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Globally optimal vaccination policies in the SIR model: smoothness of the value function and uniqueness of the optimal strategies

Laetitia LAGUZET* and Gabriel TURINICI†

March 27, 2014

Abstract

This paper focuses on optimal vaccination policies for a SIR model; the total cost of the disease is optimized with respect to the cost of a vaccination strategy. We show that the value function is the unique viscosity solution of a HJB equation. This allows to find the global optimal vaccination policy. At odds with existing literature, it is seen that the value function is not always smooth (sometimes only Lipschitz) and the optimal vaccination policies are not unique. Moreover we rigorously analyze the situation when vaccination can be modeled as instantaneous (with respect to the time evolution of the epidemic) and identify the global optimum solutions.

Keywords: optimal vaccination; SIR model; vaccination region; immunization region

1 Introduction

1.1 The model

In order to model the evolution of a epidemic, we use a SIR (Susceptible - Infected - Recovered) compartment model (cf., [3, 8, 19] for additional details).

We seek to optimize the cost of the vaccination policy; to this end denote by $V(t)$ the proportion of people vaccinated by the time t (of course $\lim_{t \rightarrow \infty} V(t) \leq 1$); we consider vaccines that confer lifetime immunity so that V is an increasing function. The evolution of the disease is described by the following equations:

$$\begin{cases} dX_1(t) = -\beta X_1(t)X_2(t)dt - dV(t), & X_1(0) = X_{10}, \\ dX_2(t) = \left(\beta X_1(t)X_2(t) - \gamma X_2(t) \right) dt, & X_2(0) = X_{20}, \\ dX_3(t) = \gamma X_2(t)dt, & X_3(0) = X_{30}, \\ X_4(t) = \int_0^t dV, & X_4(0) = 0. \end{cases} \quad (1)$$

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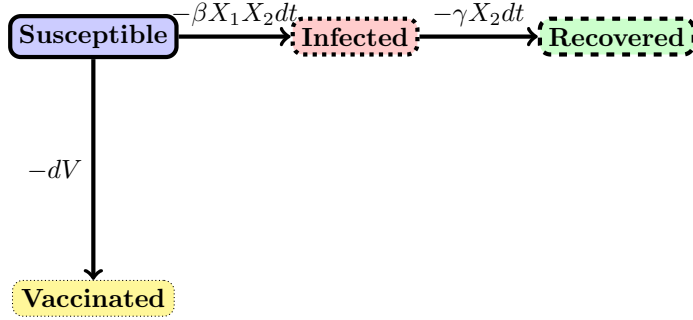


Figure 1: Graphical illustration of the SIR-V model.

Here X_1 , X_2 , X_3 , X_4 are the proportion of people in the "susceptible" respectively "infectious", "recovered" and "vaccinated" classes. Initially $X_1(0) + X_2(0) + X_3(0) = 1$ and $X_4(0) = 0$ (but X_4 need not be continuous in 0). See figure 1 for a graphical view of system (1). Note that (1) implies $X_1(t) + X_2(t) + X_3(t) + X_4(t) = 1, \forall t \geq 0$.

Here β is the transmission rate of the disease, V the control to be optimized and γ the recovery rate.

We denote r_V the unitary cost associated with vaccination including the cost of the vaccine and all possible side-effects and r_I the unitary cost incurred by infected persons. The cost of the disease is independent of the classes X_3 and X_4 (but dependent on the control $V(t)$), so we can restrict ourselves to the evolution of X_1 and X_2 . From now on a vector X will only be supposed to have two coordinates X_1 and X_2 . Denoting:

$$\Omega = \{X = (X_1, X_2) \in \mathbb{R}^2 \mid X_1, X_2 > 0, X_1 + X_2 < 1\}, \quad (2)$$

we will work under the constraints $X \in \bar{\Omega}$.

We introduce $\Phi^{Y,dV}(t) = (\Phi_1^{Y,dV}(t), \Phi_2^{Y,dV}(t))$ to denote, at time $t \geq 0$, the solution of the system (1) starting at point $X(0) = Y$ and with control dV ; in addition $Z = \Phi^{Y,dV(\cdot)}(-t)$ means $Y = \Phi^{Z,dV(t-\cdot)}(t)$ (the reverse system has a well defined mathematical meaning). To ease notations, when the measure dV is absolutely continuous with respect to the canonical Lebesgue measure dt on $[0, \infty[$ i.e., when dV can be written $dV = u(t)dt$ we will also write $\Phi^{Y,u(t)}(t)$ instead of $\Phi^{Y,u(t)dt}(t)$ (and the same for the components $\Phi_1^{Y,u(t)dt}(t)$ and $\Phi_2^{Y,u(t)dt}(t)$).

The cost of the disease is:

$$J(Y, dV) = \int_0^\infty r_I \beta \Phi_1^{Y,dV}(t) \Phi_2^{Y,dV}(t) dt + \int_0^\infty r_V dV(t). \quad (3)$$

Moreover we will use the following notation $J_0(Y) = J(Y, 0)$; note that $J_0(Y)$ is a cost proportional with the number of people infected in absence of vaccination. This number will be denoted $\zeta(Y)$ thus $J_0(Y) = r_I \zeta(Y)$ (see Appendix A for the properties of ζ).

Remark 1. Equation (1) implies

$$\begin{aligned}\Phi_2^{X,dV}(\infty) &= \Phi_2^{X,dV}(0) + \int_0^\infty d\Phi_2^{X,dV}(t) \\ &= \Phi_2^{X,dV}(0) + \int_0^\infty \left(\beta\Phi_1^{X,dV}(t)\Phi_2^{X,dV}(t) - \gamma\Phi_2^{X,dV}(t) \right) dt\end{aligned}\quad (4)$$

Thus, since $\Phi_2^{X,dV}(\infty) = 0$:

$$\int_0^\infty r_I \beta \Phi_1^{X,dV}(t) \Phi_2^{X,dV}(t) dt = \int_0^\infty r_I \gamma \Phi_2^{X,dV}(t) dt - \Phi_2^{X,dV}(0). \quad (5)$$

This allows to conclude that the cost functional

$$J^d(Y, dV) = \int_0^\infty r_I^d \Phi_2^{Y,dV}(t) dt + \int_0^\infty r_V dV(t) \quad (6)$$

with $r_I^d = r_I \gamma$ satisfies

$$J^d(Y, dV) = J(Y, dV) + Y_2. \quad (7)$$

Both J^d and J will thus have same optimal strategies (because their difference is independent of the strategy dV). Here r_I^d can be seen as the unitary cost of infection per unit time.

1.2 The admissible vaccination policies

Vaccination policy dV can be modeled in different ways. Note that the proportion $\int_0^t dV(s)$ of individuals vaccinated up to time "t" is increasing and $\int_0^t dV(s) \leq 1, \forall t \geq 0$; therefore V is a bounded variation function and $dV(t)$ is a positive measure on $[0, \infty[$; this is the most general class of vaccination strategies. A restrictive class of vaccination policies will also be considered (see also the literature review in Section 1.4 below) where the speed of vaccination is bounded; in this case $dV(t) = u(t)dt$ with $u(t) \in [0, u_{max}]$. Generic results (see e.g., [4]) suggest that considering controls with bounded speed is not restrictive because the general situation is obtained in the limit $u_{max} \rightarrow \infty$. We will rigorously prove this assertion in Section 3 and will work with the restricted class of vaccination policies until then.

Finally, in order to give a meaning to the system (1) we can write it as:

$$d \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} -\beta X_1 X_2 \\ \beta X_1 X_2 - \gamma X_2 \\ \gamma X_2 \\ 0 \end{pmatrix} dt + \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} dV(t). \quad (8)$$

Since,

- $(X_1, X_2, X_3, X_4) \mapsto (-\beta X_1 X_2, \beta X_1 X_2 - \gamma X_2, \gamma X_2, 0)^T$ and $(X_1, X_2, X_3, X_4) \mapsto (-1, 0, 0, 1)^T$ are Lipschitz functions,
- V is a bounded variation function,

results from [13] and [21] can be invoked to conclude that (8) has a solution and the solution depends smoothly on the initial data and the control V .

Let us make clear how a mathematical object such as V can be translated into vaccination policies for the unbounded case. Take for instance the trajectory $\Phi^{Y,dV}(t)$ driven by the control (here $\delta_{t=0}$ is the Dirac mass in $t = 0$):

$$dV(t) = \begin{cases} \frac{Y_1}{2} \delta_{t=0}, & t = 0 & (9a) \\ 0.10, & t \in]0, 0.5[& (9b) \\ 0, & t \geq 0.5. & (9c) \end{cases}$$

This means that half of the initial susceptible population Y_1 is vaccinated (instantaneously) at the onset $t = 0$. Then vaccination is pursued with speed of 10% percent per unit time till time $t = 0.5$; then no vaccination occurs. In particular this means that $50 + 0.5 \times 10 = 55$ percents of the population is vaccinated in all. Note that the trajectory $\Phi^{Y,dV}(t)$ is not continuous since $\Phi_1^{Y,dV}(0^+) = \Phi_1^{Y,dV}(0)/2$. This trajectory can be seen as the limit when $\epsilon \rightarrow 0$ of the trajectories $\Phi^{Y,dV_\epsilon}(t)$ corresponding to the following vaccination policies:

$$dV_\epsilon(t) = \begin{cases} \frac{Y_1}{2\epsilon}, & t \in [0, \epsilon] & (10a) \\ 0.10, & t \in]\epsilon, 0.5[& (10b) \\ 0, & t \geq 0.5. & (10c) \end{cases}$$

1.3 Notations and first remarks

We introduce the function $f : \bar{\Omega} \times \mathbb{R} \mapsto \mathbb{R}^2$:

$$(X, u) \in \bar{\Omega} \times \mathbb{R} \mapsto f(X, u) = (-\beta X_1 X_2 - u, \beta X_1 X_2 - \gamma X_2) \in \mathbb{R}^2. \quad (11)$$

Note that $f(\cdot, u)$ is a Lipschitz function with Lipschitz constant L_f independent of the second argument, i.e.,

$$\|f(Y, u) - f(Z, u)\| \leq L_f \|Y - Z\|, \quad \forall Y, Z \in \bar{\Omega}. \quad (12)$$

In order to define the admissible controls we consider a point $Y \in \Omega$; for $u_{max} < \infty$ we define:

$$\mathcal{U}_Y^{u_{max}} = \left\{ u : [0, \infty[\rightarrow [0, u_{max}] \mid u \text{ measurable, } \Phi^{Y,u(\cdot)}(t) \in \bar{\Omega}, \forall t \geq 0 \right\}. \quad (13)$$

When $u_{max} = \infty$ we define:

$$\mathcal{U}_Y^\infty = \left\{ dV \text{ positive measure on } [0, \infty[\mid \int_0^\infty dV \leq Y_1 \leq 1, \Phi^{Y,dV}(t) \in \bar{\Omega}, \forall t \geq 0 \right\}. \quad (14)$$

Irrespective of whether u_{max} is bounded or not the set $\mathcal{U}_Y^{u_{max}}$ is a closed subset of the set of (finite, positive) measures on $[0, \infty[$. Note that for any $Y \in \Omega$ and any $u_{max} : 0 \in \mathcal{U}_Y^{u_{max}}$.

To make notations easier we will not write the dependence of $\mathcal{U}_Y^{u_{max}}$ with respect to Y or u_{max} and only denote, when there is no ambiguity, by \mathcal{U}_Y or \mathcal{U} the set of admissible controls.

For $u_{max} < \infty$ we define the Hamiltonian $\mathcal{H}^{u_{max}} : \bar{\Omega} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ as:

$$\mathcal{H}^{u_{max}}(X, p) = \min_{w \in [0, u_{max}]} [p \cdot f(X, w) + r_I \beta X_1 X_2 + r_V w] \quad (15)$$

$$= -u_{max}(p_1 - r_V)_+ + \beta X_1 X_2 (r_I + p_2 - p_1) - \gamma X_2 p_2. \quad (16)$$

When $u_{max} = \infty$ the previous definition is to be replaced by $\mathcal{H}^\infty : \bar{\Omega} \times \mathbb{R}^2 \rightarrow \mathbb{R}$:

$$\mathcal{H}^\infty(X, p) = \min \{r_V - p_1, \beta X_1 X_2 (r_I + p_2 - p_1) - \gamma X_2 p_2\}. \quad (17)$$

The value function $\mathcal{V}^{u_{max}} : \bar{\Omega} \rightarrow \mathbb{R}$ is (for any u_{max} be it bounded or not):

$$\mathcal{V}^{u_{max}}(Y) = \inf_{u \in \mathcal{U}_Y^{u_{max}}} J(Y, u). \quad (18)$$

Any u such that $J(Y, u) = \mathcal{V}^{u_{max}}(Y)$ is called an optimal strategy for Y ; it is not necessarily unique. However it has been proved in [6] that if $u_{max} < \infty$ at least one optimal strategy exists in the set $\mathcal{U}_Y^{u_{max}}$ and has the form $u = u_{max} \mathbb{1}_{[0, \theta(Y)]}$ with $\theta(Y) \geq 0$. In fact since the total proportion of people susceptible to be vaccinated is at most 1 then $\theta(Y) \leq T_{max} := 1/u_{max}$. From now on we fix $\theta : \bar{\Omega} \rightarrow [0, \infty[$ to be a function (whose existence is guaranteed by the above mentioned result) such that $u_{max} \mathbb{1}_{[0, \theta(Y)]}$ is an optimal strategy for Y .

We introduce the following notations:

- $A = (\frac{\gamma}{\beta}, 0) \in \mathbb{R}^2$,
- $\Gamma_1 = \{(X_1, X_2) \in \bar{\Omega} \mid X_1 + X_2 = 1\}$,
- $\Gamma_I = \{(X_1, X_2) \in \bar{\Omega} \mid X_1 = 0\}$,
- $\Gamma_S = \{(X_1, X_2) \in \bar{\Omega} \mid X_2 = 0\}$,
- $\Gamma_{OA} = \{(X_1, X_2) \in \Gamma_S \mid 0 \leq X_1 \leq \frac{\gamma}{\beta}\}$,
- $\Gamma_{A1} = \{(X_1, X_2) \in \Gamma_S \mid \frac{\gamma}{\beta} \leq X_1 \leq 1\}$.

Note that when $\gamma/\beta > 1$: $A \notin \Omega$, $\Gamma_{OA} = \Gamma_S$ and $\Gamma_{A1} = \emptyset$.

Lemma 1.1. *The value function $\mathcal{V}^{u_{max}}$ is bounded on Ω . Moreover $\mathcal{V}^{u_{max}}|_{\Gamma_I \cup \Gamma_{OA}} = 0$ and $\mathcal{V}^{u_{max}}$ is continuous on $\Gamma_I \cup \Gamma_{OA}$.*

Proof. Choose $u = 0$ then

$$\mathcal{V}^{u_{max}}(X) \leq J(X, 0) = J_0(X) \leq r_I X_1 \leq r_I, \quad \forall X \in \Omega. \quad (19)$$

Note that $J(X, u) = 0 \quad \forall X \in \Gamma_I, \forall u \in \mathcal{U}_X$; using (19) we obtain $\mathcal{V}^{u_{max}}(X) = 0 \quad \forall u_{max} \in [0, \infty], \forall X \in \Gamma_I$ and the continuity on Γ_I . To set the value on Γ_{OA} note that when X is such that $X_1 < \gamma/\beta$ then $\Phi_2^{X,0}(t)$ tends exponentially to zero. Therefore: $J(X_n, 0) \rightarrow 0$ when $X_n \rightarrow X \in \Gamma_{OA}$.

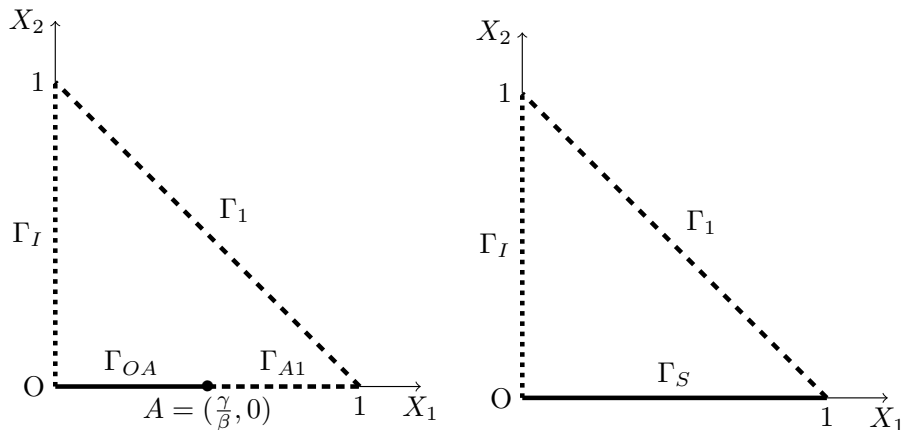


Figure 2: Boundary representation when $\frac{\gamma}{\beta} < 1$ (left) and $\frac{\gamma}{\beta} \geq 1$ (right).

1.4 Literature review

A rich diversity of epidemic models have been proposed in order to describe epidemic propagation (see [3, 8, 19] for details). Building on these models, strategies have been documented to control the propagation, among which isolating infected people or immunizing susceptible people; see [16, 6, 10] which also propose combination of these two methods. In this work we will only analyze control policies that consist in the vaccination of susceptibles (immunization). The vaccination is supposed to confer lifetime (i.e., irreversible) immunity. In the context of immunization, several facts can affect the decisions of vaccination. The reference [9] discusses this problem in general, [20] proposes an approach taking the individual point of view, and [17] introduces an extension also using game theory.

The present work is on the contrary only concerned with finding an optimal vaccination strategy. Several studies have already considered this approach recasting it as an optimal control problem.

Historically one of the first to consider this problem, Abakuks explores two alternatives: in [2] a restrictive class of vaccination policies which allows at any time to immunize either all or none of the susceptible (therefore optimal policy immunizes either at once or never); in [1] the author considers policies which at any time during the course of the epidemic allow to immunize any number of the susceptible.

Abakuks proves the existence of an immunization region: within this region it is best to vaccinate with maximum effort and outside it is optimal to do nothing. The result is only obtained for $u_{max} = \infty$; moreover the proof only applies to vaccination policies dV that are finite sums of Dirac masses and it is not indicated how the value function \mathcal{V}^∞ (assumed to be continuous) behave in the limit when Dirac masses accumulate near a point or when such masses converge to a general measure on $[0, \infty[$.

In another work (see [11]) Hethcote considers a similar problem under additional constraints on the total proportion of the population affected and the

maximum number of infected at the peak; the vaccination policies are taken to be stepwise constant functions and the cost of vaccination piecewise quadratic in the number of people vaccinated. He shows that the optimal strategy will be piecewise constant, with at most a single point of discontinuity.

In a similar work [14] authors consider $u_{max} = 1$ and define the class of admissible policies to contain function with only isolated discontinuities. They show that the optimal strategy has a single point of discontinuity and introduce the concept of vaccination border. To do this, they assume that the value function $\mathcal{V}^{u_{max}}$ is $\mathcal{C}^1(\Omega)$ which, as it will be seen in the following, is not always the case (it depends on the specific choice of parameters $\beta, \gamma, u_{max}, r_V, r_I$).

In [16] authors set $u_{max} < \infty$ for a finite horizon framework $T < \infty$ and work under the additional presence of a dumping term e^{-rt} in the cost functional which reads: $\int_0^T e^{-rt} (r_V u(t) + r_I \Phi_2^{X, u(t)}(t)) dt$; moreover the infected are supposed to pay a infection cost per unit time up to the time T and nobody recovers before time T , i.e., with our notations $\gamma = 0$. They use the maximum principle to characterize the optimal policies which turn out to be of bang-bang type with only one switch.

In [12] the existence and local optimality of singular controls is investigated and using the Maximum Principle it is shown that the optimal vaccination schedule can be singular. This corresponds to our limit $u_{max} \rightarrow \infty$. However no information is obtained on the regularity of the value function.

In the references described so far the authors focused on the optimal strategy without studying the properties of the value function. Using a similar model and an approach via optimal control [15] finds, via a Bellman equation, that the strategy is type bang-bang (only values 0 and u_{max} are taken). However they assume that the cost function is $\mathcal{C}^1(\Omega)$; finally, the results in the case where $u_{max} \rightarrow \infty$ are extrapolated and they suppose that the optimal strategy is bang-bang. As such the optimal policies are sometimes at odd with results in the stochastic case.

In a recent work H. Behncke (see [6]) proves, without using that value function is $\mathcal{C}^1(\Omega)$, that at least one optimal strategy for the trajectory starting at $X \in \bar{\Omega}$ is of the form $u_{max} \mathbf{1}_{[0, \theta(X)]}$, $\theta(X) \geq 0, \forall X \in \bar{\Omega}$. Although this information is very useful it does not allow to conclude on the regularity of the value function. As an illustration, we plot two situations: with parameters in figure 3 the function $\theta(X)$ is $\mathcal{C}^1(\bar{\Omega})$ while with parameters in figure 4 the function $\theta(X)$ is discontinuous.

Finally, without specifically entering in the context of epidemiology but using a general optimal control framework and the concept of viscosity solution the reference [18] analyzes the properties of the value function in the situation when a discount factor is present.

Considering the previous works several questions arise:

1. For which set of parameters $(\beta, \gamma, u_{max}, r_V, r_I)$ is the value function $\mathcal{V}^{u_{max}}$ of class $\mathcal{C}^1(\Omega)$ and when is it less regular; note that if the value function $\mathcal{V}^{u_{max}}$ is not \mathcal{C}^1 some vaccination strategies derived under the \mathcal{C}^1 hypothesis will not be globally optimal.
2. Are the optimal strategies unique ?
3. What happens when $u_{max} = \infty$ (i.e., when vaccination is fast with respect to the epidemic propagation).

Our work answers these questions. In particular we show that value function is not always \mathcal{C}^1 , the optimal strategies not always unique and prove rigorously what happens in the limit $u_{max} \rightarrow \infty$.

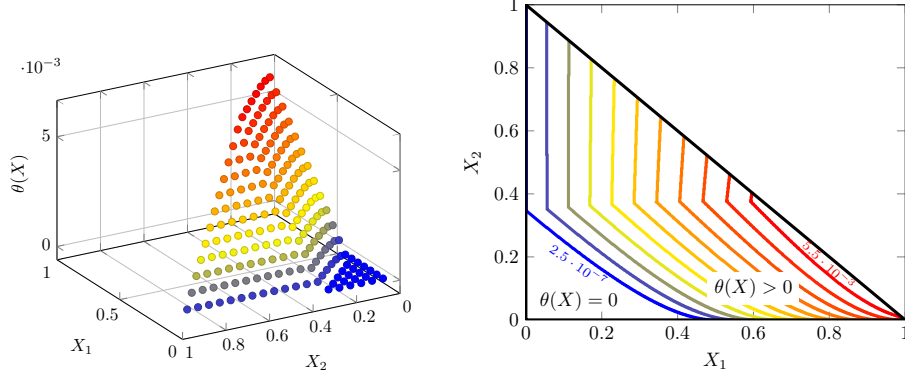


Figure 3: $\theta(X)$ for parameters $u_{max} = 100$, $r_V = 0.5$, $r_I = 1$, $\beta = 73$, $\gamma = 36.5$. **Left:** Representation as 3D function. **Right:** representation as level lines. We observe that θ is regular.

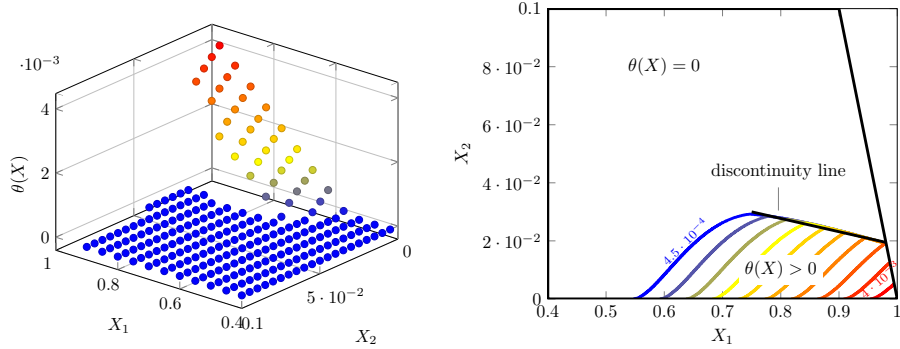


Figure 4: $\theta(X)$ for parameters $u_{max} = 100$, $r_V = 1.4$, $r_I = 1$, $\beta = 73$, $\gamma = 36.5$. **Left:** Representation as 3D function. **Right:** representation as level lines. In both cases we zoom on the discontinuity curve and plot $\Omega \cap ([0.4, 1] \times [0, 0.1])$. We observe that θ is discontinuous.

1.5 Specific mathematical difficulties of the problem

The approach proposed in this work faces specific technical difficulties among which:

- There do not exist natural boundary conditions to set on some parts of the frontier (Γ_1 and Γ_{A1}). This will pose problem when proving the uniqueness of the solution of the associated HJB equation. See section 2.3 for the technique used to mitigate this difficulty.

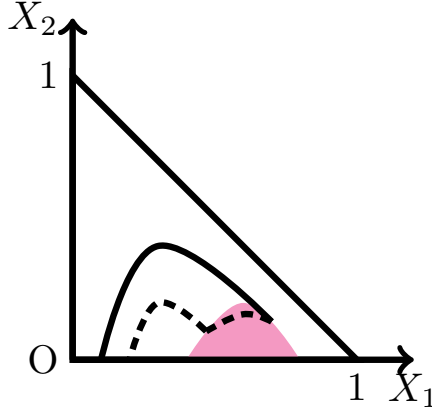


Figure 5: A typical example of non-unique optimal vaccination strategy: the solid trajectory corresponds to zero vaccination while the dashed trajectory corresponds to vaccination in the colored region followed by non vaccination. But both trajectories lead to the same, minimal, cost. In this case we expect the value function to not be of class C^1 . Non uniqueness appears when the trajectory with zero vaccination does not enter the vaccination region while the trajectory with maximal vaccination enters it. See Sections 2.4, 2.5, 3.3 and 3.4 for details.

- The state X is restricted to belong to $\bar{\Omega}$ while the controls e.g., in the form $dV = udt$, $u \in [0, u_{max}]$ can drive it outside this domain.
- The cost function $J(X_0, dV)$ has no dumping term e^{-rt} , so we need to work in infinite horizon. This is a problem when trying to obtain Lipschitz regularity for the value function. See section 2.2.
- In general, a convenient hypothesis (cf. also [7]) to prove the uniqueness of the viscosity solution of $\mathcal{F}(x, F(x), \nabla F(x)) = 0$ is that the Hamiltonian \mathcal{F} be **strictly** monotone in the second argument. But here our Hamiltonians do not depend on this argument.
- In general optimal controls are unique (and the value function differentiable). Here this is not the case (cf. figure 5) which hints that value function has regularity defects.

1.6 An introduction to viscosity solutions

This section is largely based on classical works such as [7], [4], [5] [18]. We refer the reader to these works for additional details.

Let $\xi : \mathcal{O} \rightarrow \mathbb{R}$ be a scalar function defined on an open set $\mathcal{O} \subseteq \mathbb{R}^n$.

Definition 1.1. The set of super-differentials of ξ at a point $x \in \mathcal{O}$ is:

$$D^+\xi(x) = \left\{ p \in \mathbb{R}^n; \limsup_{y \rightarrow x} \frac{\xi(y) - \xi(x) - p \cdot (y - x)}{|y - x|} \leq 0 \right\}. \quad (20)$$

Similarly, the set of sub-differentials of ξ at a point $x \in \mathcal{O}$ is:

$$D^-\xi(x) = \left\{ p \in \mathbb{R}^n; \limsup_{y \rightarrow x} \frac{\xi(y) - \xi(x) - p \cdot (y - x)}{|y - x|} \geq 0 \right\}. \quad (21)$$

We will also use the following:

Lemma 1.2. *Let $\xi \in \mathcal{C}(\mathcal{O})$. Then*

1. $p \in D^+\xi(x)$ if and only if there exists a function $\phi \in \mathcal{C}^1(\mathcal{O})$ such that $\nabla\phi(x) = p$ and $\xi - \phi$ has a local maximum at x .
2. $p \in D^-\xi(x)$ if and only if there exists a function $\phi \in \mathcal{C}^1(\mathcal{O})$ such that $\nabla\phi(x) = p$ and $\xi - \phi$ has a local minimum at x .

In the following, we consider the first order partial differential equation:

$$\mathcal{F}(x, \xi(x), \nabla\xi(x)) = 0, \quad (22)$$

defined on an open set $\mathcal{O} \in \mathbb{R}^n$. Here, $\mathcal{F} : \mathcal{O} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuous (possibly nonlinear) function.

Definition 1.2. A function $\xi \in \mathcal{C}(\mathcal{O})$ is a viscosity subsolution of (22) if

$$\mathcal{F}(x, \xi(x), p) \leq 0 \text{ for every } x \in \mathcal{O}, p \in D^+\xi(x). \quad (23)$$

Similarly, $\xi \in \mathcal{C}(\mathcal{O})$ is a viscosity supersolution of (22) if

$$\mathcal{F}(x, \xi(x), p) \geq 0 \text{ for every } x \in \mathcal{O}, p \in D^-\xi(x). \quad (24)$$

Finally, we call ξ a viscosity solution of (22) if it is both a supersolution and a subsolution in the viscosity sense.

Remark 2. For each particular problem we will explicitly specify latter the boundary conditions.

2 Bounded vaccination speed ($u_{max} < \infty$)

In this section we assume that $u_{max} < \infty$.

2.1 Properties of the value function

Theorem 2.1. *The value function $\mathcal{V}^{u_{max}} : \Omega \rightarrow \mathbb{R}$ is a Lipschitz function in Ω . It can uniquely be extended to a Lipschitz function on $\bar{\Omega}$.*

Proof. We first prove that for a fixed control u and time t the function

$$\{Y \in \Omega \mid u \in \mathcal{U}_Y\} \ni Y \mapsto \Phi^{Y,u}(t),$$

is Lipschitz with the Lipschitz constant independent of u . We write:

$$\begin{aligned} \left\| \frac{d}{dt} \Phi^{Y,u}(t) - \frac{d}{dt} \Phi^{Z,u}(t) \right\| &= \left\| f(\Phi^{Y,u}(t), u) - f(\Phi^{Z,u}(t), u) \right\| \\ &\leq L_f \left\| \Phi^{Y,u}(t) - \Phi^{Z,u}(t) \right\|. \end{aligned}$$

where L_f is the constant in equation (12). Then:

$$\frac{d}{dt} (\|\Phi^{Y,u}(t) - \Phi^{Z,u}(t)\|^2) \leq 2L_f \|\Phi^{Y,u}(t) - \Phi^{Z,u}(t)\|^2. \quad (25)$$

Using the Gronwall Lemma and taking the square root, we obtain:

$$\|\Phi^{Y,u}(t) - \Phi^{Z,u}(t)\| \leq \|Y - Z\| e^{L_f T_{max}}.$$

Fix $Y, Z \in \Omega$ and denote by $u_Y = u_{max} \cdot \mathbb{1}_{[0, \theta(Y)]}$ one optimal control of the trajectory leaving from Y . Then if $u_Y \in \mathcal{U}_Z$ and $u_Z \in \mathcal{U}_Y$:

$$\begin{aligned} \mathcal{V}^{u_{max}}(Z) &\leq J(Z, u_Y) \\ &\leq \int_0^{T_{max}} r_I \beta \Phi_1^{Z, u_Y}(t) \Phi_2^{Z, u_Y}(t) + r_V u_Y(t) dt + J_0(\Phi^{Z, u_Y}(T_{max})) \\ &\leq \int_0^{T_{max}} r_I \beta \Phi_1^{Y, u_Y}(t) \Phi_2^{Y, u_Y}(t) + r_V u_Y(t) dt + J_0(\Phi^{Y, u_Y}(T_{max})) \\ &\quad + C_{u_Y, T_{max}} \|Y - Z\| = \mathcal{V}^{u_{max}}(Y) + C_{u_Y, T_{max}} \|Y - Z\|. \end{aligned}$$

Note that u_Y is member of the compact set $\{u : [0, \infty] \rightarrow \mathbb{R} \mid u = u_{max} \mathbb{1}_{[0, \theta]}, \theta \leq T_{max}\}$. Thus the constant $C_{u_Y, T_{max}}$ only depends on T_{max} (and not on Y or Z). Changing the roles of Y and Z we obtain the reverse inequality thus the conclusion.

If $u_Y \notin \mathcal{U}_Z$ or $u_Z \notin \mathcal{U}_Y$, suppose, to fix notations, that $u_Y \notin \mathcal{U}_Z$; since $u_Y = u_{max} \cdot \mathbb{1}_{[0, \theta(Y)]}$ then $u_Y \notin \mathcal{U}_Z$ implies $\theta(Y) > \theta(Z)$ thus $u_Z \in \mathcal{U}_Y$. Take $\eta \in [\theta(Z), \theta(Y)]$ to be the maximum value such that $u_{max} \mathbb{1}_{[0, \eta]} \in \mathcal{U}_Z \cap \mathcal{U}_Y$. The maximality implies $\Phi^{Z, u_{max} \mathbb{1}_{[0, \eta]}}(\eta) \in \Gamma_I$. Using Lipschitz estimates for $\Phi^{\cdot, u_{max} \mathbb{1}_{[0, \eta]}}(t)$ we obtain as above:

$$\begin{aligned} \mathcal{V}^{u_{max}}(Z) &\leq J(Z, u_{max} \mathbb{1}_{[0, \eta]}) \leq C_{T_{max}} (\|Y - Z\|) + \mathcal{V}^{u_{max}}(\Phi^{Z, u_{max} \mathbb{1}_{[0, \eta]}}(\eta)) \\ &\quad + \mathcal{V}^{u_{max}}(Y) - \mathcal{V}^{u_{max}}(\Phi^{Y, u_{max} \mathbb{1}_{[0, \eta]}}(\eta)) \\ &\leq \mathcal{V}^{u_{max}}(Y) + C_{T_{max}} (\|Y - Z\|) + 0 + 0 \end{aligned}$$

where we used the fact that $X \in \Gamma_I$ implies $\mathcal{V}^{u_{max}}(X) = 0$ and that $\mathcal{V}^{u_{max}}$ is positive. From now on we continue as above and obtain the Lipschitz property for Y and Z .

Since $\mathcal{V}^{u_{max}}$ is a Lipschitz function on Ω with bounded Lipschitz constant it admits a unique Lipschitz extension over $\bar{\Omega}$.

2.2 The HJB equation and value function

Theorem 2.2. *The value function $\mathcal{V}^{u_{max}}$ is a viscosity solution of the Hamilton-Jacobi-Bellman (HJB) equation:*

$$(\mathcal{P}) \begin{cases} \text{Find } F : \bar{\Omega} \rightarrow \mathbb{R} \text{ such that} \\ F \text{ is Lipschitz on } \bar{\Omega}, & (26a) \\ -\mathcal{H}^{u_{max}}(X, \nabla F(X)) = 0, X \in \Omega, & (26b) \\ F(X) = 0 \text{ on } \Gamma_{O_A}, & (26c) \\ F(X) = 0 \text{ on } \Gamma_I, & (26d) \\ -\mathcal{H}^{u_{max}}(X, \nabla F(X)) = 0 \text{ on } \Gamma_1. & (26e) \end{cases}$$

Remark 3. There is no boundary condition on Γ_{A1} .

Proof. Using Lemma 1.2 and Definition 1.2 we first show that $\mathcal{V}^{u_{max}}$ is a subsolution of (26) then we will show it is also a supersolution.

Step 1. Let $Y \in \Omega$ and $\varphi \in \mathcal{C}^1(\Omega)$ such that $\mathcal{V}^{u_{max}}(Y) - \varphi(Y)$ attains a local maximum at Y . So for Z in a neighborhood of Y :

$$\mathcal{V}^{u_{max}}(Y) - \mathcal{V}^{u_{max}}(Z) \geq \varphi(Y) - \varphi(Z). \quad (27)$$

We will prove that:

$$-\mathcal{H}^{u_{max}}(Y, \nabla\varphi(Y)) \leq 0. \quad (28)$$

This is equivalent to:

$$\begin{aligned} -u_{max}(\partial_{X_1}\varphi(Y) - r_V)_+ + \beta Y_1 Y_2 (r_I + \partial_{X_2}\varphi(Y) - \partial_{X_1}\varphi(Y)) \\ - \gamma Y_2 \partial_{X_2}\varphi(Y) \geq 0. \end{aligned} \quad (29)$$

Assume that it is not the case. Then there exists, by continuity, a value $w \in [0, u_{max}]$ (see Remark 4 page 14 below) and a constant $\kappa > 0$ such that in a neighborhood of Y :

$$\begin{aligned} w(r_V - \partial_{X_1}\varphi(\Phi^{Y,u}(\cdot))) + \beta \Phi_1^{Y,u}(\cdot) \Phi_2^{Y,u}(\cdot) (r_I + \partial_{X_2}\varphi(\Phi^{Y,u}(\cdot)) - \partial_{X_1}\varphi(\Phi^{Y,u}(\cdot))) \\ - \gamma \Phi_2^{Y,u}(\cdot) \partial_{X_2}\varphi(\Phi^{Y,u}(\cdot)) \leq -\kappa, \end{aligned} \quad (30)$$

for any $u(t) \in [0, u_{max}]$. Let $u = w$ on the interval $[0, \delta]$ (since $Y \in \Omega$, for a small $\delta > 0$ $u \in \mathcal{U}_Y$) and denote $Z^0 = \Phi^{Y,w}(\delta)$. Then, choosing $Z = Z^0$ in (27) we obtain:

$$\begin{aligned} \mathcal{V}^{u_{max}}(Z^0) - \mathcal{V}^{u_{max}}(Y) &\leq \varphi(Z^0) - \varphi(Y) = \int_0^\delta \frac{d}{dt} \varphi(\Phi^{Y,w}(t)) dt \quad (31) \\ &\leq \int_0^\delta \langle \nabla\varphi(\Phi^{Y,w}(t)), f(\Phi^{Y,w}(t), w) \rangle dt \\ &\leq \int_0^\delta -\kappa - \beta \Phi_1^{Y,w}(t) \Phi_2^{Y,w}(t) r_I - w r_V dt \\ &\leq -\delta\kappa - \int_0^\delta \beta \Phi_1^{Y,w}(t) \Phi_2^{Y,w}(t) r_I + w r_V dt. \end{aligned} \quad (32)$$

Or, by the definition of the optimality of $\mathcal{V}^{u_{max}}$ in Y :

$$\begin{aligned} \mathcal{V}^{u_{max}}(Y) &\leq \int_0^\delta r_I \beta \Phi_1^{Y,w}(t) \Phi_2^{Y,w}(t) + r_V w dt + \mathcal{V}^{u_{max}}(Z^0) \\ \mathcal{V}^{u_{max}}(Y) - \mathcal{V}^{u_{max}}(Z^0) &\leq \int_0^\delta r_I \beta \Phi_1^{Y,w}(t) \Phi_2^{Y,w}(t) + r_V w dt, \end{aligned}$$

by summing the inequality we get $0 < -\kappa\delta$, which is absurd.

Therefore using Lemma 1.2 we obtain:

$$-\mathcal{H}^{u_{max}}(X, \nabla\mathcal{V}^{u_{max}}(X)) \leq 0 \text{ for all } X \in \Omega.$$

To prove (26e) we use appendix B where we prove that trajectories $\Phi^{Y,u}(\cdot)$ with $Y \in \Gamma_1$ are strictly entering the domain Ω for all $w \in [0, u_{max}]$. For this reason when $Y \in \Gamma_1$ $\Phi^{Y,w}(t) \notin \Gamma_1$ for $t \in]0, \delta]$. Moreover, we choose φ such as φ is $\mathcal{C}(\bar{\Omega})$ and $\mathcal{C}^1(\Omega)$. These arguments allow to go from equation (31) to equation (32). Therefore the proof remains true on Γ_1 so:

$$-\mathcal{H}^{u_{max}}(X, \nabla \mathcal{V}^{u_{max}}(X)) \leq 0 \text{ for all } X \in \Gamma_1.$$

By Lemma 1.1, we have that $\mathcal{V}^{u_{max}}$ is bounded on $\bar{\Omega}$ and by Theorem 2.1 $\mathcal{V}^{u_{max}}$ is a Lipschitz function. By definition of $\mathcal{V}^{u_{max}}$ we have $\mathcal{V}^{u_{max}}(X) = 0$ on Γ_I and Γ_{OA} . So $\mathcal{V}^{u_{max}}$ is a subsolution of (26).

Step 2. Now we prove that $\mathcal{V}^{u_{max}}$ is a supersolution of (26).

Let $Y \in \Omega$ and $\varphi \in \mathcal{C}^1(\Omega)$ such that $\mathcal{V}^{u_{max}}(Y) - \varphi(Y)$ attains a local minimum at Y . So for Z in a neighborhood of Y :

$$\mathcal{V}^{u_{max}}(Y) - \varphi(Y) \leq \mathcal{V}^{u_{max}}(Z) - \varphi(Z). \quad (33)$$

We will show that:

$$-\mathcal{H}^{u_{max}}(Y, \nabla \varphi(Y)) \geq 0 \quad \forall Y \in \Omega. \quad (34)$$

Assume that it is not the case. Then there exists $\kappa > 0$ such that $\mathcal{H}^{u_{max}}(Y, \nabla \varphi(Y)) > \kappa$ in a neighborhood of Y . So there exists (a small) $\delta > 0$ such that for any $u(t) \in [0, u_{max}]$:

$$\begin{aligned} -u_{max}(\partial_{X_1} \varphi(\Phi^{Y,u}(\cdot)) - r_V)_+ + \beta \Phi_1^{Y,u}(\cdot) \Phi_2^{Y,u}(\cdot) (r_I + \partial_{X_2} \varphi(\Phi^{Y,u}(\cdot)) - \partial_{X_1} \varphi(\Phi^{Y,u}(\cdot))) \\ - \gamma \Phi_2^{Y,u}(\cdot) \partial_{X_2} \varphi(\Phi^{Y,u}(\cdot)) > \kappa \quad \forall t \leq \delta. \end{aligned} \quad (35)$$

Let w be a control in \mathcal{U}_Y and $Z^0 = \Phi^{Y,w}(\delta)$ (for small δ any $w \in [0, u_{max}]$ is in \mathcal{U}_Y). Then:

$$\begin{aligned} \mathcal{V}^{u_{max}}(Z^0) - \mathcal{V}^{u_{max}}(Y) &\geq \varphi(Z^0) - \varphi(Y) \\ &= \int_0^\delta \langle \nabla \varphi(\Phi^{Y,w}(t)), f(\Phi^{Y,w}(t), w) \rangle dt \\ &\geq \delta \kappa + \int_0^\delta u_{max}(\partial_{X_1} \varphi(\Phi^{Y,w}(t)) - r_V)_+ - r_I \beta \Phi_1^{Y,w}(t) \Phi_2^{Y,w}(t) - w \partial_{X_1} \varphi(\Phi^{Y,w}(t)) dt \\ &\geq \delta \kappa - \int_0^\delta w r_V + r_I \beta \Phi_1^{Y,w}(t) \Phi_2^{Y,w}(t) dt \\ &\quad + \int_0^\delta u_{max}(\partial_{X_1} \varphi(\Phi^{Y,w}(t)) - r_V)_+ - w(\partial_{X_1} \varphi(\Phi^{Y,w}(t)) - r_V) dt \\ &\geq \delta \kappa - \int_0^\delta w r_V + r_I \beta \Phi_1^{Y,w}(t) \Phi_2^{Y,w}(t) dt. \end{aligned}$$

(because $\int_0^\delta u_{max}(\partial_{X_1} \varphi(\Phi^{Y,w}(t)) - r_V)_+ - w(\partial_{X_1} \varphi(\Phi^{Y,w}(t)) - r_V) dt \geq 0$ since $w \in [0, u_{max}]$).

So, for any w , we have:

$$\mathcal{V}^{u_{max}}(Z^0) + \int_0^\delta w r_V + r_I \beta \Phi_1^{Y,w}(t) \Phi_2^{Y,w}(t) dt \geq \mathcal{V}^{u_{max}}(Y) + \delta \kappa. \quad (36)$$

Taking the infimum with respect to w we obtain $\mathcal{V}^{u_{max}}(Y) \geq \mathcal{V}^{u_{max}}(Y) + \kappa\delta$. This is absurd, therefore $\mathcal{V}^{u_{max}}$ is a supersolution on Ω .

For the same reasons as previously, we have $-\mathcal{H}^{u_{max}}(X, \nabla \mathcal{V}^{u_{max}}(X)) \geq 0$ on Γ_1 and $\mathcal{V}^{u_{max}}$ is a supersolution of equation (26).

Step 3. To summarize this proof, we showed that:

- by Theorem 2.1, $\mathcal{V}^{u_{max}}$ is a Lipschitz function,
- $\mathcal{V}^{u_{max}}$ is both a subsolution and a supersolution of (26b) and (26e),
- $\mathcal{V}^{u_{max}}(X) = 0$ on $\Gamma_{OA} \cup \Gamma_I$ by definition of $\mathcal{V}^{u_{max}}$.

So $\mathcal{V}^{u_{max}}$ is a viscosity solution of the Hamilton-Jacobi-Bellman equation (26).

2.3 Uniqueness of the solution of the HJB problem

Theorem 2.3. *Let \mathcal{F}_1 be a subsolution of (26) and \mathcal{F}_2 a supersolution. Then:*

$$\mathcal{F}_1(X) \leq \mathcal{F}_2(X) \text{ for all } X \in \Omega.$$

Remark 4. In the following, we will use that, for any $A_1, B_1, A_2, B_2 \in \mathbb{R}$ with $\min(A_1, B_1) \leq \min(A_2, B_2)$ there exists $\rho \geq 0$ such as: $A_1 + \rho B_1 \leq A_2 + \rho B_2$.

Proof. Let $B_\alpha \in \Omega$ denote the point with coordinates $(1 - \alpha, \alpha)$ and:

$$\Gamma_{AB_\alpha} = \left\{ (X_1, X_2) \in \bar{\Omega} \mid X_2 > 0, \frac{\beta}{\gamma} X_1 + \frac{\gamma - \beta + \alpha\beta}{\alpha\gamma} X_2 = 1 \right\},$$

$$\Gamma_{B_\alpha 1} = \{(X_1, X_2) \in \bar{\Omega} \mid X_1 + X_2 = 1, X_2 > \alpha\}.$$

Let $D_\alpha \subset \bar{\Omega}$ be the domain strictly bounded by $\Gamma_I, \Gamma_{OA}, \Gamma_{AB_\alpha}$ and $\Gamma_{B_\alpha 1}$, see figure 6 for a graphical representation. When $\gamma/\beta \geq 1$ the point A will lie outside Ω , we take then $D_\alpha = \Omega, \Gamma_{AB_\alpha} = \emptyset$ and $\Gamma_{B_\alpha 1} = \Gamma_1$.

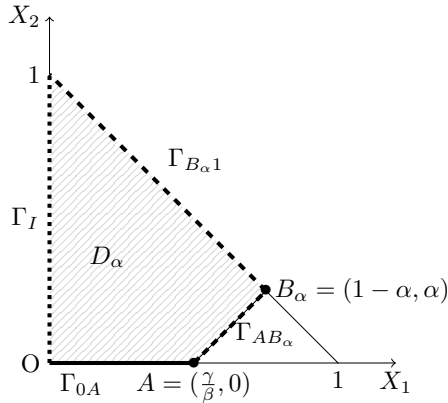


Figure 6: Boundary used in proof of the Theorem 2.3.

We prove in appendix B that for any $X_0 \in \partial D_\alpha$ the trajectory $\Phi^{X_0, w}(t)$ with $w(t) \in [0, u_{max}] \forall t$ enters D_α .

For $X \in \Gamma_{B_\alpha 1}$, $X \neq (1, 0)$, the scalar product with the incoming normal is positive:

$$\langle f(X, u), (-1, -1) \rangle = \gamma X_2 + u > 0 \quad \forall X \in \Gamma_{B_\alpha 1}, X \neq (1, 0), u \in [0, u_{max}].$$

For $X \in \Gamma_{AB_\alpha}$, $X \neq (1, 0)$, the relevant quantity is:

$$\langle f(X, u), \left(-\frac{\beta}{\gamma}, -\frac{\gamma - \beta + \alpha\beta}{\gamma\alpha}\right) \rangle = \frac{\beta}{\gamma}(\beta X_1 X_2 + u) + \gamma^{-1}(\beta X_1 - \gamma)^2 > 0.$$

We now show the Theorem for \mathcal{F}_1 and \mathcal{F}_2 restricted to D_α . To this end we make the change of variable introduced by Kruřkov (see [4]), for $X \in D_\alpha$, $\mathcal{W}(X) = 1 - e^{-\mathcal{F}(X)}$. Formally:

$$\nabla \mathcal{W}(X) = \nabla \mathcal{F}(X) e^{-\mathcal{F}(X)} = \nabla \mathcal{F}(X) (1 - \mathcal{W}(X)) \quad (37)$$

thus $\nabla \mathcal{F}(X) = \frac{\nabla \mathcal{W}(X)}{(1 - \mathcal{W}(X))}$. This motivates the introduction of the following Hamiltonian:

$$\begin{aligned} & -u_{max} \left(\frac{p_1}{1 - \mathcal{W}(X)} - r_V \right)_+ + \beta X_1 X_2 \left(r_I + \frac{p_2}{1 - \mathcal{W}(X)} \right. \\ & \left. - \frac{p_1}{1 - \mathcal{W}(X)} \right) - \gamma X_2 \frac{p_2}{1 - \mathcal{W}(X)}. \end{aligned} \quad (38)$$

Since $1 - \mathcal{W}(X)$ will always be positive, for convenience, we conclude the demonstration using the Hamiltonian: $\tilde{\mathcal{H}}^{u_{max}} : \overline{D_\alpha} \times [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$:

$$\begin{aligned} & \tilde{\mathcal{H}}^{u_{max}}(X, \mathcal{W}(X), p) \\ & = \min_{w \in [0, u_{max}]} \left[p \cdot f(X, w) + (1 - \mathcal{W}(X))(r_I \beta X_1 X_2 + r_V w) \right] \\ & = -u_{max} (p_1 - r_V (1 - \mathcal{W}(X)))_+ + \beta X_1 X_2 (r_I (1 - \mathcal{W}(X)) + p_2 - p_1) - \gamma X_2 p_2. \end{aligned}$$

So we have to prove the following:

Lemma 2.4. *Let the Hamilton-Jacobi-Bellman equation:*

$$(\mathcal{PW}) \left\{ \begin{array}{l} \text{Find } F : \overline{D_\alpha} \rightarrow \mathbb{R} \text{ such that} \\ F \text{ is Lipschitz on } \overline{D_\alpha}, \\ -\tilde{\mathcal{H}}^{u_{max}}(X, F, \nabla F(X)) = 0, \quad \forall X \in D_\alpha, \\ F(X) = 0 \text{ on } \Gamma_{OA}, \\ F(X) = 0 \text{ on } \Gamma_I, \\ -\tilde{\mathcal{H}}^{u_{max}}(X, F(X), \nabla F(X)) = 0 \text{ on } \Gamma_{AB_\alpha} \cup \Gamma_{B_\alpha 1}. \end{array} \right. \quad \begin{array}{l} (39a) \\ (39b) \\ (39c) \\ (39d) \\ (39e) \end{array}$$

If \mathcal{W}_1 is a subsolution of (39) and \mathcal{W}_2 a supersolution, then:

$$\mathcal{W}_1(X) \leq \mathcal{W}_2(X) \text{ for all } X \in D_\alpha.$$

Proof. Suppose now that the Lemma is not true, then there exists $\sigma > 0$ such that:

$$\sup_{x \in D_\alpha} [\mathcal{W}_1(x) - \mathcal{W}_2(x)] = \sigma > 0. \quad (40)$$

Consider $\Psi_\epsilon(x, y) : \overline{D_\alpha} \mapsto \mathbb{R}$ defined by:

$$\Psi_\epsilon(x, y) = \mathcal{W}_1(x) - \mathcal{W}_2(y) - \frac{|x - y|^2}{\epsilon}.$$

For any ϵ this function has a global maximum in (x^ϵ, y^ϵ) and we have for ϵ small enough: $\Psi_\epsilon(x^\epsilon, y^\epsilon) \geq \sigma/2 > 0$. Since $\mathcal{W}_1, \mathcal{W}_2$ are bounded we obtain also $\lim_{\epsilon \rightarrow 0} |x^\epsilon - y^\epsilon| = 0$.

In addition, consider the functions:

$$\varphi_1(x) = \mathcal{W}_2(y^\epsilon) + \frac{|x - y^\epsilon|^2}{\epsilon} \text{ defined on } \Omega_{\varphi_1} = \{x \in \mathbb{R}^{*+} \mid x + y^\epsilon < 1\},$$

$$\varphi_2(y) = \mathcal{W}_1(x^\epsilon) - \frac{|x^\epsilon - y|^2}{\epsilon} \text{ defined on } \Omega_{\varphi_2} = \{y \in \mathbb{R}^{*+} \mid x^\epsilon + y < 1\}.$$

These two functions are \mathcal{C}^1 on Ω_{φ_1} and Ω_{φ_2} respectively.

Then $\mathcal{W}_1(x) - \varphi_1(x)$ reaches its maximum in x^ϵ , φ_1 is $\mathcal{C}^1(\Omega_{\varphi_1})$ and \mathcal{W}_1 is a subsolution of (39). Using the Lemma (1.2), we have:

$$-\tilde{\mathcal{H}}^{u_{max}} \left(x^\epsilon, \mathcal{W}_1(x^\epsilon), \frac{2(x^\epsilon - y^\epsilon)}{\epsilon} \right) \leq 0. \quad (41)$$

Similarly, using that the application $y \mapsto \mathcal{W}_2(y) - \varphi_2(y)$ has its maximum in y^ϵ , φ_2 is $\mathcal{C}^1(\Omega_{\varphi_2})$ and \mathcal{W}_2 is a supersolution of (39), we have:

$$-\tilde{\mathcal{H}}^{u_{max}} \left(y^\epsilon, \mathcal{W}_2(y^\epsilon), \frac{2(x^\epsilon - y^\epsilon)}{\epsilon} \right) \geq 0. \quad (42)$$

Combining these two equations, we obtain:

$$-\tilde{\mathcal{H}}^{u_{max}} \left(x^\epsilon, \mathcal{W}_1(x^\epsilon), \frac{2(x^\epsilon - y^\epsilon)}{\epsilon} \right) \leq -\tilde{\mathcal{H}}^{u_{max}} \left(y^\epsilon, \mathcal{W}_2(y^\epsilon), \frac{2(x^\epsilon - y^\epsilon)}{\epsilon} \right). \quad (43)$$

We use then Remark 4, with $\tilde{\mathcal{H}}^{u_{max}}$ written as:

$$\tilde{\mathcal{H}}^{u_{max}}(X, \mathcal{W}, p) = \min(u_{max}(r_V(1 - \mathcal{W}) - p_1) + \beta X_1 X_2 (r_I(1 - \mathcal{W}) + p_2 - p_1) - \gamma X_2 p_2, \beta X_1 X_2 (r_I(1 - \mathcal{W}) + p_2 - p_1) - \gamma X_2 p_2). \quad (44)$$

So we obtain after few simplifications and factorisation (ρ is the constant given by Remark 4):

$$\begin{aligned} & -u_{max} r_V (\mathcal{W}_2(y^\epsilon) - \mathcal{W}_1(x^\epsilon)) \\ & + (1 + \rho) [\beta (-x_1^\epsilon x_2^\epsilon + y_1^\epsilon y_2^\epsilon) (r_I - p_1^\epsilon + p_2^\epsilon) - \gamma p_2^\epsilon (y_2^\epsilon - x_2^\epsilon)] \\ & - (1 + \rho) \beta r_I [y_1^\epsilon y_2^\epsilon \mathcal{W}_2(y^\epsilon) - x_1^\epsilon x_2^\epsilon \mathcal{W}_1(x^\epsilon)] \leq 0. \end{aligned} \quad (45)$$

Moreover:

$$\begin{aligned} \mathcal{W}_1(x^\epsilon) - \mathcal{W}_2(y^\epsilon) & \leq \Psi_\epsilon(x^\epsilon, y^\epsilon) \\ & \leq \mathcal{W}_1(x^\epsilon) - \mathcal{W}_2(y^\epsilon) + |\mathcal{W}_2(x^\epsilon) - \mathcal{W}_2(y^\epsilon)| - \frac{|x^\epsilon - y^\epsilon|^2}{2\epsilon}. \end{aligned}$$

Hence,

$$0 \leq |\mathcal{W}_2(x^\epsilon) - \mathcal{W}_2(y^\epsilon)| - \frac{|x^\epsilon - y^\epsilon|^2}{2\epsilon}. \quad (46)$$

Since \mathcal{W}_2 is uniformly continuous (as a continuous function on a compact) and $\lim_{\epsilon \rightarrow 0} |x^\epsilon - y^\epsilon| = 0$, we have:

$$\lim_{\epsilon \rightarrow 0} \frac{|x^\epsilon - y^\epsilon|^2}{2\epsilon} = 0. \quad (47)$$

So,

$$\begin{aligned} (-x_1^\epsilon x_2^\epsilon + y_1^\epsilon y_2^\epsilon)(-p_1^\epsilon + p_2^\epsilon) &= (-x_1^\epsilon x_2^\epsilon + x_1^\epsilon y_2^\epsilon - x_1^\epsilon y_2^\epsilon + y_1^\epsilon y_2^\epsilon) \frac{2}{\epsilon} (-x_1^\epsilon + y_1^\epsilon + x_2^\epsilon - y_2^\epsilon) \\ &= (-x_1^\epsilon (x_2^\epsilon - y_2^\epsilon) - y_2^\epsilon (x_1^\epsilon - y_1^\epsilon)) \frac{2}{\epsilon} (-x_1^\epsilon - y_1^\epsilon + x_2^\epsilon - y_2^\epsilon) \\ &\leq \frac{2}{\epsilon} (|x_1^\epsilon| + |y_2^\epsilon|) |x_2^\epsilon - y_2^\epsilon| |x_1^\epsilon - y_1^\epsilon| + \frac{2}{\epsilon} |x_1^\epsilon| |x_2^\epsilon - y_2^\epsilon|^2 + \frac{2}{\epsilon} |y_2^\epsilon| |x_1^\epsilon - y_1^\epsilon|^2 \\ &\leq \frac{4}{\epsilon} |x_2^\epsilon - y_2^\epsilon| |x_1^\epsilon - y_1^\epsilon| + \frac{2}{\epsilon} |x_2^\epsilon - y_2^\epsilon|^2 + \frac{2}{\epsilon} |x_1^\epsilon - y_1^\epsilon|^2. \end{aligned}$$

Hence,

$$\lim_{\epsilon \rightarrow 0} \beta |(-x_1^\epsilon x_2^\epsilon + y_1^\epsilon y_2^\epsilon)(-p_1^\epsilon + p_2^\epsilon)| = 0.$$

Similarly, using (47), we have:

$$\gamma p_2^\epsilon (x_2^\epsilon - y_2^\epsilon) = 0.$$

After eventually extracting a subsequence $(\epsilon_n)_{n \geq 0}$ we can suppose that $\lim_{\epsilon_n \rightarrow 0} x^{\epsilon_n} = \lim_{\epsilon_n \rightarrow 0} y^{\epsilon_n} = \bar{x}$. Note that $\bar{x}_1 = 0$ or $\bar{x}_2 = 0$ would imply $\bar{x} \in \Gamma_{OA} \cup \Gamma_I$ thus $\mathcal{W}_1(\bar{x}) = \mathcal{W}_2(\bar{x}) = 0$ in contradiction with $\Psi(x^\epsilon, y^\epsilon) \geq \frac{\sigma}{2}$ and (47). Therefore $\bar{x}_1 \neq 0$ and $\bar{x}_2 \neq 0$.

We can therefore rewrite (45) as follows:

$$-[(1 + \rho)r_I \beta \bar{x}_1 \bar{x}_2 + u_{max} r_V] [\mathcal{W}_2(\bar{x}) - \mathcal{W}_1(\bar{x})] \leq 0. \quad (48)$$

Since $r_I, r_V, \beta > 0$, $\rho \geq 0$ and $\bar{x}_1 \neq 0$, $\bar{x}_2 \neq 0$ this implies that:

$$\mathcal{W}_2(\bar{x}) \geq \mathcal{W}_1(\bar{x}). \quad (49)$$

On the other hand, for ϵ relatively small, we have $\mathcal{W}_1(x^\epsilon) \geq \mathcal{W}_2(y^\epsilon) + \frac{\sigma}{2}$. Passing to the limit, we get $\mathcal{W}_1(\bar{x}) > \mathcal{W}_2(\bar{x})$. This is in contradiction with (49) and ends the proof of the Lemma.

As $\mathcal{W}_1 \leq \mathcal{W}_2$ on D_α , we have also $\mathcal{F}_1 \leq \mathcal{F}_2$ on D_α . When $\alpha \rightarrow 0$, we obtain $\mathcal{F}_1 \leq \mathcal{F}_2$ on Ω .

This proof is also available for $X \in \Gamma_1$. For Γ_{OA} and Γ_I , we just use the value of the function.

Theorem 2.5. *The value function $\mathcal{V}^{u_{max}}$ is the unique solution of the HJB problem (26).*

Proof. Let \mathcal{F}_1 and \mathcal{F}_2 be two viscosity solutions of (26). Since \mathcal{F}_1 is a subsolution and \mathcal{F}_2 is a supersolution, we have, by Theorem 2.3 that $\mathcal{F}_1 \leq \mathcal{F}_2$ on Ω . Interchanging the roles of \mathcal{F}_1 and \mathcal{F}_2 , we can conclude $\mathcal{F}_2 \leq \mathcal{F}_1$. So $\mathcal{F}_1 = \mathcal{F}_2$ on Ω and therefore on $\bar{\Omega}$ (by continuity).

Thus the solution is unique. By Theorem 2.8 the value function $\mathcal{V}^{u_{max}}$ is the unique solution.

2.4 Solution candidate and its properties: the sub-critical case

Theorem 2.5 implies that in order to find the value function it is enough to find a solution of the HJB equation (26).

We expect the solution to lead to a partition of the domain into a vaccination region and a non-vaccination region. An important question concerns the regularity of the value function which at its turn is related to the uniqueness of the optimal strategy. The frontier of the vaccination region will be seen to be related to the level line $\mathcal{L}_{r_V/r_I}^{\partial_{X_1}\zeta}$ of $\partial_{X_1}\zeta$; see in appendix B the definition of $\mathcal{L}_{r_V/r_I}^{\partial_{X_1}\zeta}$. Thus we are about to ask a question similar to that in figure 5: does $\mathcal{L}_{r_V/r_I}^{\partial_{X_1}\zeta}$ contain points that are entering the domain for control u_{max} and exiting it for control 0. The level lines $\mathcal{L}_{r_V/r_I}^{\partial_{X_1}\zeta}$ that contain such points will lead to non unique optimal strategies (and non smooth value functions).

When $\gamma/\beta < 1$, for any $u_{max} < \infty$ we introduce the critical point $X_{u_{max}}^{crit}$ which is the unique solution of the equations:

$$\begin{cases} X \in \Gamma_1 \\ \langle f(X, u_{max}), \nabla \partial_{X_1} \zeta(X) \rangle = 0. \end{cases} \quad (50)$$

The proof of existence and uniqueness of $X_{u_{max}}^{crit}$ is left as an exercise for the reader. One can use the description of the curve $\langle f(X, u_{max}), \nabla \partial_{X_1} \zeta(X) \rangle = 0$ (see also the Appendix A for formulae involving ζ and its derivatives) to show that $X_{u_{max}}^{crit} = (x^*, 1 - x^*)$ where x^* is the solution of:

$$\frac{\gamma}{\beta} - \left(x^* - \frac{\gamma}{\beta}\right) \sqrt{\frac{u_{max}}{\beta x^*(1-x^*) + u_{max}}} = x^* e^{-\frac{\gamma}{\beta} [(1-x^*) + (x^* - \frac{\gamma}{\beta})(1 + \sqrt{\frac{u_{max}}{\beta x^*(1-x^*) + u_{max}}})]}. \quad (51)$$

Then the value $r_{V, u_{max}}^{crit}$ is defined as

$$r_{V, u_{max}}^{crit} = \partial_{X_1} \zeta(X_{u_{max}}^{crit}). \quad (52)$$

For $\gamma/\beta \geq 1$ we set $r_{V, u_{max}}^{crit} = \infty$. Note that in all situations $r_{V, u_{max}}^{crit} > 1$.

When $r_V < r_{V, u_{max}}^{crit}$ we define a partition of Ω in two regions

$$\Omega_{u_{max}}^{NV} = \{X \in \Omega \mid \partial_{X_1} \zeta(X) < r_V/r_I\} \quad (53)$$

$$\Omega_{u_{max}}^V = \{X \in \Omega \mid \partial_{X_1} \zeta(X) > r_V/r_I\}. \quad (54)$$

The level line $\mathcal{L}_{r_V/r_I}^{\partial_{X_1}\zeta}$ is situated on the common frontier $\partial\Omega_{u_{max}}^{NV} \cap \partial\Omega_{u_{max}}^V$.

For $\gamma/\beta \geq 1$ it may happen that r_V/r_I is such that $\mathcal{L}_{r_V/r_I}^{\partial_{X_1}\zeta} \cap \Omega = \emptyset$; then we take $\Omega_{u_{max}}^V = \emptyset$. This can happen for relatively small values of r_V/r_I as illustrated in figure 7.

Lemma 2.6. *Any trajectory $\Phi^{X_0, w}(t)$ with $X_0 \in \partial\Omega_{u_{max}}^{NV} \cap \partial\Omega_{u_{max}}^V$ is such that $\Phi^{X_0, w}(t) \in \Omega_{u_{max}}^{NV}$ for all $t > 0$ and $w \in \mathcal{U}_{X_0}$.*

Proof. In order to prove that the trajectory $\Phi^{X_0, w}(t)$ enters the domain $\Omega_{u_{max}}^{NV}$ it is enough to prove that the tangent to the trajectory has strictly positive scalar product with the incoming normal at X_0 to $\Omega_{u_{max}}^{NV}$ i.e.,

$$\langle f(X_0, w(0)), -\nabla \partial_{X_1} \zeta(X_0) \rangle > 0, \quad \forall X_0 \in \partial\Omega_{u_{max}}^{NV} \cap \partial\Omega_{u_{max}}^V.$$

This follows (after some straightforward computations) from the definition of $r_{V,u_{max}}^{crit}$ and the monotonicity of the derivatives of $\zeta(\cdot)$ as $r_{V,u_{max}}^{crit}$ is the smallest value r where the trajectory $u = u_{max}$ is tangent to the level line $\mathcal{L}_r^{\partial_{X_1}\zeta}$ (see in appendix B the definition of $\mathcal{L}_r^{\partial_{X_1}\zeta}$).

Introduce also the control $u_{X_0}(t)$ taken to be u_{max} as long as the trajectory $\Phi^{X_0, u_{X_0}(\cdot)}(t)$ obtained with this control $u_{X_0}(t)$ remains in $\Omega_{u_{max}}^V$ (and zero otherwise). It is a feedback control. Formally it is the solution of the equation:

$$u_{X_0}(t) = u_{max} \cdot \mathbb{1}_{\Phi^{X_0, u_{X_0}(\cdot)}(t) \in \Omega_{u_{max}}^V}. \quad (55)$$

The fact that such a solution exists is a consequence of the regularity of the boundary of $\Omega_{u_{max}}^V$ and Lemma 2.6. Note that $u_{X_0}(t)$ is of the form $u_{max} \cdot \mathbb{1}_{[0, \eta]}$ with $\eta \geq 0$. Define the function $\Pi_{u_{max}}^{r_V, r_I} : \Omega \rightarrow \mathbb{R}$ by

$$\Pi_{u_{max}}^{r_V, r_I}(X_0) = J(X_0, u_{X_0}(\cdot)). \quad (56)$$

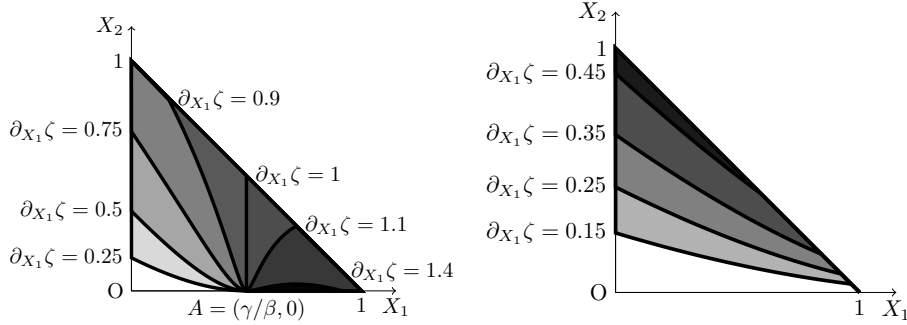


Figure 7: Illustration of level lines $\mathcal{L}_r^{\partial_{X_1}\zeta}$ of the function $\partial_{X_1}\zeta$ for $\gamma/\beta < 1$ (left) and $\gamma/\beta \geq 1$ (right).

Theorem 2.7. For $r_V < r_{V,u_{max}}^{crit} < r_I$:

1. $\Pi_{u_{max}}^{r_V, r_I} \Big|_{\Omega_{u_{max}}^{NV}} = J_0 = r_I \zeta$;
2. $\Pi_{u_{max}}^{r_V, r_I} \Big|_{\Omega_{u_{max}}^V}$ is the unique viscosity solution of the following problem:

$$(\mathcal{P}v) \begin{cases} \text{Find } F : \overline{\Omega_{u_{max}}^V} \rightarrow \mathbb{R} \text{ such that} \\ F \text{ is Lipschitz on } \overline{\Omega_{u_{max}}^V}, & (57a) \\ -\mathcal{H}^{vac, u_{max}}(X, \nabla F(X)) = 0, X \in \Omega_{u_{max}}^V, & (57b) \\ F(X) = r_I \zeta(X), X \in \overline{\Omega_{u_{max}}^{NV}} \cap \overline{\Omega_{u_{max}}^V}, & (57c) \\ -\mathcal{H}^{vac, u_{max}}(X, \nabla F(X)) = 0, X \in \partial\Omega_{u_{max}}^V \setminus (\overline{\Omega_{u_{max}}^{NV}} \cap \overline{\Omega_{u_{max}}^V}). & (57d) \end{cases}$$

Here $\mathcal{H}^{vac, u_{max}} : \overline{\Omega} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is the Hamiltonian function:

$$\begin{aligned} \mathcal{H}^{vac, u_{max}}(X, p) &= \langle f(X, u_{max}), (p_1, p_2) \rangle + r_I \beta X_1 X_2 - r_V u_{max} \\ &= -u_{max}(p_1 - r_V) + \beta X_1 X_2 (r_I + p_2 - p_1) - \gamma X_2 p_2. \end{aligned} \quad (58)$$

3. $\Pi_{u_{max}}^{r_V, r_I} \in \mathcal{C}^1(\Omega)$;
4. $\Pi_{u_{max}}^{r_V, r_I}$ is a solution of the HJB equation (26).

Proof. We only consider in this proof the circumstance when $\gamma/\beta < 1$, the proof for $\gamma/\beta \geq 1$ works in the same way.

Point 1: It is enough to show that a trajectory with initial point in $\Omega_{u_{max}}^{NV}$ remains there forever. Considering the definition of the domains for any $X \in \overline{\Omega_{u_{max}}^V} \cap \overline{\Omega_{u_{max}}^{NV}} = \overline{\mathcal{L}_{r_V/r_I}^{\partial_{X_1} \zeta}}$ the tangent direction $f(X, u)$ to the trajectory points strictly to the interior of $\Omega_{u_{max}}^{NV}$ (for any $u \in [0, u_{max}]$); this follows from Lemma 2.6.

Point 2: These properties of the function $\Pi_{u_{max}}^{r_V, r_I}$ are obtained as in the proofs of Theorems 2.1 and 2.2 once we write $\Pi_{u_{max}}^{r_V, r_I}$ as the “value function” of a trivial control problem where the control is taken in the one-element set $\{u_{max}\}$ until reaching the frontier $\Omega_{u_{max}}^V \cap \Omega_{u_{max}}^{NV}$; on the frontier the value is $r_I \zeta(X) = J_0(X)$.

Point 3: The function $\zeta(X)$ is \mathcal{C}^1 on Ω (see Appendix A); in particular $\Pi_{u_{max}}^{r_V, r_I}$ will be \mathcal{C}^1 on $\Omega_{u_{max}}^{NV}$. For $X \in \Omega_{u_{max}}^V$ we note that $\Pi_{u_{max}}^{r_V, r_I}$ is the solution of a quasi-linear first order PDE (cf. point 2) and has boundary conditions defined on a non-characteristic curve $\overline{\Omega_{u_{max}}^{NV}} \cap \overline{\Omega_{u_{max}}^V} = \overline{\mathcal{L}_{r_V/r_I}^{\partial_{X_1} \zeta}}$; the curve is non-characteristic because on $\overline{\Omega_{u_{max}}^{NV}} \cap \overline{\Omega_{u_{max}}^V}$ we have $\langle f(X, u_{max}), \partial_{X_1} \zeta(X) \rangle \neq 0$. Another way to see things is to parametrize the boundary curve with a parameter α_1 and denote α_2 the time required to reach the curve. Using the regularity properties of the ODE the function is \mathcal{C}^1 in parameters (α_1, α_2) and the change of coordinates from X to (α_1, α_2) is regular around each point in the interior of $\Omega_{u_{max}}^V$. Thus $\Pi_{u_{max}}^{r_V, r_I}$ will be \mathcal{C}^1 on $\Omega_{u_{max}}^V$.

It remains to be proved that $\Pi_{u_{max}}^{r_V, r_I}$ is also \mathcal{C}^1 around any point $X \in \mathcal{L}_{r_V/r_I}^{\partial_{X_1} \zeta} \cap \Omega$; since $\Pi_{u_{max}}^{r_V, r_I} \Big|_{\Omega_{u_{max}}^V}$ and $\Pi_{u_{max}}^{r_V, r_I} \Big|_{\Omega_{u_{max}}^{NV}}$ are both \mathcal{C}^1 , the side gradients exist and it remains only to be proved that

$$\nabla \Pi_{u_{max}}^{r_V, r_I} \Big|_{\Omega_{u_{max}}^V} (X) = \nabla \Pi_{u_{max}}^{r_V, r_I} \Big|_{\Omega_{u_{max}}^{NV}} (X), \quad \forall X \in \mathcal{L}_{r_V/r_I}^{\partial_{X_1} \zeta}.$$

Using continuity and \mathcal{C}^1 properties and the fact that $\Pi_{u_{max}}^{r_V, r_I} \Big|_{\Omega_{u_{max}}^V}$ and $\Pi_{u_{max}}^{r_V, r_I} \Big|_{\Omega_{u_{max}}^{NV}}$ coincide on the common frontier it follows that the tangential derivatives along the frontier are the same. Let us prove that the directional derivative also coincide in the direction $f(X, u_{max})$, which can be written:

$$\langle \nabla \Pi_{u_{max}}^{r_V, r_I} \Big|_{\Omega_{u_{max}}^V} (X), f(X, u_{max}) \rangle = \langle \nabla \Pi_{u_{max}}^{r_V, r_I} \Big|_{\Omega_{u_{max}}^{NV}} (X), f(X, u_{max}) \rangle. \quad (59)$$

But $\Pi_{u_{max}}^{r_V, r_I} \Big|_{\Omega_{u_{max}}^V}$ satisfies (57b) then

$$\begin{aligned} \langle \nabla \Pi_{u_{max}}^{r_V, r_I} \Big|_{\Omega_{u_{max}}^V} (X), f(X, u_{max}) \rangle &= -r_I \beta X_1 X_2 + r_V u_{max} \\ &= -\mathcal{H}^{vac, u_{max}}(X, \nabla \Pi_{u_{max}}^{r_V, r_I} \Big|_{\Omega_{u_{max}}^{NV}} (X)) + \langle \nabla \Pi_{u_{max}}^{r_V, r_I} \Big|_{\Omega_{u_{max}}^{NV}} (X), f(X, u_{max}) \rangle \\ &= \langle \nabla \Pi_{u_{max}}^{r_V, r_I} \Big|_{\Omega_{u_{max}}^{NV}} (X), f(X, u_{max}) \rangle. \end{aligned}$$

We used above the fact that $J_0 = r_I \zeta$ satisfies $\mathcal{H}^{vac,0}(X, \nabla J_0) = 0$ on Ω and that for $X \in \mathcal{L}_{r_V/r_I}^{\partial_{X_1} \zeta}$ we can add u_{max} multiplied by the null term $r_I \partial_{X_1} \zeta(X) - r_V$ to $\mathcal{H}^{vac,0}(X, \nabla J_0)$ to obtain $\mathcal{H}^{vac,u_{max}}(X, \nabla J_0) = 0$.

Note that the direction $f(X, u_{max})$ cannot be collinear with the tangent at X to the boundary $\mathcal{L}_{r_V/r_I}^{\partial_{X_1} \zeta}$ because for $r_V < r_I r_{V,u_{max}}^{crit}$ the definition of $r_{V,u_{max}}^{crit}$ ensures that $f(X, u_{max})$ has non-zero scalar product with the normal $\nabla \partial_{X_1} \zeta(X)$ to the boundary. From (59) and the coincidence of the tangential derivatives it follows that side gradients $\nabla \Pi_{u_{max}}^{r_V, r_I} \Big|_{\Omega_{u_{max}}^V}$ and $\nabla \Pi_{u_{max}}^{r_V, r_I} \Big|_{\Omega_{u_{max}}^{NV}}$ coincide on the common boundary thus $\Pi_{u_{max}}^{r_V, r_I} \in \mathcal{C}^1(\Omega)$.

Point 4: Giving what was already proved, it remains to show that

$$\partial_{X_1} \Pi_{u_{max}}^{r_V, r_I}(X) \leq r_V \quad \forall X \in \Omega_{u_{max}}^{NV}, \quad (60)$$

$$\partial_{X_1} \Pi_{u_{max}}^{r_V, r_I}(X) \geq r_V \quad \forall X \in \Omega_{u_{max}}^V, \quad (61)$$

Equation (60) is a simple consequence of (53) and Point 1. For (61) we have to analyze in detail the function $\Pi_{u_{max}}^{r_V, r_I} \Big|_{\Omega_{u_{max}}^{NV}}$, we will prove that in addition:

$$\partial_{X_2} \Pi_{u_{max}}^{r_V, r_I}(X) > 0 \quad \forall X \in \Omega_{u_{max}}^V. \quad (62)$$

Consider $X_0 \in \mathcal{L}_{r_V/r_I}^{\partial_{X_1} \zeta}$. We integrate $\partial_{X_1} \Pi_{u_{max}}^{r_V, r_I}$ on the characteristic curve $\Phi^{Y, u_{max}}(\cdot)$ issued from $Y \in \Omega_{u_{max}}^V$ that reaches the frontier at time $t > 0$ and point X_0 which can be written: $\Phi^{Y, u_{max}}(t) = X_0$. Formally

$$\begin{aligned} \partial_{X_1} \Pi_{u_{max}}^{r_V, r_I}(Y) &= \partial_{X_1} \Pi_{u_{max}}^{r_V, r_I}(X_0) \\ &- \int_0^t \langle \nabla \partial_{X_1} \Pi_{u_{max}}^{r_V, r_I}(\Phi^{Y, u_{max}}(\tau)), f(\Phi^{Y, u_{max}}(\tau), u_{max}) \rangle d\tau. \end{aligned} \quad (63)$$

From now on we will drop the notation $\Phi^{Y, u_{max}}(\tau)$ and only denote $(X_1(\tau), X_2(\tau)) = X(\tau) = \Phi^{Y, u_{max}}(\tau)$. Note that $\Pi_{u_{max}}^{r_V, r_I}$ satisfies $\mathcal{H}^{vac, u_{max}}(X, \nabla \Pi_{u_{max}}^{r_V, r_I}) = 0$ on $\Omega_{u_{max}}^V$ i.e., $\langle \nabla \Pi_{u_{max}}^{r_V, r_I}, f(X, u_{max}) \rangle + r_I \beta X_1 X_2 + r_V u_{max} = 0$ thus by differentiating formally with respect to X_1 one obtains:

$$\langle \nabla \partial_{X_1} \Pi_{u_{max}}^{r_V, r_I}(X(\tau)), f(X(\tau), u_{max}) \rangle = \beta X_2 (-r_I - \partial_{X_1} \Pi_{u_{max}}^{r_V, r_I} + \partial_{X_2} \Pi_{u_{max}}^{r_V, r_I}).$$

But this latter quantity is integrable over $[0, t]$ and after classical arguments we obtain that $\int_0^t \langle \nabla \partial_{X_1} \Pi_{u_{max}}^{r_V, r_I}(\Phi^{Y, u_{max}}(\tau)), f(\Phi^{Y, u_{max}}(\tau), u_{max}) \rangle d\tau$ is well defined and equals $\int_0^t \beta X_2(\tau) (-r_I - \partial_{X_1} \Pi_{u_{max}}^{r_V, r_I} + \partial_{X_2} \Pi_{u_{max}}^{r_V, r_I})(X(\tau)) d\tau$. Moreover using again the HJB equation satisfied by $\Pi_{u_{max}}^{r_V, r_I}$ this term can be replaced by

$$\int_0^t \frac{1}{X_1(\tau)} \left[u_{max} (\partial_{X_1} \Pi_{u_{max}}^{r_V, r_I}(X(\tau)) - r_V) + \gamma X_2(\tau) \partial_{X_2} \Pi_{u_{max}}^{r_V, r_I}(X(\tau)) \right] d\tau.$$

We obtain thus

$$\begin{aligned} \partial_{X_1} \Pi_{u_{max}}^{r_V, r_I}(Y) &= \partial_{X_1} \Pi_{u_{max}}^{r_V, r_I}(X_0) + \int_0^t \frac{1}{X_1(\tau)} \left[u_{max} (\partial_{X_1} \Pi_{u_{max}}^{r_V, r_I}(X(\tau)) - r_V) \right. \\ &\left. + \gamma X_2(\tau) \partial_{X_2} \Pi_{u_{max}}^{r_V, r_I}(X(\tau)) \right] d\tau. \end{aligned} \quad (64)$$

Similar computations allow to write:

$$\partial_{X_2} \Pi_{u_{max}}^{r_V, r_I}(Y) = \partial_{X_2} \Pi_{u_{max}}^{r_V, r_I}(X_0) + \int_0^t \frac{1}{X_2(\tau)} \left[u_{max} (\partial_{X_1} \Pi_{u_{max}}^{r_V, r_I}(X(\tau)) - r_V) \right] d\tau. \quad (65)$$

Then since $\Pi_{u_{max}}^{r_V, r_I}$ is \mathcal{C}^1 it follows from the properties of ζ that $\partial_{X_1} \Pi_{u_{max}}^{r_V, r_I}(X_0) = r_V$ and $\partial_{X_2} \Pi_{u_{max}}^{r_V, r_I}(X_0) > 0$. Combined with the identities (64)-(65) (and reasoning infinitesimally starting from X_0 along the characteristic) we obtain $\partial_{X_1} \Pi_{u_{max}}^{r_V, r_I}(Y) > r_V$ and $\partial_{X_2} \Pi_{u_{max}}^{r_V, r_I}(Y) > 0$ and equations (62) and (61) follow.

Theorem 2.8. *For $r_V < r_I r_{V, u_{max}}^{crit}$ the function $\Pi_{u_{max}}^{r_V, r_I}$ is the unique solution of the HJB equation (26) and $\Pi_{u_{max}}^{r_V, r_I} = \mathcal{V}^{u_{max}}$. As a consequence in this case the value function $\mathcal{V}^{u_{max}}$ is in $\mathcal{C}^1(\Omega)$.*

Proof. The Theorem 2.7 proves that $\Pi_{u_{max}}^{r_V, r_I}$ is a solution of (26). Furthermore, Theorem 2.5 assures the uniqueness of the solution. Then, $\mathcal{V}^{u_{max}} = \Pi_{u_{max}}^{r_V, r_I}$.

2.5 Solution candidate and its properties: the super-critical case

We work here under the hypothesis $r_V \geq r_{V, u_{max}}^{crit}$. In particular this implies $\gamma/\beta < 1$.

The simplest case is when $r_V \geq 2r_I$ and will be dealt with directly latter in Theorem 2.12. On the contrary, the situation when $r_V \in [r_{V, u_{max}}^{crit}, r_I, 2r_I[$ requires some more work. In this case the value function $\mathcal{V}^{u_{max}}$ will **not** be \mathcal{C}^1 .

Define (see also figure 8):

$$\Gamma_{sub}^{crit} = \{P \in \mathcal{L}_{r_V/r_I}^{\partial_{X_1} \zeta} \mid f(P, u_{max}), \nabla \partial_{X_1} \zeta(P) \leq 0\}. \quad (66)$$

Using the formulas for f and the derivatives of ζ one can prove with straightforward computations:

- $\mathcal{L}_{r_V/r_I}^{\partial_{X_1} \zeta} \subset \{X \in \Omega \mid X_1 > \gamma/\beta\}$ (since $r_{V, u_{max}}^{crit} > 1$);
- $\partial_{X_2 X_1} \zeta(P) < 0, \forall P \in \mathcal{L}_{r_V/r_I}^{\partial_{X_1} \zeta}$;
- Γ_{sub}^{crit} is connected; denote by $P_{r_V}^{crit}$ the other extremity of the curve; then $\langle f(P_{r_V}^{crit}, u_{max}), \nabla \partial_{X_1} \zeta(P_{r_V}^{crit}) \rangle = 0$;
- $\forall P \in \mathcal{L}_{r_V/r_I}^{\partial_{X_1} \zeta} \setminus \Gamma_{sub}^{crit}, P_1 \geq (P_{r_V}^{crit})_1$;
- the trajectories starting from points on the curve Γ_{sub}^{crit} enter the domain $\{X \in \Omega \mid \nabla \partial_{X_1} \zeta(X) \leq r_V/r_I\}$ for any $u \in [0, u_{max}]$;
- the trajectories starting from points in $\mathcal{L}_{r_V/r_I}^{\partial_{X_1} \zeta} \setminus \Gamma_{sub}^{crit}$ exit this domain for $u = u_{max}$.

For any $Y \in \Gamma_{sub}^{crit}$ introduce

$$t_Y = \sup \left\{ t \geq 0 \mid J_0(Y) + r_V t u_{max} + \int_0^t r_I \beta \Phi_1^{Y, u_{max}}(-\tau) \Phi_2^{Y, u_{max}}(-\tau) d\tau \leq J_0(\Phi^{Y, u_{max}}(-t)) \right\}. \quad (67)$$

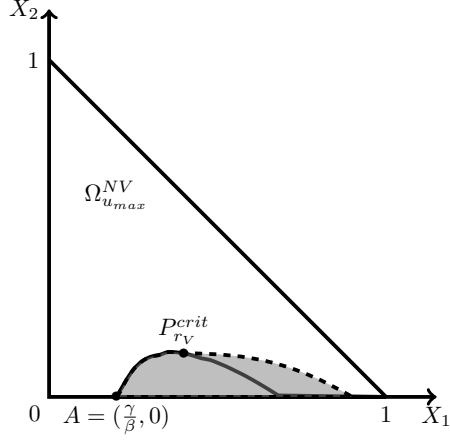


Figure 8: Illustration of the construction of the domains $\Omega_{u_{max}}^V$ and $\Omega_{u_{max}}^{NV}$. The solid curve is $\mathcal{L}_{r_V/r_I}^{\partial_{X_1}\zeta}$. The dashed curves are Γ_{sub}^{crit} (from A to $P_{r_V}^{crit}$) and Γ_{super}^{crit} (from $P_{r_V}^{crit}$). The gray domain is $\Omega_{u_{max}}^V$.

We note that the previous properties imply that $t_{X_0} > 0$; indeed, take $Z = \Phi^{Y, u_{max}}(-\epsilon)$ for ϵ small enough; then integrating over the curve $\tau \mapsto \Phi^{Z, u_{max}}(\tau)$ we obtain:

$$J_0(Y) = J_0(\Phi^{Z, u_{max}}(\epsilon)) = J_0(Z) \quad (68)$$

$$+ \int_0^\epsilon \langle \nabla J_0(\Phi^{Z, u_{max}}(\tau)), f(\Phi^{Z, u_{max}}(\tau), u_{max}) \rangle d\tau. \quad (69)$$

Developing the last term and using the HJB equation satisfied by J_0 we can write:

$$\begin{aligned} J_0(Y) &= J_0(Z) - r_V t u_{max} - \int_0^t r_I \left[\beta \Phi_1^{Z, u_{max}}(\tau) \Phi_2^{Z, u_{max}}(\tau) \right. \\ &\quad \left. + u_{max} (\partial_{X_1} J_0(\Phi^{Z, u_{max}}(\tau)) - r_V) \right] d\tau. \end{aligned} \quad (70)$$

The curve $\tau \mapsto \Phi^{Z, u_{max}}(\tau)$ belongs to the domain where $\partial_{X_1} J_0(\Phi^{Z, u_{max}}(\tau)) \geq r_V$ therefore $Z = \Phi^{Y, u_{max}}(-\epsilon)$ satisfies the inequality in the equation (67) and as such we obtain $t_Y \geq \epsilon > 0$.

We define a curve Γ_{super}^{crit} as:

$$\Gamma_{super}^{crit} = \{ \Phi^{Y, u_{max}}(-t_Y) \mid Y \in \Gamma_{sub}^{crit} \}. \quad (71)$$

The curves Γ_{sub}^{crit} and Γ_{super}^{crit} define a domain that will be denoted $\Omega_{u_{max}}^V$; set also $\Omega_{u_{max}}^{NV} = \Omega \setminus \Omega_{u_{max}}^V$ as illustrated in figure 8.

Lemma 2.9. *The following inclusion holds:*

$$\{ X \in \Omega \mid \partial_{X_1} \zeta(X) \geq r_V/r_I \} \subset \Omega_{u_{max}}^V. \quad (72)$$

Therefore we also have:

$$\Omega_{u_{max}}^{NV} \subset \{ X \in \Omega \mid \partial_{X_1} \zeta(X) \leq r_V/r_I \}. \quad (73)$$

Proof. Let $Z \in \{X \in \Omega \mid \partial_{X_1} \zeta(X) \geq r_V/r_I\}$ and consider the trajectory $\Phi^{Z, u_{max}}(t)$ starting from Z . This trajectory will exit this set at some point on the border $\mathcal{L}_{r_V/r_I}^{\partial_{X_1} \zeta}$, more precisely at some point of Γ_{sub}^{crit} (the direction tangent to the trajectory has to exit the domain, which is precisely the definition of Γ_{sub}^{crit}). Denote this point $Y = \Phi^{Z, u_{max}}(\tau^*)$. Using the same arguments as in the proof of $t_Y > 0$ above and recalling that $\partial_{X_1} J_0(\Phi^{Z, u_{max}}(\tau)) \geq r_V$ for all $\tau \leq \tau^*$ we obtain $t_Y \geq \tau^*$ and in particular $Z \in \Omega_{u_{max}}^V$.

Introduce the solution candidate $\Pi_{u_{max}}^{r_V, r_I} : \Omega \rightarrow \mathbb{R}$ defined by equation (56), but with the control $u_{X_0}(\cdot)$ defined in equation (55) depending on the newly defined set $\Omega_{u_{max}}^V$.

Theorem 2.10. For $r_V \in [r_{V, u_{max}}^{crit}, r_I, 2r_I[$:

1.

$$\Pi_{u_{max}}^{r_V, r_I}(Y) = \begin{cases} J_0(Y), & \text{if } Y \in \Omega_{u_{max}}^{NV} & (74a) \\ r_V t + \int_0^t r_I \beta \Phi_1^{Y, u_{max}}(\tau) \Phi_2^{Y, u_{max}}(\tau) d\tau + J_0(\Phi^{Y, u_{max}}(t)), & \\ \text{if } Y \in \Omega_{u_{max}}^V \text{ and } \Phi^{Y, u_{max}}(t) \in \Gamma_{sub}^{crit}, & (74b) \end{cases}$$

2. $\Pi_{u_{max}}^{r_V, r_I} \Big|_{\Omega_{u_{max}}^V}$ is the unique viscosity solution of the following problem:

$$\begin{cases} \text{Find } F : \overline{\Omega_{u_{max}}^V} \rightarrow \mathbb{R} \text{ such that} & (75a) \\ F \text{ is Lipschitz on } \overline{\Omega_{u_{max}}^V}, & (75b) \\ -\mathcal{H}^{vac, u_{max}}(X, \nabla F(X)) = 0, X \in \Omega_{u_{max}}^V, & (75c) \\ F(X) = r_I \zeta(X), X \in \overline{\Omega_{u_{max}}^{NV}} \cap \overline{\Omega_{u_{max}}^V}, & (75d) \\ -\mathcal{H}^{vac, u_{max}}(X, \nabla F(X)) = 0, X \in \partial\Omega_{u_{max}}^V \setminus (\overline{\Omega_{u_{max}}^{NV}} \cap \overline{\Omega_{u_{max}}^V}); & \end{cases}$$

3. $\Pi_{u_{max}}^{r_V, r_I}$ is Lipschitz on $\overline{\Omega}$;

4. $\Pi_{u_{max}}^{r_V, r_I}$ is a solution of the HJB equation (26).

Proof. Much of the proof uses concepts already invoked in the proof of Theorem 2.7. We will only emphasize points that are specific to this situation.

Point 1: A trajectory with initial point in $\Omega_{u_{max}}^{NV}$ remains there forever therefore we conclude as above that $\Pi_{u_{max}}^{r_V, r_I} \Big|_{\Omega_{u_{max}}^{NV}} = J_0 = r_I \zeta$; to prove the second property note that the function J satisfies

$$\begin{aligned} J(Y, u(\cdot)) &= \int_0^t r_V u(\tau) + r_I \beta \Phi_1^{Y, u}(\tau) \Phi_2^{Y, u}(\tau) d\tau \\ &+ J(\Phi^{Y, u}(\cdot), u(\cdot + t)). \end{aligned} \quad (76)$$

Thus the two definitions coincide as the control is u_{max} on $\Omega_{u_{max}}^V$ and 0 on $\Omega_{u_{max}}^{NV}$ because once the trajectory reaches the frontier Γ_{sub}^{crit} of $\Omega_{u_{max}}^{NV}$ it enters $\Omega_{u_{max}}^{NV}$ and remains there.

Point 2: The proof follows the same lines as point 2 in Theorem 2.7.

Point 3: The definition of the domain $\Omega_{u_{max}}^V$ and the previous point ensures that $\Pi_{u_{max}}^{r_V, r_I}$ is continuous in points of the common boundary $\partial\Omega_{u_{max}}^V \cap \partial\Omega_{u_{max}}^{NV}$ thus it is continuous on Ω . It is also Lipschitz on $\Omega_{u_{max}}^V$ and $\Omega_{u_{max}}^{NV}$ with Lipschitz constants that are universally bounded, thus it is Lipschitz on $\bar{\Omega}$.

Moreover, as before, one can prove that $\Pi_{u_{max}}^{r_V, r_I}$ is \mathcal{C}^1 on Γ_{sub}^{crit} .

Another alternative is to repeat the arguments used to prove that the value function is Lipschitz (here the control has the same structure: it has value u_{max} from 0 to some finite time and then 0).

Point 4: We have to prove (the analogues of) the equations (60) and (61).

Any trajectory from $Z \in \Omega_{u_{max}}^V$ (for control $u = u_{max} \mathbb{1}_{X \in \Omega_{u_{max}}^V}$) will encounter Γ_{sub}^{crit} when exiting the domain $\Omega_{u_{max}}^V$. Together with the fact that $\Pi_{u_{max}}^{r_V, r_I}$ is \mathcal{C}^1 in Γ_{sub}^{crit} this allows to use identities (64)-(65) and obtain as above that $\partial_{X_1} \Pi_{u_{max}}^{r_V, r_I}(Y) > r_V$ and $\partial_{X_2} \Pi_{u_{max}}^{r_V, r_I}(Y) > 0$ for any $Y \in \Omega_{u_{max}}^V$; then (61) follows.

To prove (60) use Lemma 2.9 and point 3 of this Theorem.

Theorem 2.11. For $r_V \in [r_I r_{V, u_{max}}^{crit}, 2r_I[$ the function $\Pi_{u_{max}}^{r_V, r_I}$ (defined by equation (56) with the control $u_{X_0}(\cdot)$ defined in equation (55) depending on the set $\Omega_{u_{max}}^V$) is the unique solution of the HJB equation (26) and $\Pi_{u_{max}}^{r_V, r_I} = \mathcal{V}^{u_{max}}$. The value function $\mathcal{V}^{u_{max}}$ is Lipschitz in Ω .

Proof. The Theorem 2.10 proves that $\Pi_{u_{max}}^{r_V, r_I}$ is a solution of (26). Furthermore, Theorem 2.5 assures the uniqueness of the solution. Then, $\mathcal{V}^{u_{max}} = \Pi_{u_{max}}^{r_V, r_I}$.

Theorem 2.12. For $r_V \geq 2r_I$ the function $J_0 = r_I \zeta$ is the unique solution of the HJB equation (26) and $\mathcal{V}^{u_{max}} = J_0$. As a consequence in this case the value function $\mathcal{V}^{u_{max}}$ is in $\mathcal{C}^1(\Omega)$.

Proof. Straightforward computation and the results from Lemma B.4 indicate that the derivative J_0 does not exceed $2r_I$ and as such $(\partial_{X_1} \zeta - r_V)_+ = 0$ and J_0 satisfies the required HJB equation.

3 Instantaneous vaccination

Recall that for $u_{max} = \infty$ the value function is denoted as \mathcal{V}^∞ ; also consult equation (17) for the definition of \mathcal{H}^∞ .

The following result connects the bounded and unbounded control problems (see also [4] pages 113-115 for generic related results):

Theorem 3.1. The sequence $(\mathcal{V}^{u_{max}})_{u_{max} \geq 0}$ is decreasing and

$$\lim_{u_{max} \rightarrow \infty} \mathcal{V}^{u_{max}} = \mathcal{V}^\infty. \quad (77)$$

Moreover the convergence is uniform over compacts of Ω and \mathcal{V}^∞ is Lipschitz over Ω .

Proof. Since for any $u_2 \geq u_1 \geq 0$ we have the inclusion $\mathcal{U}_Y^{u_1} \subset \mathcal{U}_Y^{u_2}$ the sequence $(\mathcal{V}^{u_{max}})_{u_{max} \geq 0}$ is decreasing. Therefore $\liminf_{u_{max} \rightarrow \infty} \mathcal{V}^{u_{max}} \geq \mathcal{V}^\infty$.

Let $Y \in \bar{\Omega}$ and $(dV_n)_{n \geq 0} \subset \mathcal{U}_Y^\infty$ a sequence of strategies such that

$$\lim_{n \rightarrow \infty} J(Y, dV_n) = \mathcal{V}^\infty(Y). \quad (78)$$

For each n construct an approximating sequence of admissible strategies $u_w^n \in \mathcal{U}_Y^w$ such that $\lim_{w \rightarrow \infty} u_w^n = dV_n$. Then $\mathcal{V}^w(Y) \leq J(Y, u_w^n) \rightarrow J(Y, dV_n)$ thus $\limsup_{w \rightarrow \infty} \mathcal{V}^w(Y) \leq J(Y, dV_n)$. Passing once more to the limit $n \rightarrow \infty$ we obtain $\limsup_{w \rightarrow \infty} \mathcal{V}^w(Y) \leq \mathcal{V}^\infty(Y)$.

Then $\lim_{u_{max} \rightarrow \infty} \mathcal{V}^{u_{max}} = \mathcal{V}^\infty$. Since functions $\mathcal{V}^{u_{max}}$ are Lipschitz with Lipschitz constants independent of u_{max} the limit \mathcal{V}^∞ will be Lipschitz and the convergence will hold in a neighborhood of Y (thus uniformly over compacts of Ω).

3.1 HJB equation and value function

Theorem 3.2. *The value function \mathcal{V}^∞ is a viscosity solution of the Hamilton-Jacobi-Bellman equation:*

$$(\mathcal{P}) \begin{cases} \text{Find } F : \bar{\Omega} \rightarrow \mathbb{R} \text{ such that} & (79a) \\ F \text{ is Lipschitz on } \bar{\Omega}, & (79a) \\ -\mathcal{H}^\infty(X, \nabla F(X)) = 0, X \in \Omega, & (79b) \\ F(X) = 0 \text{ on } \Gamma_{OA}, & (79c) \\ F(X) = 0 \text{ on } \Gamma_I, & (79d) \\ -\mathcal{H}^\infty(X, \nabla F(X)) = 0 \text{ on } \Gamma_1. & (79e) \end{cases}$$

Proof. We will use the same arguments and notations as in the proof of the Theorem 2.2.

Step 1. First, we prove that \mathcal{V}^∞ is a subsolution of (79b). We take the same notations and the same reasoning as in the case $u_{max} < \infty$. So equation (29) becomes:

$$\min \{r_V - p_1, \beta Y_1 Y_2 (r_I + p_2 - p_1) - \gamma Y_2 p_2\} \geq 0. \quad (80)$$

Suppose that there exists $\kappa > 0$ such that:

$$\min \{r_V - p_1, \beta Y_1 Y_2 (r_I + p_2 - p_1) - \gamma Y_2 p_2\} \leq -\kappa.$$

Remark 4 page 14 assures that there exists $\rho \geq 0$ such that:

$$\rho(r_V - \partial_{X_1} \varphi(Y)) + \beta Y_1 Y_2 (r_I + \partial_{X_2} \varphi(Y) - \partial_{X_1} \varphi(Y) - \gamma Y_2 \partial_{X_2} \varphi(Y)) \leq -\kappa.$$

Here, we choose the control ρ on the interval $[0, \delta]$ and for the same reasons as above, we obtain:

$$\mathcal{V}^\infty(Z^0) - \mathcal{V}^\infty(Y) \leq -\kappa \delta - \int_0^\delta \rho r_V + r_I \beta \Phi_1^{Y, \rho}(t) \Phi_2^{Y, \rho}(t) dt. \quad (81)$$

In particular, by the optimality of \mathcal{V}^∞ on Y , we have:

$$\mathcal{V}^\infty(Y) \leq \int_0^\delta r_I \beta \Phi_1^{Y, \rho}(t) \Phi_2^{Y, \rho}(t) + \rho r_V dt. \quad (82)$$

And we can conclude as above that \mathcal{V}^∞ is solution of equation (79).

Step 2. We prove that \mathcal{V}^∞ is a supersolution of (79).

Using the same notations and reasoning as in the proof for $u_{max} < \infty$ equation (35) becomes:

$$\min\{r_V - \partial_{X_1}\varphi(Y), \beta Y_1 Y_2 (r_I + \partial_{X_2}\varphi(Y) - \partial_{X_1}\varphi(Y)) - \gamma Y_2 \partial_{X_2}\varphi(Y)\} > \kappa. \quad (83)$$

In order to invalidate (83) we invalidate, in a neighborhood of Y :

$$\beta \Phi_1^{Y,u}(\cdot) \Phi_2^{Y,u}(\cdot) (r_I + \partial_{X_2}\varphi(\Phi^{Y,u}(\cdot)) - \partial_{X_1}\varphi(\Phi^{Y,u}(\cdot))) - \gamma \Phi_2^{Y,u}(\cdot) \partial_{X_2}\varphi(\Phi^{Y,u}(\cdot)) > \kappa. \quad (84)$$

We obtain, as above:

$$\begin{aligned} \mathcal{V}^\infty(Z^0) - \mathcal{V}^\infty(Y) &\geq \varphi(Y) - \varphi(Z^0) \geq \int_0^\delta \langle \nabla\varphi(\Phi^{Y,w}(t)) \cdot f(\Phi^{Y,w}(t), w) \rangle dt \\ &> \delta\kappa + \int_0^\delta -\beta \Phi_1^{Y,w}(t) \Phi_1^{Y,w}(t) r_I - w \partial_{X_1}\varphi dt \\ &> \delta\kappa - \int_0^\delta w r_V + r_I \beta \Phi_1^{Y,w}(t) \Phi_2^{Y,w}(t) dt + \int_0^\delta w (r_V - \partial_{X_1}\varphi(Y)) dt \\ &> \delta\kappa - \int_0^\delta w r_V + r_I \beta \Phi_1^{Y,w}(t) \Phi_2^{Y,w}(t) dt, \end{aligned}$$

because $w(r_V - \partial_{X_1}\varphi(\Phi^{Y,w}(\cdot))) \geq 0$ since $w \geq 0$ and $(r_V - \partial_{X_1}\varphi(\Phi^{Y,w}(\cdot))) \geq \mathcal{H}^\infty(\Phi^{Y,w}(\cdot), \nabla\varphi(\Phi^{Y,w}(\cdot))) \geq \kappa > 0$.

Once again, we conclude as in the proof of the Theorem 2.2.

3.2 Uniqueness of the solution of the HJB problem.

Theorem 3.3. *Let \mathcal{F}_1 a subsolution of (79) and \mathcal{F}_2 a supersolution. Then:*

$$\mathcal{F}_1(X) \leq \mathcal{F}_2(X) \text{ for all } X \in \Omega.$$

Proof. We use the same notation and reasoning as in the proof in Section 2.3. The Hamiltonian used here is:

$$\tilde{\mathcal{H}}^\infty(X, \mathcal{W}^\infty(X), p) = \min(r_V(1 - \mathcal{W}^\infty), \beta X_1 X_2 (r_I(1 - \mathcal{W}^\infty) + p_2 - p_1) - \gamma X_2 p_2). \quad (85)$$

Equation (45) becomes:

$$\begin{aligned} & -\rho r_V (\mathcal{W}_2^\infty(y^\epsilon) - \mathcal{W}_1^\infty(x^\epsilon)) \\ & + [\beta(-x_1^\epsilon x_2^\epsilon + y_1^\epsilon y_2^\epsilon)(r_I - p_1^\epsilon + p_2^\epsilon) - \gamma p_2^\epsilon (y_2^\epsilon - x_2^\epsilon)] \\ & - \beta r_I [y_1^\epsilon y_2^\epsilon \mathcal{W}_2^\infty(y^\epsilon) - x_1^\epsilon x_2^\epsilon \mathcal{W}_1^\infty(x^\epsilon)] \leq 0. \quad (86) \end{aligned}$$

And for the same reasons as in the proof for u_{max} bounded we obtain instead of (48):

$$- [r_I \beta \bar{x}_1 \bar{x}_2 + \rho r_V] [\mathcal{W}_2^\infty(\bar{x}) - \mathcal{W}_1^\infty(\bar{x})] \leq 0. \quad (87)$$

We can conclude as when u_{max} is bounded.

3.3 A candidate value function: the sub-critical case

We introduce the critical point value $r_{V,\infty}^{crit}$:

$$r_{V,\infty}^{crit} = \sup\{r \geq 0 \mid \partial_{X_1 X_1}^2 \zeta(X) > 0 \forall X \in \mathcal{L}_r^{\partial_{X_1} \zeta}\}. \quad (88)$$

We see (after some computations) that $r_{V,\infty}^{crit} < \infty$ for $\gamma/\beta < 1$ and $r_{V,\infty}^{crit} = \infty$ for $\gamma/\beta \geq 1$. Note that in all situations $r_{V,\infty}^{crit} > 1$.

We introduce the critical point X_∞^{crit} which is the unique solution of the following equation:

$$\partial_{X_1 X_1}^2 \zeta(X_\infty^{crit}) = 0, X_\infty^{crit} \in \Gamma_1. \quad (89)$$

As in (51), we show that $X_\infty^{crit} = (x^*, 1 - x^*)$ where x^* is the solution of:

$$\frac{\gamma}{\beta} - \left(x^* - \frac{\gamma}{\beta}\right) = x^* e^{-\frac{\gamma}{\beta}[(1-x^*)+2(x^*-\frac{\gamma}{\beta})]}. \quad (90)$$

When $r_V < r_{V,\infty}^{crit} r_I$ we define a partition of Ω in two regions

$$\Omega_\infty^{NV} = \{Y \in \Omega \mid \partial_{X_1} \zeta(Y) < r_V/r_I\} \quad (91)$$

$$\Omega_\infty^V = \{Y \in \Omega \mid \partial_{X_1} \zeta(Y) > r_V/r_I\}. \quad (92)$$

Note that $r_{V,\infty}^{crit} = \lim_{u_{max} \rightarrow \infty} r_{V,u_{max}}^{crit}$ and for u_{max} large enough $\Omega_{u_{max}}^V = \Omega_\infty^V$ (and $\Omega_{u_{max}}^{NV} = \Omega_\infty^{NV}$). As before we can prove the following:

Lemma 3.4. *Any trajectory $\Phi^{Y,dV}(t)$ with $Y \in \mathcal{L}_{r_V/r_I}^{\partial_{X_1} \zeta} = \partial\Omega_\infty^{NV} \cap \partial\Omega_\infty^V$ is such that $\Phi^{Y,dV}(t) \in \Omega_\infty^{NV}$ for all $t > 0$ ($dV \in \mathcal{U}_Y$).*

To any $Y \in \Omega$ associate the unique $\Delta Y \geq 0$ such that $(Y_1 - \Delta Y, Y_2) \in \mathcal{L}_{r_V/r_I}^{\partial_{X_1} \zeta}$ and define: $\Pi_\infty^{r_V, r_I}(Y) = J(Y, \Delta Y \delta_{t=0})$. If ΔY does not exist then set $\Delta Y = 0$ with the convention $0 \times \delta_{t=0} = 0$. Note that $\Delta Y = 0$ for any $Y \in \Omega_{u_{max}}^{NV}$ and moreover:

$$\Pi_\infty^{r_V, r_I}(Y) = \begin{cases} J_0(Y) & \text{if } Y \in \Omega_\infty^{NV} \\ J_0(Y_1 - \Delta Y, Y_2) + r_V(\Delta Y) & \text{if } Y \in \Omega_\infty^V, (Y_1 - \Delta Y, Y_2) \in \mathcal{L}_{r_V/r_I}^{\partial_{X_1} \zeta}, \Delta Y \geq 0. \end{cases} \quad (93)$$

For u_{max} large enough $\Pi_{u_{max}}^{r_V, r_I}$ and $\Pi_\infty^{r_V, r_I}$ coincide on $\Omega_{u_{max}}^{NV}$. Moreover since for any $Y \in \Omega_{u_{max}}^V$ and given u_{max} the optimal strategies $u_Y^{u_{max}}$ converge (when $u_{max} \rightarrow \infty$) to the Dirac delta function $\Delta Y \delta_{t=0}$ then:

$$\mathcal{V}^\infty(Y) = \lim_{u_{max} \rightarrow \infty} \mathcal{V}^{u_{max}}(Y) = \lim_{u_{max} \rightarrow \infty} J(Y, u_Y^{u_{max}}) = J(Y, \Delta Y \delta_{t=0}) = \Pi_\infty^{r_V, r_I}(Y). \quad (94)$$

Therefore we proved the following:

Theorem 3.5. *For $r_V < r_I r_{V,\infty}^{crit}$ the function $\Pi_\infty^{r_V, r_I}$ is the unique solution of the HJB equation (79) and $\Pi_\infty^{r_V, r_I} = \mathcal{V}^\infty$. As a consequence in this case the value function \mathcal{V}^∞ is in $\mathcal{C}^1(\Omega)$.*

Proof. The proof is already above.

A direct proof also can be given; for instance suppose one wants to prove e.g., that $-\mathcal{H}^\infty(Y, \nabla \Pi_\infty^{r_V, r_I}(Y)) = 0$ for $Y \in \Omega$.

The mere definition of the domain Ω_∞^{NV} imply that $r_V - \partial_{X_1} J_0 \geq 0$ on this domain; on the other hand $\beta X_1 X_2 (r_I + \partial_{X_2} J_0 - \partial_{X_1} J_0) - \gamma X_2 \partial_{X_2} J_0 = 0$ everywhere; thus $-\mathcal{H}^\infty(Y, \nabla \Pi_\infty^{r_V, r_I}(Y)) = 0$ for $Y \in \Omega_\infty^{NV}$.

For $Y \in \Omega_\infty^V$ (with $(Y_1 - \Delta Y, Y_2) \in \mathcal{L}_{r_V/r_I}^{\partial_{X_1} \zeta}$) the definition of $\Pi_\infty^{r_V, r_I}$ implies that for any $\epsilon < \Delta Y$: $\Pi_\infty^{r_V, r_I}(Y_1, Y_2) = \Pi_\infty^{r_V, r_I}(Y_1 - \epsilon, Y_2) + r_V \epsilon$ thus $\partial_{X_1} \Pi_\infty^{r_V, r_I}(Y) = r_V$; in addition $\partial_{X_2} \Pi_\infty^{r_V, r_I}(Y) = \partial_{X_2} \Pi_\infty^{r_V, r_I}(Y_1 - \Delta Y, Y_2)$ and the conclusion follows from the HJB equation of J_0 on the $\mathcal{L}_{r_V/r_I}^{\partial_{X_1} \zeta}$.

3.4 A candidate value function: the super-critical case

We consider here the situation $r_V/r_I \geq r_{V, u_{max}}^{crit}$; note that this implies $\gamma/\beta \leq 1$. Introduce

$$\Gamma_{sub}^{crit} = \{Y \in \mathcal{L}_{r_V/r_I}^{\partial_{X_1} \zeta} \mid \partial_{X_1 X_1}^2 \zeta(Y) \leq 0\}. \quad (95)$$

For any $Y \in \Gamma_{sub}^{crit}$ define:

$$Y_1^{super} = \sup\{Z_1 \geq Y_1 \mid J_0(Y) + r_V(Z_1 - Y_1) \leq J_0(Z_1, Y_2)\}. \quad (96)$$

We define a curve Γ_{super}^{crit} as:

$$\Gamma_{super}^{crit} = \{(Y_1^{super}, Y_2) \mid Y \in \Gamma_{sub}^{crit}\}. \quad (97)$$

Remark 5. We can express Γ_{sub}^{crit} in a parametric form:

$$\Gamma_{sub}^{crit} = \{(X_1^\Delta, X_2^\Delta) \in \Omega \mid 0 \leq \Delta \leq \Delta_{max}\},$$

where Δ_{max} is such that $X_2^{\Delta_{max}} = 0$ and

$$X_1^\Delta = \frac{\Delta(e^{-\frac{\gamma}{\beta} \frac{r_V}{r_I} \Delta} - 1)}{1 - \Delta \frac{\beta}{\gamma} \frac{r_V}{r_I} - e^{-\frac{\gamma}{\beta} \frac{r_V}{r_I} \Delta}} \quad (98)$$

$$X_2^\Delta = -\frac{\beta}{\gamma} \ln \left(\frac{1 - \frac{r_V}{r_I}}{1 + \frac{r_V}{r_I} \frac{\beta}{\gamma} X_1^\Delta} \right) - \frac{\frac{r_V}{r_I} X_1^\Delta (1 + \frac{\beta}{\gamma} X_1^\Delta)}{1 + \frac{r_V}{r_I} \frac{\beta}{\gamma} X_1^\Delta}. \quad (99)$$

The curves Γ_{sub}^{crit} and Γ_{super}^{crit} define a domain that will be denoted Ω_∞^V ; set also $\Omega_\infty^{NV} = \Omega \setminus \Omega_\infty^V$.

Note that when $r_V \geq 2r_I$ the sets Γ_{sub}^{crit} , Γ_{super}^{crit} and Ω_∞^V are empty.

To any $Y \in \Omega$ associate the unique $\Delta Y \geq 0$ such that $(Y_1 - \Delta Y, Y_2) \in \Gamma_{sub}^{crit}$ and define: $\Pi_\infty^{r_V, r_I}(Y) = J(Y, \Delta Y \delta_{t=0})$. If ΔY does not exist then set $\Delta Y = 0$ with the convention $0 \times \delta_{t=0} = 0$. Note that $\Delta Y = 0$ for any $Y \in \Omega_\infty^{NV}$ and moreover:

$$\Pi_\infty^{r_V, r_I}(Y) = \begin{cases} J_0(Y) & \text{if } Y \in \Omega_\infty^{NV} \\ J_0(Y_1 - \Delta Y, Y_2) + r_V(\Delta Y) & \text{if } Y \in \Omega_\infty^V, (Y_1 - \Delta Y, Y_2) \in \Gamma_{sub}^{crit}, \Delta Y \geq 0. \end{cases} \quad (100)$$

Note that for any given $Y \in \Omega_\infty^V$ for u_{max} large enough $Y \in \Omega_{u_{max}}^V$. Moreover for any $Y \in \Omega_{u_{max}}^V$ and given u_{max} the optimal strategies $u_Y^{u_{max}}$ converge (when $u_{max} \rightarrow \infty$) to a Dirac delta function $\Delta Y \delta_{t=0}$ then:

$$\mathcal{V}^\infty(Y) = \lim_{u_{max} \rightarrow \infty} \mathcal{V}^{u_{max}}(Y) = \lim_{u_{max} \rightarrow \infty} J(Y, u_Y^{u_{max}}) = J(Y, \Delta Y \delta_{t=0}) = \Pi_\infty^{r_V, r_I}(Y). \quad (101)$$

Therefore we proved the following:

Theorem 3.6. For $r_V \geq r_I r_{V,\infty}^{crit}$ the function $\Pi_\infty^{r_V, r_I}$ is the unique solution of the HJB equation (79) and $\Pi_\infty^{r_V, r_I} = \mathcal{V}^\infty$. In particular when $r_V \geq 2r_I$: $\mathcal{V}^\infty = J_0 \in C^1(\Omega)$ but when $r_V \in]r_I r_{V,\infty}^{crit}, 2r_I[$ the value function \mathcal{V}^∞ is only Lipschitz.

4 Summary of optimal strategies

When $u_{max} < \infty$ from the equations (60) and (61), the Bellman principle and the properties of the value function it follows that for $Y \in \Omega_{u_{max}}^V$ the optimal strategy is initially $u = u_{max}$ while in $\Omega_{u_{max}}^{NV}$ the optimal control is necessarily null. Similar arguments hold for other situations; thus we can completely describe the optimal strategies for an arbitrary initial point $Y \in \bar{\Omega}$:

- when $u_{max} < \infty$: if $Y \in \Omega_{u_{max}}^V$ vaccinate with rate u_{max} until reaching the frontier of domain $\Omega_{u_{max}}^V$ and then stop vaccination. If $Y \notin \Omega_{u_{max}}^V$ do not vaccinate at all;
- when $u_{max} = \infty$: if $Y \in \Omega_\infty^V$ instantaneously vaccinate the minimal number of individuals that allow to reach the frontier of domain Ω_∞^V and then stop vaccination. If $Y \notin \Omega_\infty^V$ do not vaccinate at all.

Note that when $r_V \in]r_{V,u_{max}}^{crit}, 2r_I[$ the optimal strategy may not be unique. This happens on the frontier Γ_{super}^{crit} when two different strategies give the same cost (because the value function is continuous): either vaccinate until reaching Γ_{sub}^{crit} and then stop vaccinating or do not vaccinate at all. See also figure 5 for an illustration. Otherwise the optimal strategy is unique.

We refer to the text for the definitions of the domains $\Omega_{u_{max}}^V$ and Ω_∞^V , for instance when $u_{max} < \infty$ and $r_V \leq r_{V,u_{max}}^{crit} r_I$ (for $\gamma/\beta < 1$): $\Omega_{u_{max}}^V = \{Y \in \Omega \mid \partial_{X_1} \zeta(Y) \geq r_V/r_I\}$ but when $u_{max} < \infty$ and $r_V \in [r_{V,u_{max}}^{crit} r_I, 2r_I]$ (for $\gamma/\beta < 1$) the definition of the domain $\Omega_{u_{max}}^V$ is more involved (cf. previous sections); same for Ω_∞^V and $r_{V,\infty}^{crit}$.

With respect to the existing literature the above optimal strategies are distinct in several aspects:

- when $u_{max} < \infty$: previous contributions take $\Omega_{u_{max}}^V = \{Y \in \Omega \mid \partial_{X_1} \zeta(Y) \geq r_V/r_I\}$ while our definition is different for $r_V \in [r_{V,u_{max}}^{crit} r_I, 2r_I]$. The strategies here will lead to lower costs.
- when $u_{max} = \infty$: we do not ask full vaccination but only vaccinate the minimum proportion that allows to reach the frontier of Ω_∞^V .

Finally, Remark 1 shows that the cost functional in the equation (6) has the same optimal strategies and vaccination regions.

Appendix

A Properties of the number of infected people without vaccination

We recall some properties of the number of infected people in absence of vaccination. The reader can also consult [2, 1]. Consider the model without control:

$$\begin{cases} \frac{dX_1(t)}{dt} = -\beta X_1(t)X_2(t), \\ \frac{dX_2(t)}{dt} = \beta X_1(t)X_2(t) - \gamma X_2(t), \\ \frac{dX_3(t)}{dt} = \gamma X_2(t). \end{cases} \quad (102)$$

Lemma A.1. *The size ζ of an epidemic without vaccination starting at $\Phi^{X,0}(0) = X = (X_1, X_2)$ is the unique solution in $[0, X_1[$ of the equation:*

$$1 - \frac{\zeta}{X_1} = e^{-\frac{\beta}{\gamma}(X_2 + \zeta)}. \quad (103)$$

Moreover $\zeta(X) > X_1 - \frac{\gamma}{\beta}$, $\forall X \in \Omega$ and $\zeta \in \mathcal{C}^1(\Omega)$.

Remark 6. Although ζ depends on X , when there is no ambiguity, we will just write ζ .

Proof. Denote $X_1^\infty = \lim_{t \rightarrow \infty} \Phi_1^{X,0}(t)$, $X_2^\infty = \lim_{t \rightarrow \infty} \Phi_2^{X,0}(t)$. Straightforward computations allow to prove that:

$$\Phi_1^{X,0}(t) = X_1 e^{-\frac{\beta}{\gamma}(1 - \Phi_1^{X,0}(t) - \Phi_2^{X,0}(t))}. \quad (104)$$

Or $X_2^\infty = 0$ thus $X_1^\infty = X_1 e^{-\frac{\beta}{\gamma}(X_1 + X_2 - X_1^\infty)}$. Using that $\zeta = X_1 - X_1^\infty$ we obtain equation (103).

Let $F(y, X_1, X_2) = e^{-\frac{\beta}{\gamma}(y + X_2)} - (1 - \frac{y}{X_1})$ defined on $[0, X_1] \times \Omega$. Since $F(0, X_1, X_2) = e^{-\frac{\beta}{\gamma}X_2} - 1 \leq 0$ and $F(X_1, X_1, X_2) = e^{-\frac{\beta}{\gamma}(X_1 + X_2)} > 0$ the equation (in y) $F(y, X_1, X_2) = 0$ has at least a solution in $[0, X_1[$; thus equation (103) has at least a solution in $[0, X_1[$.

Moreover $\frac{\partial F}{\partial y}(y, X_1, X_2) = -\frac{\beta}{\gamma}e^{-\frac{\beta}{\gamma}(y + X_2)} + \frac{1}{X_1}$. Since $y \leq X_1 \leq 1$ and $X_1 + X_2 \leq 1$ we obtain $\frac{\partial F}{\partial y}(y, X_1, X_2) \geq -\frac{\beta}{\gamma}e^{-\frac{\beta}{\gamma}} + 1 > 0$ (because $1 > ze^{-z}$ for any $z > 0$); therefore $F(\cdot, X_1, X_2)$ is strictly increasing in y and the solution ζ is unique.

If $X_1 \leq \frac{\gamma}{\beta}$ since $\zeta \geq 0$ we obtain immediately $\zeta \geq X_1 - \frac{\gamma}{\beta}$. If on the contrary $X_1 \geq \frac{\gamma}{\beta}$ (thus in particular $\frac{\gamma}{\beta X_1} \in]0, 1[$) we obtain:

$$\begin{aligned} F(X_1 - \frac{\gamma}{\beta}, X_1, X_2) &= e^{-\frac{\beta}{\gamma}(X_1 - \frac{\gamma}{\beta} + X_2)} - (1 - \frac{X_1 - \frac{\gamma}{\beta}}{X_1}) \\ &\leq e^{-\frac{\beta}{\gamma}(X_1 - \frac{\gamma}{\beta})} - \frac{\gamma}{\beta X_1} \leq 0 \end{aligned} \quad (105)$$

where for the last inequality we used that $e^{1-1/z} - z \leq 0$ for any $z = \frac{\gamma}{\beta X_1} \in]0, 1[$. Therefore the solution ζ is in $[(X_1 - \gamma/\beta)_+, X_1[$. When X belongs to the open set Ω then same arguments show that the inequality $\zeta > (X_1 - \gamma/\beta)$ is strict.

Remark 7. Thanks to (103) we obtain by the implicit function Theorem that ζ has continuous derivatives around any $X \in \Omega$; we can calculate first and second partial derivatives of ζ with respect to X_1 and X_2 :

$$\frac{\partial \zeta}{\partial X_1} = \frac{\zeta}{X_1 \left(1 + \frac{\beta}{\gamma}(\zeta - X_1)\right)}, \quad (106)$$

$$\frac{\partial \zeta}{\partial X_2} = \frac{\gamma/\beta}{\zeta - X_1 + \gamma/\beta} - 1, \quad (107)$$

$$\frac{\partial^2 \zeta}{\partial X_1^2} = -\frac{\gamma}{\beta} \frac{\zeta(\zeta - 2X_1 + 2\gamma/\beta)(\zeta - X_1)}{X_1^2(\zeta - X_1 + \gamma/\beta)^3}, \quad (108)$$

$$\frac{\partial^2 \zeta}{\partial X_1 \partial X_2} = \frac{\partial^2 \zeta}{\partial X_2 \partial X_1} = \frac{\gamma}{\beta} \frac{(X_1 - \gamma/\beta)(\zeta - X_1)}{X_1(\zeta - X_1 + \gamma/\beta)^3}. \quad (109)$$

Note that since $\zeta > X_1 - \frac{\gamma}{\beta}$ all fractions are well defined and ζ is even $C^2(\Omega)$.

B Properties of the trajectories

Lemma B.1. $\partial_{X_1} J_0 = \frac{\partial J_0}{\partial X_1}$ is decreasing along trajectories of the system (102).

Proof. We have to prove that:

$$\langle f(X, 0), \nabla \partial_{X_1} J_0(X) \rangle < 0. \quad (110)$$

Using the expression of ζ , we have: $J_0(X) = \int_0^\infty r_I X_1(\tau) X_2(\tau) d\tau = r_I \zeta(X)$. Equation (110) can thus be rewritten as follows:

$$\langle f(X, 0), \nabla \partial_{X_1} \zeta \rangle < 0. \quad (111)$$

Using equations (108) and (109), this gives after some computations:

$$X_1 \left(\zeta - X_1 + \frac{\gamma}{\beta} \right)^2 > 0$$

which is always true because X_1 is strictly positive and $\zeta \neq X_1 - \frac{\gamma}{\beta}$.

Lemma B.2. For all $Y \in \Gamma_1 \cup \Gamma_{A1}$ the trajectory $\Phi^{Y,u}(t)$ is incoming in $\Omega \forall u \in \mathcal{U}_Y$.

Proof. For Γ_1 , the scalar product with the incoming normal is positive:

$$\langle f(X, u), (-1, -1) \rangle = \gamma X_2 + u \geq 0 \forall u \in [0, u_{max}].$$

For Γ_{A1} :

$$\langle f(X, u), (0, 1) \rangle = X_2(\beta X_1 - \gamma) \geq 0 \forall u \in [0, u_{max}].$$

Lemma B.3. J_0 is $C^1(\Omega)$.

Proof. Since $J_0 = r_I \zeta$ the conclusion follows from Lemma A.1.

Lemma B.4. For all $X \in \Omega$, we have $\frac{\partial \zeta}{\partial X_1}(X) \leq 2$. Therefore $0 \leq \frac{\partial J_0}{\partial X_1}(X) \leq 2r_I \forall X \in \Omega$.

Proof. Using expression in (106), to prove $\partial_{X_1}\zeta \leq 2$, we just have to show that $\zeta \geq \frac{2X_1(X_1 - \frac{\gamma}{\beta})}{2X_1 - \frac{\gamma}{\beta}}$. For that, we take same notation and result as in the proof of the Lemma A.1 so $X_1 \geq \frac{\gamma}{\beta}$ and we denote $\xi = \frac{2X_1(X_1 - \frac{\gamma}{\beta})}{2X_1 - \frac{\gamma}{\beta}}$. We have to prove that $F(\xi, X_1, X_2) \leq 0$.

With these notations, we have $F(\xi, X_1, X_2) = e^{-\frac{\gamma}{\beta}(\xi + X_2)} - \frac{\frac{\gamma}{\beta}}{2X_1 - \frac{\gamma}{\beta}}$.

If we note $z = \frac{\beta}{\gamma}\xi$, we obtain, $e^{-z - \frac{\gamma}{\beta}X_2} - (z + \sqrt{z^2 + 1})$.

As $e^{-z - \frac{\gamma}{\beta}X_2} \leq e^{-z} \leq \frac{1}{z + \sqrt{z^2 + 1}}$, this proves that $F(\xi, X_1, X_2) \leq 0$.

Lemma B.5. The level lines defined by $\mathcal{L}_r^{\partial_{X_1}\zeta} = \{(X_1, X_2) \in \Omega \mid \frac{\partial \zeta}{\partial X_1}(X) = r\}$ have the parametric equation:

$$1 - \frac{1 - \frac{\beta}{\gamma}X_1}{\frac{1}{r} - \frac{\beta}{\gamma}X_1} = e^{-\frac{\beta}{\gamma}\left(X_1 \frac{1 - \frac{\beta}{\gamma}X_1}{\frac{1}{r} - \frac{\beta}{\gamma}X_1} + X_2\right)}. \quad (112)$$

and have point $A = (\frac{\gamma}{\beta}, 0)$ as limit (but $A \notin \mathcal{L}_{r_V/r_I}^{\partial_{X_1}\zeta}$).

Proof. If $X = (X_1, X_2) \in \mathcal{L}_r^{\partial_{X_1}\zeta}$ using the definition of $\mathcal{L}_r^{\partial_{X_1}\zeta}$ and (109) we have:

$$\zeta = X_1 \frac{1 - \frac{\beta}{\gamma}X_1}{\frac{1}{r} - \frac{\beta}{\gamma}X_1}. \quad (113)$$

Then, we replace in (103) to obtain the parametric equation. Note that $\nabla_X \zeta$ is not defined at A . The level line $\mathcal{L}_0^{\partial_{X_1}\zeta}$ is Γ_{O_A} and the level line $\mathcal{L}_1^{\partial_{X_1}\zeta}$ is $\{X \in \Omega \mid X_1 = \frac{\gamma}{\beta}\}$. Suppose $r \notin \{0, 1\}$, then for any $X \in \mathcal{L}_r^{\partial_{X_1}\zeta}$ we have $\frac{\partial \zeta}{\partial X_1}(X) \neq 0$. The level line $\mathcal{L}_r^{\partial_{X_1}\zeta}$ is regular in the neighborhood of any $\bar{X} = (\bar{X}_1, \bar{X}_2) \in \Omega$. Indeed if $\frac{\partial \zeta}{\partial X_1}(\bar{X}) = r$ by the implicit function Theorem in the neighborhood of \bar{X} there exists a curve $X_2 = X_2(X_1)$ such that $\frac{\partial \zeta}{\partial X_1}(X_1, X_2(X_1)) = r$. Moreover, by the same Theorem $X_2(X_1)$ is \mathcal{C}^1 locally. Thus the level line $\mathcal{L}_r^{\partial_{X_1}\zeta}$ is regular around any point in Ω . As such it does not have self-intersections either. In addition for any $r \in [0, 2]$ since $\lim_{X_1 \rightarrow \frac{\gamma}{\beta}^-} \frac{\partial \zeta}{\partial X_1}(X_1, 0) = 0$ and $\lim_{X_1 \rightarrow \frac{\gamma}{\beta}^+} \frac{\partial \zeta}{\partial X_1}(X_1, 0) = 2$ by continuity we obtain that $\mathcal{L}_r^{\partial_{X_1}\zeta}$ will be as close to A as wanted thus A is an extremity of $\mathcal{L}_r^{\partial_{X_1}\zeta}$ (but does not belong to it).

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