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# An elementary proof of Catalan-Mihailescu and Fermat-Wiles theorems and generalization to Beal conjecture

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Abstract

A proof of both Catalan and Fermat theorems is presented and a generalization to Beal conjecture is proposed. For this, we begin with Fermat, Catalan and Fermat-Catalan equations and solve them.

Keywords : Diophantine ; Fermat ; Catalan ; Fermat-Catalan ; Resolution.

## Resolution of Catalan equation

Catalan equation is  $y^p = x^q + 1$

Let  $x^{q-3}y^2 - y^{p-2}x^3 = A$

If

$$A = 0 \Rightarrow x^{q-6} = y^{p-4}$$

$$GCD(x, y) = 1 \Rightarrow p = 4$$

impossible, there is no solution, Ko Chao proved it and

If  $A^2 = 1; p \geq 3 \Rightarrow x^{q-3}y - y^{p-3}x^3 = \frac{1}{y} \in Z$  impossible thus  $p=2$

We have

$$x^{q-3}y^2 - y^{p-2}x^3 = A = Ay^{p-2}y^2 - Ax^{q-3}x^3$$

$$\Rightarrow (x^{q-3} - Ay^{p-2})y^2 = (y^{p-2} - Ax^{q-3})x^3$$

$$(y^2 + Ax^3)x^{q-3} = (x^3 + Ay^2)y^{p-2}$$

But

$$GCD(x, y) = 1$$

We have then four cases :

$$x^3 = u(-x^{q-3} + Ay^{p-2}); y^2 = u(-y^{p-2} + Ax^{q-3})$$

$$x^{q-3} = v(x^3 + Ay^2); y^{p-2} = v(y^2 + Ax^3)$$

or

$$ux^3 = -x^{q-3} + Ay^{p-2}; uy^2 = -y^{p-2} + Ax^{q-3}$$

$$vx^{q-3} = x^3 + Ay^2; vy^{p-2} = y^2 + Ax^3$$

or

$$x^3 = u(-x^{q-3} + Ay^{p-2}); y^2 = u(-y^{p-2} + Ax^{q-3})$$

$$vx^{q-3} = x^3 + Ay^2; vy^{p-2} = y^2 + Ax^3$$

or

$$ux^3 = -x^{q-3} + Ay^{p-2}; uy^2 = -y^{p-2} + Ax^{q-3}$$

$$x^{q-3} = v(x^3 + Ay^2); y^{p-2} = v(y^2 + Ax^3)$$

$$u, v \in \mathbb{Z}$$

First case

$$x^3 = u(-x^{q-3} + Ay^{p-2}); y^2 = u(-y^{p-2} + Ax^{q-3})$$

$$x^{q-3} = v(x^3 + Ay^2); y^{p-2} = v(y^2 + Ax^3)$$

We have

$$y^p = uv(-y^p + A^2x^q - A(x^3y^{p-2} - y^2x^{q-3})) = uv(-y^p + A^2x^q + A^2) = uv(A^2 - 1)y^p$$

$$uv(A^2 - 1) = 1$$

Impossible because A,u,v are integers

Second case

$$ux^3 = -x^{q-3} + Ay^{p-2}; uy^2 = -y^{p-2} + Ax^{q-3}$$

$$vx^{q-3} = x^2 + Ay^2; vy^{p-2} = y^2 + Ax^3$$

We have

$$uvy^p = -y^p + A^2x^q - A(x^3y^{p-2} - y^2x^{q-3}) = -y^p + A^2x^q + A^2 = (A^2 - 1)y^p$$

$$uv = A^2 - 1$$

And

$$uv(y^2x^{q-3} - x^3y^{p-2}) = uvA = v(-y^{2p-4} + x^{2q-6})A = u(y^4 - x^6)A$$

$$\Rightarrow v = y^4 - x^6; u = -y^{2p-4} + x^{2q-6}$$

$$uv = A^2 - 1 = (y^4 - x^6)(-y^{2p-4} + x^{2q-6})$$

$$\Rightarrow (y^2x^{q-3} - x^3y^{p-2})^2 - 1 = (y^4 - x^6)(-y^{2p-4} + x^{2q-6})$$

$$\Rightarrow x^{2q} + y^{2p} - 2x^qy^p = 1 = (x^q - y^p)^2 = (2y^p - 1)^2 = 4y^{2p} - 4y^p + 1$$

Impossible :  $\Rightarrow p = 2$

Third case

$$x^3 = u(-x^{q-3} + Ay^{p-2}); y^2 = u(-y^{p-2} + Ax^{q-3})$$
$$vx^{q-3} = x^3 + Ay^2; vy^{p-2} = y^2 + Ax^3$$

We have

$$vy^p = u(-y^p + A^2x^q - A(x^3y^{p-2} - y^2x^{q-3})) = u(-y^p + A^2x^q + A^2) = u(A^2 - 1)y^p$$
$$v = u(A^2 - 1)$$

And

$$v(y^2x^{q-3} - x^3y^{p-2}) = vA = uv(-y^{2p-4} + x^{2q-6})A = (y^4 - x^6)A$$
$$\Rightarrow u(-y^{2p-4} + x^{2q-6}) = 1 \Rightarrow p - 2 = q - 3 = 0$$

Impossible because u, A are integers

Fourth case

$$ux^3 = -x^{q-3} + Ay^{p-2}; uy^2 = -y^{p-2} + Ax^{q-3}$$
$$x^{q-3} = v(x^3 + Ay^2); y^{p-2} = v(y^2 + Ax^3)$$

We have

$$uy^p = v(-y^p + A^2x^q - A(x^2y^{p-2} - y^2x^{q-3})) = v(-y^p + A^2x^q + A^2) = v(A^2 - 1)y^p$$
$$u = v(A^2 - 1)$$

And

$$u(y^2x^{q-3} - x^3y^{p-2}) = uA = (-y^{2p-4} + x^{2q-6})A = uv(y^4 - x^6)A$$
$$\Rightarrow v(y^4 - x^6) = 1$$

Impossible, because v,A are integers !

The only solution is A=1 and p=2 this case has been studied by Ko Chao.

### The Fermat equation

Fermat equation is  $y^n = x^n \pm z^n = x^n + az^n$

Let  $x^{n-2}y^2 - y^{n-2}x^2 = Aa$

If  $A = 0 \Rightarrow x^{n-4} = y^{n-4}$  but  $GCD(x, y) = 1 \Rightarrow n = 4$  impossible, there is no solution and

If  $A^2 = z^{2n}$ ;  $n \geq 3 \Rightarrow x^{n-3}y - y^{n-2} = \frac{Aaz^n}{x} \in Z$  impossible  $\Rightarrow n = 2$

We have

$$\begin{aligned}(x^{n-2}y^2 - y^{n-2}x^2)z^n &= Aaz^n = Ay^{n-2}y^2 - Ax^{n-2}x^2 \\ \Rightarrow (x^{n-2}z^n - Ay^{n-2})y^2 &= (y^{n-2}z^n - Ax^{n-2})x^2 \\ (y^2z^n + Ax^2)x^{n-2} &= (x^2z^n + Ay^2)y^{n-2}\end{aligned}$$

$\text{GCD}(x,y)=1$

We have then four cases :

$$\begin{aligned}x^2 &= u(-x^{n-2}z^n + Ay^{n-2}); y^2 = u(-y^{n-2}z^n + Ax^{n-2}) \\ x^{n-2} &= v(x^2z^n + Ay^2); y^{n-2} = v(y^2z^n + Ax^2)\end{aligned}$$

or

$$\begin{aligned}ux^2 &= -x^{n-2}z^n + Ay^{n-2}; uy^2 = -y^{n-2}z^n + Ax^{n-2} \\ vx^{n-2} &= x^2z^n + Ay^2; vy^{n-2} = y^2z^n + Ax^2\end{aligned}$$

or

$$\begin{aligned}x^2 &= u(-x^{n-2}z^n + Ay^{n-2}); y^2 = u(-y^{n-2}z^n + Ax^{n-2}) \\ vx^{n-2} &= x^2z^n + Ay^2; vy^{n-2} = y^2z^n + Ax^2\end{aligned}$$

or

$$\begin{aligned}ux^2 &= -x^{n-2}z^n + Ay^{n-2}; uy^2 = -y^{n-2}z^n + Ax^{n-2} \\ x^{n-2} &= v(x^2z^n + Ay^2); y^{n-2} = v(y^2z^n + Ax^2)\end{aligned}$$

With  $u, v \in Z$

First case

$$\begin{aligned}x^2 &= u(-x^{n-2}z^n + Ay^{n-2}); y^2 = u(-y^{n-2}z^n + Ax^{n-2}) \\ x^{n-2} &= v(x^2z^n + Ay^2); y^{n-2} = v(y^2z^n + Ax^2)\end{aligned}$$

We have

$$\begin{aligned}y^n &= uv(-y^n z^{2n} + A^2 x^n - Az^n(x^2 y^{n-2} - y^2 x^{n-2})) = uv(-y^n z^{2n} + A^2 x^n + A^2 az^n) = uv(A^2 - z^{2n})y^n \\ uv(A^2 - z^{2n}) &= 1\end{aligned}$$

Impossible because  $A, u, v$  are integers

Second case

or

$$ux^2 = -x^{n-2}z^n + Ay^{n-2}; uy^2 = -y^{n-2}z^n + Ax^{n-2}$$

$$vx^{n-2} = x^2z^n + Ay^2; vy^{n-2} = y^2z^n + Ax^2$$

We have

$$uvy^n = -y^n z^{2n} + A^2 x^n - Az^n (x^2 y^{n-2} - y^2 x^{n-2}) = -y^n z^{2n} + A^2 x^n + A^2 az^n = (A^2 - z^{2n})y^n$$

$$uv = A^2 - z^{2n}$$

But

$$uv(y^2 x^{n-2} - x^2 y^{n-2}) = auvA = v(-y^{2n-4} + x^{2n-4})A = u(y^4 - x^4)A$$

$$\Rightarrow au = -y^{2n-4} + x^{2n-4}; av = y^4 - x^4$$

$$x < y$$

$$a(A - z^n) = (y^2 + x^2)(x^{n-2} - y^{n-2}) < 0$$

$$a(A + z^n) = y^2 x^{n-2} - x^2 y^{n-2} + y^n - x^n = (y^2 - x^2)(x^{n-2} + y^{n-2}) > 0$$

$$uv = A^2 - z^{2n} < 0$$

$$a > 0$$

$$A = a(y^2 x^{n-2} - x^2 y^{n-2}) < 0$$

$$v = a(y^4 - x^4) > 0$$

$$0 < auvA = v(x^{2n-4} - y^{2n-4}) < 0$$

And if

$$x > y$$

$$a(A - z^n) = (y^2 + x^2)(x^{n-2} - y^{n-2}) > 0$$

$$a(A + z^n) = y^2 x^{n-2} - x^2 y^{n-2} + y^n - x^n = (y^2 - x^2)(x^{n-2} + y^{n-2}) < 0$$

$$uv = A^2 - z^{2n} < 0$$

$$a < 0$$

$$A = a(y^2 x^{n-2} - x^2 y^{n-2}) < 0$$

$$v = a(y^4 - x^4) > 0$$

$$0 < auvA = v(x^{2n-4} - y^{2n-4}) < 0$$

It is impossible because  $\Rightarrow n = 2$

Third case

or

$$x^2 = u(-x^{n-2}z^n + Ay^{n-2}); y^2 = u(-y^{n-2}z^n + Ax^{n-2})$$

$$vx^{n-2} = x^2z^n + Ay^2; vy^{n-2} = y^2z^n + Ax^2$$

We have

$$vy^n = u(-y^n z^{2n} + A^2 x^n - Az^n(x^2 y^{n-2} - y^2 x^{n-2})) = u(-y^n z^{2n} + A^2 x^n + A^2 az^n) = u(A^2 - z^{2n})y^n$$

$$v = u(A^2 - z^{2n})$$

And

$$v(y^2 x^{n-2} - x^2 y^{n-2}) = vA = uv(-y^{2n-4} + x^{2n-4})A = (y^4 - x^4)A$$

$$\Rightarrow u(-y^{2n-4} + x^{2n-4}) = 1 \Rightarrow u = \infty; n = 2$$

Impossible because u, A are integers

Fourth case

$$ux^2 = -x^{n-2}z^n + Ay^{n-2}; uy^2 = -y^{n-2}z^n + Ax^{n-2}$$

$$x^{n-2} = v(x^2z^n + Ay^2); y^{n-2} = v(y^2z^n + Ax^2)$$

We have

$$uy^n = v(-y^n z^{2n} + A^2 x^n - Az^n(x^2 y^{n-2} - y^2 x^{n-2})) = v(-y^n z^{2n} + A^2 x^n + A^2 az^n) = v(A^2 - z^{2n})y^n$$

$$u = v(A^2 - z^{2n})$$

And

$$u(y^2 x^{n-2} - x^2 y^{n-2}) = uA = (-y^{2n-4} + x^{2n-4})A = uv(y^4 - x^4)A$$

$$\Rightarrow v(x^4 - y^4) = 1 \Rightarrow v = 1; x^4 = y^4 + 1$$

Impossible ! Because v,A are integers !

The only solution is A=1 and n=2

### The Fermat-Catalan equation

The equation now is  $y^p = x^q \pm z^c = x^q + az^c$

$$\text{Let } x^{q-w}y^2 - y^{p-2}x^w = aA$$

If

$$A = 0 \Rightarrow x^{q-2w} = y^{p-4}$$

$$GCD(x, y) = 1 \Rightarrow p = 4$$

It means that p=2 is the prime solution !

And

$$A^2 = z^{2c}; p \geq 3 \Rightarrow x^{q-w}y - y^{p-3}x^w = \frac{\pm z^c}{y} \in Z$$

Impossible because  $\text{GCD}(y,z)=1$ , thus  $p=2$

We have

$$\begin{aligned} (x^{q-w}y^2 - y^{p-2}x^w)z^c &= aAz^c = Ay^{p-2}y^2 - Ax^{q-w}x^w \\ \Rightarrow (x^{q-w}z^c - Ay^{p-2})y^2 &= (y^{p-2}z^c - Ax^{q-w})x^w \\ (y^2z^c + Ax^w)x^{q-w} &= (x^wz^c + Ay^2)y^{p-2} \\ \text{GCD}(x, y) &= 1 \end{aligned}$$

We have then four cases :

First case

$$\begin{aligned} x^w &= u(-x^{q-w}z^c + Ay^{p-2}); y^2 = u(-y^{p-2}z^c + Ax^{q-w}) \\ x^{q-w} &= v(x^wz^c + Ay^2); y^{p-2} = v(y^2z^c + Ax^w) \end{aligned}$$

We have

$$\begin{aligned} y^p &= uv(-y^p z^{2c} + A^2 x^q - Az^c(x^w y^{p-2} - y^2 x^{q-w})) = uv(-y^p z^{2c} + A^2 x^q + A^2 az^c) = uv(A^2 - z^{2c})y^p \\ uv(A^2 - z^{2c}) &= 1 \end{aligned}$$

Impossible because  $A, u, v$  are integers

Second case

$$\begin{aligned} ux^w &= -x^{q-w}z^c + Ay^{p-2}; uy^2 = -y^{p-2}z^c + Ax^{q-w} \\ vx^{q-w} &= x^wz^c + Ay^2; vy^{p-2} = y^2z^c + Ax^w \end{aligned}$$

We have

$$\begin{aligned} uv y^p &= -y^p z^{2c} + A^2 x^q - Az^c(x^w y^{p-2} - y^2 x^{q-w}) = -y^p z^{2c} + A^2 x^q + A^2 az^c = (A^2 - z^{2c})y^p \\ uv &= A^2 - z^{2c} \end{aligned}$$

And



$$uv(y^2 x^{q-w} - x^w y^{p-2}) = auvA = v(-y^{2p-4} + x^{2p-2w})A = u(y^4 - x^{2w})A$$

$$\Rightarrow au = -y^{2p-4} + x^{2q-2w}; av = y^4 - x^{2w}$$

$$1 < \frac{y^4}{x^{2w}} < \frac{y^p}{x^q}$$

$$a(A - z^c) = (y^2 + x^w)(x^{q-w} - y^{p-2}) < 0$$

$$a(A + z^c) = y^2 x^{q-w} - x^w y^{p-2} + y^p - x^q = (y^2 - x^w)(x^{q-w} + y^{p-2}) > 0$$

$$uv = A^2 - z^{2c} < 0$$

$$y^p > x^q \Rightarrow a > 0$$

$$A = a(y^2 x^{q-w} - x^w y^{p-2}) < 0$$

$$v = a(y^4 - x^{2w}) > 0$$

$$0 < auvA = v(x^{2q-2w} - y^{2p-4}) < 0$$

And if

$$1 > \frac{y^4}{x^{2w}} > \frac{y^p}{x^q}$$

$$a(A - z^c) = (y^2 + x^w)(x^{q-w} - y^{p-2}) > 0$$

$$a(A + z^c) = y^2 x^{q-w} - x^w y^{p-2} + y^p - x^q = (y^2 - x^w)(x^{q-w} + y^{p-2}) < 0$$

$$uv = A^2 - z^{2c} < 0$$

$$y^p < x^q \Rightarrow a < 0$$

$$A = a(y^2 x^{q-w} - x^w y^{p-2}) < 0$$

$$v = a(y^4 - x^{2w}) > 0$$

$$0 < auvA = v(x^{2q-2w} - y^{2p-4}) < 0$$

Impossible because :  $\Rightarrow p = 2$

Third case

$$x^w = u(-x^{q-w} z^c + Ay^{p-2}); y^2 = u(-y^{p-2} z^c + Ax^{q-w})$$

$$vx^{q-w} = x^w z^c + Ay^2; vy^{p-2} = y^2 z^c + Ax^w$$

We have

$$vy^p = u(-y^p z^{2c} + A^2 x^q - Az^c(x^w y^{p-2} - y^2 x^{q-w})) = u(-y^p z^{2c} + A^2 x^q + A^2 az^c) = u(A^2 - z^{2c})y^p$$

$$v = u(A^2 - z^{2c})$$

And

$$v(y^2 x^{q-w} - x^w y^{p-2}) = vA = uv(-y^{2p-4} + x^{2q-2w})A = (y^4 - x^{2w})A$$

$$\Rightarrow u(-y^{2p-4} + x^{2q-2w}) = 1 \Rightarrow p = 2$$

Impossible because u, A are integers

#### Fourth case

$$ux^w = -x^{q-w}z^c + Ay^{p-2}; uy^2 = -y^{p-2}z^c + Ax^{q-w}$$
$$x^{q-w} = v(x^wz^c + Ay^2); y^{p-2} = v(y^2z^c + Ax^w)$$

We have

$$uy^p = v(-y^p z^{2c} + A^2 x^q - Az^c (x^w y^{p-2} - y^2 x^{q-w})) = v(-y^p z^{2c} + A^2 x^q + A^2 az^c) = v(A^2 - z^{2c})y^p$$
$$u = v(A^2 - z^{2c})$$

And

$$u(y^2 x^{q-w} - x^w y^{p-2}) = uA = -y^{2p-4} + x^{2q-2w} A = uv(y^4 - x^{2w})A$$
$$\Rightarrow v(y^4 - x^{2w}) = 1$$

Impossible, because  $v, A$  are integers ! In the Fermat-Catalan equation, one of the exponents must be equal to 2 ! The Beal conjecture has been proved !

In fact, in the three precedent equations studied here, one of the exponent greater or equal to 2 must be minimum, which means that it must be 2 !

#### Conclusion

We have solved both three equations by the same method and proved two theorems and one conjecture.

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