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Maximum Vertex and Face Degree of Oblique Graphs

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Abstract

Let $G = (V, E, F)$ be a 3-connected simple graph imbedded into a surface S with vertex set V , edge set E and face set F . A face α is an $\langle a_1, a_2, \dots, a_k \rangle$ -face if α is a k -gon and the degrees of the vertices incident with α in the cyclic order are a_1, a_2, \dots, a_k . The lexicographic minimum $\langle b_1, b_2, \dots, b_k \rangle$ such that α is a $\langle b_1, b_2, \dots, b_k \rangle$ -face is called the *type* of α .

Let z be an integer. We consider z -oblique graphs, i.e. such graphs that the number of faces of each type is at most z . We show an upper bound for the maximum vertex degree of any z -oblique graph imbedded into a given surface. Moreover, an upper bound for the maximum face degree is presented. We also show that there are only finitely many oblique graphs imbedded into non-orientable surfaces.

Key words: Oblique graph, vertex degree, face degree, orientable surface, non-orientable surface

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1 Introduction

The concept of the graphs with restricted number of types of graph elements was introduced by Jendrol' and Tkáč in [2], [3]. The concept of obliqueness was defined in the paper [7] by Walther. Later, in the paper [6] the z -obliqueness was defined. Obliqueness of graphs was next studied by Walther [7], Schreyer [5] and others.

All graphs considered in this paper are simple (without multiple edges or loops), and we use the standard notation due to Diestel [1].

A graph is a pair $G = (V, E)$, where $V = V(G)$ is the set of vertices of the graph G and $E = E(G)$ is the set of edges of graph G . We shall not distinguish strictly between the graph and its vertex or edge set. For example, we may speak of a vertex $v \in G$ rather than $v \in V(G)$, and so on.

An embedding of a graph G on a surface S is a drawing of G on S without edges crossing (vertices are represented by points, edges are represented by arcs between their endvertices). If all components of $S - G$ are homeomorphic to open discs, then the embedding is cellular. In this case the components of $S - G$ are called *faces* of G . We deal only with cellular embeddings.

The *degree* $\deg(v)$ of a vertex $v \in V(G)$ is the number of vertices adjacent to v . The maximum degree in a graph G is denoted by $\Delta(G)$.

The *size* (or *degree*) $\deg(\alpha)$ of a face α is defined to be the length of the shortest closed walk containing all edges from the boundary of α , where the *boundary* of a face α is a set of all edges incident with α .

Let us remind the definition of the oblique graph. A face α is an $\langle a_1, a_2, \dots, a_k \rangle$ -face if $\deg(\alpha) = k$ and the degrees of the vertices incident with α along its boundary in the cyclic order are a_1, \dots, a_k . The lexicographic minimum $\langle b_1, b_2, \dots, b_k \rangle$ such that α is a $\langle b_1, b_2, \dots, b_k \rangle$ -face is called the *type* of α . Let z be an integer. A graph is said to be z -*oblique* if the number of faces of each type is at most z . If $z = 1$ (all faces are of different types), the graph is said to be oblique.

Voigt and Walther [6] proved that the set of all z -oblique graphs imbedded into the sphere is finite. Schreyer [5] proved that the set of all z -oblique graphs imbedded into any given orientable surface is finite as well. There is also found the upper bound for the number of all z -oblique graphs imbedded into an orientable surface with genus g . Recently Kardoš and Miškuf [4] showed that the maximum degree of an oblique triangulation imbedded into the sphere is less than 50.

2 Vertex degrees

Using the method of charging and discharging we show an upper bound for the maximum degree of a z -oblique graph imbedded into the surface with Euler's characteristic e . According to the definition of the type of a face α we are only interested in the boundary walk of this face, hence, we do not have to distinguish the orientable and the non-orientable surface.

Theorem 1 *Let G be a z -oblique graph with maximum degree Δ imbedded into the surface with Euler's characteristic e . Then*

$$\Delta \leq 35.7z^2 + 56.9z + 42 \ln(41 + 21z)z - 6e + 6.$$

Especially for every oblique graph G imbedded on the sphere we have $\Delta \leq 235$.

PROOF. Let the vertex with the maximum degree be denoted by v_Δ ; if there are more vertices with maximum degree we choose any one of them. At first let us define the initial charges of the vertices x and the faces α in the following way:

$$w(x) = \begin{cases} 0 & \text{if } \deg(x) < 42 + 21z, \\ \Delta - 6 & \text{if } x = v_\Delta, \\ 3z & \text{otherwise.} \end{cases}$$

$$w(\alpha) = 2(\deg(\alpha) - 3) + \sum_{\substack{x \in \alpha \\ x \neq v_\Delta}} \frac{\deg^*(x) - 6}{\deg(x)},$$

where $\deg^*(x)$ is defined as

$$\deg^*(x) = \begin{cases} \deg(x) & \text{if } \deg(x) < 42 + 21z, \\ \deg(x) & \text{if } x = v_\Delta, \\ \deg(x) - 3z & \text{otherwise.} \end{cases}$$

From Euler's formula we have

$$\sum_{\alpha \in F} 2(\deg(\alpha) - 3) + \sum_{x \in V} (\deg(x) - 6) = -6e,$$

so we get the sum of the initial charges

$$\begin{aligned}
& \sum_{\alpha \in F} w(\alpha) + \sum_{x \in V} w(x) = \\
&= \sum_{\alpha \in F} \left(2(\deg(\alpha) - 3) + \sum_{\substack{x \in \alpha \\ x \neq v_\Delta}} \frac{\deg^*(x) - 6}{\deg(x)} \right) + \sum_{\substack{x \neq v_\Delta \\ \deg(x) \geq 42 + 21z}} 3z + \Delta - 6 = \\
&= \sum_{\alpha \in F} 2(\deg(\alpha) - 3) + \sum_{\alpha \in F} \sum_{\substack{x \in \alpha \\ x \neq v_\Delta}} \frac{\deg^*(x) - 6}{\deg(x)} + \sum_{\substack{x \neq v_\Delta \\ \deg(x) \geq 42 + 21z}} 3z + \Delta - 6 = \\
&= -6e - \sum_{x \in V} (\deg(x) - 6) + \sum_{x \neq v_\Delta} (\deg^*(x) - 6) + \sum_{\substack{x \neq v_\Delta \\ \deg(x) \geq 42 + 21z}} 3z + \Delta - 6 = \\
&= -6e + \sum_{x \in V} (\deg^*(x) - \deg(x)) + \sum_{\substack{x \neq v_\Delta \\ \deg(x) \geq 42 + 21z}} 3z = -6e,
\end{aligned}$$

and hence,

$$\sum_{\alpha \in F} w(\alpha) + \sum_{x \in V} w(x) = -6e.$$

In the next step we relocate some of the initial charges from vertices with big degree (distinct from v_Δ) into the faces incident with them according to the following discharging rules:

- (1) For $x \neq v_\Delta$ and $\deg(x) = d \geq 42 + 21z$ we set $w'(x) := w(x)$ and
 - (a) if $x \in \alpha$ and α is a $\langle 3, 3, d \rangle$ -face, then $w'(\alpha) := w(\alpha) + \frac{8}{7}$ and $w'(x) := w'(x) - \frac{8}{7}$,
 - (b) if $x \in \alpha$ and α is a $\langle 3, 4, d \rangle$ -face, then $w'(\alpha) := w(\alpha) + \frac{5}{7}$ and $w'(x) := w'(x) - \frac{5}{7}$,
 - (c) if $x \in \alpha$ and α is a $\langle 3, 5, d \rangle$ -face, then $w'(\alpha) := w(\alpha) + \frac{3}{7}$ and $w'(x) := w'(x) - \frac{3}{7}$,
 - (d) if $x \in \alpha$ and α is a $\langle 3, 6, d \rangle$ -face, then $w'(\alpha) := w(\alpha) + \frac{1}{7}$ and $w'(x) := w'(x) - \frac{1}{7}$,
 - (e) if $x \in \alpha$ and α is a $\langle 4, 4, d \rangle$ -face, then $w'(\alpha) := w(\alpha) + \frac{1}{7}$ and $w'(x) := w'(x) - \frac{1}{7}$,
 - (f) if $x \in \alpha$ and α is a $\langle 3, 3, 3, d \rangle$ -face, then $w'(\alpha) := w(\alpha) + \frac{1}{7}$ and $w'(x) := w'(x) - \frac{1}{7}$.
- (2) For all faces not mentioned above $w'(\alpha) := w(\alpha)$ and $w'(x) := w(x)$ for all vertices x such that $\deg(x) < 42 + 21z$ or $x = v_\Delta$.

For the new charge of vertices we have

$$w'(x) \begin{cases} = 0 & \text{if } \deg(x) < 42 + 21z, \\ = \Delta - 6 & \text{if } x = v_\Delta, \\ \geq \frac{2}{7}z & \text{if } \deg(x) \geq 42 + 21z \text{ and } x \neq v_\Delta. \end{cases}$$

Observe that in the first and the third case the charge has not been changed. In the second case the vertex x would redistribute at most $\frac{19}{7}z$ of his previous charge and so the new charge is at least $\frac{2}{7}z$. Hence, the new charge of all the vertices is non-negative.

It is easy to check that for the faces with degree at least 6 we have

$$w'(\alpha) = w(\alpha) \geq 2(\deg(\alpha) - 3) + \sum_{x \in \alpha} (-1) = 2 \deg(\alpha) - 6 - \deg(\alpha) \geq 0,$$

thus only the faces with degree at most 5 can have a negative charge.

We compute the amount of the negative charge in the whole graph. Due to previous remarks we deal only with the faces with degree at most 5. According to our discharging rules we can list the types of the faces with the negative charge. They can be divided into small faces incident with many vertices with small degree (see Tables 1 and 2) and small faces incident with the vertex v_Δ (see Table 3).

Now we clarify why in the first column of Table 3 it is sufficient to have the interval $\langle 3, 41 + 21z \rangle$ instead of $\langle 3, \Delta \rangle$. Every vertex x with degree at least $42 + 21z$ has charge at least $\frac{2}{7}z$. The face of type $\langle 3, x, \Delta \rangle$ can occur at most z times. We redistribute the charge of x into these at most z faces. The vertex donates $\frac{1}{7}$ of its charge to each of those faces. Obviously, the charge of the

$a_1 =$	3	3	3	3	3	3	3	3	3
$a_2 =$	3	4	5	6	7	8	9	10	11
$a_3 <$	$42 + 21z$	$42 + 21z$	$42 + 21z$	$42 + 21z$	42	24	18	15	14
$a_1 =$	4	4	4	4	5	5			
$a_2 =$	4	5	6	7	5	6			
$a_3 <$	$42 + 21z$	20	12	10	10	8			

Table 1

The list of all the faces of types $\langle a_1, a_2, a_3 \rangle$ with a negative charge, not incident with the vertex v_Δ .

$a_1 =$	3	3	3	3	3	3	3	3
$a_2 =$	3	3	4	3	5	4	4	3
$a_3 =$	3	4	3	5	3	4	5	3
$a_4 <$	$42 + 21z$	12	12	8	8	6	5	4
$a_5 <$	-	-	-	-	-	-	-	6

Table 2

The list of all the faces of types $\langle a_1, a_2, a_3, a_4 \rangle$ or $\langle a_1, a_2, a_3, a_4, a_5 \rangle$ with a negative charge, not incident with the vertex v_Δ .

$a_1 =$	3	4	5	3	3
$a_2 =$	$\langle 3, 41 + 21z \rangle$	$\langle 4, 12 \rangle$	$\langle 5, 7 \rangle$	3	Δ
$a_3 =$	Δ	Δ	Δ	Δ	3
$a_4 <$	-	-	-	6	6

Table 3

The list of all the faces incident with the vertex v_Δ having a negative charge.

vertex is still non-negative after the second discharging. The charge of the face $\alpha = \langle 3, x, \Delta \rangle$ will be non-negative too, since $\deg(x) \geq 42 + 21z$:

$$w'(\alpha) = -1 + \frac{\deg^*(x) - 6}{\deg(x)} + \frac{1}{7} = \frac{\deg(x) - 3z - 6}{\deg(x)} - \frac{6}{7} = \frac{1}{7} - \frac{3z + 6}{\deg(x)} \geq 0.$$

The whole negative charge in the faces not incident with the vertex v_Δ is at least the sum of the charges of faces in the Tables 1 and 2, each of the face types counted z times. We show the computation only for the first column of Table 1, the others go in the same lines and for brevity we omit them. The amount of the negative charge of the faces in the Table 1 in the first column can be bounded by

$$\begin{aligned} \sum_{d=3}^{41+21z} \left(-2 + \frac{d-6}{d} \right) &= -(39 + 21z) - 6 \sum_{d=3}^{41+21z} \frac{1}{d} > \\ &> -39 - 21z - 6 \int_2^{41+21z} \frac{1}{x} dx = -39 - 21z - 6 \ln(41 + 21z) + 6 \ln 2. \end{aligned}$$

Hence, the amount of the negative charge of faces in the first column of Table 1, since they can occur z times, is at least

$$(-39 - 21z - 6 \ln(41 + 21z) + 6 \ln 2)z.$$

With similar computing, i.e. estimating of finite harmonic sums with logarithmic integrals, we obtain that the whole amount of the negative charge of the faces in the Tables 1 and 2 is at least

$$(-54.8 - 35.7z - 36 \ln(41 + 21z))z.$$

But there are also faces with negative charge incident with v_Δ – the faces listed in the Table 3. The sum of their charges can be bounded by

$$\begin{aligned} \sum_{d=3}^{41+21z} \left(-1 + \frac{d-6}{d} \right) - 6.2 &= -6 \sum_{d=3}^{41+21z} \frac{1}{d} - 6.2 > -6 \int_2^{41+21z} \frac{1}{x} - 6.2 = \\ &= -6 \ln(41 + 21z) + 6 \ln 2 - 6.2 > -6 \ln(41 + 21z) - 2.1. \end{aligned}$$

These faces can also occur z -times, thus the amount of the negative charge of them is at least $(-6 \ln(41 + 21z) - 2.1)z$. Since the sum of the charges for

whole graph is $-6e$, the (positive) charge of the vertex v_Δ can not exceed the amount of the negative charge in the whole graph increased by $-6e$:

$$\begin{aligned} w'(v_\Delta) &\leq (42 \ln(41 + 21z) + 35.7z + 56.9)z - 6e \\ \deg(v_\Delta) - 6 &\leq (42 \ln(41 + 21z) + 35.7z + 56.9)z - 6e \\ \Delta &\leq 35.7z^2 + 56.9z + 42 \ln(41 + 21z)z - 6e + 6 \end{aligned}$$

If $z = 1$, we obtain $\Delta \leq 271 - 6e$. If we sum the charge of all faces with negative charge without using logarithmic estimates, we can improve this bound to

$$\Delta \leq 247 - 6e.$$

□

Let $V(G) = \{v_1, v_2, \dots, v_n\}$ be the set of vertices of a graph G such that $\deg(v_i) \geq \deg(v_{i+1})$, $i = 1, \dots, n - 1$. We show an upper bound for the sum $\deg(v_1) + \deg(v_2)$.

Theorem 2 *Let G be a z -oblique graph imbedded into the surface with Euler's characteristic e . Let v_1 and v_2 be the vertices with the highest degree among all vertices of the graph G and let $\Delta_i = \deg(v_i)$, $i = 1, 2$. Then*

$$\Delta_1 + \Delta_2 \leq 35.7z^2 + 60.7z + 48z \ln(41 + 21z) - 6e + 12.$$

For every oblique graph imbedded on the sphere we have $\Delta_1 + \Delta_2 \leq 268$.

PROOF. We use the technique of charging and discharging again. The initial charges of the vertices and the faces are assigned in the following way:

$$\begin{aligned} w(x) &= \begin{cases} 0 & \text{if } \deg(x) < 42 + 21z, \\ \deg(x) - 6 & \text{if } x \in \{v_1, v_2\}, \\ 3z & \text{otherwise.} \end{cases} \\ w(\alpha) &= 2(\deg(\alpha) - 3) + \sum_{\substack{x \in \alpha \\ x \notin \{v_1, v_2\}}} \frac{\deg^*(x) - 6}{\deg(x)}, \end{aligned}$$

where $\deg^*(v)$ is defined as

$$\deg^*(v) = \begin{cases} \deg(x) & \text{if } \deg(x) < 42 + 21z, \\ \deg(x) & \text{if } x \in \{v_1, v_2\}, \\ \deg(x) - 3z & \text{otherwise.} \end{cases}$$

From Euler's formula we get the sum of initial charges

$$\sum_{\alpha \in F} w(\alpha) + \sum_{x \in V} w(x) = -6e.$$

In the next step we relocate some of the initial charge from vertices with big degree (distinct from v_1 and v_2) into the faces incident with them according to the following discharging rules:

- (1) if $x \notin \{v_1, v_2\}$ and $\deg(x) = d \geq 42 + 21z$ we set $w'(x) := w(x)$ and
 - (a) if $x \in \alpha$ and α is a $\langle 3, 3, d \rangle$ -face, then $w'(\alpha) := w(\alpha) + \frac{8}{7}$ and $w'(x) := w'(x) - \frac{8}{7}$,
 - (b) if $x \in \alpha$ and α is a $\langle 3, 4, d \rangle$ -face, then $w'(\alpha) := w(\alpha) + \frac{5}{7}$ and $w'(x) := w'(x) - \frac{5}{7}$,
 - (c) if $x \in \alpha$ and α is a $\langle 3, 5, d \rangle$ -face, then $w'(\alpha) := w(\alpha) + \frac{3}{7}$ and $w'(x) := w'(x) - \frac{3}{7}$,
 - (d) if $x \in \alpha$ and α is a $\langle 3, 6, d \rangle$ -face, then $w'(\alpha) := w(\alpha) + \frac{1}{7}$ and $w'(x) := w'(x) - \frac{1}{7}$,
 - (e) if $x \in \alpha$ and α is a $\langle 4, 4, d \rangle$ -face, then $w'(\alpha) := w(\alpha) + \frac{1}{7}$ and $w'(x) := w'(x) - \frac{1}{7}$,
 - (f) if $x \in \alpha$ and α is a $\langle 3, 3, 3, d \rangle$ -face, then $w'(\alpha) := w(\alpha) + \frac{1}{7}$ and $w'(x) := w'(x) - \frac{1}{7}$.
- (2) For all faces not mentioned above $w'(\alpha) := w(\alpha)$ and $w'(x) := w(x)$ for all vertices x such that $\deg(x) < 42 + 21z$ or $x \in \{v_1, v_2\}$.

The new charges of all vertices are non-negative and we have

$$w'(x) \begin{cases} = 0 & \text{if } \deg(x) < 42 + 21z, \\ = \Delta_1 - 6 & \text{if } x = v_1, \\ = \Delta_2 - 6 & \text{if } x = v_2. \\ \geq \frac{2}{7}z & \text{if } \deg(x) \geq 42 + 21z \text{ and } x \notin \{v_1, v_2\}, \end{cases}$$

Faces with degree greater than 5 can not have negative charge. Faces with negative charge can be listed like in the previous proof. There are the faces listed in Tables 1 and 2 and the faces listed in Tables 4, 5, and 6.

In the first column of the Tables 4 and 5 it is sufficient to have interval $\langle 3, 41 + 21z \rangle$, using the same arguments as in the proof of Theorem 1.

The whole negative charge of the faces not incident with the vertex v_1 or v_2 is at least $(-54.8 - 35.7z - 36 \ln(41 + 21z))z$.

But there are also faces with negative charge which are incident with vertex v_1 and/or v_2 – the faces listed in the Tables 4, 5, and 6. The sum of negative

$a_1 =$	3	4	5	3	3
$a_2 =$	$\langle 3, 41 + 21z \rangle$	$\langle 4, 12 \rangle$	$\langle 5, 7 \rangle$	3	Δ_1
$a_3 =$	Δ_1	Δ_1	Δ_1	Δ_1	3
$a_4 \leq$	-	-	-	5	5

Table 4

The list of all the faces of types $\langle a_1, a_2, a_3 \rangle$ and $\langle a_1, a_2, a_3, a_4 \rangle$ with a negative charge incident with the vertex v_1 .

$a_1 =$	3	4	5	3	3
$a_2 =$	$\langle 3, 41 + 21z \rangle$	$\langle 4, 12 \rangle$	$\langle 5, 7 \rangle$	3	Δ_2
$a_3 =$	Δ_2	Δ_2	Δ_2	Δ_2	3
$a_4 \leq$	-	-	-	5	5

Table 5

The list of all the faces of types $\langle a_1, a_2, a_3 \rangle$ and $\langle a_1, a_2, a_3, a_4 \rangle$ with a negative charge incident with the vertex v_2 .

$a_1 =$	3	4	5
$a_2 =$	Δ_2	Δ_2	Δ_2
$a_3 =$	Δ_1	Δ_1	Δ_1

Table 6

The list of all the faces of types $\langle a_1, a_2, a_3 \rangle$ with a negative charge incident with the vertices v_1 and v_2 .

charges of these faces can be bounded by the estimating of finite harmonic sums with logarithmic integrals as follows

$$(-12 \ln(41 + 21z) - 5.9)z.$$

Hence, the whole amount of the negative charge of faces of the graph is at least

$$(-60.7 - 35.7z - 48 \ln(41 + 21z))z.$$

The positive charge of the vertices v_1 and v_2 can not exceed the amount of negative charge of the faces in whole graph increased by $-6e$, therefore,

$$\begin{aligned} w'(v_1) + w'(v_2) &\leq (60.7 + 35.7z + 48 \ln(41 + 21z))z - 6e, \\ \Delta_1 + \Delta_2 &\leq (60.7 + 35.7z + 48 \ln(41 + 21z))z - 6e + 12. \end{aligned}$$

If $z = 1$, we obtain $\Delta_1 + \Delta_2 \leq 306 - 6e$. If we sum the charges of all faces with negative charge without using logarithmic estimates, we can improve this bound to

$$\Delta_1 + \Delta_2 \leq 280 - 6e.$$

□

3 Face degrees

We bound the face degree of z -oblique graphs imbedded into any given surface S .

Theorem 3 *Let G be a z -oblique graph with maximum face degree Δ_f imbedded into the surface with Euler's characteristic e . Then*

$$\Delta_f \leq 35.7z^2 + 54.8z + 36z \ln(41 + 21z) - 6e + 6.$$

For every oblique graph G imbedded on the sphere we have $\Delta_f \leq 210$.

PROOF. The proof uses the charging and discharging method. Let Δ_f denote the maximum face degree of the graph G . The initial charges of the vertices and the faces is set similarly to the charge in previous proofs:

$$w(x) = \begin{cases} 0 & \text{if } \deg(x) < 42 + 21z, \\ 3z & \text{otherwise.} \end{cases}$$

$$w(\alpha) = 2(\deg(\alpha) - 3) + \sum_{x \in \alpha} \frac{\deg^*(x) - 6}{\deg(x)},$$

where $\deg^*(v)$ is

$$\deg^*(v) = \begin{cases} \deg(x) & \text{if } \deg(x) < 42 + 21z, \\ \deg(x) - 3z & \text{otherwise.} \end{cases}$$

From Euler's formula we get the sum of the initial charges

$$\sum_{\alpha \in F} w(\alpha) + \sum_{x \in V} w(x) = -6e.$$

Discharging rules:

- (1) If $\deg(x) = d \geq 42 + 21z$ we set $w'(x) := w(x)$ and
 - (a) if $x \in \alpha$ and α is a $\langle 3, 3, d \rangle$ -face, then $w'(\alpha) := w(\alpha) + \frac{8}{7}$ and $w'(x) := w'(x) - \frac{8}{7}$,
 - (b) if $x \in \alpha$ and α is a $\langle 3, 4, d \rangle$ -face, then $w'(\alpha) := w(\alpha) + \frac{5}{7}$ and $w'(x) := w'(x) - \frac{5}{7}$,
 - (c) if $x \in \alpha$ and α is a $\langle 3, 5, d \rangle$ -face, then $w'(\alpha) := w(\alpha) + \frac{3}{7}$ and $w'(x) := w'(x) - \frac{3}{7}$,
 - (d) if $x \in \alpha$ and α is a $\langle 3, 6, d \rangle$ -face, then $w'(\alpha) := w(\alpha) + \frac{1}{7}$ and $w'(x) := w'(x) - \frac{1}{7}$,
 - (e) if $x \in \alpha$ and α is a $\langle 4, 4, d \rangle$ -face, then $w'(\alpha) := w(\alpha) + \frac{1}{7}$ and $w'(x) := w'(x) - \frac{1}{7}$,

- (f) if $x \in \alpha$ and α is a $\langle 3, 3, 3, d \rangle$ -face, then $w'(\alpha) := w(\alpha) + \frac{1}{7}$ and $w'(x) := w(x) - \frac{1}{7}$.
- (2) For all faces not mentioned above $w'(\alpha) := w(\alpha)$ and $w'(x) := w(x)$ for all vertices x such that $\deg(x) < 42 + 21z$.

The new charge of the vertices is non-negative, moreover, we have

$$w'(x) = \begin{cases} 0 & \text{if } \deg(x) < 42 + 21z, \\ \geq \frac{2}{7}z & \text{if } \deg(x) \geq 42 + 21z. \end{cases}$$

For the new charge of faces it holds $w'(\alpha) \geq \deg(\alpha) - 6$. The whole negative charge in the faces is at least $(-54.8 - 35.7z - 36 \ln(41 + 21z))z$ (see the proof of Theorem 1).

The charge of the face with the maximum degree can not be greater than the amount of negative charge in whole graph increased by $-6e$:

$$w'(\alpha) \leq (54.8 + 35.7z + 36 \ln(41 + 21z))z - 6e.$$

On the other hand

$$w'(\alpha) \geq 2\Delta_f - 6 - \Delta_f = \Delta_f - 6,$$

thus

$$\Delta_f \leq 54.8z + 35.7z^2 + 36z \ln(41 + 21z) - 6e + 6.$$

If $z = 1$, we obtain $\Delta_f \leq 245 - 6e$. But if we sum the charges of all faces with negative charge without using logarithmic estimates, we can improve this bound to

$$\Delta_f \leq 222 - 6e.$$

□

Theorem 4 *Let G be a z -oblique graph imbedded into the surface with the Euler's characteristic e . Let the α_1 and α_2 be the faces with the highest degree among all the faces of the graph G . Then*

$$\deg(\alpha_1) + \deg(\alpha_2) \leq 35.7z^2 + 54.8z + 36z \ln(41 + 21z) - 6e + 12.$$

Especially for every oblique graph G imbedded on the sphere we have $\deg(\alpha_1) + \deg(\alpha_2) \leq 228$.

This theorem can be proved using the same technique like in the proofs of the previous ones.

Corollary 5 *For any non-orientable surface N the number of all z -oblique graphs imbedded into N is finite.*

We have to mention that this result can be proven also by the technique showed by J. Schreyer in [5]. It is sufficient to observe that deciding if the graph is oblique we only need to trace the boundary walks of its faces, therefore, the proof is the same for the non-orientable surface as for the orientable surface.

PROOF. Consider all z -oblique graphs imbedded into N . From Theorem 1 and Theorem 3 we know that the vertex and the face degree of the z -oblique graphs in the surface N is bounded. Hence, also the number of face types is bounded, thus the number of faces and vertices is bounded, too. Therefore, there are only finitely many such graphs. \square

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