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An elementary proof of Fermat-Catalan conjecture.

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**An basic evidence of both Catalan-Mihailescu and
Fermat-Wiles theorems and generalization to
Fermat-Catalan and Beal conjectures
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Abstract

(MSC=11D04) We begin with an equation, for example : $Y^p = X^q \pm Z^c$ and solve it.

(Keywords : Diophantine equations, Fermat-Catalan equation; Approach)

Introduction

The goal of this document is clearly to solve the Fermat-Catalan equation $Y^b = X^q \pm Z^c$. We have some solutions, they are :

$$\begin{aligned} 1^m + 2^3 &= 3^2 \\ 2^5 + 7^2 &= 3^4 \\ 13^2 + 7^3 &= 2^9 \\ 2^7 + 17^3 &= 71^2 \\ 3^5 + 11^4 &= 122^2 \\ 33^8 + 1549034^2 &= 15613^3 \\ 1414^3 + 2213459^2 &= 65^7 \\ 9262^3 + 15312283^2 &= 113^7 \\ 17^7 + 76271^3 &= 21063928^2 \\ 43^8 + 96222^3 &= 30042907^2 \end{aligned}$$

If we study minutiously those solutions, it appears a common point, there is an exponent 2 in the formulas. It is not only the case of the Fermat-Catalan equation.

Effectively, this exponent 2 appears at least in two other diophantine equations : the Fermat equation, of course, but not only, it appears also in the Catalan equation and in some Pillai equations of the form $Y^p = X^q + a$.

Our goal, here, is to show and to prove formally, with the tools of the logic and algebra, how this exponent 2 appears in the equations !

Resolution of Fermat-Catalan equation

Let Fermat-Catalan equation :

$$Y^p = X^q + aZ^c$$

$a = \pm 1$

Now, let

$$w = \frac{\log(-aZ^c + \sqrt{Z^{2c} + 4Y^p X^q}) - \log(2Y^{p-2})}{\log(X)}$$

w exists and

$$\begin{aligned} w \log(X) &= \log(X^w) = \log(-aZ^c + \sqrt{Z^{2c} + 4Y^p X^q}) - \log(2Y^{p-2}) \\ &= \log\left(\frac{-aZ^c + \sqrt{Z^{2c} + 4Y^p X^q}}{2Y^{p-2}}\right) \end{aligned}$$

Thus

$$X^w = \frac{-aZ^c + \sqrt{Z^{2c} + 4Y^p X^q}}{2Y^{p-2}}$$

We deduce

$$2X^w Y^{p-2} + aZ^c = \sqrt{Z^{2c} + 4Y^p X^q}$$

Or

$$\begin{aligned} (2X^w Y^{p-2} + aZ^c)^2 &= Z^{2c} + 4Y^p X^q \\ &= 4X^{2w} Y^{2p-4} + Z^{2c} + 4aZ^c X^w Y^{p-2} \end{aligned}$$

And

$$X^{2w} Y^{2p-4} + aZ^c X^w Y^{p-2} = Y^p X^q$$

Or

$$X^w Y^{p-2} + aZ^c = Y^2 X^{q-w}$$

Hence

$$Y^2 X^{q-w} - X^w Y^{p-2} = aZ^c = Y^p - X^q = Y^2 Y^{p-2} - X^w X^{q-w}$$

Thus

$$\begin{aligned} Y^2 (X^{q-w} - Y^{p-2}) + X^w (X^{q-w} - Y^{p-2}) &= 0 \\ &= (Y^2 + X^w)(X^{q-w} - Y^{p-2}) = 0 \end{aligned}$$

And

$$Y^{p-2} = X^{q-w}$$

Now, let

$$w' = \frac{\log(Y^q - X^p + \sqrt{(Y^q - X^p)^2 + 4Y^q X^p}) - \log(2X^{p-2})}{\log(Y)}$$

w' exists and

$$\begin{aligned} w' \log(Y) &= \log(Y^{w'}) = \log(Y^q - X^p + \sqrt{(Y^q - X^p)^2 + 4Y^q X^p}) - \log(2X^{p-2}) \\ &= \log\left(\frac{(Y^q - X^p + \sqrt{(Y^q - X^p)^2 + 4Y^q X^p})}{2X^{p-2}}\right) \end{aligned}$$

Thus

$$Y^{w'} = \frac{Y^q - X^p + \sqrt{(Y^q - X^p)^2 + 4Y^q X^p}}{2X^{p-2}}$$

We deduce

$$2Y^{w'} Y^{p-2} - (Y^q - X^p) = \sqrt{(Y^q - X^p)^2 + 4Y^q X^p}$$

Or

$$\begin{aligned} (2Y^{w'} X^{p-2} - (Y^q - X^p))^2 &= (Y^q - X^p)^2 + 4Y^q X^p \\ &= 4Y^{2w'} X^{2p-4} + (Y^q - X^p)^2 - 4(Y^q - X^p)Y^{w'} X^{p-2} \end{aligned}$$

And

$$Y^{2w'} X^{2p-4} - (Y^q - X^p)Y^{w'} X^{p-2} = Y^q X^p$$

Or

$$Y^{w'} X^{p-2} - (Y^q - X^p) = X^2 Y^{q-w'}$$

Hence

$$X^2 Y^{q-w'} - Y^{w'} X^{p-2} = -Y^q + X^p = -Y^{w'} Y^{q-w'} + X^2 X^{p-2}$$

Thus

$$\begin{aligned} Y^{w'} (X^{p-2} - Y^{q-w'}) + X^2 (X^{p-2} - Y^{q-w'}) &= 0 \\ &= (Y^{w'} + X^2)(X^{p-2} - Y^{q-w'}) = 0 \end{aligned}$$

And

$$X^{p-2} = Y^{q-w'}$$

But

$$X^{(p-2)^2} = Y^{(p-2)(q-w')} = X^{(q-w)(q-w')}$$

Thus

$$(p-2)^2 = (q-w)(q-w')$$

And

$$X^{q-w} = Y^{p-2} = Y^{\sqrt{(q-w)(q-w')}}$$

Thus

$$X^{\sqrt{q-w}} = Y^{\sqrt{q-w'}}$$

But let

$$w_1 = q - \frac{(q-w)^2}{p-2}; \quad w_2 = q - \frac{(q-w')^2}{p-2}$$

Or

$$(q-w)^2 = (p-2)(q-w_1); \quad (q-w')^2 = (p-2)(q-w_2)$$

We have

$$(q-w)^2(q-w')^2 = (p-2)^4 = (p-2)^2(q-w_1)(q-w_2)$$

Thus

$$(q-w)(q-w') = (q-w_1)(q-w_2)$$

And if we suppose $(p-2)(q-w)(q-w') \neq 0$

$$(q-w)^2 = (p-2)(q-w_1) = \sqrt{(q-w)(q-w')}(q-w_1)$$

or

$$q-w_1 = (q-w)\sqrt{\frac{q-w}{q-w'}}$$

and

$$(q-w')^2 = (p-2)(q-w_2) = \sqrt{(q-w)(q-w')}(q-w_2)$$

$$q-w_2 = (q-w')\sqrt{\frac{q-w'}{q-w}}$$

But

$$\begin{aligned} Y^{q-w_2} &= Y^{(q-w')\sqrt{\frac{q-w'}{q-w}}} = X^{(p-2)\sqrt{\frac{q-w'}{q-w}}} \\ &= X^{\sqrt{(q-w)(q-w')}\frac{q-w'}{q-w}} = X^{q-w'} = X^{\sqrt{(p-2)(q-w_2)}} \end{aligned}$$

Hence

$$Y^{q-w_2} = X^{q-w'}$$

And

$$\begin{aligned} X^{q-w_1} &= X^{(q-w)\sqrt{\frac{q-w}{q-w'}}} = Y^{(p-2)\sqrt{\frac{q-w}{q-w'}}} \\ &= Y^{\sqrt{(q-w)(q-w')}\frac{q-w}{q-w'}} = Y^{q-w} = Y^{\sqrt{(p-2)(q-w_1)}} \end{aligned}$$

Hence

$$X^{q-w_1} = Y^{q-w}; \quad Y^{q-w_2} = X^{q-w'}$$

Then

$$X^{\sqrt[4]{q-w_1}} = Y^{\sqrt[4]{q-w_2}}$$

And

$$X^{\sqrt{p-2}} = Y^{\sqrt{q-w_2}}; \quad Y^{\sqrt{p-2}} = X^{\sqrt{q-w_1}}$$

But

$$\begin{aligned} \frac{w_2 - w'}{w - w_1} &= \frac{q - w' - (q - w_2)}{q - w_1 - (q - w)} = \frac{q - w' - (q - w')\sqrt{\frac{q-w'}{q-w}}}{(q-w)\sqrt{\frac{q-w}{q-w'}} - (q-w)} \\ &= \frac{(q-w')\sqrt{q-w'}}{(q-w)\sqrt{q-w}} \end{aligned}$$

And

$$\begin{aligned} \frac{\sqrt{q-w}(w_2 - w)}{\sqrt{q-w'}(w' - w_1)} &= \frac{\sqrt{q-w}(q-w) - \sqrt{q-w}(q-w_2)}{\sqrt{q-w'}(q-w_1) - \sqrt{q-w'}(q-w')} \\ &= \frac{\sqrt{q-w}(q-w) - \sqrt{q-w'}(q-w')}{(q-w)\sqrt{q-w} - \sqrt{q-w'}(q-w')} = 1 \end{aligned}$$

Hence

$$\begin{aligned} \frac{w_2 - w'}{w - w_1} &= \left(\frac{w_2 - w}{w' - w_1}\right)^3 \\ X^{q-w'} &= Y^{q-w_2} \end{aligned}$$

But

$$\begin{aligned} X \sqrt[3]{w_2-w'} Y \sqrt[3]{w_1-w} &= X \sqrt{\frac{q-w}{q-w'}} \sqrt[3]{w_1-w} Y^{-\sqrt[3]{w-w_1}} \\ &= Y \sqrt[3]{w-w_1} Y^{-\sqrt[3]{w-w_1}} = 1 \end{aligned}$$

But let

$$q-w = u(q-w'); \quad q-w_1 = v(q-w_2)$$

We have

$$\frac{q-w_1}{q-w_2} = v = \left(\frac{q-w}{q-w'}\right)^2 = u^2$$

Thus $v = u^2$. And

$$(q-w_1)(q-w_2) = u^2(q-w_2)^2 = (p-2)^2 = (q-w)(q-w') = u(q-w')^2$$

Thus

$$(q-w')^2 = u(q-w_2)^2$$

And

$$(q-w)^2 = u^2(q-w')^2 = u^3(q-w_2)^2$$

And

$$\frac{(q-w)^2}{(q-w_1)^2} = \frac{u^2(q-w_2)^2}{u^4(q-w_2)^2} = \frac{1}{u}$$

But

$$\begin{aligned} (q-w) \sqrt{\frac{q-w}{q-w'}} (q-w')^2 &= (q-w') \sqrt{\frac{q-w'}{q-w}} (q-w)^2 \\ &= (q-w_1)(q-w')^2 = (q-w_2)(q-w)^2 \end{aligned}$$

And

$$\begin{aligned} (p-2)^2 (q-w')^2 &= (p-2)^3 (q-w_2) = (q-w_1)(q-w_2)(q-w')^2 = (q-w_2)^2 (q-w)^2 \\ &= \frac{1}{u} (q-w_1)^2 (q-w_2)^2 = \frac{1}{u} (p-2)^4 \end{aligned}$$

And

$$\begin{aligned} (p-2)^2 (q-w)^2 &= (p-2)^3 (q-w_1) = (q-w_1)(q-w_2)(q-w)^2 = (q-w_1)^2 (q-w')^2 \\ &= u(q-w_1)^2 (q-w_2)^2 = u(p-2)^4 \end{aligned}$$

We deduce

$$p-2 = u(q-w_2) = \frac{1}{u}(q-w_1) = \sqrt{u}(q-w') = \sqrt{\frac{1}{u}}(q-w)$$

But

$$\begin{aligned} Y \sqrt{q-w'} &= X \sqrt{q-w} = X \sqrt{u(q-w')} \\ Y &= X \sqrt{u} \end{aligned}$$

And

$$X \sqrt[3]{\frac{w_2-w'}{w-w_1}} = Y = X \sqrt{u}$$

Hence

$$\begin{aligned} w_2-w' &= \sqrt{u^3}(w-w_1) = (q-w') - (q-w_2) = \frac{1}{u}(q-w) - \frac{1}{u^2}(q-w_1) = \left(\frac{1}{u} - \frac{\sqrt{u}}{u^2}\right)(q-w) \\ &= \sqrt{u^3}(q-w_1 - (q-w)) = \sqrt{u^3}(\sqrt{u}-1)(q-w) \end{aligned}$$

Hence

$$(u - \sqrt{u}) = \sqrt{u}(\sqrt{u}-1) = u^2 \sqrt{u^3}(\sqrt{u}-1)$$

Or

$$(\sqrt{u}-1)\sqrt{u}(u^3-1) = 0$$

It means that $u = 1$. But as $GCD(X, Y) = 1$ and

$$Y \sqrt{q-w'} = Y \sqrt{q-w} = X \sqrt{q-w}$$

Then $q - w = q - w' = 0$ or $p = 2$ and $q = w = w' = w_1 = w_2$.

This calculus is available if we replace $p - 2$ by $p - 3$ and it leads as this last case does not exclude the case $p - 2$ to $p = 2$ or $p = 3$. The two cases lead then to $p = 2$ and ($p = 2$ or $p = 3$) which means $p = 2$. The same calculation is available to $p = 4$ and we have $p = 2$ and ($p = 2$ or $p = 3$ or $p = 4$) and it means $p = 2$. Etc... Until infinity. The only solution is $p = 2$.

Resolution of Catalan equation

Let Catalan equation :

$$Y^p = X^q + 1$$

Let

$$w = \frac{\log(-1 + \sqrt{1 + 4Y^p X^q}) - \log(2) - (p-2)\log(Y)}{\log(X)}$$

w exists as we see. But

$$\begin{aligned} w \log(X) &= \log(X^w) = \log(-1 + \sqrt{1 + 4Y^p X^q}) - \log(2) - \log(Y^{p-2}) \\ &= \log\left(\frac{-1 + \sqrt{1 + 4Y^p X^q}}{2Y^{p-2}}\right) \end{aligned}$$

Thus

$$2X^w Y^{p-2} + 1 = \sqrt{1 + 4Y^p X^q}$$

Or

$$\begin{aligned} (2X^w Y^{p-2} + 1)^2 &= 1 + 4Y^p X^q \\ &= 1 + 4X^{2w} Y^{2p-4} + 4X^w Y^{p-2} \end{aligned}$$

We deduce

$$Y^p X^q - X^{2w} Y^{2p-4} = X^w Y^{p-2}$$

Hence

$$Y^2 X^{q-w} - X^w Y^{p-2} = 1 = Y^p - X^q = Y^{p-2} Y^2 - X^w X^{q-w}$$

Or

$$\begin{aligned} Y^2(X^{q-w} - Y^{p-2}) + X^w(X^{q-w} - Y^{p-2}) &= 0 \\ &= (Y^2 + X^w)(X^{q-w} - Y^{p-2}) = 0 \end{aligned}$$

And as $GCD(X, Y) = 1$ it leads to $p - 2 = q - w = 0$. And

$$w = q = \frac{\log(-1 + \sqrt{1 + 4Y^2 X^q}) - \log(2)}{\log(X)} \in \mathbb{N}$$

This equation leads to $(X, q) = (2, 3)$. Ko Chao has already solved the case $p = 2$.

Resolution of Fermat equation

Let Fermat equation :

$$Y^n = X^n + Z^n$$

Let here too

$$w = \frac{\log(-Z^n + \sqrt{Z^{2n} + 4Y^n X^n}) - \log(2) - (n-2)\log(Y)}{\log(X)}$$

w exists as we see. But

$$\begin{aligned} w \log(X) &= \log(X^w) = \log(-Z^n + \sqrt{Z^{2n} + 4Y^n X^n}) - \log(2) - \log(Y^{n-2}) \\ &= \log\left(\frac{-Z^n + \sqrt{Z^{2n} + 4Y^n X^n}}{2Y^{n-2}}\right) \end{aligned}$$

Thus

$$2X^w Y^{n-2} + Z^n = \sqrt{Z^{2n} + 4Y^n X^n}$$

Or

$$\begin{aligned} (2X^w Y^{n-2} + Z^n)^2 &= Z^{2n} + 4Y^n X^n \\ &= Z^{2n} + 4X^{2w} Y^{2n-4} + 4X^w Y^{n-2} Z^n \end{aligned}$$

We deduce

$$Y^n X^n - X^{2w} Y^{2n-4} = X^w Y^{n-2} Z^n$$

Hence

$$Y^2 X^{n-w} - X^w Y^{n-2} = Z^n = Y^n - X^n = Y^{n-2} Y^2 - X^w X^{n-w}$$

Or

$$\begin{aligned} Y^2(X^{n-w} - Y^{n-2}) + X^w(X^{n-w} - Y^{n-2}) &= 0 \\ &= (Y^2 + X^w)(X^{n-w} - Y^{n-2}) = 0 \end{aligned}$$

And as $GCD(X, Y) = 1$ it leads to $n - 2 = n - w = 0$. And

$$2 = n = \frac{\log(-1 + \sqrt{1 + 4Y^2 X^2}) - \log(2)}{\log(X)} \in \mathbb{N}$$

This equation leads to the solutions of Fermat equation for $n > 1$ as we will see.

Resolution of Fermat- Catalan equation

The only solution, in all cases, is $p = 2$.

And $Y^2 = X^q + aZ^c$. Thus, Fermat-Catalan equation is available for

$$\begin{aligned} q &= \frac{\log(-aZ^c + \sqrt{Z^{2c} + 4Y^2 X^q}) - \log(2)}{\log(X)} \in \mathbb{N} \\ &= \frac{\log(Y^q - X^2 + \sqrt{(Y^q - X^2)^2 + 4Y^q X^2}) - \log(2)}{\log(Y)} \in \mathbb{N} \end{aligned}$$

If we try successively $q = 3$ and $q = 4$, tc..., we will find the X, Y which satisfy the equations.

Example

$$q = \frac{Z^c = 1^c}{\log(-1 + \sqrt{1 + 4Y^2 X^q}) - \log(2)} \in \mathbb{N}$$

Implies $X = 2, Y = 3$ and $q = 3$.

$$\begin{aligned} 1^c + 2^3 &= 3^2 \\ aZ^c &= -2^5 = -32 \\ q &= \frac{\log(2^5 + \sqrt{1024 + 4Y^2 X^q}) - 0.69}{\log(X)} \in \mathbb{N} \end{aligned}$$

Implies $X = 3, q = 4$ and $Y = 7$.

$$\begin{aligned} 2^5 + 7^2 &= 3^4 \\ aZ^c &= -7^3 = -343 \\ q &= \frac{\log(343 + \sqrt{117649 + 4Y^2 X^q}) - 0.69}{\log(X)} \in \mathbb{N} \end{aligned}$$

Implies $X = 2, q = 9$ and $Y = 13$.

$$\begin{aligned} 13^2 + 7^3 &= 2^9 \\ aZ^c &= 17^3 = 4913 \\ q &= \frac{\log(-4913 + \sqrt{24137569 + 4Y^2 X^q}) - 0.69}{\log(X)} \in \mathbb{N} \end{aligned}$$

Implies $X = 2, q = 7$ and $Y = 71$.

$$\begin{aligned} 2^7 + 17^3 &= 71^2 \\ aZ^c &= 11^4 = 14641 \\ q &= \frac{\log(-14641 + \sqrt{14614^2 + 4Y^2 X^q}) - 0.69}{\log(X)} \in \mathbb{N} \end{aligned}$$

Implies $X = 3$, $q = 5$ and $Y = 122$.

$$3^5 + 11^4 = 122^2$$

$$aZ^c = -33^8$$

$$q = \frac{\log(33^8 + \sqrt{33^{16} + 4Y^2X^q}) - 0.69}{\log(X)} \in \mathbb{N}$$

Implies $X = 15613$, $q = 3$ and $Y = 1549034$.

$$33^8 + 1549034^2 = 15613^3$$

$$aZ^c = -1414^3$$

$$q = \frac{\log(1414^3 + \sqrt{1414^6 + 4Y^2X^q}) - 0.69}{\log(X)} \in \mathbb{N}$$

Implies $X = 65$, $q = 7$ and $Y = 2216459$.

$$1414^3 + 2216459^2 = 65^7$$

$$aZ^c = -9262^3$$

$$q = \frac{\log(9262^3 + \sqrt{9262^6 + 4Y^2X^q}) - 0.69}{\log(X)} \in \mathbb{N}$$

Implies $X = 113$, $q = 7$ and $Y = 15312283$.

$$9262^3 + 15312283^2 = 113^7$$

$$aZ^c = 17^7$$

$$q = \frac{\log(-17^7 + \sqrt{17^{14} + 4Y^2X^q}) - 0.69}{\log(X)} \in \mathbb{N}$$

Implies $X = 76271$, $q = 3$ and $Y = 21063928$.

$$17^7 + 76271^3 = 21063928^2$$

$$aZ^c = 43^8$$

$$q = \frac{\log(-43^8 + \sqrt{43^{16} + 4Y^2X^q}) - 0.69}{\log(X)} \in \mathbb{N}$$

Implies $X = 96222$, $q = 3$ and $Y = 30042907$.

$$43^8 + 96222^3 = 30042907^2$$

For Fermat equation, we have $q = n = 2 = w$ and

$$2 = \frac{\log(-aZ^2 + \sqrt{Z^4 + 4Y^2X^2}) - \log(2)}{\log(X)} \in \mathbb{N}$$

Example :

$$aZ^2 = 11^4$$

$$2 = \frac{\log(-11^4 + \sqrt{11^8 + 4Y^2X^2}) - 0.69}{\log(X)}$$

$$\log(2X^2) = \log(-11^4 + \sqrt{11^8 + 4Y^2X^2})$$

$$(2X^2 + 11^4)^2 = 11^8 + 4Y^2X^2 = 11^8 + 4X^4 + 4(11^4)X^2$$

$$Y^2 = X^2 + (11^4)$$

$$(Y - X)(Y + X) = 11^4$$

$$Y + X = 11^3 = 1331$$

$$Y - X = 11$$

$$2Y = 1342$$

$$2X = 1320$$

$$Y = 671$$

$$X = 660$$

Or

$$\begin{aligned} 671^2 &= 660^2 + 11^4 \\ &= (11(61))^2 = (11(60))^2 + 11^4 \\ 61^2 + 60^2 &= 11^2 \end{aligned}$$

Or

$$\begin{aligned} aZ^2 &= 13^{12} \\ Y^2 &= X^2 + 13^{12} \\ (Y - X)(Y + X) &= 13^{12} \\ Y + X &= 13^9 \\ Y - X &= 13^3 \\ 2Y &= 13^9 + 13^3 = 13^3(13^6 + 1) \\ 2X &= 13^9 - 13^3 = 13^3(13^6 - 1) \\ 4Y^2 &= (13^6)(13^6 + 1)^2 = (13^6)(13^6 - 1)^2 + 4(13^{12}) \\ (13^6 + 1)^2 &= (13^6 - 1)^2 + (2(13^3))^2 \end{aligned}$$

Etc...

Conclusion

Fermat-Catalan equation $Y^p = X^q \pm Z^c$ has solutions only for $p = 2$. We have shown a way to solve it.

References

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