Global optimal vaccination in the SIR model: properties of the value function and application to cost-effectiveness analysis

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Abstract

This work focuses on optimal vaccination policies for an Susceptible - Infected - Recovered (SIR) model; the impact of the disease is minimized with respect to the vaccination strategy. The problem is formulated as an optimal control problem and we show that the value function is the unique viscosity solution of an Hamilton - Jacobi - Bellman (HJB) equation. This allows to find the best vaccination policy. At odds with existing literature, it is seen that the value function is not always smooth (sometimes only Lipschitz) and the optimal vaccination policies are not unique. Moreover we rigorously analyze the situation when vaccination can be modeled as instantaneous (with respect to the time evolution of the epidemic) and identify the global optimum solutions. Numerical applications illustrate the theoretical results. In addition the pertussis vaccination in adults is considered from two perspectives: first the maximization of DALY averted in presence of vaccine side-effects; then the impact of the herd immunity on the cost-effectiveness analysis is discussed on a concrete example.

Keywords: optimal vaccination, SIR model, vaccination region, herd immunity and cost-effectiveness,

1 Outline of the paper

1.1 Background on vaccination strategies

The mathematical modelling of the spread of an infection disease allows to propose control strategies to decrease the cost of the epidemic. Among such control strategies we focus in this work on the vaccination. A vaccination policy indicates when and how many people should be vaccinated in order to minimize the overall impact of the epidemic. We consider here a cost that sums the cost of the infected individuals and the cost to vaccinate the individuals (see formula (3) below for the mathematical definition). We also apply the same methodology to cost-effectiveness analysis in the context of a constrained public health budget.

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1.2 State of the art and motivation

The mathematical analysis of the cost, as a function of the vaccination policy, allows to obtain an optimal vaccination strategy. Consider the epidemic in Figure 1 (see caption for the detail of the parameters) where the abscissa represents the number of the susceptible in the population, and the ordinate the proportion of infected people. In the literature several proposals for the best vaccination strategy are presented (see for example [2, 10, 23, 27]); however previous works operated under specific assumptions on the value function (see below) and consequently did not always selected the best vaccination policy.

For instance, as we illustrate in Figure 1 the solution available in the literature is, in some cases, not optimal. The two curves represent two scenarios for an epidemic starting for an initial point X_0. The solid curve represents the epidemic evolution when there is no vaccination (the state of the art solution for this set of parameters) and the dashed curve plots the epidemic evolution when there is some partial vaccination. The partial vaccination is seen to outperform the no vaccination policy.

For further information see the literature review in Section 2.4.

1.3 Methodology and results

Prompted by this remark we look in this work into the details of the calculation of the best vaccination strategy (using the technique of the "viscosity solutions") and note that all previous works used a specific assumption which is not always true; we explain precisely when the assumption is correct (and thus the previous works identified correctly the optimal vaccination policy) and when it is not (and in this case we describe the best vaccination policy).

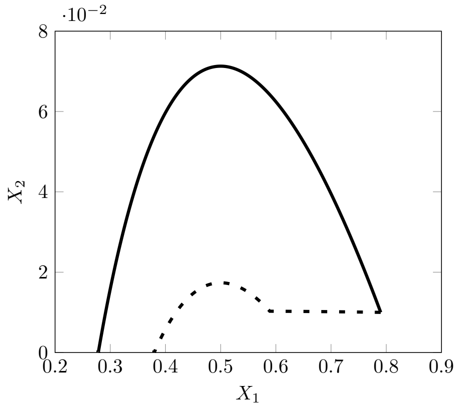


Figure 1: Two trajectories of an epidemic evolution (corresponding to the SIR model in equations (1)) are presented. The epidemic starts from $X_0 = (0.79, 0.0053)$. The parameters used are $\beta = 73$, $\gamma = 36.5$, $u_{max} = 100$, $r_I = 1$ and $r_V = 1.5$ (see formula (3) for the meaning of the parameters r_I and r_V and Section 2.2 for u_{max}). The solid curve represents the epidemic evolution when there is no vaccination (which is the state of the art solution, see [2, 23, 27]) and the dashed curve plots the epidemic evolution when there is some partial vaccination. The cost for the first trajectory is 0.51 and for the second is 0.49.

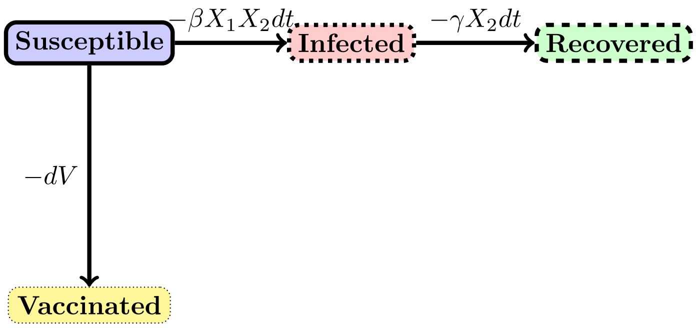


Figure 2: Graphical illustration of the SIR-V model.

1.4 Structure of the paper

The paper is organized as follows: in the next section we describe the mathematical model (section 2.1), the admissible vaccination policies (section 2.2), introduce some notations in section 2.3 and give an overview of the contributions from the literature in section 2.4; finally we present some technical obstacles in section 2.5.

In section 3 several applications of the theoretical results (proved in appendixes D and E) are presented. A summary of the numerical procedure to find the best vaccination strategy is the object of section 4.

Then in section 5 we consider two applications to the optimal pertussis vaccination in adults. Finally, conclusions are the object of section 6.

2 Model, notations and first remarks

2.1 The model

In order to model the evolution of an epidemic, we use an SIR (Susceptible - Infected - Recovered) compartment model (cf., [6, 13, 5] for additional details).

We seek to optimize the cost of the vaccination policy; to this end denote by $V(t)$ the proportion of people vaccinated by the time t (of course $\lim_{t\to\infty} V(t) \leq 1$); we consider vaccines that confer lifetime immunity so that V is an increasing function. The evolution of the disease is described by the following equations:

$$\begin{cases} dX_1(t) = -\beta X_1(t)X_2(t)dt - dV(t), & X_1(0) = X_{10}, \\ dX_2(t) = \Big(\beta X_1(t)X_2(t) - \gamma X_2(t)\Big)dt, & X_2(0) = X_{20}, \\ dX_3(t) = \gamma X_2(t)dt, & X_3(0) = X_{30}, \\ X_4(t) = \int_0^t dV, & X_4(0) = 0. \end{cases} \tag{1}$$

Here X_1, X_2, X_3, X_4 are the proportion of people in the "susceptible" respectively "infectious", "recovered" and "vaccinated" classes. Initially $X_1(0) + X_2(0) + X_3(0) = 1$ and $X_4(0) = 0$ (but X_4 need not be continuous in 0). See figure 2 for a graphical view of system (1). Note that (1) implies $X_1(t)+X_2(t)+X_3(t) + X_4(t) = 1$, $\forall t \geq 0$.

Here β is the transmission rate of the disease, V is the control to be optimized and γ is the recovery rate.

We denote r_V the unitary cost associated with vaccination including the cost of the vaccine and all possible side-effects and r_I the unitary cost incurred by infected persons. To simplify the presentation we suppose that costs are expressed in money and postpone to Section 5 the more realistic and interesting situations when costs are expressed as medical conditions.

The cost of the disease is independent of the classes X_3 and X_4 (but dependent on the control $V(t)$), so we can restrict ourselves to the evolution of X_1 and X_2. From now on a vector X will only be supposed to have two coordinates X_1 and X_2. Denoting:

$$\Omega = \{X = (X_1, X_2) \in \mathbb{R}^2 \mid X_1, X_2 > 0, X_1 + X_2 < 1\}, \tag{2}$$

we will work under the constraints $X \in \overline{\Omega}$.

We introduce $\Phi^{Y,dV}(t) = (\Phi_1^{Y,dV}(t), \Phi_2^{Y,dV}(t))$ to denote, at time $t \geq 0$, the solution of the system (1) starting at point $X(0) = Y$ and with control dV; in addition $Z = \Phi^{Y,dV(\cdot)}(-t)$ means $Y = \Phi^{Z,dV(t-\cdot)}(t)$ (the reverse system has a well defined mathematical meaning). To ease notations, when the measure dV is absolutely continuous with respect to the canonical Lebesgue measure dt on $[0,\infty[$ i.e., when dV can be written $dV = u(t)dt$ we will also write $\Phi^{Y,u(t)}(t)$ instead of $\Phi^{Y,u(t)dt}(t)$ (and the same for the components $\Phi_1^{Y,u(t)dt}(t)$ and $\Phi_2^{Y,u(t)dt}(t)$).

Remark 1. Here and in all that follows we consider the interval $[0,\infty[$ open at infinity. This simply means that ∞ is not an admissible value and no strategy can vaccinate at $t = \infty$; on the contrary instantaneous vaccination at $t = 0$ is possible.

The cost of the disease is:

$$J(Y, dV) = \int_0^\infty r_I \beta \Phi_1^{Y,dV}(t) \Phi_2^{Y,dV}(t) dt + \int_0^\infty r_V dV(t). \tag{3}$$

Moreover we will use the following notation $J_0(Y) = J(Y, 0)$; note that $J_0(Y)$ is a cost proportional with the number of people infected in absence of vaccination. This number will be denoted $\zeta(Y)$ thus $J_0(Y) = r_I \zeta(Y)$ (see Appendix A for the properties of ζ).

Remark 2. Equation (1) implies

$$\begin{aligned} \Phi_2^{X,dV}(\infty) &= \Phi_2^{X,dV}(0) + \int_0^\infty d\Phi_2^{X,dV}(t) \\ &= \Phi_2^{X,dV}(0) + \int_0^\infty \Big(\beta \Phi_1^{X,dV}(t) \Phi_2^{X,dV}(t) - \gamma \Phi_2^{X,dV}(t)\Big) dt \end{aligned} \tag{4}$$

Thus, since $\Phi_2^{X,dV}(\infty) = 0$:

$$\int_0^\infty r_I \beta \Phi_1^{X,dV}(t) \Phi_2^{X,dV}(t) dt = \int_0^\infty r_I \gamma \Phi_2^{X,dV}(t) dt - \Phi_2^{X,dV}(0). \tag{5}$$

This allows to conclude that the cost functional

$$J^d(Y, dV) = \int_0^\infty r_I^d \Phi_2^{Y,dV}(t) dt + \int_0^\infty r_V dV(t) \tag{6}$$

with $r_I^d = r_I \gamma$ satisfies

$$J^d(Y, dV) = J(Y, dV) + Y_2. \tag{7}$$

Both J^d and J will thus have same optimal strategies (because their difference is independent of the strategy dV). Here r_I^d can be seen as the unitary cost of infection per unit time.

2.2 The admissible vaccination policies

Vaccination policy dV can be modeled in different ways. Note that the proportion $\int_0^t dV(s)$ of individuals vaccinated up to time "t" is increasing and $\int_0^t dV(s) \leq 1$, $\forall t \geq 0$; therefore V is a bounded variation function and $dV(t)$ is a positive measure on $[0, \infty[$; this is the most general class of vaccination strategies. A restrictive class of vaccination policies will also be considered (see also the literature review in Section 2.4 below) where the speed of vaccination is bounded; in this case $dV(t) = u(t)dt$ with $u(t) \in [0, u_{max}]$. Generic results (see e.g., [7]) suggest that considering controls with bounded speed is not restrictive because the general situation is obtained in the limit $u_{max} \to \infty$. We will rigorously prove this assertion in appendix E and will work with the restricted class of vaccination policies until then.

We can write system (1) as:

$$d \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} -\beta X_1 X_2 \\ \beta X_1 X_2 - \gamma X_2 \\ \gamma X_2 \\ 0 \end{pmatrix} dt + \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} dV(t). \tag{8}$$

Recall that $(X_1, X_2, X_3, X_4) \mapsto (-\beta X_1 X_2, \beta X_1 X_2 - \gamma X_2, \gamma X_2, 0)^T$ and $(X_1, X_2, X_3, X_4) \mapsto (-1, 0, 0, 1)^T$ are Lipschitz functions, and V is a bounded variation function.

Then using the theoretical results in [11, Section 10, Thm. 10.2.3] it is possible to conclude that (8) has a solution and the solution depends smoothly on the initial data and the control V (in L^1 norm).

Let us make clear how a mathematical object such as V can be translated into vaccination policies for the unbounded case. Take for instance the trajectory $\Phi^{Y,dV}(t)$ driven by the control (here $\delta_{t=0}$ is the Dirac mass in $t = 0$):

$$dV(t) = \begin{cases} \frac{Y_1}{2} \delta_{t=0}, \quad t = 0 & \text{(9a)} \\ 0.10, \ t \in]0, 0.5[& \text{(9b)} \\ 0, \ t \geq 0.5. & \text{(9c)} \end{cases}$$

This means that half of the initial susceptible population Y_1 is vaccinated (instantaneously) at the onset $t = 0$. Then vaccination is pursued with speed of 10% percent per unit time till time $t = 0.5$; then no vaccination occurs. In particular this means that $50 + 0.5 \times 10 = 55$ percent of the population is vaccinated in all. Note that the trajectory $\Phi^{Y,dV}(t)$ is not continuous since $\Phi_1^{Y,dV}(0^+) = \Phi_1^{Y,dV}(0)/2$. This trajectory can be seen as the limit when $\epsilon \to 0$

of the trajectories $\Phi^{Y,dV_\epsilon}(t)$ corresponding to the following vaccination policies:

$$dV_\epsilon(t) = \begin{cases} \dfrac{Y_1}{2\epsilon}, \quad t \in [0,\epsilon] & (10a)\\ 0.10, \ \ t \in]\epsilon, 0.5[& (10b)\\ 0, \ \ t \geq 0.5. & (10c) \end{cases}$$

2.3 Notations and first remarks

We introduce the function $f : \overline{\Omega} \times \mathbb{R} \mapsto \mathbb{R}^2$:

$$(X,u) \in \overline{\Omega} \times \mathbb{R} \mapsto f(X,u) = (-\beta X_1 X_2 - u, \beta X_1 X_2 - \gamma X_2) \in \mathbb{R}^2. \tag{11}$$

Note that $f(\cdot,u)$ is a Lipschitz function with Lipschitz constant L_f independent of the second argument, i.e.,

$$\|f(Y,u) - f(Z,u)\| \leq L_f \|Y - Z\|, \ \forall \ Y, Z \in \overline{\Omega}. \tag{12}$$

In order to define the admissible controls we consider a point $Y \in \Omega$; for $u_{max} < \infty$ we define:

$$\mathcal{U}_Y^{u_{max}} = \left\{ u : [0,\infty[\to [0, u_{max}] \ \middle| \ u \text{ measurable}, \ \Phi^{Y,u(\cdot)}(t) \in \overline{\Omega}, \forall t \geq 0 \right\}. \tag{13}$$

When $u_{max} = \infty$ we define:

$$\mathcal{U}_Y^{\infty} = \left\{ dV \text{ positive measure on } [0,\infty[\ \middle| \int_0^\infty dV \leq Y_1 \leq 1, \ \Phi^{Y,dV}(t) \in \overline{\Omega}, \forall t \geq 0 \right\}. \tag{14}$$

Irrespective of whether u_{max} is bounded or not the set $\mathcal{U}_Y^{u_{max}}$ is a closed subset of the set of (finite, positive) measures on $[0,\infty[$. Note that for any $Y \in \Omega$ and any u_{max} : $0 \in \mathcal{U}_Y^{u_{max}}$.

To make notations easier we will not write the dependence of $\mathcal{U}_Y^{u_{max}}$ with respect to Y or u_{max} and only denote, when there is no ambiguity, by $\mathcal{U}_Y$ or $\mathcal{U}$ the set of admissible controls.

For $u_{max} < \infty$ we define the Hamiltonian $\mathcal{H}^{u_{max}} : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ as:

$$\mathcal{H}^{u_{max}}(X,p) = \min_{w \in [0,u_{max}]} [p \cdot f(X,w) + r_I \beta X_1 X_2 + r_V w] \tag{15}$$

$$= -u_{max}(p_1 - r_V)_+ + \beta X_1 X_2 (r_I + p_2 - p_1) - \gamma X_2 p_2. \tag{16}$$

When $u_{max} = \infty$ the previous definition is to be replaced by $\mathcal{H}^\infty : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$:

$$\mathcal{H}^\infty(X,p) = \min \{ r_V - p_1, \beta X_1 X_2 (r_I + p_2 - p_1) - \gamma X_2 p_2 \}. \tag{17}$$

The value function $\mathcal{V}^{u_{max}} : \overline{\Omega} \to \mathbb{R}$ is (for any u_{max} be it bounded or not):

$$\mathcal{V}^{u_{max}}(Y) = \inf_{u \in \mathcal{U}_Y^{u_{max}}} J(Y,u). \tag{18}$$

Any u such that $J(Y,u) = \mathcal{V}^{u_{max}}(Y)$ is called an optimal strategy for Y; it is not necessarily unique. However it has been proved in [10] that if $u_{max} < \infty$ at least one optimal strategy exists in the set $\mathcal{U}_Y^{u_{max}}$ and has the form $u = u_{max}1\!\mathrm{I}_{[0,\theta(Y)]}$ with $\theta(Y) \geq 0$. In fact since the total proportion of people susceptible to be vaccinated is at most 1 then $\theta(Y) \leq T_{max} := 1/u_{max}$. From now on we fix $\theta : \overline{\Omega} \to [0, \infty[$ to be a function (whose existence is guaranteed by the above mentioned result) such that $u_{max}1\!\mathrm{I}_{[0,\theta(Y)]}$ is an optimal strategy for Y.

We introduce the following notations:

- $A = (\frac{\gamma}{\beta}, 0) \in \mathbb{R}^2$,
- $\Gamma_1 = \{(X_1, X_2) \in \overline{\Omega} \mid X_1 + X_2 = 1\}$,
- $\Gamma_I = \{(X_1, X_2) \in \overline{\Omega} \mid X_1 = 0\}$,
- $\Gamma_S = \{(X_1, X_2) \in \overline{\Omega} \mid X_2 = 0\}$,
- $\Gamma_{OA} = \{(X_1, X_2) \in \Gamma_S \mid 0 \leq X_1 \leq \frac{\gamma}{\beta}\}$,
- $\Gamma_{A1} = \{(X_1, X_2) \in \Gamma_S \mid \frac{\gamma}{\beta} \leq X_1 \leq 1\}$.

Note that when $\gamma/\beta > 1$: $A \notin \Omega$, $\Gamma_{OA} = \Gamma_S$ and $\Gamma_{A1} = \emptyset$.

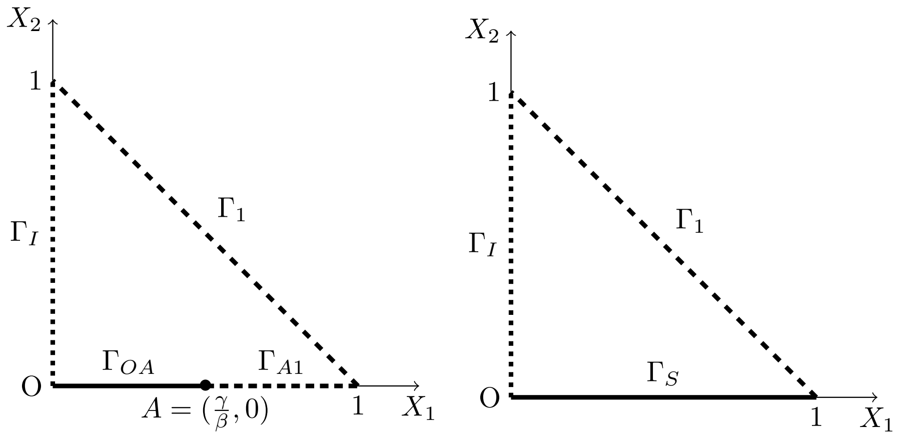


Figure 3: Boundary representation when $\frac{\gamma}{\beta} < 1$ (left) and $\frac{\gamma}{\beta} \geq 1$ (right).

Lemma 2.1. *The value function $\mathcal{V}^{u_{max}}$ is bounded on Ω. Moreover $\mathcal{V}^{u_{max}}|_{\Gamma_I \cup \Gamma_{OA}} = 0$ and $\mathcal{V}^{u_{max}}$ is continuous on $\Gamma_I \cup \Gamma_{OA}$.*

Proof. Choose $u = 0$ then

$$\mathcal{V}^{u_{max}}(X) \leq J(X, 0) = J_0(X) \leq r_I X_1 \leq r_I, \ \forall X \in \Omega. \tag{19}$$

Note that $J(X, u) = 0 \ \forall X \in \Gamma_I, \forall u \in \mathcal{U}_X$; using (19) we obtain $\mathcal{V}^{u_{max}}(X) = 0$ $\forall u_{max} \in [0, \infty]$, $\forall X \in \Gamma_I$ and the continuity on Γ_I. To set the value on Γ_{OA} note that when X is such that $X_1 < \gamma/\beta$ then $\Phi_2^{X,0}(t)$ tends exponentially to zero. Therefore: $J(X_n, 0) \to 0$ when $X_n \to X \in \Gamma_{OA}$.

2.4 Literature review

Many epidemic models have been proposed in order to describe epidemic propagation (see [6, 13, 5] for details). These models can be adapted in order to help controlling the propagation; several control options are available such as isolating infected people or immunizing susceptible people (see also [29, 10, 17] which propose combinations of these two methods). In this work we only analyze control policies that consist in the vaccination of susceptibles (immunization). The vaccination is supposed to confer lifetime (i.e., irreversible) immunity. In the context of immunization, several facts can affect the decisions of vaccination. The reference [16] discusses this problem in general, [9] proposes an approach taking the individual point of view, and [31] introduces an extension also using game theory.

The present work is on the contrary only concerned with finding an optimal vaccination strategy. Several studies have already considered this approach recasting it as an optimal control problem.

Historically one of the first to consider this problem, Abakuks explores two alternatives: in [2] a restrictive class of vaccination policies which allows at any time to immunize either all or none of the susceptible (therefore optimal policy immunizes either at once or never); in [1] the author considers policies which at any time during the course of the epidemic allow to immunize any number of the susceptible.

Abakuks proves the existence of an immunization region: within this region it is best to vaccinate with maximum effort and outside it is optimal to do nothing. The result is only obtained for $u_{max} = \infty$; moreover the proof only applies to vaccination policies dV that are finite sums of Dirac masses and it is not indicated how the value function $\mathcal{V}^{\infty}$ (assumed to be continuous) behaves in the limit when Dirac masses accumulate near a point or when such masses converge to a general measure on $[0, \infty[$.

In another work (see [19]) Hethcote considers a similar problem under additional constraints on the total proportion of the population affected and the maximum number of infected at the peak; the vaccination policies are taken to be stepwise constant functions and the cost of vaccination piecewise quadratic in the number of people vaccinated. He shows that the optimal strategy will be piecewise constant, with at most a single point of discontinuity.

In a similar work [23] authors consider $u_{max} = 1$ and define the class of admissible policies to contain function with only isolated discontinuities. They show that the optimal strategy has a single point of discontinuity and introduce the concept of vaccination border. To do this, they assume that the value function $\mathcal{V}^{u_{max}}$ is $\mathcal{C}^1(\Omega)$ which, as it will be seen in the following, is not always the case (it depends on the specific choice of parameters β, γ, u_{max}, r_V, r_I).

In [29] authors set $u_{max} < \infty$ for a finite horizon framework $T < \infty$ and work under the additional presence of a dumping term e^{-rt} in the cost functional which reads: $\int_0^T e^{-rt}\Big(r_V u(t) + r_I \Phi_2^{X,u(t)}(t)\Big)dt$; moreover the infected are supposed to pay an infection cost per unit time up to the time T and nobody recovers before time T, i.e., with our notations $\gamma = 0$. They use the maximum principle to characterize the optimal policies which turn out to be of bang-bang type with only one switch.

In [20] the existence and local optimality of singular controls is investigated and using the Maximum Principle it is shown that the optimal vaccination schedule can be singular. This corresponds to our limit $u_{max} \to \infty$. However no information is obtained on the regularity of the value function.

In the references described so far the authors focused on the optimal strategy without studying the properties of the value function. Using a similar model and an approach via optimal control [27] finds, via a Bellman equation, that the strategy is type bang-bang (only values 0 and u_{max} are taken). However they assume that the cost function is $\mathcal{C}^1(\Omega)$; finally, the results in the case where $u_{max} \to \infty$ are extrapolated and they suppose that the optimal strategy is bang-bang. As such the optimal policies are sometimes at odd with results in the stochastic case.

In a recent work H. Behncke (see [10]) proves, without using that value function is $\mathcal{C}^1(\Omega)$, that at least one optimal strategy for the trajectory starting at $X \in \overline{\Omega}$ is of the form $u_{max} 1\!\mathrm{I}_{[0,\theta(X)]}$, $\theta(X) \geq 0$, $\forall X \in \overline{\Omega}$. Although this information is very useful it does not allow to conclude on the regularity of the value function. As an illustration, we plot two situations: with parameters in figure 4 the function $\theta(X)$ is $\mathcal{C}^1(\overline{\Omega})$ while with parameters in figure 5 the function $\theta(X)$ is discontinuous.

Finally, without specifically entering in the context of epidemiology but using a general optimal control framework and the concept of viscosity solution the reference [32] analyzes the properties of the value function in the situation when a discount factor is present.

Considering the previous works several questions arise:

1. For which set of parameters (β, γ, u_{max}, r_V, r_I) is the value function $\mathcal{V}^{u_{max}}$ of class $\mathcal{C}^1(\Omega)$ and when is it less regular; note that if the value function $\mathcal{V}^{u_{max}}$ is not $\mathcal{C}^1$ some vaccination strategies derived under the $\mathcal{C}^1$ hypothesis will not be globally optimal.

2. Are the optimal strategies unique ?

3. What happens when $u_{max} = \infty$ (i.e., when vaccination is fast with respect to the epidemic propagation).

Our work answers these questions. In particular we show that value function is not always $\mathcal{C}^1$, the optimal strategies not always unique and prove rigorously what happens in the limit $u_{max} \to \infty$.

2.5 Specific mathematical difficulties of the problem

The approach proposed in this work faces specific technical difficulties among which:

- There do not exist natural boundary conditions to set on some parts of the frontier (Γ_1 and Γ_{A1}). This will pose problem when proving the uniqueness of the solution of the associated HJB equation. See section D.3 for the technique used to mitigate this difficulty.

- The state X is restricted to $\overline{\Omega}$ while the controls e.g., in the form $dV = udt$, $u \in [0, u_{max}]$ can drive it outside this domain.

- The cost function $J(X_0, dV)$ has no dumping term e^{-rt}, so we need to work in infinite horizon. This is a problem when trying to obtain Lipschitz regularity for the value function. See section D.2.

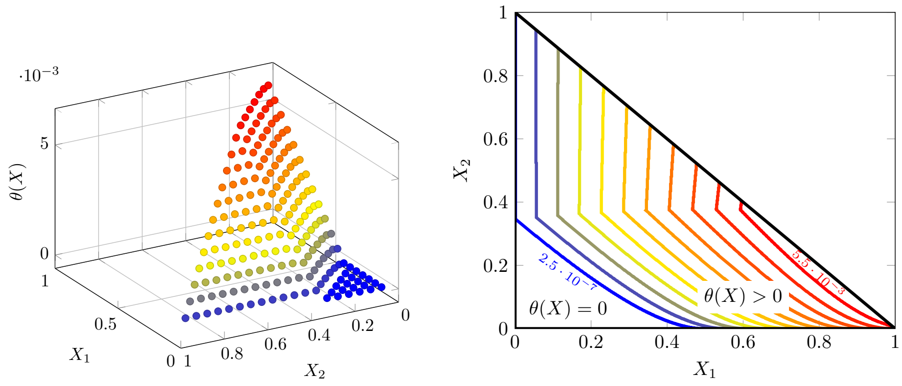


Figure 4: $\theta(X)$ for parameters $u_{max} = 100$, $r_V = 0.5$, $r_I = 1$, $\beta = 73$, $\gamma = 36.5$. **Left**: Representation as 3D function. **Right**: representation as level lines. We observe that θ is regular.

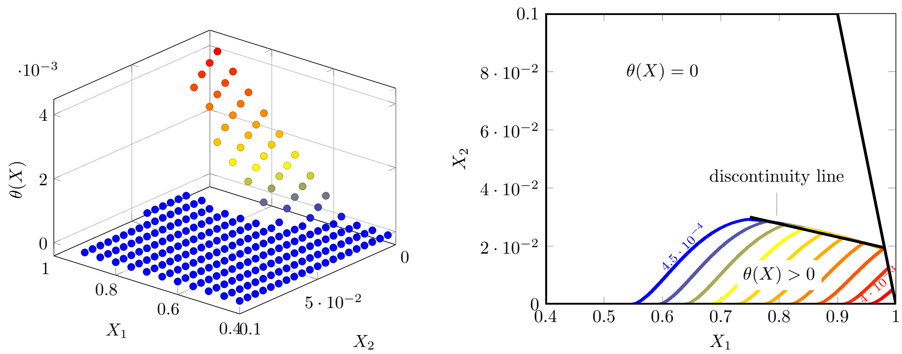


Figure 5: $\theta(X)$ for parameters $u_{max} = 100$, $r_V = 1.4$, $r_I = 1$, $\beta = 73$, $\gamma = 36.5$. **Left**: Representation as 3D function. **Right**: representation as level lines. In both cases we zoom on the discontinuity curve and plot $\Omega \cap ([0.4, 1] \times [0, 0.1])$. We observe that θ is discontinuous.

- In general, a convenient hypothesis (cf. also [12]) to prove the uniqueness of the viscosity solution of $\mathcal{F}(x, F(x), \nabla F(x)) = 0$ is that the Hamiltonian $\mathcal{F}$ is **strictly** monotone in the second argument. But here our Hamiltonians do not depend on this argument.
- In general optimal controls are unique (and the value function differentiable). Here this is not the case (cf. figure 6) which hints that value function has regularity defects.

3 Applications

In this section we apply the theoretical results obtained in appendixes D and E to several values of the parameters describing the epidemic propagation and vaccination policies. We refer to the appendixes for all notations used.

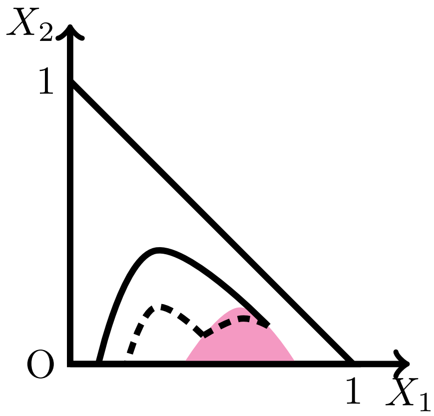


Figure 6: A typical example of non-unique optimal vaccination strategy: the solid trajectory corresponds to zero vaccination while the dashed trajectory corresponds to vaccination in the colored region followed by non vaccination. But both trajectories lead to the same, minimal, cost. In this case we expect the value function to not be of class $\mathcal{C}^1$. Non uniqueness appears when the trajectory with zero vaccination does not enter the vaccination region while the trajectory with maximal vaccination enters it. See subsections D.4, D.5, E.3 and E.4 for details.

3.1 A classical situation: $\beta = 73$, $\gamma = 36.5$, $u_{max} = 10$, $r_I = 1$, $r_V = 0.5$, $X_0 = (0.3, 0.05)$

Here $u_{max} < \infty$, we are thus in the situation described by appendix D. Using equation (60) we obtain $x^* = 0.59$ therefore $X^{crit}_{u_{max}} = (0.59, 0.41)$ and using equation (61) (and formulas involving $\partial_{X_1}\zeta$ available in the Appendix A) we obtain $r^{crit}_{V,u_{max}} = 1.036$. Therefore we are in the situation $r_V < r^{crit}_{V,u_{max}} r_I$ treated in Theorem D.7. The Theorem instructs us to plot the level line $\mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I}$ (whose equation as a curve is given in formula (31)). Plotting this curve (and zooming around the starting point X_0) we obtain the image in figure 7: the grey area corresponds to points on level lines $\mathcal{L}^{\partial_{X_1}\zeta}_r$ with $r \geq r_V/r_I$ while in the white area are situated points on level lines $\mathcal{L}^{\partial_{X_1}\zeta}_r$ with $r \leq r_V/r_I$. Our starting point X_0 is in the white domain. The theoretical result states that the optimal strategy is to not vaccinate at all. To illustrate this choice we compare numerically in figure 7 the no vaccination strategy with a partial vaccination strategy. As expected the no vaccination policy is better; this result is consistent with the existing literature.

3.2 A non classical situation: $\beta = 73$, $\gamma = 36.5$, $u_{max} = 10$, $r_I = 1$, $r_V = 1.5$, $X_0 = (0.7, 0.01)$

The parameters β, γ, u_{max} are the same as in the previous section therefore $r^{crit}_{V,u_{max}}$ is the same. But here $r_V > r^{crit}_{V,u_{max}} r_I$, situation treated in Theorem D.10. Although the theoretical results gives, as before, a precise description of the vaccination and no vaccination region, we advocate here an even simpler

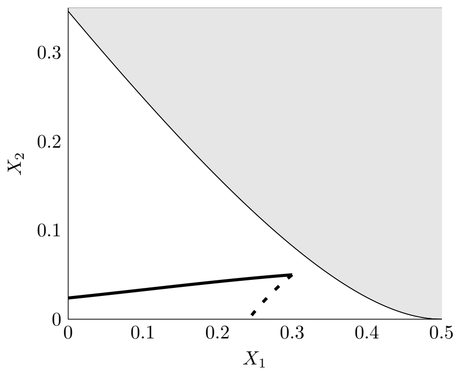


Figure 7: Two trajectories of an SIR evolution starting from $X_0 = (0.3, 0.05)$ with $\beta = 73$, $\gamma = 36.5$, $u_{max} = 10$, $r_I = 1$ and $r_V = 0.5$. The grey region is the vaccination region while the white region is the no vaccination domain. We are in the situation $r_V < r^{crit}_{V,u_{max}} r_I$ and the optimal strategy corresponds to the dashed curve with a cost equal to 0.05. The solid curve is an example of a partial vaccination strategy with cost 0.15.

approach: compute first the cost of a no vaccination strategy starting from X_0, denoted J_n. Compute then the cost of a strategy that vaccinates with maximum intensity (here u_{max}) from the initial time until the time θ where such that $\partial_{X_1}\zeta(X(\theta)) \leq r_V/r_I$ and $\langle f(X(\theta)), u_{max}), \nabla\partial_{X_1}\zeta(X(\theta))\rangle \leq 0$; denote by J_v this cost. Compare the two values: if $J_n \leq J_v$ the best strategy is to not vaccinate at all; otherwise the second strategy is the best. Numerical details are given in figure 8. In this case the initial point was in the vaccination region; previous works (see discussion in section 2.4) indicated that this point is in the no vaccination region.

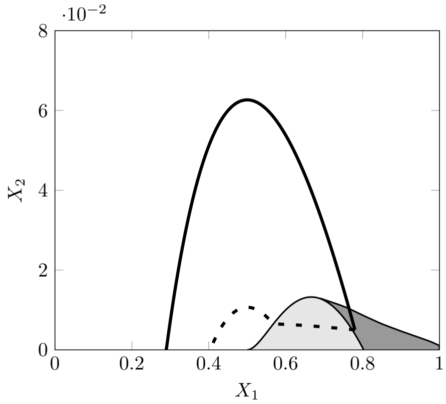
3.3 Instantaneous vaccination: $\beta = 73$, $\gamma = 36.5$, $u_{max} = \infty$, $r_I = 1$, $r_V = 1.5$, $X_0 = (0.7, 0.01)$ and $X_0 = (0.8, 0.009)$

We now illustrate the case $u_{max} = \infty$ in figures 9 and 10.

Using equation (98) we find $X^{crit}_\infty = (0.57, 0.43)$ and computing $\partial_{X_1}\zeta(X^{crit}_\infty)$ allows to obtain $r^{crit}_{V,\infty} = 1.0284$. We are therefore in the situation $r_V > r^{crit}_{V,u_{max}} r_I$, treated in Theorem E.6. The parametric equations of the frontier between vaccination and no vaccination are given in equations (106) and (107) (for Γ^{crit}_{sub}) and (108) (for Γ^{crit}_{super}). We plot both curves.

In both figures, the light gray region is the region delimited by the level line $\mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I}$ and the dark gray region is the additional vaccination region (not appearing in the literature, delimited by Γ^{crit}_{super} and $\mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I} \setminus \Gamma^{crit}_{sub}$). The union of those two regions is the vaccination region.

In figure 9 the initial point is in the vaccination region delimited by $\mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I}$. The solid path corresponds to total vaccination, and the dashed path is partial vaccination (until trajectory exits the vaccination area). The theoretical result in appendix E.4 states that the total vaccination cost will be larger than partial vaccination, which is verified numerically (see the figure).



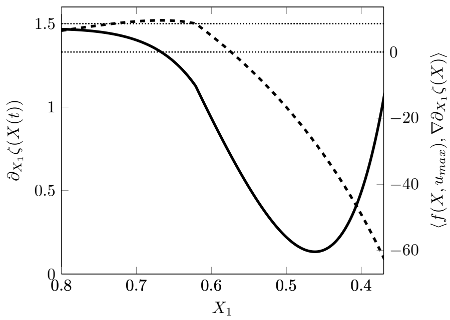


Figure 8: **Left:** Two trajectories of an SIR evolution starts on $X_0 = (0.78, 0.005)$ with $\beta = 73$, $\gamma = 36.5$, $u_{max} = 10$, $r_I = 1$ and $r_V = 1.5$. The cost for the solid trajectory is $J_n = 0.49$ and for the vaccination strategy (dashed) is $J_v = 0.48$. **Right:** In order to decide when the dashed trajectory stops vaccinating we compute $\partial_{X_1}\zeta(X(t))$ and $\langle f(X(t)), u_{max}), \nabla\partial_{X_1}\zeta(X(t))\rangle$; we plot $\partial_{X_1}\zeta(X(t))$ and the reference value r_V/r_I (left axis of the plot) and $\langle f(X(t)), u_{max}), \nabla\partial_{X_1}\zeta(X(t))\rangle$ and the reference value 0 (right axis of the plot). Vaccination stops at $X_1 = 0.62$ when $\partial_{X_1}\zeta(X(t)) \leq r_V/r_I$ and $\langle f(X(t)), u_{max}), \nabla\partial_{X_1}\zeta(X(t))\rangle \leq 0$.

The figure 10 illustrates a situation when X_0 is in the vaccination region (but outside the region delimited by $\mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I}$). This case is not correctly treated in the existing literature. We see in the figure that the optimal vaccination is a partial vaccination.

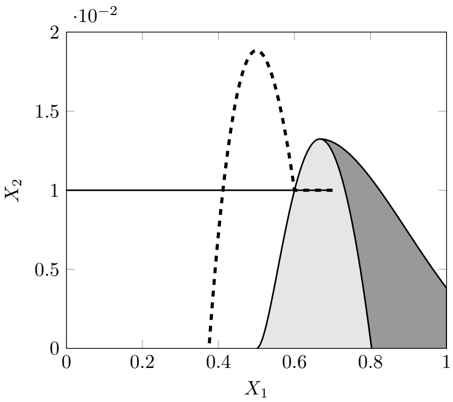


Figure 9: Two trajectories of an SIR evolution starts in $X_0 = (0.7, 0.01)$ with $\beta = 73$, $\gamma = 36.5$, $u_{max} = \infty$, $r_I = 1$ and $r_V = 1.5$. The solid curve corresponds to a trajectory with vaccination for all susceptible and the dashed trajectory with partial vaccination. The cost of the first strategy is 1.05 and the cost of the second strategy is 0.37.

Remark 3. Note that when $r_V \in]r^{crit}_{V,u_{max}} r_I, 2r_I[$ the optimal strategy may not

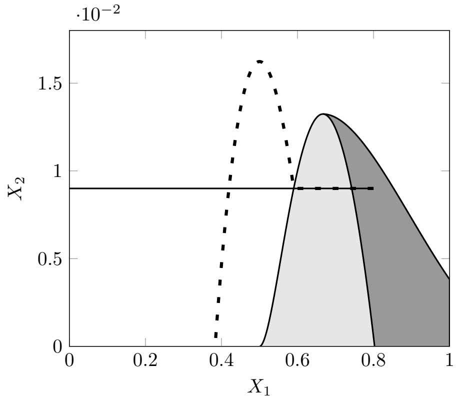


Figure 10: Two trajectories of an SIR evolution starts in $X_0 = (0.8, 0.009)$ with $\beta = 73$, $\gamma = 36.5$, $u_{max} = \infty$, $r_I = 1$ and $r_V = 1.5$. The solid curve corresponds to a trajectory with vaccination for all susceptible and the dashed trajectory with partial vaccination. The cost of the first strategy is 1.2 and the cost of the second strategy is 0.52.

be unique. This happens on the frontier Γ^{crit}_{super} when two different strategies give the same cost (because the value function is continuous): either vaccinate until reaching Γ^{crit}_{sub} and then stop vaccinating or do not vaccinate at all. See also figure 6 for an illustration. Otherwise the optimal strategy is unique.

4 Summary of optimal strategies

The previous sections show that the domain Ω is decomposed in two disjoint regions: a vaccination region and a no vaccination region. The optimal policy is to vaccinate (only) when the dynamics $X(t)$ is in the vaccination region. In principle in order to find precisely the vaccination domain one has to solve the associated HJB equation. But, in this situation, we can build a simpler algorithm to compute the optimal vaccination policy. This algorithm is described below. It uses as inputs the values β, γ, u_{max}, r_I, r_V, X_0.

We recall that the function ζ and its derivatives are easily computed as indicated in Appendix A.

1. When $u_{max} < \infty$:
   (a) If $r_V/r_I \geq 2$ the optimal vaccination policy is to not vaccinate. The overall cost is $r_I\zeta(X_0)$.
   (b) Otherwise, using equation (60), compute x^* then $X^{crit}_{u_{max}} = (x^*, 1-x^*)$ and $r^{crit}_{V,u_{max}}$ using equation (61).
      i. If $r_V \leq r^{crit}_{V,u_{max}} r_I$ then compute $\partial_{X_1}\zeta(X_0)$.
         A. If $\partial_{X_1}\zeta(X_0) \leq r_V/r_I$ the optimal vaccination policy is to not vaccinate. The overall cost is $r_I\zeta(X_0)$.
         B. Otherwise the optimal vaccination policy is to vaccinate: solve numerically equation (1) with $dV = u_{max}dt$ and monitor $\partial_{X_1}\zeta(X(t))$; at the time θ when $\partial_{X_1}\zeta(X(\theta)) = r_V/r_I$ stop vaccination.

ii. If $r_V > r^{crit}_{V,u_{max}} r_I$ then compute first $r_I \zeta(X_0)$ and denote $J_n = r_I \zeta(X_0)$ (the cost of the no vaccination policy). Also solve numerically equation (1) with $dV = u_{max} dt$ and monitor $\partial_{X_1}\zeta(X(t))$ and $\langle f(X(t), u_{max}), \nabla \partial_{X_1}\zeta(X(t))\rangle$; at the first time θ when $\partial_{X_1}\zeta(X(\theta)) \leq r_V/r_I$ and $\langle f(X(\theta), u_{max}), \nabla \partial_{X_1}\zeta(X(\theta))\rangle \leq 0$ stop vaccination. Denote J_v this cost. Compare J_n and J_v and decide which cost is the best and adopt the corresponding vaccination policy.

2. When $u_{max} = \infty$:
   (a) If $r_V/r_I \geq 2$ the optimal vaccination policy is to not vaccinate. The overall cost is $r_I \zeta(X_0)$.
   (b) Otherwise, using equation (98), compute x^* then $X^{crit}_\infty = (x^*, 1-x^*)$ and $r^{crit}_{V,u_{max}}$ (from $\partial_{X_1}\zeta(X^{crit}_\infty)$).
      i. If $r_V \leq r^{crit}_{V,u_{max}} r_I$ then compute $\partial_{X_1}\zeta(X_0)$.
         A. If $\partial_{X_1}\zeta(X_0) \leq r_V/r_I$ the optimal vaccination policy is to not vaccinate. The overall cost is $r_I \zeta(X_0)$.
         B. Otherwise the optimal vaccination policy is to vaccinate: find numerically (using (31)) the quantity Δ such that $\partial_{X_1}\zeta(X_0 - (\Delta, 0)) = r_V/r_I$. Vaccinate Δ percent of individuals and then stop vaccination. The optimum cost is $\Delta r_V + r_I \zeta(X_0 - (\Delta, 0))$.
      ii. If $r_V > r^{crit}_{V,u_{max}} r_I$ then compute $r_I \zeta(X_0)$ and denote $J_n = r_I \zeta(X_0)$ (the cost of the no vaccination policy). Also find numerically (using (31)) the quantity Δ such that $\partial_{X_1}\zeta(X_0 - (\Delta, 0)) = r_V/r_I$ and $\partial^2_{X_1 X_1}\zeta(X_0 - (\Delta, 0)) \geq 0$. Vaccinate Δ percent of individuals and then stop vaccination; denote $J_v = \Delta r_V + r_I \zeta(X_0 - (\Delta, 0))$. Compare J_n and J_v and decide which cost is the best and adopt the corresponding vaccination policy.

Remark 4. In all situations the algorithm above solves at most once the evolution equation (1).

Finally, Remark 2 shows that the cost functional in the equation (6) has the same optimal strategies and vaccination regions.

With respect to the existing literature the above optimal strategies are distinct in several aspects:

- when $u_{max} < \infty$: previous contributions take the vaccination region to be $\{Y \in \Omega \mid \partial_{X_1}\zeta(Y) \geq r_V/r_I\}$ while our definition is different for $r_V \in [r^{crit}_{V,u_{max}} r_I, 2r_I]$. The strategies here will lead to lower costs.
- when $u_{max} = \infty$: we do not ask full vaccination but only vaccinate the minimum proportion that allows to reach the frontier of the vaccination region.

5 Pertussis vaccination in adults: maximization of DALYs averted and cost-effectiveness

In this section we explain how the theoretical results apply to additional situations. We explore first a situation when the vaccine has known side-effects.

Then we present an application to cost-effectiveness analysis.

When the vaccine has known side effects or when the illness generates severe medical conditions the money alone cannot be the only decision dimension. In this situation other techniques have to be employed. Following works in the literature we use the *Quality Adjusted Life Year* (QALY) and *Disability-Adjusted Life Year* (DALY) scales that measure the disease burden; see [33, 28, 3] and related literature for an introduction and criticism.

In the QALY scale each health state is given an utility between 1 (one year of perfect health) and 0 (death). Each individual has a number of QALY equivalent to its life expectancy in perfect health. A medical condition can reduce both the life expectancy and the quality of life and in general the QALY will combine the expected length of life and quality of life. The effect of any illness is therefore to reduce the QALY of an individual. The goal of a treatment is to increase QALY.

The DALY scale, on the contrary, measures the disease burden as disability, with 0 being no disability (perfect health) and 1 (a full year of life lost). The DALY is usually computed for an entire population and takes into account the average life expectancy at age of death in years. The goal of a health policy is to reduce the DALYs at the level of the population. DALY was introduced and is the scale of choice of the World Health Organization (WHO), see [25, 24, 14].

Although both scales are similar, in general slight differences in numerical values are expected for a given health policy.

5.1 Optimal vaccination in presence of vaccine side-effects

We consider here an application to the optimal vaccination of pertussis with a vaccine that has identified side effects, see [26][Chapters 1,4,5,6] and [21]. We focus more specifically on the combined tetanus toxoid, reduced diphtheria toxoid, and acellular pertussis vaccine (Tdap).

The vaccine side effects for adults and associated induced utility (or disabilities) are taken from [21] and reproduced in Table 1 together with the same information for the disease. Note that it is assumed that there are no deaths among adults due to pertussis (see arguments in the references for further discussion).

To summarize, the (average) DALYs induced by the vaccine are $r_V = \frac{3.2605}{100'000}$ and the DALYs of the disease are $r_I = \frac{3511}{100'000}$. As a remark, the illness seems to be $\simeq 1000$ times less desirable than the vaccine.

The goal is to find a vaccination strategy that minimizes the overall DALY burden, which is equivalent to minimizing functional J in equation (3).

As an illustration we consider an outbreak of pertussis. The generally admitted propagation parameters are $\gamma = 1/21$, $R_0 = \beta/\gamma = 15.7$ (thus $\beta = 0.75$), see for instance [4][pages 1055-1056] and [18][pages 640-641].

Consider now an epidemic starting from a pool of 100 infected individuals in a susceptible population of 65 Millions individuals among which 10% are susceptible. Thus $X_{10} = 0.1$ and $X_{20} = 100/(6.5 * 10^7) = 1.54 * 10^{-6}$.

We consider first that the vaccination can be implemented very fast which, with our notations, means $u_{max} = \infty$. Using the theoretical results of previous sections it appears that it is optimal to vaccinate 4.657% percent of the population. At the end of the vaccination there is still $5.343\% \leq 1/R_0 = 6.36\%$ percent of the population susceptible.

Health State	Utility	Probability of occurence	Duration (in years)	contribution to DALY
Vaccine side effects	-	-	-	3.2605/100′000
local reaction	0.95	2%	7/365	1.9178/100′000
systemic reaction	0.93	1%	7/365	1.3425/100′000
anaphylaxis	0.6	0.0001%	2/365	0.00021912/100′000
Disease states	-	-	-	3511/100′000
mild cough illness	0.9	38%	87/365	905.753/100′000
moderate cough illness	0.85	21%	87/365	750.822/100′000
severe cough illness	0.81	40%	87/365	1811.507/100′000
pneumonia	0.82	1%	87/365	42.904/100′000

Table 1: The vaccine side effects for adults and disease health states parameters, from [21]. The utility U of a given health state can be used to compute the disability D of the state by the formula $D = 1 - U$. The DALY contribution of a given state is computed by multiplying the duration with the disability of the state weighted by the probability of occurrence. For instance for *local reaction* one obtains $2\% * (1 - 0.95) * (7/365) = \frac{1.9178}{100'000}$.

If on the contrary only 1% of the population can be vaccinated in a month, then $u_{max} = 0.12$ and it is optimal to vaccinate until the susceptible population is 5.344%. In this case 4.656% percent (Susceptible) have been vaccinated and $5.987 * 10^{-5}\%$ percent were infected before vaccination stopped.

In both cases vaccination avoids 142708 DALY.

5.2 Cost-effectiveness analysis

A different perspective in vaccination programs arises when a vaccine without notable side effects (but an economic cost, expressed in \$) is to be compared with other possible public health programs. In this case the money allocated to the vaccine campaign cannot be allocated to other projects. The optimal vaccination is found through a cost-effectiveness analysis, adapted below to our SIR model. We emphasize that what follows is a simple deterministic description and in practice additional tools, related to societal parameters and uncertainties have to be taken into account.

Suppose that the available public health budget is $B^{\$}$ and that other projects spend $\rho\$$ in order to avert one DALY. The goal is to find the optimal vaccination policy which, combined with all other health programs, maximize the total DALY averted for the given budget $B^{\$}$. As above, r_I will be the DALY lost by an infected individual and r_V^Q to be the (economic) cost of implementing one vaccine; the total DALYs averted with budget $B^{\$}$ including vaccination with policy dV are:

$$J^Q(Y, dV) = \frac{1}{\rho}\left(B^{\$} - \int_0^{\infty} r_V^Q dV(t)\right) + r_I\left(\zeta(Y) - \int_0^{\infty} \beta \Phi_1^{Y,dV}(t)\Phi_2^{Y,dV}(t)dt\right). \quad (20)$$

Algebraic manipulations indicate that the maximization of J^Q is equivalent to the minimization of the cost functional J in equation (3) if we set $r_V = r_V^Q/\rho$. Note that in this case both r_I and r_V are expressed in DALY (not \$).

As an application we consider again the pertussis vaccination for adults as an addition to the traditional multi-valent vaccines administered during the childhood.

Although no consensus for the value of ρ exists, the World Health Organization, through its CHOICE program (see [14, 30, 15]) considers that for a country with a Gross Domestic Product (GDP) per capita of g\$, a public health project is considered cost-efficient when it saves at least one DALY for each g\$ invested. Many countries in the African "low income" zone have g around 400\$ (United Nations 2013 data: Ethiopia, Madagascar,...). We set the threshold at $\rho = 400\$$ per DALY averted. For the cost of the implementation of the vaccine we follow [22][Chapter 2, page 44 and Table 20.4 page 400] and set $r_V^Q = 20\$$ (a mean value). Parameters r_I, β, γ are maintained as before.

Susceptibles are set initially to $X_{10} = 15\%$ and the proportion of Infected class $X_{20} = 0.001\%$. Note that we are here in the super-critical region $r_V/r_I = 1.42 \geq r_{V,\infty}^{crit}$.

When the vaccination can be implemented very fast ($u_{max} = \infty$) the theoretical results indicate that it is optimal to vaccinate 8.576% percent of the population. At the end of the vaccination there is still 6.424% percent of the population susceptible. For every million individuals in the population 696 DALYs are averted.

Note that the initial point is precisely in a region where previous analyses in the literature would conclude that optimal strategy is no vaccination.

At the end of the vaccination period the epidemic is still not contained because $6.424\% > 1/R_0 = 6.36\%$. Why does the vaccination stops while the epidemic still expands ? To understand this, consider first the quotient $r_V^Q/r_I = 569.63$ which is above the threshold value $\rho = 400$. This means that vaccination, seen as "treatment", is not cost-effective. But vaccination, even at high costs, can reduce the propagation of the epidemic, creating herd immunity and saving more than the vaccinated individual. As such, when the epidemic is large in size, vaccination becomes, temporarily, more cost-efficient than other public health programs. This is precisely what happens here. On the contrary, when the Susceptible approach $1/R_0$ the vaccination creates less herd immunity and its cost becomes a limitation.

The figure 11 illustrates the optimal vaccination policy in terms of classical cost-effectiveness analysis. For each vaccination level $x\%$ two criterions are plotted: the marginal cost per marginal DALY averted $x \mapsto \frac{r_V^Q}{r_I \partial_{X_1} \zeta(X_{10}-x, X_{20})}$ and the cumulative cost per DALY averted $x \mapsto \frac{r_V^Q x}{r_I(\zeta(X_{10},X_{20}) - \zeta(X_{10}-x, X_{20}))}$. In this very particular setting, both costs are initially **above** the threshold ρ. The theoretical result guarantees that, if the available budget is large enough to traverse the initial, "above the threshold" region, both curves will be below the threshold ρ at the end of the vaccination. In fact the vaccination stops when the marginal cost reaches ρ the second time.

6 Conclusion

We analyze in this work the optimal vaccination policy in a SIR model. The theoretical results allow to compute the global optimum without any smoothness hypothesis; from the technical point of view we show that the value function

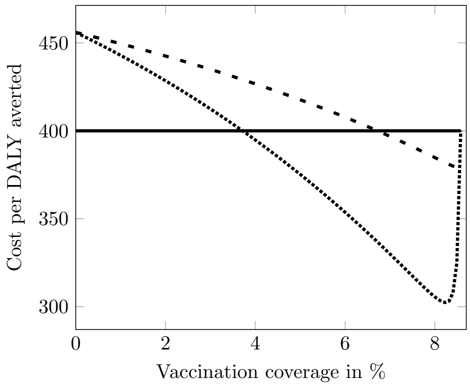


Figure 11: Illustration of the cost-effectiveness criterions for $u_{max} = \infty$. Solid line: the threshold ρ. Dashed line: the cumulative cost per DALY averted $x \mapsto \frac{r_V^Q x}{r_I(\zeta(X_{10},X_{20})-\zeta(X_{10}-x,X_{20}))}$. Dotted line: the marginal cost per marginal DALY averted $x \mapsto \frac{r_V^Q}{r_I \partial_{X_1}\zeta(X_{10}-x,X_{20})}$.

is the unique solution of a Hamilton-Jacobi-Bellman equation. As previous studies indicate, the Susceptible-Infected plane is divided in two regions: one vaccination region and one non-vaccination region. This partition is proven to be globally optimal.

Several applications are considered: first some toy examples when the costs are expressed as economic values. Then we consider pertussis vaccination in adults when the vaccine has side-effects and the optimal policy maximizes the DALYs averted. A final application, still in the framework of pertussis vaccination in adults, considers the optimal vaccination with constraints on the public health budget. The theoretical results are particularly relevant in this situation not adequately considered in the literature: the performance of a vaccination policy does not only depend on the marginal cost per DALY averted, but also on the long term herd immunity effects created. The model is able to predict when the long term effects will offset the initial expense to the point that makes vaccination cost-effective.

Appendix

A Properties of the number of infected people without vaccination

We recall some properties of the number of infected people in absence of vaccination. The reader can also consult [2, 1]. Consider the model without control:

$$\begin{cases} \frac{dX_1(t)}{dt} = -\beta X_1(t) X_2(t), \\ \frac{dX_2(t)}{dt} = \beta X_1(t) X_2(t) - \gamma X_2(t), \\ \frac{dX_3(t)}{dt} = \gamma X_2(t). \end{cases} \tag{21}$$

Lemma A.1. *The size ζ of an epidemic without vaccination starting at $\Phi^{X,0}(0) = X = (X_1, X_2)$ is the unique solution in $[0, X_1[$ of the equation:*

$$1 - \frac{\zeta}{X_1} = e^{-\frac{\beta}{\gamma}(X_2+\zeta)}. \tag{22}$$

Moreover $\zeta(X) > X_1 - \frac{\gamma}{\beta}$, $\forall X \in \Omega$ and $\zeta \in \mathcal{C}^1(\Omega)$.

Remark 5. Although ζ depends on X, when there is no ambiguity, we will just write ζ.

Proof. Denote $X_1^\infty = \lim_{t\to\infty} \Phi_1^{X,0}(t)$, $X_2^\infty = \lim_{t\to\infty} \Phi_2^{X,0}(t)$. Straightforward computations allow to prove that:

$$\Phi_1^{X,0}(t) = X_1 e^{-\frac{\beta}{\gamma}(1-\Phi_1^{X,0}(t)-\Phi_2^{X,0}(t))}. \tag{23}$$

Or $X_2^\infty = 0$ thus $X_1^\infty = X_1 e^{-\frac{\beta}{\gamma}(X_1+X_2-X_1^\infty)}$. Using that $\zeta = X_1 - X_1^\infty$ we obtain equation (22).

Let $F(y, X_1, X_2) = e^{-\frac{\beta}{\gamma}(y+X_2)} - (1 - \frac{y}{X_1})$ defined on $[0, X_1] \times \Omega$. Since $F(0, X_1, X_2) = e^{-\frac{\beta}{\gamma}X_2} - 1 \le 0$ and $F(X_1, X_1, X_2) = e^{-\frac{\beta}{\gamma}(X_1+X_2)} > 0$ the equation (in y) $F(y, X_1, X_2) = 0$ has at least a solution in $[0, X_1[$; thus equation (22) has at least a solution in $[0, X_1[$.

Moreover $\frac{\partial F}{\partial y}(y, X_1, X_2) = -\frac{\beta}{\gamma} e^{-\frac{\beta}{\gamma}(y+X_2)} + \frac{1}{X_1}$. Since $y \le X_1 \le 1$ and $X_1 + X_2 \le 1$ we obtain $\frac{\partial F}{\partial y}(y, X_1, X_2) \ge -\frac{\beta}{\gamma} e^{-\frac{\beta}{\gamma}} + 1 > 0$ (because $1 > ze^{-z}$ for any $z > 0$); therefore $F(\cdot, X_1, X_2)$ is strictly increasing in y and the solution ζ is unique.

If $X_1 \le \frac{\gamma}{\beta}$ since $\zeta \ge 0$ we obtain immediately $\zeta \ge X_1 - \frac{\gamma}{\beta}$. If on the contrary $X_1 \ge \frac{\gamma}{\beta}$ (thus in particular $\frac{\gamma}{\beta X_1} \in]0, 1]$) we obtain:

$$\begin{aligned} F(X_1 - \frac{\gamma}{\beta}, X_1, X_2) &= e^{-\frac{\beta}{\gamma}(X_1-\frac{\gamma}{\beta}+X_2)} - (1 - \frac{X_1 - \frac{\gamma}{\beta}}{X_1}) \\ &\le e^{-\frac{\beta}{\gamma}(X_1-\frac{\gamma}{\beta})} - \frac{\gamma}{\beta X_1} \le 0 \end{aligned} \tag{24}$$

where for the last inequality we used that $e^{1-1/z} - z \le 0$ for any $z = \frac{\gamma}{\beta X_1} \in]0, 1]$. Therefore the solution ζ is in $[(X_1 - \gamma/\beta)_+, X_1[$. When X belongs to the open set Ω then same arguments show that the inequality $\zeta > (X_1 - \gamma/\beta)$ is strict.

Remark 6. Thanks to (22) we obtain by the implicit function Theorem that ζ has continuous derivatives around any $X \in \Omega$; we can calculate first and second partial derivatives of ζ with respect to X_1 and X_2:

$$\frac{\partial \zeta}{\partial X_1} = \frac{\zeta}{X_1\left(1 + \frac{\beta}{\gamma}(\zeta - X_1)\right)}, \tag{25}$$

$$\frac{\partial \zeta}{\partial X_2} = \frac{\gamma/\beta}{\zeta - X_1 + \gamma/\beta} - 1, \tag{26}$$

$$\frac{\partial^2 \zeta}{\partial X_1^2} = -\frac{\gamma}{\beta} \frac{\zeta(\zeta - 2X_1 + 2\gamma/\beta)(\zeta - X_1)}{X_1^2(\zeta - X_1 + \gamma/\beta)^3}, \tag{27}$$

$$\frac{\partial^2\zeta}{\partial X_1\partial X_2} = \frac{\partial^2\zeta}{\partial X_2\partial X_1} = \frac{\gamma}{\beta}\frac{(X_1-\gamma/\beta)(\zeta-X_1)}{X_1(\zeta-X_1+\gamma/\beta)^3}. \tag{28}$$

Note that since $\zeta > X_1 - \frac{\gamma}{\beta}$ all fractions are well defined and ζ is even $C^2(\Omega)$.

B Properties of the trajectories

Lemma B.1. $\partial_{X_1} J_0 = \frac{\partial J_0}{\partial X_1}$ *is decreasing along trajectories of the system* (21).

Proof. We have to prove that:

$$\langle f(X,0), \nabla\partial_{X_1} J_0(X)\rangle < 0. \tag{29}$$

Using the expression of ζ, we have: $J_0(X) = \int_0^\infty r_I X_1(\tau)X_2(\tau)d\tau = r_I\zeta(X)$. Equation (29) can thus be rewritten as follows:

$$\langle f(X,0), \nabla\partial_{X_1}\zeta\rangle < 0. \tag{30}$$

Using equations (27) and (28), this gives after some computations: $X_1\left(\zeta - X_1 + \frac{\gamma}{\beta}\right)^2 > 0$ which is always true because X_1 is strictly positive and $\zeta \neq X_1 - \frac{\gamma}{\beta}$.

Lemma B.2. *For all* $Y \in \Gamma_1 \cup \Gamma_{A1}$ *the trajectory* $\Phi^{Y,u}(t)$ *is incoming in* Ω $\forall u \in \mathcal{U}_Y$*.*

Proof. For Γ_1, the scalar product with the incoming normal is positive:

$$\langle f(X,u), (-1,-1)\rangle = \gamma X_2 + u \geq 0 \ \forall u \in [0, u_{max}].$$

For Γ_{A1}:

$$\langle f(X,u), (0,1)\rangle = X_2(\beta X_1 - \gamma) \geq 0 \ \forall u \in [0, u_{max}].$$

Lemma B.3. J_0 *is* $\mathcal{C}^1(\Omega)$*.*

Proof. Since $J_0 = r_I\zeta$ the conclusion follows from Lemma A.1.

Lemma B.4. *For all* $X \in \Omega$*, we have* $\frac{\partial\zeta}{\partial X_1}(X) \leq 2$*. Therefore* $0 \leq \frac{\partial J_0}{\partial X_1}(X) \leq 2r_I$ $\forall X \in \Omega$*.*

Proof. Using expression in (25), to prove $\partial_{X_1}\zeta \leq 2$, we just have to show that $\zeta \geq \frac{2X_1(X_1-\frac{\gamma}{\beta})}{2X_1-\frac{\gamma}{\beta}}$. For that, we take same notation and result as in the proof of the Lemma A.1 so $X_1 \geq \frac{\gamma}{\beta}$ and we denote $\xi = \frac{2X_1(X_1-\frac{\gamma}{\beta})}{2X_1-\frac{\gamma}{\beta}}$. We have to prove that $F(\xi, X_1, X_2) \leq 0$.

With these notations, we have $F(\xi, X_1, X_2) = e^{-\frac{\gamma}{\beta}(\xi+X_2)} - \frac{\frac{\gamma}{\beta}}{2X_1-\frac{\gamma}{\beta}}$.

If we note $z = \frac{\beta}{\gamma}\xi$, we obtain, $e^{-z-\frac{\gamma}{\beta}X_2} - (z+\sqrt{z^2+1})$.

As $e^{-z-\frac{\gamma}{\beta}X_2} \leq e^{-z} \leq \frac{1}{z+\sqrt{z^2+1}}$, this proves that $F(\xi, X_1, X_2) \leq 0$.

Lemma B.5. *The level lines defined by* $\mathcal{L}_r^{\partial_{X_1}\zeta} = \{(X_1, X_2) \in \Omega \mid \frac{\partial \zeta}{\partial X_1}(X) = r\}$ *have the parametric equation:*

$$1 - \frac{1 - \frac{\beta}{\gamma}X_1}{\frac{1}{r} - \frac{\beta}{\gamma}X_1} = e^{-\frac{\beta}{\gamma}\left(X_1 \frac{1-\frac{\beta}{\gamma}X_1}{\frac{1}{r}-\frac{\beta}{\gamma}X_1} + X_2\right)}. \tag{31}$$

and have point $A = (\frac{\gamma}{\beta}, 0)$ *as limit (but* $A \notin \mathcal{L}_{r_V/r_I}^{\partial_{X_1}\zeta}$ *).*

Proof. If $X = (X_1, X_2) \in \mathcal{L}_r^{\partial_{X_1}\zeta}$ using the definition of $\mathcal{L}_r^{\partial_{X_1}\zeta}$ and (28) we have:

$$\zeta = X_1 \frac{1 - \frac{\beta}{\gamma}X_1}{\frac{1}{r} - \frac{\beta}{\gamma}X_1}. \tag{32}$$

Then, we replace in (22) to obtain the parametric equation. Note that $\nabla_X \zeta$ is not defined at A. The level line $\mathcal{L}_0^{\partial_{X_1}\zeta}$ is Γ_{OA} and the level line $\mathcal{L}_1^{\partial_{X_1}\zeta}$ is $\{X \in \Omega \mid X_1 = \frac{\gamma}{\beta}\}$. Suppose $r \notin \{0, 1\}$, then for any $X \in \mathcal{L}_r^{\partial_{X_1}\zeta}$ we have $\frac{\partial \zeta}{\partial X_1}(X) \neq 0$. The level line $\mathcal{L}_r^{\partial_{X_1}\zeta}$ is regular in the neighborhood of any $\overline{X} = (\overline{X_1}, \overline{X_2}) \in \Omega$. Indeed if $\frac{\partial \zeta}{\partial X_1}(\overline{X}) = r$ by the implicit function Theorem in the neighborhood of $\overline{X}$ there exists a curve $X_2 = X_2(X_1)$ such that $\frac{\partial \zeta}{\partial X_1}\Big(X_1, X_2(X_1)\Big) = r$. Moreover, by the same Theorem $X_2(X_1)$ is $\mathcal{C}^1$ locally. Thus the level line $\mathcal{L}_r^{\partial_{X_1}\zeta}$ is regular around any point in Ω. As such it does not have self-intersections either. In addition for any $r \in [0, 2]$ since $\lim_{X_1 \to \frac{\gamma}{\beta}^-} \frac{\partial \zeta}{\partial X_1}(X_1, 0) = 0$ and $\lim_{X_1 \to \frac{\gamma}{\beta}^+} \frac{\partial \zeta}{\partial X_1}(X_1, 0) = 2$ by continuity we obtain that $\mathcal{L}_r^{\partial_{X_1}\zeta}$ will be as close to A as wanted thus A is an extremity of $\mathcal{L}_r^{\partial_{X_1}\zeta}$ (but does not belong to it).

C An introduction to viscosity solutions

This section is largely based on classical works such as [12], [7], [8] [32]. We refer the reader to these works for additional details.

Let $\xi : \mathcal{O} \to \mathbb{R}$ be a scalar function defined on an open set $\mathcal{O} \subseteq \mathbb{R}^n$.

Definition C.1. The set of super-differentials of ξ at a point $x \in \mathcal{O}$ is:

$$D^+\xi(x) = \left\{ p \in \mathbb{R}^n ; \limsup_{y \to x} \frac{\xi(y) - \xi(x) - p \cdot (y - x)}{|y - x|} \leq 0 \right\}. \tag{33}$$

Similarly, the set of sub-differentials of ξ at a point $x \in \mathcal{O}$ is:

$$D^-\xi(x) = \left\{ p \in \mathbb{R}^n ; \liminf_{y \to x} \frac{\xi(y) - \xi(x) - p \cdot (y - x)}{|y - x|} \geq 0 \right\}. \tag{34}$$

We will also use the following:

Lemma C.1. *Let* $\xi \in \mathcal{C}(\mathcal{O})$*. Then*

1. $p \in D^+\xi(x)$ *if and only if there exists a function* $\phi \in \mathcal{C}^1(\mathcal{O})$ *such that* $\nabla\phi(x) = p$ *and* $\xi - \phi$ *has a local maximum at* x*.*

2. *$p \in D^-\xi(x)$ if and only if there exists a function $\phi \in \mathcal{C}^1(\mathcal{O})$ such that $\nabla\phi(x) = p$ and $\xi - \phi$ has a local minimum at x.*

In the following, we consider the first order partial differential equation:

$$\mathcal{F}(x, \xi(x), \nabla\xi(x)) = 0, \tag{35}$$

defined on an open set $\mathcal{O} \in \mathbb{R}^n$. Here, $\mathcal{F} : \mathcal{O} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a continuous (possibly nonlinear) function.

Definition C.2. A function $\xi \in \mathcal{C}(\mathcal{O})$ is a viscosity subsolution of (35) if

$$\mathcal{F}(x, \xi(x), p) \leq 0 \text{ for every } x \in \mathcal{O}, p \in D^+\xi(x). \tag{36}$$

Similarly, $\xi \in \mathcal{C}(\mathcal{O})$ is a viscosity supersolution of (35) if

$$\mathcal{F}(x, \xi(x), p) \geq 0 \text{ for every } x \in \mathcal{O}, p \in D^-\xi(x). \tag{37}$$

Finally, we call ξ a viscosity solution of (35) if it is both a supersolution and a subsolution in the viscosity sense.

Remark 7. For each particular problem we explicitly specify the boundary conditions.

D Bounded vaccination speed $(u_{max} < \infty)$

In this section we assume that $u_{max} < \infty$.

D.1 Properties of the value function

Theorem D.1. *The value function $\mathcal{V}^{u_{max}} : \Omega \to \mathbb{R}$ is a Lipschitz function in Ω. It can uniquely be extended to a Lipschitz function on $\overline{\Omega}$.*

Proof. We first prove that for a fixed control u and time t the function

$$\{Y \in \Omega \mid u \in \mathcal{U}_Y\} \ni Y \mapsto \Phi^{Y,u}(t),$$

is Lipschitz with the Lipschitz constant independent of u. We write: $||\frac{d}{dt}\Phi^{Y,u}(t) - \frac{d}{dt}\Phi^{Z,u}(t)|| = ||f(\Phi^{Y,u}(t), u) - f(\Phi^{Z,u}(t), u)|| \leq L_f ||\Phi^{Y,u}(t) - \Phi^{Z,u}(t)||$ where L_f is the constant in equation (12). Then:

$$\frac{d}{dt}\left(||\Phi^{Y,u}(t) - \Phi^{Z,u}(t)||^2\right) \leq 2L_f ||\Phi^{Y,u}(t) - \Phi^{Z,u}(t)||^2. \tag{38}$$

Using the Gronwall Lemma and taking the square root, we obtain:

$$||\Phi^{Y,u}(t) - \Phi^{Z,u}(t)|| \leq ||Y - Z|| e^{L_f T_{max}}.$$

Fix $Y, Z \in \Omega$ and denote by $u_Y = u_{max} \cdot \mathbb{I}_{[0,\theta(Y)]}$ one optimal control of the trajectory leaving from Y. Then if $u_Y \in \mathcal{U}_Z$ and $u_Z \in \mathcal{U}_Y$ we can obtain the following estimates:

$$\begin{aligned}
&\mathcal{V}^{u_{max}}(Z) \leq J(Z, u_Y) \\
&\leq \int_0^{T_{max}} r_I \beta \Phi_1^{Z,u_Y}(t) \Phi_2^{Z,u_Y}(t) + r_V u_Y(t) dt + J_0(\Phi^{Z,u_Y}(T_{max})) \\
&\leq \int_0^{T_{max}} r_I \beta \Phi_1^{Y,u_Y}(t) \Phi_2^{Y,u_Y}(t) + r_V u_Y(t) dt + J_0(\Phi^{Y,u_Y}(T_{max})) \\
&+C_{u_Y,T_{max}} \|Y - Z\| = \mathcal{V}^{u_{max}}(Y) + C_{u_Y,T_{max}} \|Y - Z\|.
\end{aligned}$$

Note that u_Y is member of the compact set $\{u : [0,\infty] \to \mathbb{R} \mid u = u_{max} \mathbb{1}_{[0,\theta]}, \theta \leq T_{max}\}$. Thus the constant $C_{u_Y,T_{max}}$ only depends on T_{max} (and not on Y or Z). Changing the roles of Y and Z we obtain the reverse inequality thus the conclusion.

If $u_Y \notin \mathcal{U}_Z$ or $u_Z \notin \mathcal{U}_Y$, suppose, to fix notations, that $u_Y \notin \mathcal{U}_Z$; since $u_Y = u_{max} \cdot \mathbb{1}_{[0,\theta(Y)]}$ then $u_Y \notin \mathcal{U}_Z$ implies $\theta(Y) > \theta(Z)$ thus $u_Z \in \mathcal{U}_Y$. Take $\eta \in [\theta(Z), \theta(Y)]$ to be the maximum value such that $u_{max} \mathbb{1}_{[0,\eta]} \in \mathcal{U}_Z \cap \mathcal{U}_Y$. The maximality implies $\Phi^{Z,u_{max}\mathbb{1}_{[0,\eta]}}(\eta) \in \Gamma_I$. Using Lipschitz estimates for $\Phi^{\cdot,u_{max}\mathbb{1}_{[0,\eta]}}(t)$ we obtain as above:

$$\begin{aligned}
&\mathcal{V}^{u_{max}}(Z) \leq J(Z, u_{max}\mathbb{1}_{[0,\eta]}) \leq C_{T_{max}}(\|Y - Z\|) + \mathcal{V}^{u_{max}}(\Phi^{Z,u_{max}\mathbb{1}_{[0,\eta]}}(\eta)) \\
&+\mathcal{V}^{u_{max}}(Y) - \mathcal{V}^{u_{max}}(\Phi^{Y,u_{max}\mathbb{1}_{[0,\eta]}}(\eta)) \leq \mathcal{V}^{u_{max}}(Y) + C_{T_{max}}(\|Y - Z\|) + 0 + 0
\end{aligned}$$

where we used the fact that $X \in \Gamma_I$ implies $\mathcal{V}^{u_{max}}(X) = 0$ and that $\mathcal{V}^{u_{max}}$ is positive. From now on we continue as above and obtain the Lipschitz property for Y and Z.

Since $\mathcal{V}^{u_{max}}$ is a Lipschitz function on Ω with bounded Lipschitz constant it admits a unique Lipschitz extension over $\overline{\Omega}$.

D.2 The HJB equation and value function

Theorem D.2. *The value function $\mathcal{V}^{u_{max}}$ is a viscosity solution of the Hamilton-Jacobi-Bellman (HJB) equation:*

$$(\mathcal{P}) \begin{cases}
\text{Find } F : \overline{\Omega} \to \mathbb{R} \text{ such that} & \\
F \text{ is Lipschitz on } \overline{\Omega}, & (39a) \\
-\mathcal{H}^{u_{max}}(X, \nabla F(X)) = 0, \ X \in \Omega, & (39b) \\
F(X) = 0 \text{ on } \Gamma_{OA}, & (39c) \\
F(X) = 0 \text{ on } \Gamma_I, & (39d) \\
-\mathcal{H}^{u_{max}}(X, \nabla F(X)) = 0 \text{ on } \Gamma_1. & (39e)
\end{cases}$$

Remark 8. There is no boundary condition on Γ_{A1}.

Remark 9. See appendix C for an introduction to viscosity solutions.

Proof. Using Lemma C.1 and Definition C.2 we first show that $\mathcal{V}^{u_{max}}$ is a subsolution of (39) then we will show it is also a supersolution.

Step 1. Let $Y \in \Omega$ and $\varphi \in \mathcal{C}^1(\Omega)$ such that $\mathcal{V}^{u_{max}}(Y) - \varphi(Y)$ attains a local maximum at Y. So for Z in a neighborhood of Y:

$$\mathcal{V}^{u_{max}}(Y) - \mathcal{V}^{u_{max}}(Z) \geq \varphi(Y) - \varphi(Z). \tag{40}$$

We will prove that:

$$-\mathcal{H}^{u_{max}}(Y, \nabla\varphi(Y)) \leq 0. \tag{41}$$

This is equivalent to:

$$-u_{max}(\partial_{X_1}\varphi(Y) - r_V)_+ + \beta Y_1 Y_2(r_I + \partial_{X_2}\varphi(Y) - \partial_{X_1}\varphi(Y)) - \gamma Y_2 \partial_{X_2}\varphi(Y) \geq 0. \tag{42}$$

Assume that it is not the case. Then there exists, by continuity, a value $w \in [0, u_{max}]$ (see Remark 10 page 27 below) and a constant $\kappa > 0$ such that in a neighborhood of Y:

$$w(r_V - \partial_{X_1}\varphi(\Phi^{Y,u}(\cdot))) + \beta\Phi_1^{Y,u}(\cdot)\Phi_2^{Y,u}(\cdot)(r_I + \partial_{X_2}\varphi(\Phi^{Y,u}(\cdot)) - \partial_{X_1}\varphi(\Phi^{Y,u}(\cdot))) - \gamma\Phi_2^{Y,u}(\cdot)\partial_{X_2}\varphi(\Phi^{Y,u}(\cdot)) \leq -\kappa, \tag{43}$$

for any $u(t) \in [0, u_{max}]$. Let $u = w$ on the interval $[0, \delta]$ (since $Y \in \Omega$, for a small $\delta > 0$ $u \in \mathcal{U}_Y$) and denote $Z^0 = \Phi^{Y,w}(\delta)$. Then, choosing $Z = Z^0$ in (40) we obtain:

$$\mathcal{V}^{u_{max}}(Z^0) - \mathcal{V}^{u_{max}}(Y) \leq \varphi(Z^0) - \varphi(Y) = \int_0^\delta \frac{d}{dt}\varphi(\Phi^{Y,w}(t))dt \tag{44}$$

$$\leq \int_0^\delta \langle\nabla\varphi(\Phi^{Y,w}(t)), f(\Phi^{Y,w}(t), w)\rangle dt \leq \int_0^\delta -\kappa - \beta\Phi_1^{Y,w}(t)\Phi_2^{Y,w}(t)r_I - wr_V dt$$

$$\leq -\delta\kappa - \int_0^\delta \beta\Phi_1^{Y,w}(t)\Phi_2^{Y,w}(t)r_I + wr_V dt. \tag{45}$$

Or, by the definition of the optimality of $\mathcal{V}^{u_{max}}$ in Y:

$$\mathcal{V}^{u_{max}}(Y) \leq \int_0^\delta r_I\beta\Phi_1^{Y,w}(t)\Phi_2^{Y,w}(t) + r_V w dt + \mathcal{V}^{u_{max}}(Z^0)$$

$$\mathcal{V}^{u_{max}}(Y) - \mathcal{V}^{u_{max}}(Z^0) \leq \int_0^\delta r_I\beta\Phi_1^{Y,w}(t)\Phi_2^{Y,w}(t) + r_V w dt,$$

by summing the inequality we get $0 < -\kappa\delta$, which is absurd.

Therefore using Lemma C.1 we obtain:

$$-\mathcal{H}^{u_{max}}(X, \nabla\mathcal{V}^{u_{max}}(X)) \leq 0 \text{ for all } X \in \Omega.$$

To prove (39e) we use appendix B where we prove that trajectories $\Phi^{Y,u}(\cdot)$ with $Y \in \Gamma_1$ are strictly entering the domain Ω for all $w \in [0, u_{max}]$. For this reason when $Y \in \Gamma_1$ $\Phi^{Y,w}(t) \notin \Gamma_1$ for $t \in]0, \delta]$. Moreover, we choose φ such that φ is $\mathcal{C}(\overline{\Omega})$ and $\mathcal{C}^1(\Omega)$. These arguments allow to prove equation (45) from equation (44). Moreover the same proof can be used for all $X \in \Gamma_1$ and we obtain:

$$-\mathcal{H}^{u_{max}}(X, \nabla\mathcal{V}^{u_{max}}(X)) \leq 0 \text{ for all } X \in \Gamma_1.$$

By Lemma 2.1, we have that $\mathcal{V}^{u_{max}}$ is bounded on $\overline{\Omega}$ and by Theorem D.1 $\mathcal{V}^{u_{max}}$ is a Lipschitz function. By definition of $\mathcal{V}^{u_{max}}$ we have $\mathcal{V}^{u_{max}}(X) = 0$ on Γ_I and Γ_{OA}. So $\mathcal{V}^{u_{max}}$ is a subsolution of (39).

Step 2. Now we prove that $\mathcal{V}^{u_{max}}$ is a supersolution of (39).

Let $Y \in \Omega$ and $\varphi \in \mathcal{C}^1(\Omega)$ such that $\mathcal{V}^{u_{max}}(Y) - \varphi(Y)$ attains a local minimum at Y. So for Z in a neighborhood of Y: $\mathcal{V}^{u_{max}}(Y) - \varphi(Y) \leq \mathcal{V}^{u_{max}}(Z) - \varphi(Z)$. We will show that: $-\mathcal{H}^{u_{max}}(Y, \nabla\varphi(Y)) \geq 0 \; \forall \; Y \in \Omega$.

Assume that it is not the case. Then there exists $\kappa > 0$ such that $\mathcal{H}^{u_{max}}(Y, \nabla\varphi(Y)) > \kappa$ in a neighborhood of Y. So there exists (a small) $\delta > 0$ such that for any $u(t) \in [0, u_{max}]$:

$$-u_{max}(\partial_{X_1}\varphi(\Phi^{Y,u}(\cdot)) - r_V)_+ + \beta\Phi_1^{Y,u}(\cdot)\Phi_2^{Y,u}(\cdot)(r_I + \partial_{X_2}\varphi(\Phi^{Y,u}(\cdot)) - \partial_{X_1}\varphi(\Phi^{Y,u}(\cdot))) - \gamma\Phi_2^{Y,u}(\cdot)\partial_{X_2}\varphi(\Phi^{Y,u}(\cdot)) > \kappa \; \forall t \leq \delta. \quad (46)$$

Let w be a control in $\mathcal{U}_Y$ and $Z^0 = \Phi^{Y,w}(\delta)$ (for small δ any $w \in [0, u_{max}]$ is in $\mathcal{U}_Y$). Then:

$$\begin{aligned}
&\mathcal{V}^{u_{max}}(Z^0) - \mathcal{V}^{u_{max}}(Y) \geq \varphi(Z^0) - \varphi(Y) = \int_0^\delta \langle \nabla\varphi(\Phi^{Y,w}(t)), f(\Phi^{Y,w}(t), w)\rangle dt \\
&\geq \delta\kappa + \int_0^\delta u_{max}(\partial_{X_1}\varphi(\Phi^{Y,w}(t)) - r_V)_+ - r_I\beta\Phi_1^{Y,w}(t)\Phi_2^{Y,w}(t) - w\partial_{X_1}\varphi(\Phi^{Y,w}(t))dt \\
&\geq \delta\kappa - \int_0^\delta wr_V + r_I\beta\Phi_1^{Y,w}(t)\Phi_2^{Y,w}(t)dt \\
&\quad + \int_0^\delta u_{max}(\partial_{X_1}\varphi(\Phi^{Y,w}(t)) - r_V)_+ - w(\partial_{X_1}\varphi(\Phi^{Y,w}(t)) - r_V)dt \\
&\geq \delta\kappa - \int_0^\delta wr_V + r_I\beta\Phi_1^{Y,w}(t)\Phi_2^{Y,w}(t)dt.
\end{aligned}$$

(because $\int_0^\delta u_{max}(\partial_{X_1}\varphi(\Phi^{Y,w}(t)) - r_V)_+ - w(\partial_{X_1}\varphi(\Phi^{Y,w}(t)) - r_V)dt \geq 0$ since $w \in [0, u_{max}]$).

So, for any w, we have:

$$\mathcal{V}^{u_{max}}(Z^0) + \int_0^\delta wr_V + r_I\beta\Phi_1^{Y,w}(t)\Phi_2^{Y,w}(t)dt \geq \mathcal{V}^{u_{max}}(Y) + \delta\kappa. \quad (47)$$

Taking the infimum with respect to w we obtain $\mathcal{V}^{u_{max}}(Y) \geq \mathcal{V}^{u_{max}}(Y) + \kappa\delta$. This is absurd, therefore $\mathcal{V}^{u_{max}}$ is a supersolution on Ω.

For the same reasons as previously, we have $-\mathcal{H}^{u_{max}}(X, \nabla\mathcal{V}^{u_{max}}(X)) \geq 0$ on Γ_1 and $\mathcal{V}^{u_{max}}$ is a supersolution of equation (39).

Step 3. To summarize this proof, we showed that:

- by Theorem D.1, $\mathcal{V}^{u_{max}}$ is a Lipschitz function,
- $\mathcal{V}^{u_{max}}$ is both a subsolution and a supersolution of (39b) and (39e),
- $\mathcal{V}^{u_{max}}(X) = 0$ on $\Gamma_{OA} \cup \Gamma_I$ by definition of $\mathcal{V}^{u_{max}}$.

So $\mathcal{V}^{u_{max}}$ is a viscosity solution of the Hamilton-Jacobi-Bellman equation (39).

D.3 Uniqueness of the solution of the HJB problem

Theorem D.3. *Let $\mathcal{F}_1$ be a subsolution of* (39) *and $\mathcal{F}_2$ a supersolution. Then:*

$$\mathcal{F}_1(X) \leq \mathcal{F}_2(X) \text{ for all } X \in \Omega.$$

Remark 10. In the following, we will use that, for any $A_1, B_1, A_2, B_2 \in \mathbb{R}$ with $\min(A_1, B_1) \leq \min(A_2, B_2)$ there exists $\rho \geq 0$ such as: $A_1 + \rho B_1 \leq A_2 + \rho B_2$.

Proof. Let $B_\alpha \in \Omega$ denote the point with coordinates $(1-\alpha, \alpha)$ and:

$$\Gamma_{AB_\alpha} = \left\{ (X_1, X_2) \in \overline{\Omega} \mid X_2 > 0,\ \frac{\beta}{\gamma} X_1 + \frac{\gamma - \beta + \alpha\beta}{\alpha\gamma} X_2 = 1 \right\},$$

$$\Gamma_{B_\alpha 1} = \left\{ (X_1, X_2) \in \overline{\Omega} \mid X_1 + X_2 = 1\ ,\ X_2 > \alpha \right\}.$$

Let $D_\alpha \subset \overline{\Omega}$ be the domain strictly bounded by Γ_I, Γ_{OA}, Γ_{AB_α} and $\Gamma_{B_\alpha 1}$, see figure 12 for a graphical representation. When $\gamma/\beta \geq 1$ the point A will lie outside Ω, we take $D_\alpha = \Omega$, $\Gamma_{AB_\alpha} = \emptyset$ and $\Gamma_{B_\alpha 1} = \Gamma_1$.

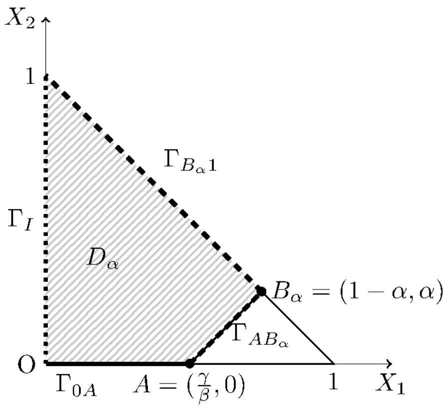


Figure 12: Boundary used in proof of the Theorem D.3.

We prove in appendix B that for any $X_0 \in \partial D_\alpha$ the trajectory $\Phi^{X_0,w}(t)$ with $w(t) \in [0, u_{max}]$ $\forall t$ enters D_α.

For $X \in \Gamma_{B_\alpha 1}$, $X \neq (1,0)$, the scalar product with the incoming normal is positive:

$$\langle f(X,u), (-1,-1) \rangle = \gamma X_2 + u > 0\ \forall X \in \Gamma_{B_\alpha 1}, X \neq (1,0), u \in [0, u_{max}].$$

For $X \in \Gamma_{AB_\alpha}$, $X \neq (1,0)$, the relevant quantity is:

$$\langle f(X,u), (-\frac{\beta}{\gamma}, -\frac{\gamma - \beta + \alpha\beta}{\gamma\alpha}) \rangle = \frac{\beta}{\gamma}(\beta X_1 X_2 + u) + \gamma^{-1}(\beta X_1 - \gamma)^2 > 0.$$

We now show the Theorem for $\mathcal{F}_1$ and $\mathcal{F}_2$ restricted to D_α. To this end we make the change of variable introduce by Kružkov (see [7]), for $X \in D_\alpha$, $\mathcal{W}(X) = 1 - e^{-\mathcal{F}(X)}$. Formally:

$$\nabla \mathcal{W}(X) = \nabla \mathcal{F}(X) e^{-\mathcal{F}(X)} = \nabla \mathcal{F}(X)(1 - \mathcal{W}(X)) \tag{48}$$

thus $\nabla \mathcal{F}(X) = \frac{\nabla \mathcal{W}(X)}{(1-\mathcal{W}(X))}$. This motivates the introduction of the following Hamiltonian:

$$
\begin{aligned}
&-u_{max}(\frac{p_1}{1-\mathcal{W}(X)} - r_V)_+ + \beta X_1 X_2 (r_I + \frac{p_2}{1-\mathcal{W}(X)} \\
&-\frac{p_1}{1-\mathcal{W}(X)}) - \gamma X_2 \frac{p_2}{1-\mathcal{W}(X)}.
\end{aligned} \tag{49}
$$

Since $1-\mathcal{W}(X)$ will always be positive, for convenience, we conclude the demonstration using the Hamiltonian: $\tilde{\mathcal{H}}^{u_{max}} : \overline{D_\alpha} \times [0,1] \times \mathbb{R}^2 \to \mathbb{R}$:

$$
\begin{aligned}
&\tilde{\mathcal{H}}^{u_{max}}(X, \mathcal{W}(X), p) \\
&= \min_{w\in[0,u_{max}]} \Big[p \cdot f(X,w) + (1-\mathcal{W}(X))(r_I \beta X_1 X_2 + r_V w) \Big] \\
&= -u_{max}(p_1 - r_V(1-\mathcal{W}(X)))_+ + \beta X_1 X_2 (r_I(1-\mathcal{W}(X)) + p_2 - p_1) - \gamma X_2 p_2.
\end{aligned}
$$

So we have to prove the following:

Lemma D.4. *Let the Hamilton-Jacobi-Bellman equation:*

$$
(\mathcal{PW}) \begin{cases}
\text{Find } F : \overline{D_\alpha} \to \mathbb{R} \text{ such that} & \\
F \text{ is Lipschitz on } \overline{D_\alpha}, & \text{(50a)} \\
-\tilde{\mathcal{H}}^{u_{max}}(X, F, \nabla F(X)) = 0, \ \forall X \in D_\alpha, & \text{(50b)} \\
F(X) = 0 \text{ on } \Gamma_{OA}, & \text{(50c)} \\
F(X) = 0 \text{ on } \Gamma_I, & \text{(50d)} \\
-\tilde{\mathcal{H}}^{u_{max}}(X, F(X), \nabla F(X)) = 0 \text{ on } \Gamma_{AB_\alpha} \cup \Gamma_{B_\alpha 1}. & \text{(50e)}
\end{cases}
$$

If $\mathcal{W}_1$ is a subsolution of (50) *and $\mathcal{W}_2$ a supersolution, then $\mathcal{W}_1(X) \leq \mathcal{W}_2(X)$ for all $X \in D_\alpha$.*

Proof. Suppose now that the Lemma is not true, then there exists $\sigma > 0$ such that:

$$
\sup_{x\in D_\alpha} [\mathcal{W}_1(x) - \mathcal{W}_2(x)] = \sigma > 0. \tag{51}
$$

Consider $\Psi_\epsilon(x,y) : \overline{D_\alpha} \mapsto \mathbb{R}$ defined by $\Psi_\epsilon(x,y) = \mathcal{W}_1(x) - \mathcal{W}_2(y) - \frac{|x-y|^2}{\epsilon}$.

For any ϵ this function has a global maximum in (x^ϵ, y^ϵ) and we have for ϵ small enough: $\Psi_\epsilon(x^\epsilon, y^\epsilon) \geq \sigma/2 > 0$. Since $\mathcal{W}_1$, $\mathcal{W}_2$ are bounded we obtain also $\lim_{\epsilon\to 0} |x^\epsilon - y^\epsilon| = 0$.

In addition, consider the functions:

$\varphi_1(x) = \mathcal{W}_2(y^\epsilon) + \frac{|x-y^\epsilon|^2}{\epsilon}$ defined on $\Omega_{\varphi_1} = \{x \in \mathbb{R}^{*+} \mid x + y^\epsilon < 1\}$,

$\varphi_2(y) = \mathcal{W}_1(x^\epsilon) - \frac{|x^\epsilon-y|^2}{\epsilon}$ defined on $\Omega_{\varphi_2} = \{y \in \mathbb{R}^{*+} \mid x^\epsilon + y < 1\}$.

These two functions are $\mathcal{C}^1$ on Ω_{φ_1} and Ω_{φ_2} respectively.

Then $\mathcal{W}_1(x) - \varphi_1(x)$ reaches its maximum in x^ϵ, φ_1 is $\mathcal{C}^1(\Omega_{\varphi_1})$ and $\mathcal{W}_1$ is a subsolution of (50). Using the Lemma (C.1), we have:

$$
-\tilde{\mathcal{H}}^{u_{max}}\left(x^\epsilon, \mathcal{W}_1(x^\epsilon), \frac{2(x^\epsilon - y^\epsilon)}{\epsilon}\right) \leq 0. \tag{52}
$$

Similarly, using that the application $y \mapsto \mathcal{W}_2(y) - \varphi_2(y)$ has its maximum in y^ϵ, φ_2 is $\mathcal{C}^1(\Omega_{\varphi_2})$ and $\mathcal{W}_2$ is a supersolution of (50), we have $-\tilde{\mathcal{H}}^{u_{max}}\left(y^\epsilon, \mathcal{W}_2(y^\epsilon), \frac{2(x^\epsilon-y^\epsilon)}{\epsilon}\right) \geq 0.$

Combining these two equations, we obtain:

$$-\tilde{\mathcal{H}}^{u_{max}}\left(x^\epsilon, \mathcal{W}_1(x^\epsilon), \frac{2(x^\epsilon - y^\epsilon)}{\epsilon}\right) \leq -\tilde{\mathcal{H}}^{u_{max}}\left(y^\epsilon, \mathcal{W}_2(y^\epsilon), \frac{2(x^\epsilon - y^\epsilon)}{\epsilon}\right). \quad (53)$$

We use then Remark 10, with $\tilde{\mathcal{H}}^{u_{max}}$ written as:

$$\tilde{\mathcal{H}}^{u_{max}}(X, \mathcal{W}, p) = \min\left(u_{max}(r_V(1-\mathcal{W}) - p_1) + \beta X_1 X_2(r_I(1-\mathcal{W}) + p_2 - p_1)\right.$$
$$\left. -\gamma X_2 p_2, \beta X_1 X_2(r_I(1-\mathcal{W}) + p_2 - p_1) - \gamma X_2 p_2\right). \quad (54)$$

So we obtain after few simplifications and factorisation (ρ is the constant given by Remark 10):

$$- u_{max} r_V(\mathcal{W}_2(y^\epsilon) - \mathcal{W}_1(x^\epsilon))$$
$$+ (1+\rho)\left[\beta(-x_1^\epsilon x_2^\epsilon + y_1^\epsilon y_2^\epsilon)(r_I - p_1^\epsilon + p_2^\epsilon) - \gamma p_2^\epsilon(y_2^\epsilon - x_2^\epsilon)\right]$$
$$- (1+\rho)\beta r_I\left[y_1^\epsilon y_2^\epsilon \mathcal{W}_2(y^\epsilon) - x_1^\epsilon x_2^\epsilon \mathcal{W}_1(x^\epsilon)\right] \leq 0. \quad (55)$$

Moreover, $\mathcal{W}_1(x^\epsilon) - \mathcal{W}_2(y^\epsilon) \leq \Psi_\epsilon(x^\epsilon, y^\epsilon) \leq \mathcal{W}_1(x^\epsilon) - \mathcal{W}_2(y^\epsilon) + |\mathcal{W}_2(x^\epsilon) - \mathcal{W}_2(y^\epsilon)| - \frac{|x^\epsilon - y^\epsilon|^2}{2\epsilon}$.

Hence, $0 \leq |\mathcal{W}_2(x^\epsilon) - \mathcal{W}_2(y^\epsilon)| - \frac{|x^\epsilon - y^\epsilon|^2}{2\epsilon}$.

Since $\mathcal{W}_2$ is uniformly continuous (as a continuous function on a compact) and $\lim_{\epsilon\to 0} |x^\epsilon - y^\epsilon| = 0$, we have:

$$\lim_{\epsilon \to 0} \frac{|x^\epsilon - y^\epsilon|^2}{2\epsilon} = 0. \quad (56)$$

So,

$$(-x_1^\epsilon x_2^\epsilon + y_1^\epsilon y_2^\epsilon)(-p_1^\epsilon + p_2^\epsilon) = (-x_1^\epsilon x_2^\epsilon + x_1^\epsilon y_2^\epsilon - x_1^\epsilon y_2^\epsilon + y_1^\epsilon y_2^\epsilon)\frac{2}{\epsilon}(-x_1^\epsilon + y_1^\epsilon + x_2^\epsilon - y_2^\epsilon)$$
$$= \left(-x_1^\epsilon(x_2^\epsilon - y_2^\epsilon) - y_2^\epsilon(x_1^\epsilon - y_2^\epsilon)\right)\frac{2}{\epsilon}\left(-(x_1^\epsilon - y_1^\epsilon) + x_2^\epsilon - y_2^\epsilon\right)$$
$$\leq \frac{2}{\epsilon}(|x_1^\epsilon| + |y_2^\epsilon|)|x_2^\epsilon - y_2^\epsilon||x_1^\epsilon - y_1^\epsilon| + \frac{2}{\epsilon}|x_1^\epsilon||x_2^\epsilon - y_2^\epsilon|^2 + \frac{2}{\epsilon}|y_2^\epsilon||x_1^\epsilon - y_1^\epsilon|^2$$
$$\leq \frac{4}{\epsilon}|x_2^\epsilon - y_2^\epsilon||x_1^\epsilon - y_1^\epsilon| + \frac{2}{\epsilon}|x_2^\epsilon - y_2^\epsilon|^2 + \frac{2}{\epsilon}|x_1^\epsilon - y_1^\epsilon|^2.$$

Hence, $\lim_{\epsilon\to 0}\ \beta|(-x_1^\epsilon x_2^\epsilon + y_1^\epsilon y_2^\epsilon)(-p_1^\epsilon + p_2^\epsilon)| = 0$.

Similarly, using (56), we have $\gamma p_2^\epsilon(x_2^\epsilon - y_2^\epsilon) = 0$.

After eventually extracting a subsequence $(\epsilon_n)_{n\geq 0}$ we can suppose that $\lim_{\epsilon_n\to 0} x^{\epsilon_n} = \lim_{\epsilon_n \to 0} y^{\epsilon_n} = \overline{x}$. Note that $\overline{x}_1 = 0$ or $\overline{x}_2 = 0$ would imply $\overline{x} \in \Gamma_{OA} \cup \Gamma_I$ thus $\mathcal{W}_1(\overline{x}) = \mathcal{W}_2(\overline{x}) = 0$ in contradiction with $\Psi(x^\epsilon, y^\epsilon) \geq \frac{\sigma}{2}$ and (56). Therefore $\overline{x}_1 \neq 0$ and $\overline{x}_2 \neq 0$.

We can therefore rewrite (55) as follows:

$$-\left[(1+\rho)r_I\beta\overline{x}_1\overline{x}_2 + u_{max}r_V\right]\left[\mathcal{W}_2(\overline{x}) - \mathcal{W}_1(\overline{x})\right] \leq 0. \quad (57)$$

Since $r_I, r_V, \beta > 0$, $\rho \geq 0$ and $\overline{x}_1 \neq 0$, $\overline{x}_2 \neq 0$ this implies that:

$$\mathcal{W}_2(\overline{x}) \geq \mathcal{W}_1(\overline{x}). \quad (58)$$

On the other hand, for ϵ relatively small, we have $\mathcal{W}_1(x^\epsilon) \geq \mathcal{W}_2(y^\epsilon) + \frac{\sigma}{2}$. Passing to the limit, we get $\mathcal{W}_1(\overline{x}) > \mathcal{W}_2(\overline{x})$. This is in contradiction with (58) and ends the proof of the Lemma.

As $\mathcal{W}_1 \leq \mathcal{W}_2$ on D_α, we have also $\mathcal{F}_1 \leq \mathcal{F}_2$ on D_α. When $\alpha \to 0$, we obtain $\mathcal{F}_1 \leq \mathcal{F}_2$ on Ω.

This proof is also available for $X \in \Gamma_1$. For Γ_{OA} and Γ_I, we just use the value of the function.

Theorem D.5. *The value function $\mathcal{V}^{u_{max}}$ is the unique solution of the HJB problem* (39).

Proof. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two viscosity solutions of (39). Since $\mathcal{F}_1$ is a subsolution and $\mathcal{F}_2$ is a supersolution, we have, by Theorem D.3 that $\mathcal{F}_1 \leq \mathcal{F}_2$ on Ω. Interchanging the roles of $\mathcal{F}_1$ and $\mathcal{F}_2$, we can conclude $\mathcal{F}_2 \leq \mathcal{F}_1$. So $\mathcal{F}_1 = \mathcal{F}_2$ on Ω and therefore on $\overline{\Omega}$ (by continuity).

Thus the solution is unique. By Theorem D.8 the value function $\mathcal{V}^{u_{max}}$ is the unique solution.

D.4 Solution candidate and its properties: the sub-critical case

Theorem D.5 implies that in order to find the value function it is enough to find a solution of the HJB equation (39).

We expect the solution to lead to a partition of the domain into a vaccination region and a non-vaccination region. An important question concerns the regularity of the value function which at its turn is related to the uniqueness of the optimal strategy. The frontier of the vaccination region will be seen to be related to the level line $\mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I}$ of $\partial_{X_1}\zeta$; see in appendix B the definition of $\mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I}$.Thus we are about to ask a question similar to that in figure 6: does $\mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I}$ contain points that are entering the domain for control u_{max} and exiting it for control 0. The level lines $\mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I}$ that contain such points will lead to non unique optimal strategies (and non smooth value functions).

When $\gamma/\beta < 1$, for any $u_{max} < \infty$ we introduce the critical point $X^{crit}_{u_{max}}$ which is the unique solution of the equations:

$$\begin{cases} X \in \Gamma_1 \\ \langle f(X, u_{max}), \nabla \partial_{X_1}\zeta(X)\rangle = 0. \end{cases} \tag{59}$$

The proof of existence and uniqueness of $X^{crit}_{u_{max}}$ is left as an exercise for the reader. One can use the description of the curve $\langle f(X, u_{max}), \nabla \partial_{X_1}\zeta(X)\rangle = 0$ (see also the Appendix A for formulaes involving ζ and its derivatives) to show that $X^{crit}_{u_{max}} = (x^*, 1 - x^*)$ where x^* is the solution of:

$$\frac{\gamma}{\beta} - \left(x^* - \frac{\gamma}{\beta}\right)\sqrt{\frac{u_{max}}{\beta x^*(1-x^*) + u_{max}}} = x^* e^{-\frac{\beta}{\gamma}\left[(1-x^*) + \left(x^* - \frac{\gamma}{\beta}\right)\left(1 + \sqrt{\frac{u_{max}}{\beta x^*(1-x^*)+u_{max}}}\right)\right]}. \tag{60}$$

Then the value $r^{crit}_{V,u_{max}}$ is defined as

$$r^{crit}_{V,u_{max}} = \partial_{X_1}\zeta(X^{crit}_{u_{max}}). \tag{61}$$

For $\gamma/\beta \geq 1$ we set $r^{crit}_{V,u_{max}} = \infty$. Note that in all situations $r^{crit}_{V,u_{max}} > 1$.

When $r_V < r^{crit}_{V,u_{max}} r_I$ we define a partition of Ω in two regions

$$\Omega^{NV}_{u_{max}} = \{X \in \Omega \mid \partial_{X_1}\zeta(X) < r_V/r_I\} \tag{62}$$
$$\Omega^{V}_{u_{max}} = \{X \in \Omega \mid \partial_{X_1}\zeta(X) > r_V/r_I\}. \tag{63}$$

The level line $\mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I}$ is situated on the common frontier $\partial\Omega^{NV}_{u_{max}} \cap \partial\Omega^{V}_{u_{max}}$. For $\gamma/\beta \geq 1$ it may happen that r_V/r_I is such that $\mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I} \cap \Omega = \emptyset$; then we take $\Omega^{V}_{u_{max}} = \emptyset$. This can happen for relatively small values of r_V/r_I as illustrated in figure 13.

Lemma D.6. *Any trajectory $\Phi^{X_0,w}(t)$ with $X_0 \in \partial\Omega^{NV}_{u_{max}} \cap \partial\Omega^{V}_{u_{max}}$ is such that $\Phi^{X_0,w}(t) \in \Omega^{NV}_{u_{max}}$ for all $t > 0$ and $w \in \mathcal{U}_{X_0}$.*

Proof. In order to prove that the trajectory $\Phi^{X_0,w}(t)$ enters the domain $\Omega^{NV}_{u_{max}}$ it is enough to prove that the tangent to the trajectory has strictly positive scalar product with the incoming normal at X_0 to $\Omega^{NV}_{u_{max}}$ i.e.,

$$\langle f(X_0, w(0)), -\nabla\partial_{X_1}\zeta(X_0)\rangle > 0,\ \forall X_0 \in \partial\Omega^{NV}_{u_{max}} \cap \partial\Omega^{V}_{u_{max}}.$$

This follows (after some straightforward computations) from the definition of $r^{crit}_{V,u_{max}}$ and the monotonicity of the derivatives of $\zeta(\cdot)$ as $r^{crit}_{V,u_{max}}$ is the smallest value r where the trajectory $u = u_{max}$ is tangent to the level line $\mathcal{L}^{\partial_{X_1}\zeta}_{r}$ (see in appendix B the definition of $\mathcal{L}^{\partial_{X_1}\zeta}_{r}$).

Introduce also the control $u_{X_0}(t)$ taken to be u_{max} as long as the trajectory $\Phi^{X_0,u_{X_0}(\cdot)}(t)$ obtained with this control $u_{X_0}(t)$ remains in $\Omega^{V}_{u_{max}}$ (and zero otherwise). It is a feedback control. Formally it is the solution of the equation:

$$u_{X_0}(t) = u_{max} \cdot \mathbb{1}_{\Phi^{X_0,u_{X_0}(\cdot)}(t) \in \Omega^{V}_{u_{max}}}. \tag{64}$$

The fact that such a solution exists is a consequence of the regularity of the boundary of $\Omega^{V}_{u_{max}}$ and Lemma D.6. Note that $u_{X_0}(t)$ is of the form $u_{max} \cdot \mathbb{1}_{[0,\eta]}$ with $\eta \geq 0$. Define the function $\Pi^{r_V,r_I}_{u_{max}} : \Omega \to \mathbb{R}$ by

$$\Pi^{r_V,r_I}_{u_{max}}(X_0) = J(X_0, u_{X_0}(\cdot)). \tag{65}$$

Theorem D.7. *For $r_V < r^{crit}_{V,u_{max}} r_I$:*

1. $\Pi^{r_V,r_I}_{u_{max}}\Big|_{\Omega^{NV}_{u_{max}}} = J_0 = r_I\zeta$*;*

2. $\Pi^{r_V,r_I}_{u_{max}}\Big|_{\Omega^{V}_{u_{max}}}$ *is the unique viscosity solution of the following problem:*

$$(\mathcal{P}v)\begin{cases} \text{Find } F : \overline{\Omega^{V}_{u_{max}}} \to \mathbb{R} \text{ such that} \\ F \text{ is Lipschitz on } \overline{\Omega^{V}_{u_{max}}}, & (66a) \\ -\mathcal{H}^{vac,u_{max}}(X, \nabla F(X)) = 0,\ X \in \Omega^{V}_{u_{max}}, & (66b) \\ F(X) = r_I\zeta(X),\ \ X \in \overline{\Omega^{NV}_{u_{max}}} \cap \overline{\Omega^{V}_{u_{max}}}, & (66c) \\ -\mathcal{H}^{vac,u_{max}}(X, \nabla F(X)) = 0, X \in \partial\Omega^{V}_{u_{max}} \setminus (\overline{\Omega^{NV}_{u_{max}}} \cap \overline{\Omega^{V}_{u_{max}}}). & (66d) \end{cases}$$

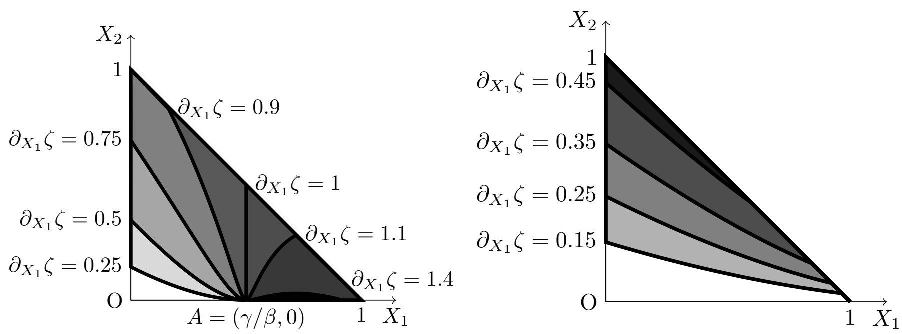


Figure 13: Illustration of level lines $\mathcal{L}_r^{\partial_{X_1}\zeta}$ of the function $\partial_{X_1}\zeta$ for $\gamma/\beta < 1$ (left) and $\gamma/\beta \geq 1$ (right).

Here $\mathcal{H}^{vac,u_{max}} : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ *is the Hamiltonian function:*

$$\begin{aligned}\mathcal{H}^{vac,u_{max}}(X,p) &= \langle f(X,u_{max}),(p_1,p_2)\rangle + r_I\beta X_1 X_2 - r_V u_{max} \\ &= -u_{max}(p_1 - r_V) + \beta X_1 X_2 (r_I + p_2 - p_1) - \gamma X_2 p_2. \end{aligned} \tag{67}$$

3. $\Pi^{r_V,r_I}_{u_{max}} \in \mathcal{C}^1(\Omega)$*;*

4. $\Pi^{r_V,r_I}_{u_{max}}$ *is a solution of the HJB equation* (39).

Proof. We only consider in this proof the circumstance when $\gamma/\beta < 1$, because proof for $\gamma/\beta \geq 1$ is similar to the proof for $\gamma/\beta < 1$.
Point 1: It is enough to show that a trajectory with initial point in $\Omega^{NV}_{u_{max}}$ remains there forever. Considering the definition of the domains for any $X \in \overline{\Omega^V_{u_{max}}} \cap \overline{\Omega^{NV}_{u_{max}}} = \overline{\mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I}}$ the tangent direction $f(X,u)$ to the trajectory points strictly to the interior of $\Omega^{NV}_{u_{max}}$ (for any $u \in [0,u_{max}]$); this follows from Lemma D.6.
Point 2: These properties of the function $\Pi^{r_V,r_I}_{u_{max}}$ are obtained as in the proofs of Theorems D.1 and D.2 once we write $\Pi^{r_V,r_I}_{u_{max}}$ as the "value function" of a trivial control problem where the control is taken in the one-element set $\{u_{max}\}$ until reaching the frontier $\Omega^V_{u_{max}} \cap \Omega^{NV}_{u_{max}}$; on the frontier the value is $r_I\zeta(X) = J_0(X)$.
Point 3: The function $\zeta(X)$ is $\mathcal{C}^1$ on Ω (see Appendix A); in particular $\Pi^{r_V,r_I}_{u_{max}}$ will be $\mathcal{C}^1$ on $\Omega^{NV}_{u_{max}}$. For $X \in \Omega^V_{u_{max}}$ we note that $\Pi^{r_V,r_I}_{u_{max}}$ is the solution of a quasi-linear first order PDE (cf. point 2) and has boundary conditions defined on a non-characteristic curve $\overline{\Omega^{NV}_{u_{max}}} \cap \overline{\Omega^V_{u_{max}}} = \overline{\mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I}}$; the curve is non-characteristic because on $\overline{\Omega^{NV}_{u_{max}}} \cap \overline{\Omega^V_{u_{max}}}$ we have $\langle f(X,u_{max}), \partial_{X_1}\zeta(X)\rangle \neq 0$. Another way to prove the result is to parametrize the boundary curve with a parameter α_1 and denote α_2 the time required to reach the curve. Using the regularity properties of the ODE the function is $\mathcal{C}^1$ in parameters (α_1,α_2) and the change of coordinates from X to (α_1,α_2) is regular around each point in the interior of $\Omega^V_{u_{max}}$. Thus $\Pi^{r_V,r_I}_{u_{max}}$ will be $\mathcal{C}^1$ on $\Omega^V_{u_{max}}$.

It remains to be proved that $\Pi^{r_V,r_I}_{u_{max}}$ is also $\mathcal{C}^1$ around any point $X \in \mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I} \cap \Omega$; since $\Pi^{r_V,r_I}_{u_{max}}\Big|_{\Omega^V_{u_{max}}}$ and $\Pi^{r_V,r_I}_{u_{max}}\Big|_{\Omega^{NV}_{u_{max}}}$ are both $\mathcal{C}^1$, the side gradients exist and it remains only to be proved that

$$\nabla \Pi^{r_V,r_I}_{u_{max}}\Big|_{\Omega^V_{u_{max}}}(X) = \nabla \Pi^{r_V,r_I}_{u_{max}}\Big|_{\Omega^{NV}_{u_{max}}}(X),\ \forall X \in \mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I}.$$

Using continuity and $\mathcal{C}^1$ properties and the fact that $\Pi^{r_V,r_I}_{u_{max}}\Big|_{\Omega^V_{u_{max}}}$ and $\Pi^{r_V,r_I}_{u_{max}}\Big|_{\Omega^{NV}_{u_{max}}}$ coincide on the common frontier it follows that the tangential derivatives along the frontier are the same. Let us prove that the directional derivative also coincide in the direction $f(X, u_{max})$, which can be written:

$$\langle \nabla \Pi^{r_V,r_I}_{u_{max}}\Big|_{\Omega^V_{u_{max}}}(X), f(X, u_{max})\rangle = \langle \nabla \Pi^{r_V,r_I}_{u_{max}}\Big|_{\Omega^{NV}_{u_{max}}}(X), f(X, u_{max})\rangle. \quad (68)$$

But $\Pi^{r_V,r_I}_{u_{max}}\Big|_{\Omega^V_{u_{max}}}$ satisfies (66b) then

$$\begin{aligned}
&\langle \nabla \Pi^{r_V,r_I}_{u_{max}}\Big|_{\Omega^V_{u_{max}}}(X), f(X, u_{max})\rangle = -r_I\beta X_1 X_2 + r_V u_{max} \\
&= -\mathcal{H}^{vac,u_{max}}(X, \nabla \Pi^{r_V,r_I}_{u_{max}}\Big|_{\Omega^{NV}_{u_{max}}}(X)) + \langle \nabla \Pi^{r_V,r_I}_{u_{max}}\Big|_{\Omega^{NV}_{u_{max}}}(X), f(X, u_{max})\rangle \\
&= \langle \nabla \Pi^{r_V,r_I}_{u_{max}}\Big|_{\Omega^{NV}_{u_{max}}}(X), f(X, u_{max})\rangle.
\end{aligned}$$

We used above the fact that $J_0 = r_I\zeta$ satisfies $\mathcal{H}^{vac,0}(X, \nabla J_0) = 0$ on Ω and that for $X \in \mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I}$ we can add u_{max} multiplied by the null term $r_I\partial_{X_1}\zeta(X) - r_V$ to $\mathcal{H}^{vac,0}(X, \nabla J_0)$ to obtain $\mathcal{H}^{vac,u_{max}}(X, \nabla J_0) = 0$.

Note that the direction $f(X, u_{max})$ cannot be collinear with the tangent at X to the boundary $\mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I}$ because for $r_V < r_I r^{crit}_{V,u_{max}}$ the definition of $r^{crit}_{V,u_{max}}$ ensures that $f(X, u_{max})$ has non-zero scalar product with the normal $\nabla\partial_{X_1}\zeta(X)$ to the boundary. From (68) and the coincidence of the tangential derivatives it follows that side gradients $\nabla \Pi^{r_V,r_I}_{u_{max}}\Big|_{\Omega^V_{u_{max}}}$ and $\nabla \Pi^{r_V,r_I}_{u_{max}}\Big|_{\Omega^{NV}_{u_{max}}}$ coincide on the common boundary thus $\Pi^{r_V,r_I}_{u_{max}} \in \mathcal{C}^1(\Omega)$.

Point 4: Given what was already proved, it remains to show that

$$\partial_{X_1}\Pi^{r_V,r_I}_{u_{max}}(X) \le r_V\ \forall X \in \Omega^{NV}_{u_{max}}, \quad (69)$$

$$\partial_{X_1}\Pi^{r_V,r_I}_{u_{max}}(X) \ge r_V\ \forall X \in \Omega^{V}_{u_{max}}, \quad (70)$$

Equation (69) is a simple consequence of (62) and Point 1. For (70) we have to analyze in detail the function $\Pi^{r_V,r_I}_{u_{max}}\Big|_{\Omega^{NV}_{u_{max}}}$, we will prove that in addition:

$$\partial_{X_2}\Pi^{r_V,r_I}_{u_{max}}(X) > 0\ \forall X \in \Omega^{V}_{u_{max}}. \quad (71)$$

Consider $X_0 \in \mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I}$. We integrate $\partial_{X_1}\Pi^{r_V,r_I}_{u_{max}}$ on the characteristic curve $\Phi^{Y,u_{max}}(\cdot)$ issued from $Y \in \Omega^V_{u_{max}}$ that reaches the frontier at time $t > 0$ and point X_0 which can be written: $\Phi^{Y,u_{max}}(t) = X_0$. Formally

$$\begin{aligned}
&\partial_{X_1}\Pi^{r_V,r_I}_{u_{max}}(Y) = \partial_{X_1}\Pi^{r_V,r_I}_{u_{max}}(X_0) \\
&- \int_0^t \langle \nabla\partial_{X_1}\Pi^{r_V,r_I}_{u_{max}}(\Phi^{Y,u_{max}}(\tau)), f(\Phi^{Y,u_{max}}(\tau), u_{max})\rangle d\tau. \quad (72)
\end{aligned}$$

From now on we will drop the notation $\Phi^{Y,u_{max}}(\tau)$ and only denote $(X_1(\tau), X_2(\tau)) = X(\tau) = \Phi^{Y,u_{max}}(\tau)$. Note that $\Pi^{r_V,r_I}_{u_{max}}$ satisfies $\mathcal{H}^{vac,u_{max}}(X, \nabla\Pi^{r_V,r_I}_{u_{max}}) = 0$ on $\Omega^V_{u_{max}}$ i.e., $\langle \nabla\Pi^{r_V,r_I}_{u_{max}}, f(X, u_{max})\rangle + r_I\beta X_1X_2 + r_V u_{max} = 0$ thus by differentiating formally with respect to X_1 one obtains:

$$\langle \nabla\partial_{X_1}\Pi^{r_V,r_I}_{u_{max}}(X(\tau)), f(X(\tau), u_{max})\rangle = \beta X_2(-r_I - \partial_{X_1}\Pi^{r_V,r_I}_{u_{max}} + \partial_{X_2}\Pi^{r_V,r_I}_{u_{max}}).$$

But this latter quantity is integrable over $[0, t]$ and after classical arguments we obtain that $\int_0^t \langle \nabla\partial_{X_1}\Pi^{r_V,r_I}_{u_{max}}(\Phi^{Y,u_{max}}(\tau)), f(\Phi^{Y,u_{max}}(\tau), u_{max})\rangle d\tau$ is well defined and equals $\int_0^t \beta X_2(\tau)(-r_I - \partial_{X_1}\Pi^{r_V,r_I}_{u_{max}} + \partial_{X_2}\Pi^{r_V,r_I}_{u_{max}})(X(\tau))d\tau$. Moreover using again the HJB equation satisfied by $\Pi^{r_V,r_I}_{u_{max}}$ this term can be replaced by

$$\int_0^t \frac{1}{X_1(\tau)}\Big[u_{max}(\partial_{X_1}\Pi^{r_V,r_I}_{u_{max}}(X(\tau)) - r_V) + \gamma X_2(\tau)\partial_{X_2}\Pi^{r_V,r_I}_{u_{max}}(X(\tau))\Big]d\tau.$$

Thus, we obtain

$$\partial_{X_1}\Pi^{r_V,r_I}_{u_{max}}(Y) = \partial_{X_1}\Pi^{r_V,r_I}_{u_{max}}(X_0) + \int_0^t \frac{1}{X_1(\tau)}\Big[u_{max}(\partial_{X_1}\Pi^{r_V,r_I}_{u_{max}}(X(\tau)) - r_V) + \gamma X_2(\tau)\partial_{X_2}\Pi^{r_V,r_I}_{u_{max}}(X(\tau))\Big]d\tau. \tag{73}$$

Similar computations allow to write:

$$\partial_{X_2}\Pi^{r_V,r_I}_{u_{max}}(Y) = \partial_{X_2}\Pi^{r_V,r_I}_{u_{max}}(X_0) + \int_0^t \frac{1}{X_2(\tau)}\Big[u_{max}(\partial_{X_1}\Pi^{r_V,r_I}_{u_{max}}(X(\tau)) - r_V)\Big]d\tau. \tag{74}$$

Since $\Pi^{r_V,r_I}_{u_{max}}$ is $\mathcal{C}^1$ it follows from the properties of ζ that $\partial_{X_1}\Pi^{r_V,r_I}_{u_{max}}(X_0) = r_V$ and $\partial_{X_2}\Pi^{r_V,r_I}_{u_{max}}(X_0) > 0$. Combined with the identities (73)-(74) (and reasoning infinitesimally starting from X_0 along the characteristic) we obtain $\partial_{X_1}\Pi^{r_V,r_I}_{u_{max}}(Y) > r_V$ and $\partial_{X_2}\Pi^{r_V,r_I}_{u_{max}}(Y) > 0$ and equations (71) and (70) follow.

Theorem D.8. *For $r_V < r_I r^{crit}_{V,u_{max}}$ the function $\Pi^{r_V,r_I}_{u_{max}}$ is the unique solution of the HJB equation (39) and $\Pi^{r_V,r_I}_{u_{max}} = \mathcal{V}^{u_{max}}$. As a consequence in this case the value function $\mathcal{V}^{u_{max}}$ is in $\mathcal{C}^1(\Omega)$.*

Proof. Theorem D.7 proves that $\Pi^{r_V,r_I}_{u_{max}}$ is a solution of (39). Furthermore, Theorem D.5 assures the uniqueness of the solution. Then, $\mathcal{V}^{u_{max}} = \Pi^{r_V,r_I}_{u_{max}}$.

D.5 Solution candidate and its properties: the super-critical case

We work here under the hypothesis $r_V \geq r^{crit}_{V,u_{max}}$. In particular this implies $\gamma/\beta < 1$.

The simplest case is when $r_V \geq 2r_I$ and will be dealt with directly later in Theorem D.12. On the contrary, the situation when $r_V \in [r^{crit}_{V,u_{max}} r_I, 2r_I[$ requires some more work. In this case the value function $\mathcal{V}^{u_{max}}$ will **not** be $\mathcal{C}^1$.

Define (see also figure 14):

$$\Gamma^{crit}_{sub} = \{P \in \mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I} | \langle f(P, u_{max}), \nabla\partial_{X_1}\zeta(P)\rangle \leq 0\}. \tag{75}$$

Using the formulas for f and the derivatives of ζ one can prove with straightforward computations:

- $\mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I} \subset \{X \in \Omega | X_1 > \gamma/\beta\}$ (since $r^{crit}_{V,u_{max}} > 1$);
- $\partial_{X_2X_1}\zeta(P) < 0$, $\forall P \in \mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I}$;
- Γ^{crit}_{sub} is connected; denote by $P^{crit}_{r_V}$ the other extremity of the curve; then $\langle f(P^{crit}_{r_V}, u_{max}), \nabla\partial_{X_1}\zeta(P^{crit}_{r_V})\rangle = 0$;
- $\forall P \in \mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I} \setminus \Gamma^{crit}_{sub}$, $P_1 \geq (P^{crit}_{r_V})_1$;
- the trajectories starting from points on the curve Γ^{crit}_{sub} enter the domain $\{X \in \Omega | \nabla\partial_{X_1}\zeta(X) \leq r_V/r_I\}$ for any $u \in [0, u_{max}]$;
- the trajectories starting from points in $\mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I} \setminus \Gamma^{crit}_{sub}$ exit this domain for $u = u_{max}$.

For any $Y \in \Gamma^{crit}_{sub}$ introduce

$$t_Y = \sup\Big\{t \geq 0 \;\Big|\; J_0(Y) + r_V t u_{max} + \int_0^t r_I \beta \Phi_1^{Y,u_{max}}(-\tau)\Phi_2^{Y,u_{max}}(-\tau)d\tau \leq J_0(\Phi^{Y,u_{max}}(-t))\Big\}. \tag{76}$$

We note that the previous properties imply that $t_{X_0} > 0$; indeed, take $Z = \Phi^{Y,u_{max}}(-\epsilon)$ for ϵ small enough; then integrating over the curve $\tau \mapsto \Phi^{Z,u_{max}}(\tau)$ we obtain:

$$J_0(Y) = J_0(\Phi^{Z,u_{max}}(\epsilon)) = J_0(Z) \tag{77}$$

$$+ \int_0^\epsilon \langle \nabla J_0(\Phi^{Z,u_{max}}(\tau)), f(\Phi^{Z,u_{max}}(\tau), u_{max})\rangle d\tau. \tag{78}$$

Developing the last term and using the HJB equation satisfied by J_0 we can write:

$$J_0(Y) = J_0(Z) - r_V t u_{max} - \int_0^t r_I \Big[\beta \Phi_1^{Z,u_{max}}(\tau)\Phi_2^{Z,u_{max}}(\tau) + u_{max}(\partial_{X_1} J_0(\Phi^{Z,u_{max}}(\tau)) - r_V)\Big] d\tau. \tag{79}$$

The curve $\tau \mapsto \Phi^{Z,u_{max}}(\tau)$ belongs to the domain where $\partial_{X_1} J_0(\Phi^{Z,u_{max}}(\tau)) \geq r_V$ therefore $Z = \Phi^{Y,u_{max}}(-\epsilon)$ satisfies the inequality in the equation (76) and as such we obtain $t_Y \geq \epsilon > 0$.

We define a curve Γ^{crit}_{super} as:

$$\Gamma^{crit}_{super} = \{\Phi^{Y,u_{max}}(-t_Y) \mid Y \in \Gamma^{crit}_{sub}\}. \tag{80}$$

The curves Γ^{crit}_{sub} and Γ^{crit}_{super} define a domain that will be denoted $\Omega^V_{u_{max}}$; set also $\Omega^{NV}_{u_{max}} = \Omega \setminus \Omega^V_{u_{max}}$ as illustrated in figure 14.

Lemma D.9. *The following inclusion holds:*

$$\{X \in \Omega \mid \partial_{X_1}\zeta(X) \geq r_V/r_I\} \subset \Omega^V_{u_{max}}. \tag{81}$$

Therefore we also have:

$$\Omega^{NV}_{u_{max}} \subset \{X \in \Omega \mid \partial_{X_1}\zeta(X) \leq r_V/r_I\}. \tag{82}$$

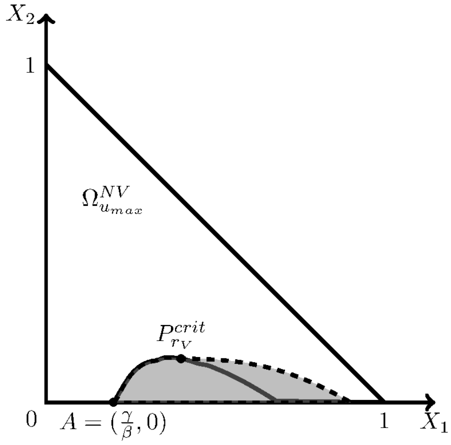


Figure 14: Illustration of the construction of the domains $\Omega^V_{u_{max}}$ and $\Omega^{NV}_{u_{max}}$. The solid curve is $\mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I}$. The dashed curves are Γ^{crit}_{sub} (from A to $P^{crit}_{r_V}$) and Γ^{crit}_{super} (from $P^{crit}_{r_V}$). The gray domain is $\Omega^V_{u_{max}}$.

Proof. Let $Z \in \{X \in \Omega \mid \partial_{X_1}\zeta(X) \geq r_V/r_I\}$ and consider the trajectory $\Phi^{Z,u_{max}}(t)$ starting from Z. This trajectory will exit this set at some point on the border $\mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I}$, more precisely at some point of Γ^{crit}_{sub} (the direction tangent to the trajectory has to exit the domain, which is precisely the definition of Γ^{crit}_{sub}). Denote this point $Y = \Phi^{Z,u_{max}}(\tau^*)$. Using the same arguments as in the proof of $t_Y > 0$ above and recalling that $\partial_{X_1} J_0(\Phi^{Z,u_{max}}(\tau)) \geq r_V$ for all $\tau \leq \tau^*$ we obtain $t_Y \geq \tau^*$ and in particular $Z \in \Omega^V_{u_{max}}$.

Introduce the solution candidate $\Pi^{r_V,r_I}_{u_{max}} : \Omega \to \mathbb{R}$ defined by equation (65), but with the control $u_{X_0}(\cdot)$ defined in equation (64) depending on the newly defined set $\Omega^V_{u_{max}}$.

Theorem D.10. *For $r_V \in [r^{crit}_{V,u_{max}} r_I, 2r_I[$:*

1.

$$\Pi^{r_V,r_I}_{u_{max}}(Y) = \begin{cases} J_0(Y), \text{ if } Y \in \Omega^{NV}_{u_{max}} & (83a) \\ r_V t + \int_0^t r_I \beta \Phi_1^{Y,u_{max}}(\tau)\Phi_2^{Y,u_{max}}(\tau)d\tau + J_0(\Phi^{Y,u_{max}}(t)), \\ \text{if } Y \in \Omega^V_{u_{max}} \text{ and } \Phi^{Y,u_{max}}(t) \in \Gamma^{crit}_{sub}; & (83b) \end{cases}$$

2. $\Pi^{r_V,r_I}_{u_{max}}\big|_{\Omega^V_{u_{max}}}$ *is the unique viscosity solution of the following problem:*

$$(\mathcal{P}v) \begin{cases} \text{Find } F : \overline{\Omega^V_{u_{max}}} \to \mathbb{R} \text{ such that} \\ F \text{ is Lipschitz on } \overline{\Omega^V_{u_{max}}}, & (84a) \\ -\mathcal{H}^{vac,u_{max}}(X, \nabla F(X)) = 0,\ X \in \Omega^V_{u_{max}}, & (84b) \\ F(X) = r_I \zeta(X),\ X \in \overline{\Omega^{NV}_{u_{max}}} \cap \overline{\Omega^V_{u_{max}}}, & (84c) \\ -\mathcal{H}^{vac,u_{max}}(X, \nabla F(X)) = 0, X \in \partial\Omega^V_{u_{max}} \setminus (\overline{\Omega^{NV}_{u_{max}}} \cap \overline{\Omega^V_{u_{max}}}); & (84d) \end{cases}$$

3. $\Pi^{r_V,r_I}_{u_{max}}$ *is Lipschitz on* $\overline{\Omega}$*;*

4. $\Pi^{r_V,r_I}_{u_{max}}$ *is a solution of the HJB equation* (39).

Proof. Much of the proof uses concepts already invoked in the proof of Theorem D.7. We will only emphasize points that are specific to this situation.
Point 1: A trajectory with initial point in $\Omega^{NV}_{u_{max}}$ remains there forever therefore we conclude as above that $\Pi^{r_V,r_I}_{u_{max}}\Big|_{\Omega^{NV}_{u_{max}}} = J_0 = r_I\zeta$; to prove the second property note that the function J satisfies

$$J(Y,u(\cdot)) = \int_0^t r_V u(\tau) + r_I\beta\Phi_1^{Y,u}(\tau)\Phi_2^{Y,u}(\tau)d\tau + J(\Phi^{Y,u(\cdot)}(t), u(\cdot+t)). \quad (85)$$

Thus the two definitions coincide as the control is u_{max} on $\Omega^V_{u_{max}}$ and 0 on $\Omega^{NV}_{u_{max}}$ because once the trajectory reaches the frontier Γ^{crit}_{sub} of $\Omega^{NV}_{u_{max}}$ it enters $\Omega^{NV}_{u_{max}}$ and remains there.
Point 2: The proof follows the same lines as point 2 in Theorem D.7.
Point 3: The definition of the domain $\Omega^V_{u_{max}}$ and the previous point ensures that $\Pi^{r_V,r_I}_{u_{max}}$ is continuous in points of the common boundary $\partial\Omega^V_{u_{max}} \cap \partial\Omega^{NV}_{u_{max}}$ thus it is continuous on Ω. It is also Lipschitz on $\Omega^V_{u_{max}}$ and $\Omega^{NV}_{u_{max}}$ with Lipschitz constants that are universally bounded, thus it is Lipschitz on $\overline{\Omega}$.

Moreover, as before, one can prove that $\Pi^{r_V,r_I}_{u_{max}}$ is $\mathcal{C}^1$ on Γ^{crit}_{sub}.

Another alternative is to repeat the arguments used to prove that the value function is Lipschitz (here the control has the same structure: it has value u_{max} from 0 to some finite time and then 0).
Point 4: We have to prove (the analogues of) the equations (69) and (70).

Any trajectory from $Z \in \Omega^V_{u_{max}}$ (for control $u = u_{max}\mathbb{1}_{X\in\Omega^V_{u_{max}}}$) will encounter Γ^{crit}_{sub} when exiting the domain $\Omega^V_{u_{max}}$. Together with the fact that $\Pi^{r_V,r_I}_{u_{max}}$ is $\mathcal{C}^1$ in Γ^{crit}_{sub} this allows to use identities (73)-(74) and obtain as above that $\partial_{X_1}\Pi^{r_V,r_I}_{u_{max}}(Y) > r_V$ and $\partial_{X_2}\Pi^{r_V,r_I}_{u_{max}}(Y) > 0$ for any $Y \in \Omega^V_{u_{max}}$; then (70) follows.

To prove (69) use Lemma D.9 and point 3 of this Theorem.

Theorem D.11. *For* $r_V \in [r_I r^{crit}_{V,u_{max}}, 2r_I[$ *the function* $\Pi^{r_V,r_I}_{u_{max}}$ *(defined by equation* (65) *with the control* $u_{X_0}(\cdot)$ *defined in equation* (64) *depending on the set* $\Omega^V_{u_{max}}$*) is the unique solution of the HJB equation* (39) *and* $\Pi^{r_V,r_I}_{u_{max}} = \mathcal{V}^{u_{max}}$*. The value function* $\mathcal{V}^{u_{max}}$ *is Lipschitz in* Ω*.*

Proof. The Theorem D.10 proves that $\Pi^{r_V,r_I}_{u_{max}}$ is a solution of (39). Furthermore, Theorem D.5 assures the uniqueness of the solution. Then, $\mathcal{V}^{u_{max}} = \Pi^{r_V,r_I}_{u_{max}}$.

Theorem D.12. *For* $r_V \geq 2r_I$ *the function* $J_0 = r_I\zeta$ *is the unique solution of the HJB equation* (39) *and* $\mathcal{V}^{u_{max}} = J_0$*. As a consequence in this case the value function* $\mathcal{V}^{u_{max}}$ *is in* $\mathcal{C}^1(\Omega)$*.*

Proof. Straightforward computation and the results from Lemma B.4 indicate that the derivative J_0 does not exceed $2r_I$ and as such $(\partial_{X_1}\zeta - r_V)_+ = 0$ and J_0 satisfies the required HJB equation.

E Instantaneous vaccination

Recall that for $u_{max} = \infty$ the value function is denoted as $\mathcal{V}^\infty$; also consult equation (17) for the definition of $\mathcal{H}^\infty$.

The following result connects the bounded and unbounded control problems (see also [7] pages 113-115 for generic related results):

Theorem E.1. *The sequence $(\mathcal{V}^{u_{max}})_{u_{max}\geq 0}$ is decreasing and*

$$\lim_{u_{max}\to\infty} \mathcal{V}^{u_{max}} = \mathcal{V}^\infty. \tag{86}$$

Moreover the convergence is uniform over compacts of Ω and $\mathcal{V}^\infty$ is Lipschitz over Ω.

Proof. Since for any $u_2 \geq u_1 \geq 0$ we have the inclusion $\mathcal{U}_Y^{u_1} \subset \mathcal{U}_Y^{u_2}$ the sequence $(\mathcal{V}^{u_{max}})_{u_{max}\geq 0}$ is decreasing. Therefore $\liminf_{u_{max}\to\infty} \mathcal{V}^{u_{max}} \geq \mathcal{V}^\infty$.

Let $Y \in \Omega$ and $(dV_n)_{n\geq 0} \subset \mathcal{U}_Y^\infty$ a sequence of strategies such that $\lim_{n\to\infty} J(Y, dV_n) = \mathcal{V}^\infty(Y)$. For each n construct an approximating sequence of admissible strategies $u_w^n \in \mathcal{U}_Y^w$ such that $\lim\limits_{w\to\infty} u_w^n = dV_n$. Then $\mathcal{V}^w(Y) \leq J(Y, u_w^n) \to J(Y, dV_n)$ thus $\limsup_{w\to\infty} \mathcal{V}^w(Y) \leq J(Y, dV_n)$. Passing once more to the limit $n \to \infty$ we obtain $\limsup_{w\to\infty} \mathcal{V}^w(Y) \leq \mathcal{V}^\infty(Y)$.

Then $\lim_{u_{max}\to\infty} \mathcal{V}^{u_{max}} = \mathcal{V}^\infty$. Since functions $\mathcal{V}^{u_{max}}$ are Lipschitz with Lipschitz constants independent of u_{max} the limit $\mathcal{V}^\infty$ will be Lipschitz and the convergence will hold in a neighborhood of Y (thus uniformly over compacts of Ω).

E.1 HJB equation and value function

Theorem E.2. *The value function $\mathcal{V}^\infty$ is a viscosity solution of the Hamilton-Jacobi-Bellman equation:*

$$(\mathcal{P}) \begin{cases} \text{Find } F : \overline{\Omega} \to \mathbb{R} \text{ such that} \\ F \text{ is Lipschitz on } \overline{\Omega}, & \text{(87a)} \\ -\mathcal{H}^\infty(X, \nabla F(X)) = 0, \ X \in \Omega, & \text{(87b)} \\ F(X) = 0 \text{ on } \Gamma_{OA}, & \text{(87c)} \\ F(X) = 0 \text{ on } \Gamma_I, & \text{(87d)} \\ -\mathcal{H}^\infty(X, \nabla F(X)) = 0 \text{ on } \Gamma_1. & \text{(87e)} \end{cases}$$

Proof. We will use the same arguments and notations as in the proof of the Theorem D.2.

Step 1. First, we prove that $\mathcal{V}^\infty$ is a subsolution of (87b). We take the same notations and the same reasoning as in the case $u_{max} < \infty$. So equation (42) becomes:

$$\min\{r_V - p_1, \beta Y_1 Y_2(r_I + p_2 - p_1) - \gamma Y_2 p_2\} \geq 0. \tag{88}$$

Suppose that there exists $\kappa > 0$ such that:

$$\min\{r_V - p_1, \beta Y_1 Y_2(r_I + p_2 - p_1) - \gamma Y_2 p_2\} \leq -\kappa.$$

Remark 10 page 27 assures that there exists $\rho \geq 0$ such that:

$$\rho(r_V - \partial_{X_1}\varphi(Y)) + \beta Y_1 Y_2 (r_I + \partial_{X_2}\varphi(Y) - \partial_{X_1}\varphi(Y) - \gamma Y_2 \partial_{X_2}\varphi(Y) \leq -\kappa.$$

Here, we choose the control ρ on the interval $[0, \delta]$ and for the same reasons as above, we obtain:

$$\mathcal{V}^\infty(Z^0) - \mathcal{V}^\infty(Y) \leq -\kappa\delta - \int_0^\delta \rho r_V + r_I \beta \Phi_1^{Y,\rho}(t)\Phi_2^{Y,\rho}(t)dt. \tag{89}$$

In particular, by the optimality of $\mathcal{V}^\infty$ on Y, we have:

$$\mathcal{V}^\infty(Y) \leq \int_0^\delta r_I \beta \Phi_1^{Y,\rho}(t)\Phi_2^{Y,\rho}(t) + \rho r_V dt. \tag{90}$$

And we can conclude as above that $\mathcal{V}^\infty$ is solution of equation (87).

Step 2. We prove that $\mathcal{V}^\infty$ is a supersolution of (87).

Using the same notations and reasoning as in the proof for $u_{max} < \infty$ equation (46) becomes:

$$\min\{r_V - \partial_{X_1}\varphi(Y), \beta Y_1 Y_2 (r_I + \partial_{X_2}\varphi(Y) - \partial_{X_1}\varphi(Y)) - \gamma Y_2 \partial_{X_2}\varphi(Y)\} > \kappa. \tag{91}$$

In order to invalidate (91) we invalidate, in a neighborhood of Y:

$$\beta\Phi_1^{Y,u}(\cdot)\Phi_2^{Y,u}(\cdot)(r_I + \partial_{X_2}\varphi(\Phi^{Y,u}(\cdot)) - \partial_{X_1}\varphi(\Phi^{Y,u}(\cdot))) - \gamma\Phi_2^{Y,u}(\cdot)\partial_{X_2}\varphi(\Phi^{Y,u}(\cdot))) > \kappa. \tag{92}$$

We obtain, as above:

$$\begin{aligned}
\mathcal{V}^\infty(Z^0) - \mathcal{V}^\infty(Y) \geq \varphi(Y) - \varphi(Z^0) &\geq \int_0^\delta \langle \nabla\varphi(\Phi^{Y,w}(t)) \cdot f(\Phi^{Y,w}(t), w)\rangle dt \\
&> \delta\kappa + \int_0^\delta -\beta\Phi_1^{Y,w}(t)\Phi_1^{Y,w}(t) r_I - w\partial_{X_1}\varphi dt \\
&> \delta\kappa - \int_0^\delta w r_V + r_I\beta\Phi_1^{Y,w}(t)\Phi_2^{Y,w}(t)dt + \int_0^\delta w(r_V - \partial_{X_1}\varphi(Y))dt \\
&> \delta\kappa - \int_0^\delta w r_V + r_I\beta\Phi_1^{Y,w}(t)\Phi_2^{Y,w}(t)dt,
\end{aligned}$$

because $w(r_V - \partial_{X_1}\varphi(\Phi^{Y,w}(\cdot))) \geq 0$ since $w \geq 0$ and $(r_V - \partial_{X_1}\varphi(\Phi^{Y,w}(\cdot))) \geq \mathcal{H}^\infty(\Phi^{Y,w}(\cdot), \nabla\varphi(\Phi^{Y,w}(\cdot)) \geq \kappa > 0$.

Once again, we conclude as in the proof of the Theorem D.2.

E.2 Uniqueness of the solution of the HJB problem.

Theorem E.3. *Let $\mathcal{F}_1$ a subsolution of* (87) *and $\mathcal{F}_2$ a supersolution. Then:*

$$\mathcal{F}_1(X) \leq \mathcal{F}_2(X) \text{ for all } X \in \Omega.$$

Proof. We use the same notation and reasoning as in the proof in Section D.3. The Hamiltonian used here is:

$$\tilde{\mathcal{H}}^\infty(X, \mathcal{W}^\infty(X), p) = \min(r_V(1-\mathcal{W}^\infty), \beta X_1 X_2(r_I(1-\mathcal{W}^\infty)+p_2-p_1)-\gamma X_2 p_2). \tag{93}$$

Equation (55) becomes:

$$\begin{aligned} &- \rho r_V(\mathcal{W}_2^\infty(y^\epsilon) - \mathcal{W}_1^\infty(x^\epsilon)) \\ &\qquad + [\beta(-x_1^\epsilon x_2^\epsilon + y_1^\epsilon y_2^\epsilon)(r_I - p_1^\epsilon + p_2^\epsilon) - \gamma p_2^\epsilon(y_2^\epsilon - x_2^\epsilon)] \\ &\qquad\qquad - \beta r_I\,[y_1^\epsilon y_2^\epsilon \mathcal{W}_2^\infty(y^\epsilon) - x_1^\epsilon x_2^\epsilon \mathcal{W}_1^\infty(x^\epsilon)] \leq 0. \end{aligned} \tag{94}$$

And for the same reasons as in the proof for u_{max} bounded we obtain instead of (57):

$$-\left[r_I \beta \overline{x}_1 \overline{x}_2 + \rho r_V\right]\left[\mathcal{W}_2^\infty(\overline{x}) - \mathcal{W}_1^\infty(\overline{x})\right] \leq 0. \tag{95}$$

We can conclude as when u_{max} is bounded.

E.3 A candidate value function: the sub-critical case

We introduce the critical point value $r_{V,\infty}^{crit}$:

$$r_{V,\infty}^{crit} = \sup\{r \geq 0 \mid \partial^2_{X_1 X_1}\zeta(X) > 0 \;\forall X \in \mathcal{L}_r^{\partial_{X_1}\zeta}\}. \tag{96}$$

We see (after some computations) that $r_{V,\infty}^{crit} < \infty$ for $\gamma/\beta < 1$ and $r_{V,\infty}^{crit} = \infty$ for $\gamma/\beta \geq 1$. Note that in all situations $r_{V,\infty}^{crit} > 1$.

We introduce the critical point X_∞^{crit} which is the unique solution of the following equation:

$$\partial^2_{X_1 X_1}\zeta(X_\infty^{crit}) = 0, X_\infty^{crit} \in \Gamma_1. \tag{97}$$

As in (60), we show that $X_\infty^{crit} = (x^*, 1 - x^*)$ where x^* is the solution of:

$$\frac{\gamma}{\beta} - \left(x^* - \frac{\gamma}{\beta}\right) = x^* e^{-\frac{\beta}{\gamma}\left[(1-x^*)+2\left(x^*-\frac{\gamma}{\beta}\right)\right]}. \tag{98}$$

When $r_V < r_{V,\infty}^{crit} r_I$ we define a partition of Ω in two regions

$$\Omega_\infty^{NV} = \{Y \in \Omega \mid \partial_{X_1}\zeta(Y) < r_V/r_I\} \tag{99}$$

$$\Omega_\infty^{V} = \{Y \in \Omega \mid \partial_{X_1}\zeta(Y) > r_V/r_I\}. \tag{100}$$

Note that $r_{V,\infty}^{crit} = \lim_{u_{max}\to\infty} r_{V,u_{max}}^{crit}$ and for u_{max} large enough $\Omega_{u_{max}}^V = \Omega_\infty^V$ (and $\Omega_{u_{max}}^{NV} = \Omega_\infty^{NV}$). As before we can prove the following:

Lemma E.4. *Any trajectory $\Phi^{Y,dV}(t)$ with $Y \in \mathcal{L}_{r_V/r_I}^{\partial_{X_1}\zeta} = \partial\Omega_\infty^{NV} \cap \partial\Omega_\infty^V$ is such that $\Phi^{Y,dV}(t) \in \Omega_\infty^{NV}$ for all $t > 0$ ($dV \in \mathcal{U}_Y$).*

To any $Y \in \Omega$ associate the unique $\Delta Y \geq 0$ such that $(Y_1 - \Delta Y, Y_2) \in \mathcal{L}_{r_V/r_I}^{\partial_{X_1}\zeta}$ and define: $\Pi_\infty^{r_V,r_I}(Y) = J(Y, \Delta Y \delta_{t=0})$. If ΔY does not exist then set $\Delta Y = 0$

with the convention $0 \times \delta_{t=0} = 0$. Note that $\Delta Y = 0$ for any $Y \in \Omega^{NV}_{u_{max}}$ and moreover:

$$\Pi^{r_V,r_I}_\infty(Y) = \begin{cases} J_0(Y) \text{ if } Y \in \Omega^{NV}_\infty \\ J_0(Y_1 - \Delta Y, Y_2) + r_V(\Delta Y) \ \text{ if } Y \in \Omega^V_\infty,\ (Y_1 - \Delta Y, Y_2) \in \mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I},\ \Delta Y \geq 0. \end{cases} \tag{101}$$

For u_{max} large enough $\Pi^{r_V,r_I}_{u_{max}}$ and Π^{r_V,r_I}_∞ coincide on $\Omega^{NV}_{u_{max}}$. Moreover since for any $Y \in \Omega^V_{u_{max}}$ and given u_{max} the optimal strategies $u^{u_{max}}_Y$ converge (when $u_{max} \to \infty$) to the Dirac delta function $\Delta Y \delta_{t=0}$ then:

$$\mathcal{V}^\infty(Y) = \lim_{u_{max}\to\infty} \mathcal{V}^{u_{max}}(Y) = \lim_{u_{max}\to\infty} J(Y, u^{u_{max}}_Y) = J(Y, \Delta Y \delta_{t=0}) = \Pi^{r_V,r_I}_\infty(Y). \tag{102}$$

Therefore we proved the following:

Theorem E.5. *For $r_V < r_I r^{crit}_{V,\infty}$ the function Π^{r_V,r_I}_∞ is the unique solution of the HJB equation (87) and $\Pi^{r_V,r_I}_\infty = \mathcal{V}^\infty$. As a consequence in this case the value function $\mathcal{V}^\infty$ is in $\mathcal{C}^1(\Omega)$.*

Proof. The proof is already above.

A direct proof also can be given; for instance suppose one wants to prove e.g., that $-\mathcal{H}^\infty(Y, \nabla\Pi^{r_V,r_I}_\infty(Y)) = 0$ for $Y \in \Omega$.

The mere definition of the domain Ω^{NV}_∞ imply that $r_V - \partial_{X_1} J_0 \geq 0$ on this domain; on the other hand $\beta X_1 X_2 (r_I + \partial_{X_2} J_0 - \partial_{X_1} J_0) - \gamma X_2 \partial_{X_2} J_0 = 0$ everywhere; thus $-\mathcal{H}^\infty(Y, \nabla\Pi^{r_V,r_I}_\infty(Y)) = 0$ for $Y \in \Omega^{NV}_\infty$.

For $Y \in \Omega^V_\infty$ (with $(Y_1 - \Delta Y, Y_2) \in \mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I}$) the definition of Π^{r_V,r_I}_∞ implies that for any $\epsilon < \Delta Y$: $\Pi^{r_V,r_I}_\infty(Y_1, Y_2) = \Pi^{r_V,r_I}_\infty(Y_1 - \epsilon, Y_2) + r_V \epsilon$ thus $\partial_{X_1}\Pi^{r_V,r_I}_\infty(Y) = r_V$; in addition $\partial_{X_2}\Pi^{r_V,r_I}_\infty(Y) = \partial_{X_2}\Pi^{r_V,r_I}_\infty(Y_1 - \Delta Y, Y_2)$ and the conclusion follows from the HJB equation of J_0 on the $\mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I}$.

E.4 A candidate value function: the super-critical case

We consider here the situation $r_V/r_I \geq r^{crit}_{V,\infty}$; note that this implies $\gamma/\beta \leq 1$. Introduce

$$\Gamma^{crit}_{sub} = \{Y \in \mathcal{L}^{\partial_{X_1}\zeta}_{r_V/r_I} \mid \partial^2_{X_1X_1}\zeta(Y) \leq 0\}. \tag{103}$$

For any $Y \in \Gamma^{crit}_{sub}$ define:

$$Y^{super}_1 = \sup\{Z_1 \geq Y_1 \mid J_0(Y) + r_V(Z_1 - Y_1) \leq J_0(Z_1, Y_2)\}. \tag{104}$$

We define a curve Γ^{crit}_{super} as:

$$\Gamma^{crit}_{super} = \{(Y^{super}_1, Y_2) \mid Y \in \Gamma^{crit}_{sub}\}. \tag{105}$$

Remark 11. We can express Γ^{crit}_{sub} in a parametric form:

$$\Gamma^{crit}_{sub} = \{(X^\Delta_1, X^\Delta_2) \in \Omega \mid \ 0 \leq \Delta \leq \Delta_{max}\},$$

where: $X^{\Delta_{max}}_1 = \frac{\gamma}{\beta}$, $X^{\Delta_{max}}_2 = 0$, $X^0_1 = 2\frac{\gamma r_I}{\beta r_V}$ and $X^0_2 = -\frac{\gamma}{\beta}\ln[\frac{r_V}{r_I} - 1] + 2\frac{\gamma}{\beta}(1 - 2\frac{r_I}{r_V})$ and

$$X^\Delta_1 = \frac{\Delta(e^{-\frac{\beta}{\gamma}\frac{r_V}{r_I}\Delta} - 1)}{1 - \Delta\frac{\beta}{\gamma}\frac{r_V}{r_I} - e^{-\frac{\beta}{\gamma}\frac{r_V}{r_I}\Delta}}, \text{ for } \Delta > 0 \tag{106}$$

$$X_2^\Delta = -\frac{\gamma}{\beta}\ln\left(1 - \frac{\frac{r_V}{r_I}(1-\frac{\beta}{\gamma}X_1^\Delta)}{1-\frac{r_V}{r_I}\frac{\beta}{\gamma}X_1^\Delta}\right) - \frac{\frac{r_V}{r_I}X_1^\Delta(1-\frac{\beta}{\gamma}X_1^\Delta)}{1-\frac{r_V}{r_I}\frac{\beta}{\gamma}X_1^\Delta}. \tag{107}$$

With these notations,

$$\Gamma_{super}^{crit} = \{(Y_1^\Delta, Y_2^\Delta) \in \overline{\Omega} | \; Y_1^\Delta = X_1^\Delta + \Delta, Y_2^\Delta = X_2^\Delta\}. \tag{108}$$

The curves Γ_{sub}^{crit} and Γ_{super}^{crit} define a domain that will be denoted Ω_∞^V; set also $\Omega_\infty^{NV} = \Omega \setminus \Omega_\infty^V$.

Note that when $r_V \geq 2r_I$ the sets Γ_{sub}^{crit}, Γ_{super}^{crit} and Ω_∞^V are empty.

To any $Y \in \Omega$ associate the unique $\Delta Y \geq 0$ such that $(Y_1 - \Delta Y, Y_2) \in \Gamma_{sub}^{crit}$ and define: $\Pi_\infty^{r_V,r_I}(Y) = J(Y, \Delta Y \delta_{t=0})$. If ΔY does not exist then set $\Delta Y = 0$ with the convention $0 \times \delta_{t=0} = 0$. Note that $\Delta Y = 0$ for any $Y \in \Omega_\infty^{NV}$ and moreover:

$$\Pi_\infty^{r_V,r_I}(Y) = \begin{cases} J_0(Y) \text{ if } Y \in \Omega_\infty^{NV} \\ J_0(Y_1 - \Delta Y, Y_2) + r_V(\Delta Y) \;\text{ if } Y \in \Omega_\infty^V, \; (Y_1 - \Delta Y, Y_2) \in \Gamma_{sub}^{crit}, \; \Delta Y \geq 0. \end{cases} \tag{109}$$

Note that for any given $Y \in \Omega_\infty^V$ for u_{max} large enough $Y \in \Omega_{u_{max}}^V$. Moreover for any $Y \in \Omega_{u_{max}}^V$ and given u_{max} the optimal strategies $u_Y^{u_{max}}$ converge (when $u_{max} \to \infty$) to a Dirac delta function $\Delta Y \delta_{t=0}$ then:

$$\mathcal{V}^\infty(Y) = \lim_{u_{max}\to\infty} \mathcal{V}^{u_{max}}(Y) = \lim_{u_{max}\to\infty} J(Y, u_Y^{u_{max}}) = J(Y, \Delta Y \delta_{t=0}) = \Pi_\infty^{r_V,r_I}(Y). \tag{110}$$

Therefore we proved the following:

Theorem E.6. *For $r_V \geq r_I r_{V,\infty}^{crit}$ the function $\Pi_\infty^{r_V,r_I}$ is the unique solution of the HJB equation* (87) *and $\Pi_\infty^{r_V,r_I} = \mathcal{V}^\infty$. In particular when $r_V \geq 2r_I$: $\mathcal{V}^\infty = J_0 \in \mathcal{C}^1(\Omega)$ but when $r_V \in]r_I r_{V,\infty}^{crit}, 2r_I[$ the value function $\mathcal{V}^\infty$ is only Lipschitz.*

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