QUANTUM PAINLEVÉ II SOLUTION AND YUKAWA POTENTIAL APPROXIMATION

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Abstract. In this paper, it has been shown that one dimensional non-stationary Schrödinger equation with a specific choice of potential reduces to the quantum Painlevé II equation and the solution of its Riccati form appears as a dominant term of that potential. Further, this article shows that the square of quantum Painlevé II Riccati solution is equivalent to the centrifugal expression of radial schrödinger potential which reduces to the approximated form of Yukawa potential that can be used to solve the radial Schrödinger equation by applying the Nikiforov-Uvarov method.. Finally, I express the approximated form of Yukawa potential explicitly in terms of qunatume Painlevé II solution.

1. Introduction

The Painlevé equations were discovered by Painlevé and his colleagues while classifying the nonlinear second-order ordinary differential equations with respect to their solutions [1]. The study of Painlevé equations is important because of the wide applications of these equations in various areas of mathematics and physics. For example, in hydrodynamics and plasma physics the Painlevé equations are usually obtained as reduced ODEs of some PDEs that describe the evolution of the flow or convective flows with viscous dissipation [2]. In nonlinear optics, the nonlinear Schrödinger equation plays an important role to explain the wave propagation in media, the ODE reduction of this equation is PIV equation [2]. It has been shown that the description of two dimensional quantum gravity involves PI equation [3] - [5] and one of the applications of Painlevé second equation has been studied in [6] for the wave collapse in the three-dimensional nonlinear Schrödinger equation
Further it has been show that the exactly solvable models in the field of statistical physics and quantum field theory can be expressed in terms of Painlevé transcendents [7] - [9]. The classical Painlevé equations are regarded as completely integrable equations and obeyed the Painlevé test [10] - [12]. These equations admit some properties such as linear representations, hierarchies, they possess Darboux transformations(DTs) and Hamiltonian structure. These equations also arise as ordinay differential equations (ODEs) reduction of some integrable systems, i.e, the ODE reduction of the KdV equation is Painlevé II (PII) equation [13] and [14]. Further it seems quite interetsing to derive the

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quantum analogue of these equation and to explore their physical aspects in various areas of physics and mathematics.
In this paper, we construct a connection of quantum Painlevé II equation [15] and its Riccati solution to the quantum mechanical systems involves the approximated form of Yukawa potential. We show that one dimensional non-stationary Schrödinger equation with a specific choice of potential reduces to the quantum Painlevé II equation and the dominant term of that potential can be obtained as the solution of quantum Pianlevé II Riccati equation. Further we observe that quantum Painlevé II solution coincides with centrifugal expression [16] of the radial schrödinger potential that expression was applied to derive the approximated form of Yukawa potential [16] - [19].

2. Quantum Painlevé II equation and its Riccati form

The quantum Painlevé II equation in the following form

$$\begin{cases} f^{''} = 2f^3 - 2[z, f]_+ + c \\ zf - fz = i\hbar f \end{cases} \tag{1}$$

can be derived from the compatibility condition of following linear system

$$\Psi_\lambda = A(z;\lambda)\Psi, \quad \Psi_z = B(z;\lambda)\Psi \tag{2}$$

and the Lax matrices A and B are given by

$$\begin{cases} A = (8i\lambda^2 + if^2 - 2iz)\sigma_3 + f^{'}\sigma_2 + (\frac{1}{4}c\lambda^{-1} - 4\lambda f)\sigma_1 + i\hbar\sigma_2 \\ B = -2i\lambda\sigma_3 + f\sigma_1 + fI \end{cases} \tag{3}$$

where $\hbar$ is Planck constant and σ_1, σ_2, σ_3 are Pauli spin matrices and $[z, f]_+$ is anti-commutator of z and f. The quantum Painlevé II equation (1) was derived in [15] by taking the quantum commutation relations [20] as a kind of quantization for Noumi-Yamada Painlevé II system [21]. This quatization can be taken aa a particular case of deformed space that involves the star product and the version of pure non-commutative Painlevé II under the star product has been derived by V. Retakh and V. Roubtsov [22]. By using the commutation relation $zf - fz = i\hbar f$ the first equation in system (1) can be written as

$$f^{''} = 2f^3 - 4zf - 2i\hbar f + c. \tag{4}$$

The quantum Painlevé II equation (4) reduces to the ordinary Painlevé II equation under the classical limit when $\hbar \to 0$. Substituting the eigenvector $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ in linear system (1) and setting $\Delta = \psi_1\psi_2^{-1}$, we obtain the quantum PII equation in Riccati form as follow

$$\Delta^{'} = -4i\Delta + f + [f, \Delta]_- - \Delta f \Delta. \tag{5}$$

3. Non-Stationary Shrödinger equation and Quantum PII equation

In this section we will observe that how the quantum Painlevé II equation is connected to the quantum system described by one dimensional non-stationary Schrödinger equation with a specific choice of potential $V(x,t)$ and eigenfunction $\psi(x,t)$ that has been explained in the following proposition.

Proposition 1.1.
The time dependent Shrödinger equation for a particle of mass m with potential $V(x,t)$

$$-i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x,t)\psi(x,t) \tag{6}$$

reduces to quantum PII equation (4) at constant $c = 0$ with the choice of potential $V(x,t)$ as follow

$$V(x,t) = \gamma x - 2\psi^*(x,t)\psi(x,t). \tag{7}$$

where $\gamma = 4i(\frac{2m}{\hbar})^{\frac{1}{2}}$ and the eigenfunction

$$\psi(x,t) = f(z,\lambda)e^{i\alpha t}, \quad z = i(\frac{2m}{\hbar^2})^{\frac{1}{2}}x \tag{8}$$

where function $f(z,\lambda)$ satisfies the quantum PII equation (4).

Proof:
For the Schrödinger equation (6) we can easily evaluate $\frac{\partial\psi(x,t)}{\partial t}$ and $\frac{\partial^2\psi(x,t)}{\partial x^2}$ in terms of z variable as follow

$$\frac{\partial\psi(x,t)}{\partial t} = i\alpha f(z,\lambda)e^{i\alpha t} \tag{9}$$

and

$$\frac{\partial^2\psi(x,t)}{\partial x^2} = -\frac{2m}{\hbar^2}f''(z,\lambda)e^{i\alpha t}. \tag{10}$$

Now substituting the potential from (7) and using the results from (9) and (10) in Schrödinger equation (6). After simplifications, we obtain the following expression

$$f'' = 2f^3 - 4zf - \alpha\hbar f. \tag{11}$$

The expression (11) represents quantum PII equation (4) at constant $c = 0$ and $\alpha = 2i$.

4. Solution to the Quantum PII Riccati equation

This section consists the derivation of solution for the quantum Painlevé II Riccati form (5) that has been described in proposition 1.2 and further, in next section we will observe that how this Riccati solution coincides to the centrifugal expression [16] that is helpful to construct approximated analytic solution of Yukawa potential for the radial Schrödinger equation [16] - [19].

Proposition 1.2.
We can show that the following choice of $f(z;\lambda)$ and Δ

$$\begin{cases} f = \beta(1-e^{-8\lambda z})^{-1}e^{-4\lambda z} \\ \Delta = e^{4\lambda z} \end{cases} \tag{12}$$

satisfies quantum PII Riccati equation (5) where β is a complex parameter.
Proof:
By using system (12) we can show that

$$\Delta f \Delta = \beta e^{4\lambda z}(1-e^{-8\lambda z})^{-1},\ [f,\Delta]_- = 0. \tag{13}$$

Now after using the valus of $f(z;\lambda)$ and Δ from (12) in Riccati equation (5) we get

$$4\lambda e^{4\lambda z} - 4\lambda e^{-4\lambda z} = -4ie^{4\lambda z} + 4ie^{-4\lambda z} + \beta e^{-4\lambda z} - \beta e^{4\lambda z}$$

or

$$4(\lambda+i)e^{4\lambda z} - 4(\lambda+i)e^{-4\lambda z} = \beta e^{-4\lambda z} - \beta e^{4\lambda z}. \tag{14}$$

We can show that system (12) satisfies the Riccati equation (5) by taking $\beta = -4(\lambda+i)$ in equation (14).

5. Quantum PII solution and Yukawa potential approximation

In this section we will observe that how the quantum Painlevé II solution f coincides to the centrifugal expression of [16] and finally we express approximated form of Yukawa potential [16] - [19] in terms of quantum Painlevé II solution f, the procedure has been explained in following proposition. The explicit expression of Yukawa Potential [23] in radial form can be expressed as follow

$$V(r) = -V_0\frac{e^{-2ar}}{r}. \tag{15}$$

where where $V_0 = \alpha Z$, $\alpha = (137.037)^{-1}$ is the fine-structure constant and Z is the atomic number of neutral atom. This potential can be applied to evaluate the normalized bound-state and the energy levels of neutral atoms.
Proposition 1.3.
The potential (7) can be expressed in terms of z and spectral parameter λ by using the transformation from (8) as follow

$$V(z,\lambda) = 4z - 2f^2 = 4z - 2\beta^2(e^{-4\lambda z} - e^{4\lambda z})^{-2} \tag{16}$$

where $\beta^2 = \beta^*\beta$ and solution to the non-stationary Schrödinger equation (6) can written as

$$\psi(x,t) = \beta(1-e^{-8\lambda z})^{-1}e^{-4\lambda z + i\alpha t}. \tag{17}$$

The quantum Painlevé II solution f in potential (16) is equivalent to the centrifugal expression [16] that has been applied to derive the approximated form of Yukawa potential [16] - [19] to solve radial Schrödinger equation by using Nikiforov-Uvarov method.
Proof:

The dominant term f^2 in potential (16) represents the square of quantum Painlmevé II solution and explicitly can be written as and given by

$$f^2 = \beta^2 (e^{-4\lambda z} - e^{4\lambda z})^{-1} \tag{18}$$

or

$$f^2 = \beta^2 \frac{e^{-8\lambda z}}{(1 - e^{-8\lambda z})^2}. \tag{19}$$

The expression (18) is equivalent to the following centrifugal term [16] of radial Schrödinger equation

$$\frac{1}{r^2} \approx 4a^2 \frac{e^{-2ar}}{(1 - e^{-2ar}})^2. \tag{20}$$

by setting the parameters β and a of (19) and (20) as follow

$$\begin{cases} \beta^2 = 4a^2 \\ a = 4\lambda \end{cases} \tag{21}$$

and for one dimensional case we can take r along $z - axis$. Now after comparing the equation (19) and equation (20) we can express $\frac{1}{r^2}$ in terms of quantum Painlevé II solution f as follow

$$\frac{1}{r^2} = f^2 \tag{22}$$

and after taking the square root of both sides we obtain

$$\frac{1}{r} = \pm f. \tag{23}$$

We only consider the positive value of (23) because r represents separation and finally we get

$$\frac{1}{r} = f. \tag{24}$$

the result (24) holds for $ar \ll 1$ and also equivalently for $\lambda z \ll 1$. Finally we can express the approximated form [16] - [19] of Yukawa potential (15) in terms of quantum Painlevé II solution with the help of (24) in the following form

$$V(z) = -V_0 |\beta| e^{-4\lambda z} \frac{e^{-4\lambda z}}{(1 - e^{-8\lambda z})} \tag{25}$$

or

$$V(z) = -V_0 |\beta| \frac{e^{-8\lambda z}}{(1 - e^{-8\lambda z})} \tag{26}$$

where $|\beta| = 4(\lambda^2 + 1)^{\frac{1}{2}}$.

6. Conclusion

In this article, we have derived a connection of one dimensional non-stationary Schrödinger equation to the quantum Painlevé II equation and also we have constructed the solution of Schrödinger equation by using quantum Painleé II Riccati solution. We have expressed the centrifugal term $\frac{1}{r^2}$ of the radial Schrödinger potential [16] in terms of quantum Painlevé II solution. We observed that the quantum Painlevé II solution $f(z;\lambda)$ coincides to the results of [16] for centrifugal expression by setting the parameters β of quantum Painlevé II solution and a screening parameter of Yukawa potential, as $\beta^2 = 4a^2$. The radial centrifugal expression has been applied to derive the approximated analytic solutions of radial Schrödinger equation by using Nikiforov-Uvarov method [24]. Further we can extend our results to derive an explicit expression for Killingbeeck potential [25] and [26] in terms of Painlevé II solution for smaal values of β^2 and also it seems straight forward to derive the equivalent expression of Cornell potential [27] and [28] in terms of quantum Painlevé II transcendent.

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(1)-Department of Mathematics, Université d'Angers,, 2 Boulevard Lavoisier, 49045 Angers Cedex 01, France, (2)- University of the Punjab, 54590 Lahore, Pakistan

E-mail address: mahirfan@yahoo.com