An elementary proof of Fermat-Catalan conjecture.

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An elementary proof of Catalan-Mihailescu and Fermat-Wiles theorems and generalization to Beal conjecture

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Abstract

A proof of both Catalan and Fermat theorems is presented and a generalization to Beal conjecture is proposed. For this, we begin with Fermat, Catlan and Fermat-Catalan equations and solve them.

Keywords : Diophantine ; Fermat ; Catalan ; Fermat-Catalan ; Resolution.

Resolution of Catalan equation

Catalan equation is $y^p = x^q + 1$

Let $x^{q-3}y^2 - y^{p-2}x^3 = A$

If

$$A = 0 \Rightarrow x^{q-6} = y^{p-4}$$
$$GCD(x, y) = 1 \Rightarrow p = 4$$

impossible, there is no solution, Ko Chao proved it and

If $A^2 = 1; p \geq 3 \Rightarrow x^{q-3}y - y^{p-3}x^3 = \frac{1}{y} \in Z$ impossible thus p=2

We have

$$x^{q-3}y^2 - y^{p-2}x^3 = A = Ay^{p-2}y^2 - Ax^{q-3}x^3$$
$$\Rightarrow (x^{q-3} - Ay^{p-2})y^2 = (y^{p-2} - Ax^{q-3})x^3$$
$$(y^2 + Ax^3)x^{q-3} = (x^3 + Ay^2)y^{p-2}$$

But

$$GCD(x, y) = 1$$

We have then four cases :

$x^3 = u(-x^{q-3} + Ay^{p-2}); y^2 = u(-y^{p-2} + Ax^{q-3})$
$x^{q-3} = v(x^3 + Ay^2); y^{p-2} = v(y^2 + Ax^3)$
or
$ux^3 = -x^{q-3} + Ay^{p-2}; uy^2 = -y^{p-2} + Ax^{q-3}$
$vx^{q-3} = x^3 + Ay^2; vy^{p-2} = y^2 + Ax^3$
or
$x^3 = u(-x^{q-3} + Ay^{p-2}); y^2 = u(-y^{p-2} + Ax^{q-3})$
$vx^{q-3} = x^3 + Ay^2; vy^{p-2} = y^2 + Ax^3$
or
$ux^3 = -x^{q-3} + Ay^{p-2}; uy^2 = -y^{p-2} + Ax^{q-3}$
$x^{q-3} = v(x^3 + Ay^2); y^{p-2} = v(y^2 + Ax^3)$
$u, v \in Z$

First case

$x^3 = u(-x^{q-3} + Ay^{p-2}); y^2 = u(-y^{p-2} + Ax^{q-3})$
$x^{q-3} = v(x^3 + Ay^2); y^{p-2} = v(y^2 + Ax^3)$

We have

$y^p = uv(-y^p + A^2x^q - A(x^3y^{p-2} - y^2x^{q-3})) = uv(-y^p + A^2x^q + A^2) = uv(A^2 - 1)y^p$
$uv(A^2 - 1) = 1$

Impossible because A,u,v are integers

Second case

$ux^3 = -x^{q-3} + Ay^{p-2}; uy^2 = -y^{p-2} + Ax^{q-3}$
$vx^{q-3} = x^2 + Ay^2; vy^{p-2} = y^2 + Ax^3$

We have

$uvy^p = -y^p + A^2x^q - A(x^3y^{p-2} - y^2x^{q-3}) = -y^p + A^2x^q + A^2 = (A^2 - 1)y^p$
$uv = A^2 - 1$

And

$uv(y^2x^{q-3} - x^3y^{p-2}) = uvA = v(-y^{2p-4} + x^{2q-6})A = u(y^4 - x^6)A$
$\Rightarrow v = y^4 - x^6; u = -y^{2p-4} + x^{2q-6}$
$uv = A^2 - 1 = (y^4 - x^4)(-y^{2p-4} + x^{2q-6})$
$\Rightarrow (y^2x^{q-3} - x^3y^{p-2})^2 - 1 = (y^4 - x^6)(-y^{2p-4} + x^{2q-6})$
$\Rightarrow x^{2q} + y^{2p} - 2x^qy^p = 1 = (x^q - y^p)^2 = (2y^p - 1)^2 = 4y^{2p} - 4y^p + 1$

Impossible : $\Rightarrow\ p = 2$

Third case

$$x^3 = u(-x^{q-3} + Ay^{p-2});\, y^2 = u(-y^{p-2} + Ax^{q-3})$$
$$vx^{q-3} = x^3 + Ay^2;\, vy^{p-2} = y^2 + Ax^3$$

We have

$$vy^p = u(-y^p + A^2x^q - A(x^3y^{p-2} - y^2x^{q-3})) = u(-y^p + A^2x^q + A^2) = u(A^2 - 1)y^p$$
$$v = u(A^2 - 1)$$

And

$$v(y^2x^{q-3} - x^3y^{p-2}) = vA = uv(-y^{2p-4} + x^{2q-6})A = (y^4 - x^6)A$$
$$\Rightarrow\ u(-y^{2p-4} + x^{2q-6}) = 1 \Rightarrow\ p - 2 = q - 3 = 0$$

Impossible because u, A are integers

Fourth case

$$ux^3 = -x^{q-3} + Ay^{p-2};\, uy^2 = -y^{p-2} + Ax^{q-3}$$
$$x^{q-3} = v(x^3 + Ay^2);\, y^{p-2} = v(y^2 + Ax^3)$$

We have

$$uy^p = v(-y^p + A^2x^q - A(x^2y^{p-2} - y^2x^{q-3})) = v(-y^p + A^2x^q + A^2) = v(A^2 - 1)y^p$$
$$u = v(A^2 - 1)$$

And

$$u(y^2x^{q-3} - x^3y^{p-2}) = uA = (-y^{2p-4} + x^{2q-6})A = uv(y^4 - x^6)A$$
$$\Rightarrow\ v(y^4 - x^6) = 1$$

Impossible, because v,A are integers !

The only solution is A=1 and p=2 this case has been studied by Ko Chao.

The Fermat equation

Fermat equation is $y^n = x^n \pm z^n = x^n + az^n$

Let $x^{n-2}y^2 - y^{n-2}x^2 = Aa$

If $A = 0 \Rightarrow x^{n-4} = y^{n-4}$ but $GCD(x, y) = 1 \Rightarrow n = 4$ impossible, there is no solution and

If $A^2 = z^{2n}; n \geq 3 \Rightarrow x^{n-3}y - y^{n-2} = \dfrac{Aaz^n}{x} \in Z$ impossible $\Rightarrow n = 2$

We have

$$(x^{n-2}y^2 - y^{n-2}x^2)z^n = Aaz^n = Ay^{n-2}y^2 - Ax^{n-2}x^2$$
$$\Rightarrow (x^{n-2}z^n - Ay^{n-2})y^2 = (y^{n-2}z^n - Ax^{n-2})x^2$$
$$(y^2z^n + Ax^2)x^{n-2} = (x^2z^n + Ay^2)y^{n-2}$$

GCD(x,y)=1

We have then four cases :

$$x^2 = u(-x^{n-2}z^n + Ay^{n-2}); y^2 = u(-y^{n-2}z^n + Ax^{n-2})$$
$$x^{n-2} = v(x^2z^n + Ay^2); y^{n-2} = v(y^2z^n + Ax^2)$$
$$or$$
$$ux^2 = -x^{n-2}z^n + Ay^{n-2}; uy^2 = -y^{n-2}z^n + Ax^{n-2}$$
$$vx^{n-2} = x^2z^n + Ay^2; vy^{n-2} = y^2z^n + Ax^2$$

$$or$$
$$x^2 = u(-x^{n-2}z^n + Ay^{n-2}); y^2 = u(-y^{n-2}z^n + Ax^{n-2})$$
$$vx^{n-2} = x^2z^n + Ay^2; vy^{n-2} = y^2z^n + Ax^2$$

$$or$$
$$ux^2 = -x^{n-2}z^n + Ay^{n-2}; uy^2 = -y^{n-2}z^n + Ax^{n-2}$$
$$x^{n-2} = v(x^2z^n + Ay^2); y^{n-2} = v(y^2z^n + Ax^2)$$

With $u, v \in Z$

First case

$$x^2 = u(-x^{n-2}z^n + Ay^{n-2}); y^2 = u(-y^{n-2}z^n + Ax^{n-2})$$
$$x^{n-2} = v(x^2z^n + Ay^2); y^{n-2} = v(y^2z^n + Ax^2)$$

We have

$$y^n = uv(-y^nz^{2n} + A^2x^n - Az^n(x^2y^{n-2} - y^2x^{n-2})) = uv(-y^nz^{2n} + A^2x^n + A^2az^n) = uv(A^2 - z^{2n})y^n$$
$$uv(A^2 - z^{2n}) = 1$$

Impossible because A,u,v are integers

Second case

or

$ux^2 = -x^{n-2}z^n + Ay^{n-2}; uy^2 = -y^{n-2}z^n + Ax^{n-2}$

$vx^{n-2} = x^2z^n + Ay^2; vy^{n-2} = y^2z^n + Ax^2$

We have

$uvy^n = -y^nz^{2n} + A^2x^n - Az^n(x^2y^{n-2} - y^2x^{n-2}) = -y^nz^{2n} + A^2x^n + A^2az^n = (A^2 - z^{2n})y^n$

$uv = A^2 - z^{2n}$

But

$uv(y^2x^{n-2} - x^2y^{n-2}) = auvA = v(-y^{2n-4} + x^{2n-4})A = u(y^4 - x^4)A$

$\Rightarrow au = -y^{2n-4} + x^{2n-4}; av = y^4 - x^4$

$x < y$

$a(A - z^n) = (y^2 + x^2)(x^{n-2} - y^{n-2}) < 0$

$a(A + z^n) = y^2x^{n-2} - x^2y^{n-2} + y^n - x^n = (y^2 - x^2)(x^{n-2} + y^{n-2}) > 0$

$uv = A^2 - z^{2n} < 0$

$a > 0$

$A = a(y^2x^{n-2} - x^2y^{n-2}) < 0$

$v = a(y^4 - x^4) > 0$

$0 < auvA = v(x^{2n-4} - y^{2n-4}) < 0$

And if

$x > y$

$a(A - z^n) = (y^2 + x^2)(x^{n-2} - y^{n-2}) > 0$

$a(A + z^n) = y^2x^{n-2} - x^2y^{n-2} + y^n - x^n = (y^2 - x^2)(x^{n-2} + y^{n-2}) < 0$

$uv = A^2 - z^{2n} < 0$

$a < 0$

$A = a(y^2x^{n-2} - x^2y^{n-2}) < 0$

$v = a(y^4 - x^4) > 0$

$0 < auvA = v(x^{2n-4} - y^{2n-4}) < 0$

It is impossible because $\Rightarrow n = 2$

Third case

$$or$$
$$x^2 = u(-x^{n-2}z^n + Ay^{n-2}); y^2 = u(-y^{n-2}z^n + Ax^{n-2})$$
$$vx^{n-2} = x^2z^n + Ay^2; vy^{n-2} = y^2z^n + Ax^2$$

We have

$$vy^n = u(-y^nz^{2n} + A^2x^n - Az^n(x^2y^{n-2} - y^2x^{n-2})) = u(-y^nz^{2n} + A^2x^n + A^2az^n) = u(A^2 - z^{2n})y^n$$
$$v = u(A^2 - z^{2n})$$

And

$$v(y^2x^{n-2} - x^2y^{n-2}) = vA = uv(-y^{2n-4} + x^{2n-4})A = (y^4 - x^4)A$$
$$\Rightarrow u(-y^{2n-4} + x^{2n-4}) = 1 \Rightarrow u = \infty; n = 2$$

Impossible because u, A are integers

Fourth case

$$ux^2 = -x^{n-2}z^n + Ay^{n-2}; uy^2 = -y^{n-2}z^n + Ax^{n-2}$$
$$x^{n-2} = v(x^2z^n + Ay^2); y^{n-2} = v(y^2z^n + Ax^2)$$

We have

$$uy^n = v(-y^nz^{2n} + A^2x^n - Az^n(x^2y^{n-2} - y^2x^{n-2})) = v(-y^nz^{2n} + A^2x^n + A^2az^n) = v(A^2 - z^{2n})y^n$$
$$u = v(A^2 - z^{2n})$$

And

$$u(y^2x^{n-2} - x^2y^{n-2}) = uA = (-y^{2n-4} + x^{2n-4})A = uv(y^4 - x^4)A$$
$$\Rightarrow v(x^4 - y^4) = 1 \Rightarrow v = 1; x^4 = y^4 + 1$$

Impossible ! Because v,A are integers !

The only solution is A=1 and n=2

The Fermat-Catalan equation

The equation now is $y^p = x^q \pm z^c = x^q + az^c$

Let $x^{q-w}y^2 - y^{p-2}x^w = aA$

If

$$A = 0 \Rightarrow x^{q-2w} = y^{p-4}$$
$$GCD(x, y) = 1 \Rightarrow p = 4$$

It means that p=2 is the prime solution !

And

$$A^2 = z^{2c};\ p \geq 3 \Rightarrow\ x^{q-w}y - y^{p-3}x^w = \frac{\pm z^c}{y} \in Z$$

Impossible because GCD(y,z)=1, thus p=2

We have

$$(x^{q-w}y^2 - y^{p-2}x^w)z^c = aAz^c = Ay^{p-2}y^2 - Ax^{q-w}x^w$$
$$\Rightarrow (x^{q-w}z^c - Ay^{p-2})y^2 = (y^{p-2}z^c - Ax^{q-w})x^w$$
$$(y^2z^c + Ax^w)x^{q-w} = (x^wz^c + Ay^2)y^{p-2}$$
$$GCD(x,y) = 1$$

We have then four cases :

First case

$$x^w = u(-x^{q-w}z^c + Ay^{p-2});\ y^2 = u(-y^{p-2}z^c + Ax^{q-w})$$
$$x^{q-w} = v(x^wz^c + Ay^2);\ y^{p-2} = v(y^2z^c + Ax^w)$$

We have

$$y^p = uv(-y^pz^{2c} + A^2x^q - Az^c(x^wy^{p-2} - y^2x^{q-w})) = uv(-y^pz^{2c} + A^2x^q + A^2az^c) = uv(A^2 - z^{2c})y^p$$
$$uv(A^2 - z^{2c}) = 1$$

Impossible because A,u,v are integers

Second case

$$ux^w = -x^{q-w}z^c + Ay^{p-2}; uy^2 = -y^{p-2}z^c + Ax^{q-w}$$
$$vx^{q-w} = x^wz^c + Ay^2; vy^{p-2} = y^2z^c + Ax^w$$

We have

$$uvy^p = -y^pz^{2c} + A^2x^q - Az^c(x^wy^{p-2} - y^2x^{q-w}) = -y^pz^{2c} + A^2x^q + A^2az^c) = (A^2 - z^{2c})y^p$$
$$uv = A^2 - z^{2c}$$

And

$$uv(y^2x^{q-w} - x^w y^{p-2}) = auvA = v(-y^{2p-4} + x^{2p-2w})A = u(y^4 - x^{2w})A$$

$$\Rightarrow\ au = -y^{2p-4} + x^{2q-2w};av = y^4 - x^{2w}$$

$$1 < \frac{y^4}{x^{2w}} < \frac{y^p}{x^q}$$

$$a(A - z^c) = (y^2 + x^w)(x^{q-w} - y^{p-2}) < 0$$

$$a(A + z^c) = y^2x^{q-w} - x^w y^{p-2} + y^p - x^q = (y^2 - x^w)(x^{q-w} + y^{p-2}) > 0$$

$$uv = A^2 - z^{2c} < 0$$

$$y^p > x^q \Rightarrow a > 0$$

$$A = a(y^2x^{q-w} - x^w y^{p-2}) < 0$$

$$v = a(y^4 - x^{2w}) > 0$$

$$0 < auvA = v(x^{2q-2w} - y^{2p-4}) < 0$$

And if

$$1 > \frac{y^4}{x^{2w}} > \frac{y^p}{x^q}$$

$$a(A - z^c) = (y^2 + x^w)(x^{q-w} - y^{p-2}) > 0$$

$$a(A + z^c) = y^2x^{q-w} - x^w y^{p-2} + y^p - x^q = (y^2 - x^w)(x^{q-w} + y^{p-2}) < 0$$

$$uv = A^2 - z^{2c} < 0$$

$$y^p < x^q \Rightarrow a < 0$$

$$A = a(y^2x^{q-w} - x^w y^{p-2}) < 0$$

$$v = a(y^4 - x^{2w}) > 0$$

$$0 < auvA = v(x^{2q-2w} - y^{2p-4}) < 0$$

Impossible because : $\Rightarrow\ p = 2$

Third case

$$x^w = u(-x^{q-w}z^c + Ay^{p-2}); y^2 = u(-y^{p-2}z^c + Ax^{q-w})$$

$$vx^{q-w} = x^w z^c + Ay^2;vy^{p-2} = y^2z^c + Ax^w$$

We have

$$vy^p = u(-y^p z^{2c} + A^2x^q - Az^c(x^w y^{p-2} - y^2x^{q-w})) = u(-y^p z^{2c} + A^2x^q + A^2az^c) = u(A^2 - z^{2c})y^p$$

$$v = u(A^2 - z^{2c})$$

And

$$v(y^2x^{q-w} - x^w y^{p-2}) = vA = uv(-y^{2p-4} + x^{2q-2w})A = (y^4 - x^{2w})A$$

$$\Rightarrow u(-y^{2p-4} + x^{2q-2w}) = 1 \Rightarrow\ p = 2$$

Impossible because u, A are integers

Fourth case

$$ux^{w} = -x^{q-w}z^{c} + Ay^{p-2}; uy^{2} = -y^{p-2}z^{c} + Ax^{q-w}$$
$$x^{q-w} = v(x^{w}z^{c} + Ay^{2}); y^{p-2} = v(y^{2}z^{c} + Ax^{w})$$

We have

$$uy^{p} = v(-y^{p}z^{2c} + A^{2}x^{q} - Az^{c}(x^{w}y^{p-2} - y^{2}x^{q-w})) = v(-y^{p}z^{2c} + A^{2}x^{q} + A^{2}az^{c}) = v(A^{2} - z^{2c})y^{p}$$
$$u = v(A^{2} - z^{2c})$$

And

$$u(y^{2}x^{q-w} - x^{w}y^{p-2}) = uA = -y^{2p-4} + x^{2q-2w}A = uv(y^{4} - x^{2w})A$$
$$\Rightarrow v(y^{4} - x^{2w}) = 1$$

Impossible, because v,A are integers ! In the Fermat-Catalan equation, one of the exponents must be equal to 2 ! The Beal conjecture has been proved !

In fact, in the three precedent equations studied here, one of the exponent greater or equal to 2 must be minimum, which means that it must be 2 !

Conclusion

We have solved both three equations by the same method and proved two theorems and one conjecture.

Bibliography

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