}, \Omega)$.

Let ν be the length of the chain $\mathbb{X}$. By Proposition 3.29, we can always reduce to $\nu \le 3$. We write

$$\mathbb{X} = (\mathbf{x}_0, \dots, \mathbf{x}_{\nu-1}) \quad \text{with} \quad \nu \in \{1, 2, 3\}.$$

Our quasimode $\varphi_h^{[0]}$ will have distinct features according to the value of ν: We will need ν intermediaries $\varphi_h^{[j]}$ between $\varphi_h^{[0]}$ and the truncated A.G.E. $\chi_h \Psi_h^{\mathbb{X}}$ given in (9.6): In all cases, the final object is

$$\varphi_h^{[\nu]} = \chi_h \Psi_h^{\mathbb{X}}. \tag{9.9}$$

At a glance

- $\nu = 1$: The quasimode $\varphi_h^{[0]}$ is deduced from $\varphi_h^{[1]} = \chi_h \Psi_h^{\mathbb{X}}$ through the local map $\mathrm{U}^{\mathbf{x}_0}$. This is the classical construction: We say that the quasimode is *sitting* because as $h \to 0$ the supports of $\varphi_h^{[0]}$ are included in each other and concentrate to $\mathbf{x}_0$.
- $\nu = 2$: The quasimode $\varphi_h^{[0]}$ is deduced from $\varphi_h^{[1]}$ through the local map $\mathrm{U}^{\mathbf{x}_0}$, and $\varphi_h^{[1]}$ is itself deduced from $\varphi_h^{[2]} = \chi_h \Psi_h^{\mathbb{X}}$ through another local map $\mathrm{U}^{\mathbf{v}_1}$ connected to the second element $\mathbf{x}_1$ of the chain. We say that the quasimode is *sliding* because as $h \to 0$ the supports of $\varphi_h^{[0]}$ are shifted along a direction $\boldsymbol{\tau}_1$ determined by $\mathbf{x}_1$.
- $\nu = 3$: The quasimode $\varphi_h^{[0]}$ is still deduced from $\varphi_h^{[1]}$ through $\mathrm{U}^{\mathbf{x}_0}$, and $\varphi_h^{[1]}$ from $\varphi_h^{[2]}$ through $\mathrm{U}^{\mathbf{v}_1}$. Finally $\varphi_h^{[2]}$ is itself deduced from $\varphi_h^{[3]} = \chi_h \Psi_h^{\mathbb{X}}$ through a third local map $\mathrm{U}^{\mathbf{v}_2}$ connected to the third element $\mathbf{x}_2$ of the chain. We say that the quasimode is *doubly sliding* because as $h \to 0$ the supports of $\varphi_h^{[0]}$ are shifted along two directions $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ determined by $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively.

Let us introduce some notation.

Notation 9.2. *(i)* If U is a diffeomorphism, let U_* be the operator of composition:

$$\mathrm{U}_*(f) = f \circ \mathrm{U}.$$

(ii) If $\zeta_h^{\mathbf{v}}$ is a phase, let $\mathrm{Z}_h^{\mathbf{v}}$ be the operator of multiplication

$$\mathrm{Z}_h^{\mathbf{v}}(f) = f\, \overline{\zeta_h^{\mathbf{v}}}.$$

We are going to construct recursively functions $\varphi_h^{[j]}$ assuming that $\varphi_h^{[j+1]}$ is known. In this construction, $\varphi_h^{[0]}$ is a quasimode defined in Ω and $\varphi_h^{[j]}$ is defined in $\Pi_{\mathbf{x}_0,\ldots,\mathbf{x}_{j-1}}$ if $0 < j < \nu$. Typically, these relations will take the form

$$\varphi_h^{[j]} = \mathrm{Z}_h^{\mathbf{v}_j} \circ \mathrm{U}_*^{\mathbf{v}_j}(\varphi_h^{[j+1]}).$$

Remark 9.3. Since $\mathbf{x}_0$ is determined, we can always assume that $\mathbf{x}_0$ belongs to the reference set $\mathfrak{X}$ of an admissible atlas. The error rate that we will obtain in the end will depend on whether $\nu = 1$ or is larger, and on whether $\mathbf{x}_0$ is a conical point or not.

9.2. **First level of construction and sitting quasimodes.** We perform the first change of variables as in Section 4.1: The local diffeomorphism $\mathrm{U}^{\mathbf{x}_0}$ sends (a neighborhood of) $\mathbf{x}_0$ in $\overline{\Omega}$ to (a neighborhood of) $\mathbf{0}$ in $\overline{\Pi}_{\mathbf{x}_0}$.

- [a1]. Let $\mathbf{A}^{\mathbf{x}_0}$ be the new potential (4.1) deduced from $\mathbf{A} - \mathbf{A}(\mathbf{x}_0)$ by the local map $\mathrm{U}^{\mathbf{x}_0}$.

- [b1]. Let $\zeta_h^{\mathbf{x}_0}(\mathbf{x}) = \mathrm{e}^{i\langle \mathbf{A}(\mathbf{x}_0),\, \mathbf{x}/h\rangle}$, for $\mathbf{x} \in \Omega$.

- [c1]. Let us introduce the relation

$$\varphi_h^{[0]} = \mathrm{Z}_h^{\mathbf{x}_0} \circ \mathrm{U}_*^{\mathbf{x}_0}(\varphi_h^{[1]}), \tag{9.10}$$

and let $r_h^{[1]}$ be the radius of the smallest ball centered at $\mathbf{0}$ containing the support of $\varphi_h^{[1]}$ in $\overline{\Pi}_{\mathbf{x}_0}$. The number $r_h^{[1]}$ is intended to converge to 0 as h tends to 0.

Using (4.5), we have

$$\mathscr{Q}_h[\mathbf{A},\Omega](\varphi_h^{[0]}) = \mathscr{Q}_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}, \mathrm{G}^{\mathbf{x}_0}](\varphi_h^{[1]}). \tag{9.11}$$

Combining with Lemma 4.5, case (i), we find the relation between the Rayleigh quotients

$$\mathscr{Q}_h[\mathbf{A},\Omega](\varphi_h^{[0]}) = \mathscr{Q}_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\varphi_h^{[1]})\left(1+\mathcal{O}(r_h^{[1]})\right), \tag{9.12}$$

which implies

$$\left|\mathscr{Q}_h[\mathbf{A},\Omega](\varphi_h^{[0]}) - \mathscr{Q}_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\varphi_h^{[1]})\right| \le C\, r_h^{[1]}\, \mathscr{Q}_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\varphi_h^{[1]}). \tag{9.13}$$

- [d1]. We recall that $\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}$ is the linear part of $\mathbf{A}^{\mathbf{x}_0}$ at $\mathbf{0}$. Using relation (A.7) with $\mathbf{A} = \mathbf{A}^{\mathbf{x}_0}$ and $\mathbf{A}' = \mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}$ and a Cauchy-Schwarz inequality, we obtain

$$\left|q_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\varphi_h^{[1]}) - q_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\varphi_h^{[1]})\right| \le C\Big(a_h^{[1]}\sqrt{\mu_h^{[1]}} + \big(a_h^{[1]}\big)^2\Big)\|\varphi_h^{[1]}\|^2, \tag{9.14}$$

where we have set

$$\mu_h^{[1]} = \mathscr{Q}_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\varphi_h^{[1]}) \quad \text{and} \quad a_h^{[1]} = \frac{\|(\mathbf{A}^{\mathbf{x}_0} - \mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0})\varphi_h^{[1]}\|}{\|\varphi_h^{[1]}\|}. \tag{9.15}$$

By Lemmas 4.6–4.7, case (i), and since $\varphi_h^{[1]}$ is supported in the ball $\mathcal{B}(\mathbf{0}, r_h^{[1]})$, we have

$$a_h^{[1]} \le C\,\big(r_h^{[1]}\big)^2. \tag{9.16}$$

Putting together (9.14)–(9.16), we obtain

$$\left|\mathscr{Q}_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\varphi_h^{[1]}) - \mathscr{Q}_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\varphi_h^{[1]})\right| \le C\Big(\big(r_h^{[1]}\big)^2\sqrt{\mu_h^{[1]}} + \big(r_h^{[1]}\big)^4\Big). \tag{9.17}$$

Using the above estimate (9.17), we have

$$r_h^{[1]}\, \mathscr{Q}_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\varphi_h^{[1]}) \le r_h^{[1]}\, \bigg(\mathscr{Q}_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\varphi_h^{[1]}) + C\Big(\big(r_h^{[1]}\big)^2\sqrt{\mu_h^{[1]}} + \big(r_h^{[1]}\big)^4\Big)\bigg).$$

Combining this last inequality, (9.17) and (9.13), we have for $r_h^{[1]}$ small enough

$$\left|\mathscr{Q}_h[\mathbf{A},\Omega](\varphi_h^{[0]}) - \mathscr{Q}_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\varphi_h^{[1]})\right| \le C\Big(\mu_h^{[1]} r_h^{[1]} + \big(r_h^{[1]}\big)^2\sqrt{\mu_h^{[1]}} + \big(r_h^{[1]}\big)^4\Big). \tag{9.18}$$

• [e1]. If $\nu = 1$, we set, as already mentioned, $\varphi_h^{[1]} = \chi_h \Psi_h^{\mathbb{X}}$. Note that $\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}$ coincides with $\mathbf{A}_{\mathbb{X}}$. To tune the cut-off χ_h, we choose the exponent δ as δ_0 and the radius R as 1. Therefore $r_h^{[1]} = \mathcal{O}(h^{\delta_0})$ and by (9.7) $\mu_h^{[1]} = \mathcal{O}(h)$. Using (9.18) and again (9.7), we deduce

$$\mathcal{Q}_h[\mathbf{A},\Omega](\varphi_h^{[0]}) \le h\Lambda_{\mathbb{X}} + C\big(h^{2-2\delta_0} + h^{1+\delta_0} + h^{\frac12+2\delta_0} + h^{4\delta_0}\big). \tag{9.19}$$

So we can conclude in the sitting case. Choosing $\delta_0 = 3/8$, we optimize remainders and we get the upper bound

$$\lambda_h(\mathbf{B},\Omega) \le h\mathscr{E}(\mathbf{B},\Omega) + C(\Omega)\big(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)h^{5/4}\ .$$

• *Case when* $\mathbf{B}(\mathbf{x}_0) = 0$. If $\mathbf{B}(\mathbf{x}_0) = 0$, the function $\Psi^{\mathbb{X}} \equiv 1$ is an A.G.E. on $\Pi_{\mathbf{x}_0}$ associated with the value $\Lambda_{\mathbb{X}} = 0$. We are in the sitting case $\nu = 1$ and the estimate (9.18) is still valid. But now (9.7) (combined with Remark 7.10) yields

$$\mathcal{Q}_h[\mathbf{A}_{\mathbb{X}},\Pi_{\mathbb{X}}](\chi_h\Psi_h^{\mathbb{X}}) \le Ch^{2-2\delta}.$$

Choosing δ as δ_0 as above, we deduce $\mu_h^{[1]} = \mathcal{O}(h^{2-2\delta_0})$. Hence

$$\mathcal{Q}_h[\mathbf{A},\Omega](\varphi_h^{[0]}) \le C\big(h^{2-2\delta_0} + h^{2-2\delta_0+\delta_0} + h^{1-\delta_0+2\delta_0} + h^{4\delta_0}\big). \tag{9.20}$$

Choosing $\delta_0 = 1/3$, we optimize remainders and we get the upper bound

$$\lambda_h(\mathbf{B},\Omega) \le C(\Omega)\big(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)h^{4/3}\ .$$

9.3. **Second level of construction and sliding quasimodes.** We have now to deal with the case $\nu \ge 2$. So $\mathbb{X} = (\mathbf{x}_0,\mathbf{x}_1)$ or $(\mathbf{x}_0,\mathbf{x}_1,\mathbf{x}_2)$.

Here we use the same notation as the introduction of singular chains in Section 3.4. Let $\underline{\mathrm{U}}^0 \in \mathfrak{O}_3$ such that $\Pi_{\mathbf{x}_0} = \underline{\mathrm{U}}^0(\mathbb{R}^{3-d_0}\times\Gamma_{\mathbf{x}_0})$ where $\Gamma_{\mathbf{x}_0}$ is the reduced cone of $\Pi_{\mathbf{x}_0}$. Let $\Omega_{\mathbf{x}_0} = \Gamma_{\mathbf{x}_0}\cap\mathbb{S}^{d_0-1}$ be the section of $\Gamma_{\mathbf{x}_0}$. By definition of chains, $\mathbf{x}_1$ belongs to $\overline{\Omega}_{\mathbf{x}_0}$ and let $C_{\mathbf{x}_0,\mathbf{x}_1}$ be the tangent cone to $\Omega_{\mathbf{x}_0}$ at $\mathbf{x}_1$. Then the substructure $\Pi_{\mathbf{x}_0,\mathbf{x}_1}$ is determined by the formula

$$\Pi_{\mathbf{x}_0,\mathbf{x}_1} = \underline{\mathrm{U}}^0\big(\mathbb{R}^{3-d_0}\times\langle\mathbf{x}_1\rangle\times C_{\mathbf{x}_0,\mathbf{x}_1}\big)\ .$$

Let us define the unitary vector $\boldsymbol{\tau}_1$ by the formulas

$$\underline{\boldsymbol{\tau}_1} := (0,\mathbf{x}_1)\in\mathbb{R}^{3-d_0}\times\Gamma_{\mathbf{x}_0} \quad\text{and}\quad \boldsymbol{\tau}_1 = \underline{\mathrm{U}}^0\,\underline{\boldsymbol{\tau}_1}\in\overline{\Pi}_{\mathbf{x}_0}\cap\mathbb{S}^2. \tag{9.21}$$

With this definition, the substructure $\Pi_{\mathbf{x}_0,\mathbf{x}_1}$ is the tangent cone to $\Pi_{\mathbf{x}_0}$ at the point $\boldsymbol{\tau}_1$. Note that in the case when $\mathbf{x}_0$ is a vertex of Ω, the above formulas simplify: $\Pi_{\mathbf{x}_0}$ is its own reduced cone, $\Pi_{\mathbf{x}_0,\mathbf{x}_1} = \langle\mathbf{x}_1\rangle\times C_{\mathbf{x}_0,\mathbf{x}_1}$, and $\boldsymbol{\tau}_1$ coincides with $\mathbf{x}_1$.

Note also that the cone $\Pi_{\mathbf{x}_0,\mathbf{x}_1}$ can be the full space, a half-space or a wedge, and that $\boldsymbol{\tau}_1$ gives a direction associated with $\Pi_{\mathbf{x}_0,\mathbf{x}_1}$ starting from the origin $\mathbf{0}$ of $\Pi_{\mathbf{x}_0}$:

(1) If $\Pi_{\mathbf{x}_0,\mathbf{x}_1}\equiv\mathbb{R}^3$, $\boldsymbol{\tau}_1$ belongs to the interior of $\Pi_{\mathbf{x}_0}$.
(2) If $\Pi_{\mathbf{x}_0,\mathbf{x}_1}\equiv\mathbb{R}^3_+$, $\boldsymbol{\tau}_1$ belongs to a face of $\Pi_{\mathbf{x}_0}$.
(3) If $\Pi_{\mathbf{x}_0,\mathbf{x}_1}\equiv\mathcal{W}_\alpha$, $\boldsymbol{\tau}_1$ belongs to an edge of $\Pi_{\mathbf{x}_0}$.

Unless we are in the latter case ($\Pi_{\mathbf{x}_0,\mathbf{x}_1}$ is a wedge), the choice of $\boldsymbol{\tau}_1$ is not unique.

Set $\mathbf{v}_1 = d_h^{[1]}\boldsymbol{\tau}_1$ where $d_h^{[1]}$ is a positive quantity intended to converge to 0 with h. The vector $\mathbf{v}_1$ is a shift that allows to pass from the cone $\Pi_{\mathbf{x}_0}$ to the substructure $\Pi_{\mathbf{x}_0,\mathbf{x}_1}$, which is also the tangent cone to $\Pi_{\mathbf{x}_0}$ at the point $\mathbf{v}_1$. Let $\mathrm{U}^{\mathbf{v}_1}$ be a local diffeomorphism that sends (a neighborhood of) $\mathbf{v}_1$ in $\Pi_{\mathbf{x}_0}$ to (a neighborhood of) $\mathbf{0}$ in $\Pi_{\mathbf{x}_0,\mathbf{x}_1}$. We can assume without restriction that $\mathrm{U}^{\mathbf{v}_1}$ is part of an admissible atlas on $\Pi_{\mathbf{x}_0}$.

- [a2]. By the change of variable $\mathrm{U}^{\mathbf{v}_1}$, the potential $\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0} - \mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}(\mathbf{v}_1)$ becomes $\mathbf{A}^{\mathbf{v}_1}$ (cf. (4.1))

$$\mathbf{A}^{\mathbf{v}_1} = (\mathrm{J}^{\mathbf{v}_1})^\top\Big(\big(\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0} - \mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}(\mathbf{v}_1)\big)\circ(\mathrm{U}^{\mathbf{v}_1})^{-1}\Big) \quad \text{with} \quad \mathrm{J}^{\mathbf{v}_1} = \mathrm{d}(\mathrm{U}^{\mathbf{v}_1})^{-1}.$$

- [b2]. Let $\zeta_h^{\mathbf{v}_1}(\mathbf{x}) = \mathrm{e}^{i\langle \mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}(\mathbf{v}_1),\, \mathbf{x}/h\rangle}$, for $\mathbf{x} \in \Pi_{\mathbf{x}_0}$.
- [c2]. We introduce the relation

$$\varphi_h^{[1]} = \mathrm{Z}_h^{\mathbf{v}_1} \circ \mathrm{U}_*^{\mathbf{v}_1}(\varphi_h^{[2]}), \tag{9.22}$$

and let $r_h^{[2]}$ be the radius of the smallest ball centered at $\mathbf{0}$ containing the support of $\varphi_h^{[2]}$ in $\overline{\Pi}_{\mathbf{x}_0,\mathbf{x}_1}$. This new quantity is also intended to converge to 0 with h.

Using (4.5) and (4.13) in Lemma 4.5, we find a relation between Rayleigh quotients of the same form as (9.12), with $\mathcal{O}(r_h^{[1]})$ replaced by $\mathcal{O}(r_h^{[2]}/d_h^{[1]})$ or $\mathcal{O}(r_h^{[2]})$ according whether $\mathbf{x}_0$ is a conical point or not. Like for (9.13), we deduce

(9.23)

$$\Big|\mathscr{Q}_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\varphi_h^{[1]}) - \mathscr{Q}_h[\mathbf{A}^{\mathbf{v}_1}, \Pi_{\mathbf{x}_0,\mathbf{x}_1}](\varphi_h^{[2]})\Big| \lesssim \begin{cases} \dfrac{r_h^{[2]}}{d_h^{[1]}}\mathscr{Q}_h[\mathbf{A}^{\mathbf{v}_1}, \Pi_{\mathbf{x}_0,\mathbf{x}_1}](\varphi_h^{[2]}) & \text{if } \ \mathbf{x}_0 \in \mathfrak{V}^\circ, \\ r_h^{[2]}\mathscr{Q}_h[\mathbf{A}^{\mathbf{v}_1}, \Pi_{\mathbf{x}_0,\mathbf{x}_1}](\varphi_h^{[2]}) & \text{if } \ \mathbf{x}_0 \notin \mathfrak{V}^\circ. \end{cases}$$

- [d2]. Let $\mathbf{A}_{\mathbf{0}}^{\mathbf{v}_1}$ be the linear part of $\mathbf{A}^{\mathbf{v}_1}$ at $\mathbf{0} \in \Pi_{\mathbf{x}_0,\mathbf{x}_1}$. Thus, by relation (A.7) and a Cauchy-Schwarz inequality, we have

$$\Big|q_h[\mathbf{A}^{\mathbf{v}_1}, \Pi_{\mathbf{x}_0,\mathbf{x}_1}](\varphi_h^{[2]}) - q_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{v}_1}, \Pi_{\mathbf{x}_0,\mathbf{x}_1}](\varphi_h^{[2]})\Big| \le C\Big(a_h^{[2]}\sqrt{\mu_h^{[2]}} + \big(a_h^{[2]}\big)^2\Big)\|\varphi_h^{[2]}\|^2, \tag{9.24}$$

with

$$\mu_h^{[2]} = \mathscr{Q}_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{v}_1}, \Pi_{\mathbf{x}_0,\mathbf{x}_1}](\varphi_h^{[2]}) \quad \text{and} \quad a_h^{[2]} = \frac{\|(\mathbf{A}^{\mathbf{v}_1} - \mathbf{A}_{\mathbf{0}}^{\mathbf{v}_1})\varphi_h^{[2]}\|}{\|\varphi_h^{[2]}\|}. \tag{9.25}$$

By Lemmas 4.6–4.7, case (ii), and since $\varphi_h^{[2]}$ is supported in the ball $\mathcal{B}(\mathbf{0}, r_h^{[2]})$, we have

$$a_h^{[2]} \lesssim \frac{\big(r_h^{[2]}\big)^2}{d_h^{[1]}} \ \text{ if } \ \mathbf{x}_0 \in \mathfrak{V}^\circ \quad \text{and} \quad a_h^{[2]} \lesssim \big(r_h^{[2]}\big)^2 \ \text{ if } \ \mathbf{x}_0 \notin \mathfrak{V}^\circ. \tag{9.26}$$

Using (9.23)–(9.26), we find, if $\mathbf{x}_0$ belongs to $\mathfrak{V}^\circ$ and $r_h^{[2]}/d_h^{[1]}$ is small enough,

$$\Big|\mathscr{Q}_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\varphi_h^{[1]}) - \mathscr{Q}_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{v}_1},\Pi_{\mathbf{x}_0,\mathbf{x}_1}](\varphi_h^{[2]})\Big| \lesssim \mu_h^{[2]}\frac{r_h^{[2]}}{d_h^{[1]}} + \frac{\big(r_h^{[2]}\big)^2}{d_h^{[1]}}\sqrt{\mu_h^{[2]}} + \frac{\big(r_h^{[2]}\big)^4}{\big(d_h^{[1]}\big)^2}\,, \tag{9.27}$$

and, if $\mathbf{x}_0$ does not belong to $\mathfrak{V}^\circ$ and $r_h^{[2]}$ is small enough

$$\Big|\mathscr{Q}_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\varphi_h^{[1]}) - \mathscr{Q}_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{v}_1},\Pi_{\mathbf{x}_0,\mathbf{x}_1}](\varphi_h^{[2]})\Big| \lesssim \mu_h^{[2]}r_h^{[2]} + \big(r_h^{[2]}\big)^2\sqrt{\mu_h^{[2]}} + \big(r_h^{[2]}\big)^4. \tag{9.28}$$

- [e2]. If $\nu = 2$, we set, as already mentioned, $\varphi_h^{[2]} = \chi_h\Psi_h^{\mathbb{X}}$. Note that $\mathbf{A}_{\mathbf{0}}^{\mathbf{v}_1}$ coincides with $\mathbf{A}_{\mathbb{X}}$. We have now to distinguish two cases, according as $\mathbf{x}_0$ is or not a conical point:
  (a) If $\mathbf{x}_0$ is not a conical point, i.e., $\mathbf{x}_0 \notin \mathfrak{V}^\circ$, the local map $\mathrm{U}^{\mathbf{v}_1}$ is simply the translation $\mathbf{x} \mapsto \mathbf{x} - \mathbf{v}_1$. To tune the cut-off χ_h, we choose the exponent δ as δ_0 and the shift $d_h^{[1]}$ as h^{δ_0}. We choose the radius R for the cut-off χ_h so that the support of $\underline{\chi}_R$, cf. (7.8), is contained in a map neighborhood $\mathcal{V}_{\boldsymbol{\tau}_1}$ of $\mathbf{0}$ in $\overline{\Pi}_{\mathbf{x}_0,\mathbf{x}_1}$, i.e., a neighborhood such that:
  $$\mathrm{U}^{\boldsymbol{\tau}_1}(\mathcal{U}_{\boldsymbol{\tau}_1}\cap\Pi_{\mathbf{x}_0}) = \mathcal{V}_{\boldsymbol{\tau}_1}\cap\Pi_{\mathbf{x}_0,\mathbf{x}_1}$$
  where $\mathrm{U}^{\boldsymbol{\tau}_1}(\mathbf{x}) = \mathbf{x} - \boldsymbol{\tau}_1$ and $\mathcal{U}_{\boldsymbol{\tau}_1} = \mathcal{V}_{\boldsymbol{\tau}_1} + \boldsymbol{\tau}_1$. Then the quantities $r_h^{[1]}$ and $r_h^{[2]}$ are both $\mathcal{O}(h^{\delta_0})$ and we can combine (9.28) with (9.18) and the cut-off estimate (9.7). In this way we find that for h small enough, the quantities $\mu_h^{[1]}$ and $\mu_h^{[2]}$ are both $\mathcal{O}(h)$, and we deduce the estimate (9.19) as in the case $\nu = 1$, which leads, like in the sitting case, to the upper bound
  $$\lambda_h(\mathbf{B},\Omega) \le h\mathscr{E}(\mathbf{B},\Omega) + C(\Omega)\big(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)h^{5/4}\,.$$
  (b) If $\mathbf{x}_0$ is a conical point, to tune the cut-off χ_h, we choose the exponent δ as $\delta_0+\delta_1$ and the shift $d_h^{[1]}$ as h^{δ_0}, with $\delta_0,\delta_1 > 0$ such that $\delta_0+\delta_1 < \frac12$. We choose the radius R equal to 1. Therefore $r_h^{[2]} = \mathcal{O}(h^{\delta_0+\delta_1})$ and $r_h^{[1]} = \mathcal{O}(h^{\delta_0})$. By (9.7) $\mu_h^{[2]} = \mathcal{O}(h)$ and, since for h small enough, $r_h^{[2]}/d_h^{[1]}$ is arbitrarily small, we also deduce with the help of (9.27) that $\mu_h^{[1]} = \mathcal{O}(h)$. Putting this together with (9.18) and (9.27), and using (9.7) once more, we deduce the estimate

$$\begin{aligned}\mathscr{Q}_h[\mathbf{A},\Omega](\varphi_h^{[0]}) \le h\Lambda_{\mathbb{X}} &+ C\left(h^{1+\delta_0} + h^{\frac12+2\delta_0} + h^{4\delta_0}\right)\\ &+ C\left(h^{2-2\delta_0-2\delta_1} + h^{1+\delta_1} + h^{\frac12+\delta_0+2\delta_1} + h^{2\delta_0+4\delta_1}\right).\end{aligned} \tag{9.29}$$

The exponents that appear here are the same as for the lower bound (5.27). Thus taking $\delta_0 = 3/10$ and $\delta_1 = 3/20$, we optimize remainders and deduce

$$\lambda_h(\mathbf{B},\Omega) \le h\mathscr{E}(\mathbf{B},\Omega) + C(\Omega)\big(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)h^{11/10}\,.$$

The latter step ends in particular the handling of the polyhedral case since we can always reduce to chains of length $\nu \le 2$ in polyhedral domains, cf Proposition 3.29.

9.4. **Third level of construction and doubly sliding quasimodes.** It remains to deal the case $\nu = 3$. In that case, the chain $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2)$ is such that

- $\mathbf{x}_0$ is a conical point,
- $\mathbf{x}_1$ is a vertex of $\Omega_{\mathbf{x}_0}$, $\boldsymbol{\tau}_1$ coincides with $\mathbf{x}_1$, the corresponding edge of $\Pi_{\mathbf{x}_0}$ is generated by $\mathbf{x}_1$, and $\Pi_{\mathbf{x}_0,\mathbf{x}_1}$ is a wedge,
- $\mathbf{x}_2$ is an end of the interval $\Omega_{\mathbf{x}_0,\mathbf{x}_1}$, it corresponds to a point $\boldsymbol{\tau}_2$ on a face of $\Pi_{\mathbf{x}_0,\mathbf{x}_1}$, defined as in (9.21). Finally $\Pi_{\mathbf{x}_1,\mathbf{x}_1,\mathbf{x}_2} = \Pi_{\mathbb{X}}$ is a half-space.

Set $\mathbf{v}_2 = d_h^{[2]}\boldsymbol{\tau}_2$ where $d_h^{[2]}$ is a positive quantity intended to converge to 0 with h. Let $\mathrm{U}^{\mathbf{v}_2}$ be the translation that sends (a neighborhood of) $\mathbf{v}_2$ in $\Pi_{\mathbf{x}_0,\mathbf{x}_1}$ to (a neighborhood of) $\mathbf{0}$ in $\Pi_{\mathbb{X}} = \Pi_{\mathbf{x}_0,\mathbf{x}_1,\mathbf{x}_2}$.

- [a3]. By the change of variable $\mathrm{U}^{\mathbf{v}_2}$, since $\mathrm{J}^{\mathbf{v}_2} = \mathbb{I}_3$, the potential $\mathbf{A}_\mathbf{0}^{\mathbf{v}_1} - \mathbf{A}_\mathbf{0}^{\mathbf{v}_1}(\mathbf{v}_2)$ becomes

$$\mathbf{A}^{\mathbf{v}_2} = \big(\mathbf{A}_\mathbf{0}^{\mathbf{v}_1} - \mathbf{A}_\mathbf{0}^{\mathbf{v}_1}(\mathbf{v}_2)\big) \circ (\mathrm{U}^{\mathbf{v}_2})^{-1},$$

and it coincides with its linear part $\mathbf{A}_\mathbf{0}^{\mathbf{v}_2}$.

- [b3]. Let $\zeta_h^{\mathbf{v}_2}(\mathbf{x}) = \mathrm{e}^{i\langle \mathbf{A}_\mathbf{0}^{\mathbf{v}_1}(\mathbf{v}_2),\, \mathbf{x}/h\rangle}$, for $\mathbf{x} \in \Pi_{\mathbf{x}_0,\mathbf{x}_1}$.
- [c3]. We define

$$\varphi_h^{[2]} = Z_h^{\mathbf{v}_2} \circ \mathrm{U}_*^{\mathbf{v}_2}(\varphi_h^{[3]}). \tag{9.30}$$

- [d3]. Since $\mathrm{G}^{\mathbf{v}_2} = \mathbb{I}_3$, we have

$$\mathscr{Q}_h[\mathbf{A}_\mathbf{0}^{\mathbf{v}_1}, \Pi_{\mathbf{x}_0,\mathbf{x}_1}](\varphi_h^{[2]}) = \mathscr{Q}_h[\mathbf{A}_\mathbf{0}^{\mathbf{v}_2}, \Pi_{\mathbb{X}}](\varphi_h^{[3]}). \tag{9.31}$$

- [e3]. We set, as already mentioned $\varphi_h^{[3]} = \chi_h \Psi_h^{\mathbb{X}}$. We have $\mathbf{A}_\mathbf{0}^{\mathbf{v}_2} = \mathbf{A}_{\mathbb{X}}$. We choose the exponent δ as $\delta_0 + \delta_1$, the shifts $d_h^{[2]}$ as $h^{\delta_0+\delta_1}$ and $d_h^{[1]}$ as h^{δ_0}, with $\delta_0, \delta_1 > 0$ such that $\delta_0 + \delta_1 < \frac{1}{2}$. We conclude as the conical case at level 2 and obtain again (9.29). We deduce

$$\lambda_h(\mathbf{B}, \Omega) \leq h\mathscr{E}(\mathbf{B}, \Omega) + C(\Omega)\big(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)h^{11/10}\ .$$

The outcome of the last four sections is the achievement of the proof of Theorem 9.1. We may notice that there is only one configuration where we cannot prove the convergence rate $h^{5/4}$: This is the case when all points with minimal local energy $\mathbf{x}_0$ satisfy all the following conditions

(1) $\mathbf{x}_0$ is a conical point ($\mathbf{x}_0 \in \mathfrak{V}^\circ$),
(2) the corresponding model operator $H(\mathbf{A}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0})$ has no eigenvalue below its essential spectrum,
(3) the geometry around $\mathbf{x}_0$ is not trivial.

9.5. **Improvement in case of corner concentration.** When the geometry minimizing the energy is given by a corner whose tangent problem has an eigenvalue under its essential spectrum, we get a better upper bound by improving the estimate on the quotient $a_h^{[1]} = \|(\mathbf{A}^{\mathbf{x}_0} - \mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0})\varphi_h^{[1]}\|/\|\varphi_h^{[1]}\|$ which quantifies the linearization error, cf. (9.15).

Proposition 9.4. *Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ be a corner domain, $\mathbf{A} \in W^{2,\infty}(\overline{\Omega})$ be a twice differentiable magnetic potential such that the associated magnetic field $\mathbf{B}$ does not vanish on $\overline{\Omega}$. We assume moreover that there exists a corner $\mathbf{x}_0 \in \overline{\Omega}$ such that*

$$\mathscr{E}(\mathbf{B},\Omega) = E(\mathbf{B}_{\mathbf{x}_0},\Pi_{\mathbf{x}_0}) < \mathscr{E}^*(\mathbf{B}_{\mathbf{x}_0},\Pi_{\mathbf{x}_0}).$$

Then there exist $C(\Omega) > 0$ and $h_0 > 0$ such that

$$\forall h \in (0,h_0), \quad \lambda_h(\mathbf{B},\Omega) \le h\mathscr{E}(\mathbf{B},\Omega) + C(\Omega)(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})\, h^{3/2}|\log h|\,.$$

Proof. Since $E(\mathbf{B}_{\mathbf{x}_0},\Pi_{\mathbf{x}_0}) < \mathscr{E}^*(\mathbf{B}_{\mathbf{x}_0},\Pi_{\mathbf{x}_0})$ and $\lambda_{\mathrm{ess}}(\mathbf{B}_{\mathbf{x}_0},\Pi_{\mathbf{x}_0}) = \mathscr{E}^*(\mathbf{B}_{\mathbf{x}_0},\Pi_{\mathbf{x}_0})$ by Theorem 6.6, the generalized eigenfunction $\Psi^{\mathbb{X}}$ of $H(\mathbf{A}_{\mathbf{x}_0},\Pi_{\mathbf{x}_0})$ provided by Theorem 7.2 is an eigenfunction and has exponential decay when $|\mathbf{x}| \to +\infty$. Here $\mathbb{X} = (\mathbf{x}_0)$, the quasimode $\varphi_h^{[0]}$ is sitting and defined by (9.10), (9.9), (9.6). Using (4.14) and Lemma 4.7, case (i), we get $C(\Omega) > 0$ such that

$$\forall \mathbf{x} \in \mathrm{supp}(\varphi_h^{[1]}), \quad |(\mathbf{A}^{\mathbf{x}_0} - \mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0})(\mathbf{x})| \le C(\Omega)\|\mathbf{A}^{\mathbf{x}_0}\|_{W^{2,\infty}(\mathrm{supp}(\varphi_h^{[1]}))}|\mathbf{x}|^2\,.$$

Using the change of variable $\mathbf{X} = \mathbf{x}h^{-1/2}$ and the exponential decay of $\Psi^{\mathbb{X}}$ we get

$$a_h^{[1]} = \frac{\|(\mathbf{A}^{\mathbf{x}_0} - \mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0})\varphi_h^{[1]}\|}{\|\varphi_h^{[1]}\|} \le C(\Omega)\|\mathbf{A}^{\mathbf{x}_0}\|_{W^{2,\infty}(\mathrm{supp}(\varphi_h^{[1]}))}h. \tag{9.32}$$

Using (9.14) with estimate (9.32) and Lemma 7.9, for any $\delta \in (0,\frac12]$, we get

$$\begin{aligned}\mathscr{Q}[\mathbf{A}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\varphi_h^{[1]}) &\le h\Lambda_{\mathbb{X}} + C\Big(h^{2-2\delta}\mathrm{e}^{-ch^{\delta-\frac12}} + \|\mathbf{A}\|_{W^{2,\infty}(\Omega)}h^{\frac32} + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}h^2\Big)\\ &\le h\Lambda_{\mathbb{X}} + C\big(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)\,\big(h^{2-2\delta}\mathrm{e}^{-ch^{\delta-\frac12}} + h^{\frac32}\big).\end{aligned}$$

Thanks to (9.13), the quasimode $\varphi_h^{[0]}$ satisfies

$$\begin{aligned}\mathscr{Q}_h[\mathbf{A},\Omega](\varphi_h^{[0]}) &\le \big(1+\mathcal{O}(h^\delta)\big)\big\{h\Lambda_{\mathbb{X}} + C\big(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)\big(h^{2-2\delta}\mathrm{e}^{-ch^{\delta-\frac12}} + h^{3/2}\big)\big\}\\ &\le h\Lambda_{\mathbb{X}} + C\big(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)\big\{h^{1+\delta} + h^{2-2\delta}\mathrm{e}^{-ch^{\delta-\frac12}} + h^{3/2}\big\}.\end{aligned}$$

Here C denotes various constants depending on Ω but independent from $h \le h_0$ and $\delta \le \frac12$. We optimize this by taking $\delta = \frac12 - \varepsilon(h)$ with $\varepsilon(h)$ so that $h^{1+\delta} = h^{2-2\delta}\mathrm{e}^{-ch^{\delta-\frac12}}$, i.e.

$$h^{\frac32-\varepsilon(h)} = h^{1+2\varepsilon(h)}\mathrm{e}^{-ch^{-\varepsilon(h)}}.$$

We find

$$\mathrm{e}^{ch^{-\varepsilon(h)}} = h^{-\frac12+3\varepsilon(h)}, \quad \text{i.e.} \quad h^{-\varepsilon(h)} = \tfrac{1}{c}(-\tfrac12 + 3\varepsilon(h))\log h\,.$$

The latter equation has one solution $\varepsilon(h)$ which tends to 0 as h tends to 0. Replacing $h^{-\varepsilon(h)}$ by the value above in $h^{\frac{3}{2}-\varepsilon(h)}$, we find that the remainder is a $O(h^{3/2}|\log h|)$ and the min-max principle provides the proposition. ☐

Appendix A. Technical lemmas

A.1. Gauge transform.

Lemma A.1. *Let $\mathcal{O} \subset \mathbb{R}^n$ be a domain and let ϑ be a regular function on $\overline{\mathcal{O}}$. Let $\mathbf{A}$ be a regular potential. Then*

$$\forall \psi \in \mathrm{Dom}(q_h[\mathbf{A}, \mathcal{O}]), \quad q_h[\mathbf{A} + \nabla\vartheta, \mathcal{O}](\mathrm{e}^{-i\vartheta/h}\psi) = q_h[\mathbf{A}, \mathcal{O}](\psi).$$

Furthermore a function ψ is an eigenfunction for the operator $H_h(\mathbf{A}, \mathcal{O})$ if an only if $\mathrm{e}^{-i\vartheta/h}\psi$ is an eigenfunction for $H_h(\mathbf{A} + \nabla\vartheta, \mathcal{O})$ associated with the same eigenvalue.

Proof. The commutation formula

$$(-ih\nabla + \mathbf{A} + \nabla\vartheta)\left(\mathrm{e}^{-i\vartheta/h}\psi\right) = \mathrm{e}^{-i\vartheta/h}(-ih\nabla + \mathbf{A})\psi$$

yields the result. ☐

For the sake of completeness we provide the following standard lemma describing the effect of a translation when the potential is affine:

Lemma A.2 (Translation)**.** *Let $\mathcal{O} \subset \mathbb{R}^3$ be a domain and $\mathbf{A}$ be an affine magnetic potential. Let $\mathbf{d} \in \mathbb{R}^3$ be a vector and $\mathcal{O}_\mathbf{d} := \mathcal{O} + \mathbf{d}$ be the translated domain. For $\psi \in \mathrm{Dom}(H_h(\mathbf{A}, \mathcal{O}))$, we define the translated function on $\mathcal{O}_\mathbf{d}$ by*

$$\psi_\mathbf{d} : \mathbf{x} \mapsto \mathrm{e}^{-i\langle \mathbf{A}(\mathbf{d}) - \mathbf{A}(\mathbf{0}), \mathbf{x}\rangle/h}\psi(\mathbf{x} - \mathbf{d})\,.$$

Then $q_h[\mathbf{A}, \mathcal{O}_\mathbf{d}](\psi_\mathbf{d}) = q_h[\mathbf{A}, \mathcal{O}](\psi)$ and ψ is an eigenfunction of $H_h(\mathbf{A}, \mathcal{O})$ if and only if $\psi_\mathbf{d}$ is an eigenfunction of $H_h(\mathbf{A}, \mathcal{O}_\mathbf{d})$.

Lemma A.3. *Let $\mathcal{O}$ be a bounded domain such that $\mathbf{0} \in \overline{\mathcal{O}}$. Let $\mathbf{A} \in W^{3,\infty}(\mathcal{O})$ be a magnetic potential such that $\mathbf{A}(\mathbf{0}) = 0$. Let $\mathbf{A_0}$ denote the linear part of $\mathbf{A}$ at $\mathbf{0}$. Let ℓ be an index in $\{1, 2, 3\}$.*

(a) There exists a change of gauge ∇F where F is a polynomial function of degree 3, so that

(1) The linear part of $\mathbf{A} - \nabla F$ at $\mathbf{0}$ is still $\mathbf{A_0}$,

(2) The second derivative of $\mathbf{A} - \nabla F$ with respect to u_ℓ cancels at $\mathbf{0}$:

$$\partial^2_{u_\ell}(\mathbf{A} - \nabla F)(\mathbf{0}) = 0.$$

(3) The coefficients of F are bounded by $\|\mathbf{A}\|_{W^{2,\infty}(\mathcal{O})}$.

(b) Let us choose $\ell = 1$ for instance. We have the estimate

$$|\mathbf{A}(\mathbf{u}) - \mathbf{A_0}(\mathbf{u}) - \nabla F(\mathbf{u})| \le C(\mathcal{O})\, \|\mathbf{A}\|_{W^{3,\infty}(\mathcal{O})} \big(|u_1|^3 + |u_1u_2| + |u_1u_3| + |u_2|^2 + |u_3|^2\big)\,, \tag{A.1}$$

where the constant $C(\mathcal{O})$ depends only on the outer diameter of $\mathcal{O}$.

Proof. The Taylor expansion of $\mathbf{A}$ at $\mathbf{0}$ takes the form

$$\mathbf{A} = \mathbf{A_0} + \mathbf{A}^{(2)} + \mathbf{A}^{(\mathrm{rem},3)},$$

where $\mathbf{A}^{(2)}$ is a homogeneous polynomial of degree 2 with 3 components and $\mathbf{A}^{(\mathrm{rem},3)}$ is a remainder:

$$|\mathbf{A}^{(\mathrm{rem},3)}(\mathbf{u})| \le \|\mathbf{A}\|_{W^{3,\infty}(\mathcal{O})} |\mathbf{u}|^3 \quad \text{for} \quad \mathbf{u} \in \mathcal{O}. \tag{A.2}$$

Let us write the m-th component $A_m^{(2)}$ of $\mathbf{A}^{(2)}$ as

$$A_m^{(2)}(\mathbf{u}) = \sum_{|\alpha|=2} a_{m,\alpha} u_1^{\alpha_1} u_2^{\alpha_2} u_3^{\alpha_3} \quad \text{for} \quad \mathbf{u} = (u_1, u_2, u_3) \in \mathcal{O}.$$

(a) Now, the polynomial F can be explicitly determined. It suffices to take

$$F(\mathbf{u}) = u_\ell^2 \big(a_{1,\alpha^*} u_1 + a_{2,\alpha^*} u_2 + a_{3,\alpha^*} u_3 - \tfrac{2}{3} a_{\ell,\alpha^*} u_\ell\big),$$

where α^* is such that $\alpha^*_\ell = 2$ (and the other components are 0). Then

$$\nabla F(\mathbf{u}) = u_\ell^2 \begin{pmatrix} a_{1,\alpha^*} \\ a_{2,\alpha^*} \\ a_{3,\alpha^*} \end{pmatrix}$$

and point (a) of the lemma is proved.

(b) Choosing $\ell = 1$, we see that the m-th components of $\mathbf{A}^{(2)} - \nabla F$ is

$$A_m^{(2)}(\mathbf{u}) - (\nabla F)_m(\mathbf{u}) \\ = a_{m,(1,1,0)} u_1u_2 + a_{m,(1,0,1)} u_1u_3 + a_{m,(0,1,1)} u_2u_3 + a_{m,(0,2,0)} u_2^2 + a_{m,(0,0,2)} u_3^2\,.$$

Hence $\mathbf{A}^{(2)} - \nabla F$ satisfies the estimate

$$|(\mathbf{A}^{(2)}(\mathbf{u}) - \nabla F(\mathbf{u})| \le \|\mathbf{A}\|_{W^{2,\infty}(\mathcal{O})} \big(|u_1u_2| + |u_1u_3| + |u_2|^2 + |u_3|^2\big).$$

But

$$\mathbf{A} - \mathbf{A_0} - \nabla F = \mathbf{A}^{(2)} - \nabla F + \mathbf{A}^{(\mathrm{rem},3)}.$$

Therefore, with (A.2)

$$|\mathbf{A}(\mathbf{u}) - \mathbf{A_0}(\mathbf{u}) - \nabla F(\mathbf{u})| \le \|\mathbf{A}\|_{W^{2,\infty}(\mathcal{O})} \big(|u_1u_2| + |u_1u_3| + |u_2|^2 + |u_3|^2\big) + \|\mathbf{A}\|_{W^{3,\infty}(\mathcal{O})} |\mathbf{u}|^3.$$

Using finally that $|\mathbf{u}|^3 \le 12(|u_1|^3 + |u_2|^3 + |u_3|^3) \le C(\mathcal{O})(|u_1|^3 + |u_2|^2 + |u_3|^2)$, we conclude the proof of estimate (A.1). $\square$

A.2. **Change of variables.** Let G be a metric of $\mathbb{R}^3$, that is a 3×3 positive symmetric matrix with regular coefficients. For a smooth magnetic potential, the quadratic form of the associated magnetic Laplacian with the metric G is denoted by $q_h[\mathbf{A},\mathcal{O},\mathrm{G}]$ and is defined in (1.20). The following lemma describes how this quadratic form is involved when using a change of variables:

Lemma A.4. *Let $\mathrm{U}:\mathcal{O}\to\mathcal{O}'$, $\mathbf{u}\mapsto\mathbf{v}$ be a diffeomorphism with $\mathcal{O},\mathcal{O}'$ domains. We denote by*

$$\mathrm{J} := \mathrm{d}(\mathrm{U}^{-1})$$

the jacobian matrix of the inverse of U. Let $\mathbf{A}$ be a smooth magnetic potential and $\mathbf{B}=\operatorname{curl}\mathbf{A}$ the associated magnetic field. Let f be a function of $\operatorname{Dom}(q_h[\mathbf{A},\mathcal{O}])$ and $\psi := f\circ\mathrm{U}^{-1}$ defined in $\mathcal{O}'$. For any $h>0$ we have

$$q_h[\mathbf{A},\mathcal{O}](f) = q_h[\widetilde{\mathbf{A}},\mathcal{O}',\mathrm{G}](\psi) \quad \text{and} \quad \|f\|_{L^2(\mathcal{O})} = \|\psi\|_{L^2_{\mathrm{G}}(\mathcal{O}')} \tag{A.3}$$

where the new magnetic potential and the metric are respectively given by

$$\widetilde{\mathbf{A}} := \mathrm{J}^{\top}\big(\mathbf{A}\circ\mathrm{U}^{-1}\big) \quad \text{and} \quad \mathrm{G} := \mathrm{J}^{-1}(\mathrm{J}^{-1})^{\top}\ . \tag{A.4}$$

The magnetic field $\widetilde{\mathbf{B}}=\operatorname{curl}\widetilde{\mathbf{A}}$ in the new variables is given by

$$\widetilde{\mathbf{B}} := |\det \mathrm{J}|\,\mathrm{J}^{-1}\big(\mathbf{B}\circ\mathrm{U}^{-1}\big). \tag{A.5}$$

Let $\rho>0$, using the previous Lemma with the scaling $\mathrm{U}^{\rho} := \mathbf{x}\mapsto\sqrt{\rho}\,\mathbf{x}$ we get

Lemma A.5. *Let $\mathcal{O}\subset\mathbb{R}^3$ be a domain and for $r>0$, we denote by $r\mathcal{O}$ the domain $\{\mathbf{x}\in\mathbb{R}^3,\ \mathbf{x}=r\mathbf{x}' \text{ with } \mathbf{x}'\in\mathcal{O}\}$. Let $\mathbf{B}$ be a constant magnetic field and $\mathbf{A}$ be an associated linear potential associated. For any $\psi\in\operatorname{Dom}(q[\mathbf{A},\mathcal{O}])$, we define*

$$\psi_\rho(\mathbf{x}) := \rho^{-3/4}\psi\Big(\frac{\mathbf{x}}{\sqrt{\rho}}\Big), \qquad \forall\mathbf{x}\in\mathcal{O}.$$

Then $\psi_\rho\in\operatorname{Dom}(q_\rho[\mathbf{A},\sqrt{\rho}\mathcal{O}])$ and we have

(1) *$\forall\rho>0$, $q[\mathbf{A},\mathcal{O}](\psi) = \rho\, q[\rho^{-1}\mathbf{A},\sqrt{\rho}\mathcal{O}](\psi_\rho) = \rho^{-1}q_\rho[\mathbf{A},\sqrt{\rho}\mathcal{O}](\psi_\rho)$.*

(2) *$\forall\rho>0$, $E(\mathbf{B},\mathcal{O}) = \rho\, E\Big(\rho^{-1}\mathbf{B},\sqrt{\rho}\,\mathcal{O}\Big)$.*

(3) *ψ is a generalized eigenfunction of $H(\mathbf{A},\mathcal{O})$ associated with $E(\mathbf{B},\mathcal{O})$ if and only if ψ_ρ is a generalized eigenfunction of $H(\rho^{-1}\mathbf{A},\sqrt{\rho}\,\mathcal{O})$ associated with $\rho E(\rho^{-1}\mathbf{B},\sqrt{\rho}\,\mathcal{O})$ if and only if ψ_ρ is a generalized eigenfunction of $H_\rho(\mathbf{A},\sqrt{\rho}\,\mathcal{O})$ associated with $\rho E(\mathbf{B},\mathcal{O})$.*

A.3. **Miscellaneous.**

• *Orientation of the magnetic field.* Let $\mathbf{B}$ be a magnetic field. It is known that changing $\mathbf{B}$ into $-\mathbf{B}$ does not affect the spectrum of the associated magnetic Laplacian. More precisely we have:

Lemma A.6. *Let $\mathcal{O} \subset \mathbb{R}^3$ be a domain, $\mathbf{B}$ be a magnetic field and $\mathbf{A}$ an associated potential. Then $H_h(-\mathbf{A}, \mathcal{O})$ and $H_h(\mathbf{A}, \mathcal{O})$ are unitary equivalent. We have*

$$\forall \psi \in \mathrm{Dom}(q_h[\mathbf{A}, \mathcal{O}]), \quad q_h[-\mathbf{A}, \mathcal{O}](\overline{\psi}) = q_h[\mathbf{A}, \mathcal{O}](\psi)$$

and ψ is an eigenfunction of $H_h(\mathbf{A}, \mathcal{O})$ if and only if $\overline{\psi}$ is an eigenfunction of $H_h(-\mathbf{A}, \mathcal{O})$.

• *Model linear potential.* Let us remark that if $\mathbf{B}$ is a constant magnetic field, an associated magnetic potential is given by

$$\mathbf{A}^{\mathrm{S}}(\mathbf{x}) := \frac{1}{3}\mathbf{B} \wedge \mathbf{x} \,. \tag{A.6}$$

Indeed we have

$$\mathrm{curl}\, \mathbf{A}^{\mathrm{S}} = \frac{1}{3}\nabla \wedge (\mathbf{B} \wedge \mathbf{x}) = \frac{1}{3}\left((\nabla \cdot \mathbf{x})\mathbf{B} - (\nabla \cdot \mathbf{B})\mathbf{x}\right) = \mathbf{B} \,.$$

• *Comparison between two potentials.* Let $\mathcal{O} \subset \mathbb{R}^3$ be a domain and let $\mathbf{A}$ and $\mathbf{A}'$ be two magnetic potentials. Then, for any function ψ of $\mathrm{Dom}(q_h[\mathbf{A}, \mathcal{O}]) \cap \mathrm{Dom}(q_h[\mathbf{A}', \mathcal{O}])$, we have:

$$q_h[\mathbf{A}, \mathcal{O}](\psi) = q_h[\mathbf{A}', \mathcal{O}](\psi) + 2\,\mathrm{Re}\left\langle(-ih\nabla + \mathbf{A}')\psi, (\mathbf{A} - \mathbf{A}')\psi\right\rangle_{\mathcal{O}} + \|(\mathbf{A} - \mathbf{A}')\psi\|^2 \,. \tag{A.7}$$

A.4. **Cut-off effect.** In this section we recall standard IMS[10] formulas. This kind of formulas appear for Schrödinger operators in [11], but they can also be found in older works like [30]. In this section $\mathbf{A}$ denotes a regular magnetic potential and $\mathcal{O}$ a generic domain of $\mathbb{R}^3$.

The first formula describes the effect of a partition of the unity on the energy of a function which is in the form domain, see for example [44, Lemma 3.1]:

Lemma A.7 (IMS formula)**.** *Assume that $\chi_1, \ldots, \chi_L \in \mathscr{C}^\infty(\overline{\mathcal{O}})$ are such that*

$$\sum_{\ell=1}^{L} \chi_\ell^2 \equiv 1 \quad \text{on} \quad \mathcal{O}.$$

Then, for any $\psi \in \mathrm{Dom}(q_h[\mathbf{A}, \mathcal{O}])$

$$q_h[\mathbf{A}, \mathcal{O}](\psi) = \sum_{\ell=1}^{L} q_h[\mathbf{A}, \mathcal{O}](\chi_\ell \psi) - h^2 \sum_{\ell=1}^{L} \|\psi \nabla \chi_\ell\|^2_{L^2(\mathcal{O})}$$

[10]IMS stands for Ismagilov-Morgan-Sigal (or Simon)

Recall that $\mathrm{Dom}_{\mathrm{loc}}(H_h(\mathbf{A},\mathcal{O}))$ denotes the functions that are locally in the domain of the operator, see (1.22). In particular they satisfy the Neumann boundary condition. The second formula describes the energy of such a function when applying a cut-off function, see for example [20, (6.11)]:

Lemma A.8. *Let $\chi \in \mathscr{C}_0^\infty(\overline{\mathcal{O}})$ a real smooth function. Then for any $\psi \in \mathrm{Dom}_{\mathrm{loc}}(H_h(\mathbf{A},\mathcal{O}))$*

$$q_h[\mathbf{A},\mathcal{O}](\chi\psi) = \mathrm{Re}\,\big\langle \chi^2 H_h(\mathbf{A},\mathcal{O})\psi,\psi\big\rangle_{\mathcal{O}} + h^2\|\,|\nabla\chi|\,\psi\|^2_{L^2(\mathcal{O})}\,. \tag{A.8}$$

Appendix B. Partition of unity suitable for IMS formulas

We need a preliminary definition.

Definition B.1. Let $\Omega \in \mathfrak{D}(M)$ with $M=\mathbb{R}^n$ or $M=\mathbb{S}^n$. Let $\mathbf{x}\in\overline{\Omega}$ and $\mathcal{U}$ be an open neighborhood of $\mathbf{x}$ in M. We say that *$\mathcal{U}$ is a map-neighborhood of $\mathbf{x}$ for Ω* if there exists a local smooth diffeomorphism $\mathrm{U}^{\mathbf{x}}$ which maps the neighborhood $\mathcal{U}$ onto a neighborhood $\mathcal{V}$ of 0 in $\mathbb{R}^n$ and such that

$$\mathrm{U}^{\mathbf{x}}(\mathcal{U}\cap\Omega) = \mathcal{V}\cap\Pi_{\mathbf{x}} \quad\text{and}\quad \mathrm{U}^{\mathbf{x}}(\mathcal{U}\cap\partial\Omega) = \mathcal{V}\cap\partial\Pi_{\mathbf{x}}\,, \tag{B.1}$$

where $\Pi_{\mathbf{x}}$ is the tangent cone to Ω at $\mathbf{x}$ (compare with (3.1)). ■

Lemma B.2. *Let $n\geq 1$ be the space dimension. M denotes $\mathbb{R}^n$ or $\mathbb{S}^n$. Let $\Omega\in\mathfrak{D}(M)$ and $K>1$. There exist a positive integer L and two positive constants $\rho_{\max}$ and $\kappa\leq 1$ (depending on Ω and K) such that for all $\rho\in(0,\rho_{\max}]$, there exists a (finite) set $\mathscr{X}\subset\overline{\Omega}\times[\kappa\rho,\rho]$ satisfying the following three properties*

1. *We have the inclusion $\overline{\Omega}\subset\cup_{(\mathbf{x},r)\in\mathscr{X}}\overline{\mathcal{B}}(\mathbf{x},r)$*
2. *For any $(\mathbf{x},r)\in\mathscr{X}$, the ball $\mathcal{B}(\mathbf{x},Kr)$ is a map-neighborhood of $\mathbf{x}$ for Ω*
3. *Each point $\mathbf{x}_0$ of $\overline{\Omega}$ belongs to at most L different balls $\mathcal{B}(\mathbf{x},Kr)$.*

Here are preparatory notations and lemmas.

Let $\Omega\in\mathfrak{D}(M)$ and $K>1$. If the assertions of Lemma B.2 are true for this Ω and this K, we say that Property $\mathscr{P}(\Omega,K)$ holds. We may also specify that the assertion by the sentence

Property $\mathscr{P}(\Omega,K)$ holds with parameters $(L,\rho_{\max},\kappa)$.

Let $\mathcal{U}^*\subset\subset\mathcal{U}$ be two nested open sets. We say that the property $\mathscr{P}(\Omega,K;\mathcal{U}^*,\mathcal{U})$ holds[11] if the assertions of Lemma B.2 are true for this Ω and this K, with discrete sets $\mathscr{X}\subset(\mathcal{U}^*\cap\overline{\Omega})\times[\kappa_\Omega\rho,\rho]$ and with (1)-(3) replaced by

1. We have the inclusion $\mathcal{U}^*\cap\overline{\Omega}\subset\cup_{(\mathbf{x},r)\in\mathscr{X}}\overline{\mathcal{B}}(\mathbf{x},r)$
2. For any $(\mathbf{x},r)\in\mathscr{X}$, the ball $\mathcal{B}(\mathbf{x},Kr)$ is included in $\mathcal{U}$ and is a map-neighborhood of $\mathbf{x}$ for Ω
3. Each point $\mathbf{x}_0$ of $\mathcal{U}\cap\overline{\Omega}$ belongs to at most L different balls $\mathcal{B}(\mathbf{x},Kr)$.

[11] This is the localized version of property $\mathscr{P}(\Omega,K)$.

Like above the specification is

$$\text{Property } \mathscr{P}(\Omega, K; \mathcal{U}^*, \mathcal{U}) \text{ holds with parameters } (L, \rho_{\max}, \kappa).$$

In the process of proof, we will construct coverings which are not exactly balls, but domains uniformly comparable to balls. Let us introduce the local version of this new assertion. For $0 < a \le a'$ we say that

$$\text{Property } \mathscr{P}[a, a'](\Omega, K; \mathcal{U}^*, \mathcal{U}) \text{ holds with parameters } (L, \rho_{\max}, \kappa)$$

if for all $\rho \in (0, \rho_{\max}]$, there exists a finite set $\mathscr{Z} \subset (\mathcal{U}^* \cap \overline{\Omega}) \times [\kappa_\Omega \rho, \rho]$ and open sets $\mathcal{D}(\mathbf{x}, r)$ satisfying the following four properties

(1) We have the inclusion $\mathcal{U}^* \cap \overline{\Omega} \subset \cup_{(\mathbf{x},r)\in\mathscr{Z}} \overline{\mathcal{D}}(\mathbf{x}, r)$
(2) For any $(\mathbf{x}, r) \in \mathscr{Z}$, the set[12] $\mathcal{D}(\mathbf{x}, Kr)$ is included in $\mathcal{U}$ and is a map-neighborhood of $\mathbf{x}$ for Ω
(3) Each point $\mathbf{x}_0$ of $\mathcal{U} \cap \overline{\Omega}$ belongs to at most L different sets $\mathcal{D}(\mathbf{x}, Kr)$
(4) For any $(\mathbf{x}, r) \in \mathscr{Z}$, we have the inclusions $\mathcal{B}(\mathbf{x}, ar) \subset \mathcal{D}(\mathbf{x}, r) \subset \mathcal{B}(\mathbf{x}, a'r)$.

Note that $\mathscr{P}[1,1](\Omega, K; \mathcal{U}^*, \mathcal{U}) = \mathscr{P}(\Omega, K; \mathcal{U}^*, \mathcal{U})$.

Lemma B.3. *If Property $\mathscr{P}[a, a'](\Omega, K; \mathcal{U}^*, \mathcal{U})$ holds with parameters $(L, \rho_{\max}, \kappa)$, then*

$$\textit{Property } \mathscr{P}(\Omega, \tfrac{a}{a'}K; \mathcal{U}^*, \mathcal{U}) \textit{ holds with parameters } (L, a'\rho_{\max}, \kappa).$$

Proof. Starting from the covering of $\mathcal{U}^* \cap \overline{\Omega}$ by the sets $\overline{\mathcal{D}}(\mathbf{x}, r)$ and using condition (4), we can consider the covering of $\mathcal{U}^* \cap \overline{\Omega}$ by the balls $\mathcal{B}(\mathbf{x}, a'r)$. Then $r' := a'r \in [\kappa a'\rho, a'\rho] = [\kappa\rho', \rho']$ with $\rho' < a'\rho_{\max}$.

Concerning conditions (2) and (3), it suffices to note the inclusions

$$\mathcal{B}(\mathbf{x}, \frac{a}{a'}Kr') \subset \mathcal{D}(\mathbf{x}, \frac{1}{a'}r'K) = \mathcal{D}(\mathbf{x}, rK)\,.$$

The lemma is proved. □

Proof. of Lemma B.2. The principle of the proof is a recursion on the dimension n.

Step 1. Explicit construction when $n = 1$.
The domain Ω and the localizing open sets $\mathcal{U}^*$ and $\mathcal{U}$ are then open intervals. Let us assume for example that $\mathcal{U}^* = (-\ell, \ell)$, $\mathcal{U} = (-\ell - \delta, \ell + \delta)$ and $\Omega = (0, \ell + \delta')$ with $\ell, \delta > 0$ and $\delta' > \delta$. Let $K \ge 1$. We can take

$$\rho_{\max} = \min\left\{\frac{\ell}{K}, \delta\right\}$$

and for any $\rho \le \rho_{\max}$ the following set of couples $(\mathbf{x}_j, r_j)$, $j = 0, 1, \dots, J$

$$\mathbf{x}_0 = 0,\ \ r_0 = \rho \quad \text{and} \quad \mathbf{x}_j = \rho + \frac{2j-1}{K}\rho,\ \ r_j = \frac{\rho}{K} \quad \text{for} \quad j = 1, \dots, J$$

[12] Here $\mathcal{D}(\mathbf{x}, Kr)$ is the set of $\mathbf{y}$ such that $\mathbf{x} + (\mathbf{y} - \mathbf{x})/K \in \mathcal{D}(\mathbf{x}, r)$.

with J such that $\mathbf{x}_J < \ell$ and $\rho + \frac{2J+1}{K}\rho \geq \ell$. If $\mathbf{x}_J < \ell - \frac{\rho}{K}$, we add the point $\mathbf{x}_{J+1} = \rho + \frac{2J}{K}\rho$. The covering condition (1) is obvious.

Concerning condition (2), we note that the bound $\rho_{\max} \leq \frac{\ell}{K}$ implies that $[0, Kr_0) = [0, K\rho)$ is a map-neighborhood for the boundary of Ω, and the bound $\rho_{\max} \leq \delta$ implies that when $j \geq 1$, the "balls" $(\mathbf{x}_j - Kr_j, \mathbf{x}_j + Kr_j) = (\mathbf{x}_j - \rho, \mathbf{x}_j + \rho)$ are map-neighborhoods for the interior of Ω.

Concerning condition (3), we can check that $L = K + 2$ is suitable.

Step 2. Localization.
Let $\Omega \in \mathfrak{D}(\mathbb{R}^n)$ or $\Omega \in \mathfrak{D}(\mathbb{S}^n)$. For any $\mathbf{x} \in \overline{\Omega}$, there exists a ball $\mathcal{B}(\mathbf{x}, r_x)$ with positive radius $r_\mathbf{x}$ that is a map-neighborhood for Ω. We extract a finite covering of $\overline{\Omega}$ by open sets $\mathcal{B}(\mathbf{x}^{(\ell)}, \frac{1}{2}r^{(\ell)})$. We set

$$\mathcal{U}_\ell^* = \mathcal{B}(\mathbf{x}^{(\ell)}, \frac{1}{2}r^{(\ell)}) \quad \text{and} \quad \mathcal{U}_\ell = \mathcal{B}(\mathbf{x}^{(\ell)}, r^{(\ell)}).$$

The map $\mathrm{U}^\ell := \mathrm{U}^{\mathbf{x}^{(\ell)}}$ transforms $\mathcal{U}_\ell^*$ and $\mathcal{U}_\ell$ into neighborhoods $\mathcal{V}_\ell^*$ and $\mathcal{V}_\ell$ of 0 in the tangent cone $\Pi_\ell := \Pi_{\mathbf{x}^{(\ell)}}$. Thus we are reduced to prove the local property $\mathscr{P}(\Pi_\ell, K; \mathcal{V}_\ell^*, \mathcal{V}_\ell)$ for any ℓ. Indeed

- The local diffeomorphism U^ℓ allows to deduce Property $\mathscr{P}(\Omega, K; \mathcal{U}_\ell^*, \mathcal{U}_\ell)$ from Property $\mathscr{P}(\Pi_\ell, K'; \mathcal{V}_\ell^*, \mathcal{V}_\ell)$ for a ratio K'/K that only depends on U^ℓ (this relies on Lemma B.3).
- Properties $\mathscr{P}(\Omega, K; \mathcal{U}_\ell^*, \mathcal{U}_\ell)$ imply Property $\mathscr{P}(\Omega, K; \cup_\ell \mathcal{U}_\ell^*, \cup_\ell \mathcal{U}_\ell) = \mathscr{P}(\Omega, K)$ (it suffices to merge the (finite) union of the sets $\mathscr{X}$ corresponding to each $\mathcal{U}_\ell$).

Step 3. Core recursive argument: If Ω_0 is the section of the cone Π, Property $\mathscr{P}(\Omega_0, K)$ implies Property $\mathscr{P}(\Pi, K'; \mathcal{B}(0, 1), \mathcal{B}(0, 2))$ for a suitable ratio K'/K. We are going to prove this separately in several lemmas (B.4 to B.6). Then the proof Lemma B.2 will be complete. □

Lemma B.4. *Let Γ be a cone in $\mathfrak{P}^{n-1}$. For $\ell = 1, 2$, let $\mathcal{B}_\ell$ and $\mathcal{I}_\ell$ be the ball $\mathcal{B}(0, \ell)$ of $\mathbb{R}^{n-1}$ and the interval $(-\ell, \ell)$, respectively. We assume that Property $\mathscr{P}(\Gamma, K; \mathcal{B}_1, \mathcal{B}_2)$ holds (with parameters $(L, \rho_{\max}, \kappa)$). Then Property $\mathscr{P}[1, \sqrt{2}](\Gamma \times \mathbb{R}, K; \mathcal{B}_1 \times \mathcal{I}_1, \mathcal{B}_2 \times \mathcal{I}_2)$ holds.*

Proof. Let us denote by $\mathbf{y}$ and z coordinates in Γ and $\mathbb{R}$, respectively. For $\rho \leq \rho_{\max}$, let $\mathscr{X}_\Gamma$ be an associate set of couples $(\mathbf{y}, r_\mathbf{y})$. For each $\mathbf{y}$ we consider the unique set of equidistant points $\mathscr{Z}_\mathbf{y} = \{z_j \in [-1, 1],\ j = 1, \ldots, J_\mathbf{y}\}$ such that

$$z_j - z_{j-1} = 2r_\mathbf{y} \quad \text{and} \quad z_1 + 1 = 1 - z_{J_\mathbf{y}} < r_\mathbf{y}\,.$$

Then we define

$$\mathscr{X}^{(\rho)} = \big\{(\mathbf{x}, r_\mathbf{x}), \quad \text{for } \ \mathbf{x} = (\mathbf{y}, z) \text{ with } (\mathbf{y}, r_\mathbf{y}) \in \mathscr{X}_\Gamma,\ z \in \mathscr{Z}_\mathbf{y} \text{ and } r_\mathbf{x} = r_\mathbf{y}\big\}. \tag{B.2}$$

The associate open set $\mathcal{D}(\mathbf{x}, r_\mathbf{x})$ is the product

$$\mathcal{D}(\mathbf{x}, r_\mathbf{x}) = \mathcal{B}(\mathbf{y}, r_\mathbf{y}) \times (z - r_\mathbf{y}, z + r_\mathbf{y})\,.$$

We have the inclusions $\mathcal{B}(\mathbf{x}, r_{\mathbf{x}}) \subset \mathcal{D}(\mathbf{x}, r_{\mathbf{x}}) \subset \mathcal{B}(\mathbf{x}, \sqrt{2}\, r_{\mathbf{x}})$ and it is easy to check that Property $\mathscr{P}[1, \sqrt{2}](\Gamma \times \mathbb{R}, K; \mathcal{B}(0,1) \times \mathcal{I}_1, \mathcal{B}(0,2) \times \mathcal{I}_2)$ holds with parameters $(L', \rho_{\max}, \kappa)$ with $L' = LK$. □

Lemma B.5. *Let Ω be a section in $\mathfrak{D}(\mathbb{S}^{n-1})$, let Π be the corresponding cone, and let $\mathcal{I}_\ell$ be the interval $(2^{-\ell}, 2^{\ell})$ for $\ell = 1, 2$. We define the annuli*

$$\mathcal{A}_\ell = \Big\{ \mathbf{x} \in \Pi, \quad |\mathbf{x}| \in \mathcal{I}_\ell \ \ \textit{and}\ \ \frac{\mathbf{x}}{|\mathbf{x}|} \in \Omega \Big\}.$$

We assume that Property $\mathscr{P}(\Omega, K)$ holds (with parameters $(L, \rho_{\max}, \kappa)$). Then Property $\mathscr{P}[a, a'](\Pi, K; \mathcal{A}_1, \mathcal{A}_2)$ holds for suitable constants a and a' (independent of Ω and K).

Proof. Let us consider the diffeomorphism

$$\text{(B.3)} \qquad \begin{array}{rccc} \mathrm{T}: & \Omega \times (-2,2) & \longrightarrow & \mathcal{A}_2 \\ & \mathbf{x} = (\mathbf{y}, z) & \longmapsto & \breve{\mathbf{x}} = 2^z \mathbf{y} \end{array}$$

in view of proving Property $\mathscr{P}[a, a'](\Pi, K; \mathcal{A}_1, \mathcal{A}_2)$, for a given $\rho \le \rho_{\max}$, we define a suitable set $\breve{\mathscr{Z}}^{(\rho)}$ using the set $\mathscr{Z}^{(\rho)}$ introduced in (B.2)

$$\text{(B.4)} \qquad \breve{\mathscr{Z}}^{(\rho)} = \big\{ (\breve{\mathbf{x}}, r_{\mathbf{x}}), \quad \text{for}\ \ \breve{\mathbf{x}} = \mathrm{T}\mathbf{x}\ \ \text{with}\ \ (\mathbf{x}, r_{\mathbf{x}}) \in \mathscr{Z}^{(\rho)} \big\},$$

and the associated open sets

$$\breve{\mathcal{D}}(\breve{\mathbf{x}}, r_{\mathbf{x}}) = \mathrm{T}\big(\mathcal{D}(\mathbf{x}, r_{\mathbf{x}})\big).$$

We can check that

$$\mathcal{B}(\breve{\mathbf{x}}, a r_{\mathbf{x}}) \subset \breve{\mathcal{D}}(\breve{\mathbf{x}}, r_{\mathbf{x}}) \subset \mathcal{B}(\breve{\mathbf{x}}, a' r_{\mathbf{x}})$$

with $a = \frac{1}{8}\log 2$ and $a' = 8\sqrt{2}\log 2$ and that Property $\mathscr{P}[a, a'](\Pi, K; \mathcal{A}_1, \mathcal{A}_2)$ holds with parameters $(L', \rho_{\max}, \kappa)$ for $L' = NLK$ with an integer N independent of L and K. □

Lemma B.6. *Let Ω be a section in $\mathfrak{D}(\mathbb{S}^{n-1})$, let Π be the corresponding cone, and let $\mathcal{B}_\ell$ be the balls $\mathcal{B}(0, \ell)$ of $\mathbb{R}^n$ for $\ell = 1, 2$. We assume that Property $\mathscr{P}(\Omega, K)$ holds with parameters $(L, \rho_{\max}, \kappa)$ for a $\rho_{\max} \le 1$. Then Property $\mathscr{P}[a, a'](\Pi, K; \mathcal{B}_1, \mathcal{B}_2)$ holds for suitable constants a and a' (independent of Ω and K) and with parameters $(L', 1, \kappa\rho_{\max})$.*

Proof. Let $\rho \le 1$ and let M be the natural number such that

$$2^{-M-1} < \rho \le 2^{-M}.$$

On the model of (B.3)-(B.4), we set

$$\breve{\mathscr{Z}}^{m} = \big\{ (2^{-m}\mathrm{T}\mathbf{x}, 2^{-m} r_{\mathbf{x}}), \ \ \text{with}\ \ (\mathbf{x}, r_{\mathbf{x}}) \in \mathscr{Z}^{(2^m \rho_{\max} \rho)} \big\}, \qquad m = 0, \dots, M,$$

and the associated open sets are

$$\text{(B.5)} \qquad 2^{-m}\mathrm{T}\big(\mathcal{D}(\mathbf{x}, r_{\mathbf{x}})\big) \ \ \text{with}\ \ (\mathbf{x}, r_{\mathbf{x}}) \in \mathscr{Z}^{(2^m \rho_{\max} \rho)}.$$

The set $\breve{\mathscr{Z}}$ associated with the cone Π in the ball $\mathcal{B}_1$ is

$$\{(0, \rho)\} \cup \bigcup_{m=0}^{M} \breve{\mathscr{Z}}^{m}$$

and the associated open sets are the reunion of the sets (B.5) for $m = 0, \dots, M$ and of the ball $\mathcal{B}(0, \rho)$. As the radii $r_{\mathbf{x}}$ belong to $[\kappa 2^m \rho_{\max}\rho, 2^m\rho_{\max}\rho]$, we have $2^{-m} r_{\mathbf{x}} \in [\kappa\rho_{\max}\rho, \rho_{\max}\rho]$. Since ρ itself belongs to the full collection of radii r, we finally find $r \in [\kappa\rho_{\max}\rho, \rho]$. The finite covering holds with $L' = 3NLK + 1$ for the same integer N appearing at the end of the proof of Lemma B.5. □

Lemma B.7. *Let $\Omega \in \mathfrak{D}(\mathbb{R}^n)$. Let $(L, \rho_{\max}, \kappa)$ be the parameters provided by Lemma B.2, for the Property $\mathscr{P}(\Omega, 2)$ to hold. For any $\rho \in (0, \rho_{\max}]$ let $\mathscr{X} \subset \overline{\Omega} \times [\kappa\rho, \rho]$ be an associate set of pairs (center, radius). Then there exists a collection of smooth functions $(\chi_{(\mathbf{x},r)})_{(\mathbf{x},r)\in\mathscr{X}}$ with $\chi_{(\mathbf{x},r)} \in \mathscr{C}_0^\infty(\mathcal{B}(\mathbf{x}, 2r))$ satisfying the identity (partition of unity)*

$$\sum_{(\mathbf{x},r)\in\mathscr{X}} \chi^2_{(\mathbf{x},r)} = 1 \quad \text{on} \quad \overline{\Omega}$$

and the uniform estimate of gradients

$$\exists C > 0, \quad \forall (\mathbf{x}, r) \in \mathscr{X}, \quad \|\nabla\chi_{(\mathbf{x},r)}\|_{L^\infty(\Omega)} \le C\rho^{-1}\,,$$

where C only depends on Ω. By construction any ball $\mathcal{B}(\mathbf{x}, 2r)$ is a map-neighborhood of $\mathbf{x}$ for Ω.

Proof. Let $\xi_{(\mathbf{x},r)} \in \mathscr{C}_0^\infty(\mathcal{B}(\mathbf{x}, 2r))$, with the property that $\xi_{(\mathbf{x},r)} \equiv 1$ in $\mathcal{B}(\mathbf{x}, r)$, and satisfying

$$\|\nabla\xi_{(\mathbf{x},r)}\|_{L^\infty(\mathbb{R}^3)} \le Cr^{-1}$$

where C is a universal constant. Then we set for each $(\mathbf{x}_0, r_0) \in \mathscr{X}$

$$\chi_{(\mathbf{x}_0,r_0)} = \frac{\xi_{(\mathbf{x}_0,r_0)}}{(\sum_{(\mathbf{x},r)\in\mathscr{X}} \xi^2_{(\mathbf{x},r)})^{1/2}}\,.$$

Due to property (1) in Lemma B.2, $\sum_{(\mathbf{x},r)\in\mathscr{X}} \xi^2_{(\mathbf{x},r)} \ge 1$ and due to property (3),

$$\|\sum_{(\mathbf{x},r)\in\mathscr{X}} \nabla\xi^2_{(\mathbf{x},r)}\|_{L^\infty(\mathbb{R}^3)} \le CL_\Omega\,.$$

We deduce the lemma. □

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