Magnetic WKB Constructions

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Abstract

This paper is devoted to the semiclassical magnetic Laplacian. Until now WKB expansions for the eigenfunctions were only established in presence of a non-zero electric potential. Here we tackle the pure magnetic case. Thanks to Feynman-Hellmann type formulas and coherent states decomposition, we develop here a magnetic Born-Oppenheimer theory. Exploiting the multiple scales of the problem, we are led to solve an effective eikonal equation in pure magnetic cases and to obtain WKB expansions. We also investigate explicit examples for which we can improve our general theorem: global WKB expansions, quasi-optimal estimates of Agmon and upper bound of the tunelling effect (in symmetric cases). We also apply our strategy to get more accurate descriptions of the eigenfunctions in a wide range of situations analyzed in the last two decades.

Keywords. WKB expansion, magnetic laplacian, Born-Oppenheimer approximation, coherent states, Agmon estimates, tunnel effect.

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Contents

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1 Motivation and main results

1.1 Context and motivation

This paper is devoted to the analysis of the self-adjoint operators on $\mathrm{L}^2(\mathbb{R}^m_s \times \mathbb{R}^n_t, \mathrm{d}s\,\mathrm{d}t)$ of the following type

$$\mathfrak{L}_h = (hD_s + A_1(s,t))^2 + (D_t + A_2(s,t))^2, \tag{1.1}$$

where A_1 and A_2 are polynomial, $D = -i\nabla$, and where the space $\mathrm{L}^2(\mathbb{R}^m_s \times \mathbb{R}^n_t, \mathrm{d}s\,\mathrm{d}t)$ is equipped with the standard scalar product:

$$\langle \psi_1, \psi_2 \rangle_{\mathrm{L}^2(\mathbb{R}^m_s \times \mathbb{R}^n_t, \mathrm{d}s\,\mathrm{d}t)} = \int_{\mathbb{R}^m \times \mathbb{R}^n} \psi_1 \overline{\psi_2}\, \mathrm{d}s\,\mathrm{d}t.$$

The corresponding quadratic form is denoted by $\mathfrak{Q}_h$. We would like to describe the lowest eigenpairs (eigenvalues and eigenfunctions) of this operator in the limit $h \to 0$ under elementary confining assumptions.

1.1.1 The Born-Oppenheimer strategy

The problem of considering partial semiclassical problems appears for instance in the context of [34, 32] where the main issue is to approximate the eigenpairs of operators with electrical potentials in the form:

$$-h^2\Delta_s - \Delta_t + V(s,t). \tag{1.2}$$

The main idea, due to Born and Oppenheimer in [11], is to replace, for fixed s, the operator $-\Delta_t + V(s,t)$ by its eigenvalues $\mu_k(s)$. Then we are led to consider for instance the reduced operator (called Born-Oppenheimer approximation)

$$-h^2\Delta_s + \mu_1(s),$$

and to apply the semiclassical techniques *à la* Helffer-Sjöstrand [28, 29] to analyze in particular the tunnel effect when the potential μ_1 admits symmetries. The main point is to make the reduction of dimension rigorous. Note that we have always the following lower bound

$$-h^2\Delta_s - \Delta_t + V(s,t) \geq -h^2\Delta_s + \mu_1(s), \tag{1.3}$$

which involves accurate estimates of Agmon with respect to s.

Our paper aims at understanding the analogy between magnetic case (1.1) and electric case (1.2). In particular even the formal dimensional reduction seems to be a little more problematic than in the electric case. Let us write the operator valued symbol of $\mathfrak{L}_h$. For $(x,\xi) \in \mathbb{R}^m \times \mathbb{R}^m$, we introduce the electro-magnetic Laplacian acting on $\mathrm{L}^2(\mathbb{R}^n, \mathrm{d}t)$:

$$\mathcal{M}_{x,\xi} = (D_t + A_2(x,t))^2 + (\xi + A_1(x,t))^2.$$

Denoting by $\mu_1(x,\xi)$ its lowest eigenvalue we would like to replace $\mathfrak{L}_h$ by the m-dimensional pseudo-differential operator:

$$\mu_1(s, hD_s).$$

Such reductions are considered in [36, Theorem 2.1 and remark thereafter] where it is proved that the spectrum of $\mathfrak{L}_h$ is completely determined by an effective Hamiltonian (a matrix of pseudo-differential operators) whose principal symbol can be described thanks to the spectral invariants of the operator valued symbol of $\mathfrak{L}_h$. For the present situation the low lying spectrum of $\mathfrak{L}_h$ could be described by the one of $\mu_1(s, hD_s)$ modulo $O(h)$ and we will see that, under generic assumptions, $O(h)$ is precisely the order of the spectral gap between the first eigenvalues in the simple well case.

1.1.2 Multiple scales induced by the fully semiclassical magnetic Laplacian

Another important motivation to analyze partially semiclassical problems with magnetic fields comes in fact from the fully semiclassical case (*i.e.* when the parameter h lies in front of all derivatives). Let us now explain in which sense. The study of the discrete spectrum magnetic Laplacian $(-ih\nabla + \mathbf{A})^2$ has given rise to many contributions in the last twenty years, especially in the semiclassical limit. To have an overview on the subject one may refer to the book by Fournais and Helffer [19], the survey by Helffer and Kordyukov [24] and the lecture notes by Raymond [47]. Many papers are concerned with finding approximations of the first eigenmodes. Such approximations are difficult to obtain due to the geometry of a possible boundary (carrying in general a Neumann type condition) and to the eventual variations of the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. In dimension two the case of the disk is investigated in [3, 5, 4, 14] and generalized to smooth domains in [26] where it is proved that

$$\lambda_1(h) = \Theta_0 h - C_1 \kappa_{max} h^{3/2} + o(h^{3/2}), \tag{1.4}$$

where κ_{max} is the maximal curvature of the boundary and where $\Theta_0 > 0$ and $C_1 > 0$ will be defined later. An important point to notice is that, in the above mentioned papers, nothing is told about the simplicity of the first eigenvalue or about the approximation of the eigenfunctions. A reason for this is that the spirit of the analysis is essentially variational: it is based on the construction of appropriate test functions for the first Rayleigh quotient so that, even if the simplicity of the eigenfunctions were known, nothing could be deduced for the approximation of the eigenfunctions.

The paper [18] is the first one to establish, in a smooth case and under non-degeneracy assumptions, the approximation of the eigenfunctions and the simplicity of the lowest eigenvalues. The crucial idea to get such results is to understand a double scale structure due to the inhomogeneity of the pure magnetic Laplacian, which is specific to problems with smooth boundaries or without boundary, and to apply the spectral mapping theorem. In such situations it appears that the microlocalization (on possibly different scales) of the eigenfunctions plays an important role in the determination of the spectral asymptotics. In particular the papers [45, 16, 43], which are concerned with varying magnetic fields, establish full asymptotic expansions of the low lying eigenvalues and eigenfunctions by the reduction to the (electric) Born-Oppenheimer approximation which naturally involves different scales. The analysis of [16], related to vanishing magnetic fields, is motivated by the papers [38, 25, 21, 23] and solves one of their conjectures on the asymptotic simplicity of the eigenvalues. This paper will provide simple examples suggested by all the above mentioned models. Moreover, as we will see, the proof of the simplicity of the eigenvalues as well as the expansions of the eigenfunctions strongly relies on microlocal considerations in the spirit of hypoellipticity. We will see that an analysis *à la Born-Oppenheimer* will allow us to deal with all the above mentioned situations.

1.1.3 Magnetic WKB expansions and estimates of Agmon

In all the papers about asymptotic expansions of the magnetic eigenfunctions, one of the methods consists in using a formal power series expansion. It turns out that these constructions are never in the famous WKB form, but in a weaker and somehow more flexible one. When there is an additional electric potential, the WKB expansions are possible as we can see in [30] and [37]. The reason for which we would like to have a WKB description of the eigenfunctions is to get a precise estimate of the magnetic tunnel effect in the case of symmetries. Until now, such estimates are only investigated in two dimensional corner domains in [7] and [8] for the numerical counterpart. It turns out that the crucial point to get an accurate estimate of the exponentially small splitting of the eigenvalues is to establish exponential decay estimates of Agmon type. These localization estimates are rather easy to obtain (at least to get the good scale in the exponential decay) in the corner cases due to the fact that the operator is "more elliptic" than in the regular case in the following sense: the spectral asymptotics is completely drifted by the principal symbol. Nevertheless, let us notice here that, on the one hand, the numerics suggests that the eigenvalues do not seem to be simple (see for instance the case of the square in [8]) and, on the other hand, that establishing the optimal estimates of Agmon is still an open problem. In smooth cases, due to a lack of ellipticity and to the multiple scales, the localization estimates obtained in the literature are in general not optimal or rely on the presence of an electric potential (see [39, 40]): the principal symbol provides only a partial confinement whereas the precise localization of the eigenfunctions seem to be determined by the subprincipal terms. As far as we know, the present paper provides the first examples of WKB expansions in pure magnetic situations as well as quasi-optimal – optimal in terms of power of h but with no exhibited distance of Agmon – estimates of Agmon in model situations. In particular, we prove for a wide range of situations analyzed in the past decades that the magnetic eigenfunctions are in the WKB form under generic assumptions. This paper can be considered as the first necessary step (WKB expansions and rather accurate Agmon estimates) towards the complete comprehension of the magnetic tunnel effect.

1.2 Main results and strategy of the proofs

1.2.1 Spectrum of the simple magnetic wells

In the simple well situation, we will work under the following assumptions. The first assumption states that the lowest eigenvalue of the operator symbol of $\mathfrak{L}_h$ admits a unique and non degenerate minimum and the second one concerns the simplicity of the spectrum of the effective harmonic oscillator.

Assumption 1.1 *We assume that, for all $(x,\xi) \in \mathbb{R}^m \times \mathbb{R}^m$, the bottom of the spectrum of $\mathcal{M}_{x,\xi}$ is a simple eigenvalue denoted by $\mu_1(x,\xi)$ and that $(x,\xi) \mapsto \mu_1(x,\xi)$ is analytic and associated with a L^2-normalized eigenfunction $u_{x,\xi} \in \mathcal{S}(\mathbb{R}^n)$ which also analytically depends on (x,ξ). Moreover we assume that μ_1 admits a unique and non degenerate minimum at point denoted by (x_0,ξ_0). We let $\mu_0 = \mu_1(x_0,\xi_0)$.*

Remark 1.2 *In a neighborhood of (x_0,ξ_0), we still denote by $u_{x,\xi}$ and $\mu_1(x,\xi)$ the analytic continuations of u and μ_1 and we have locally:*

$$\int_{\mathbb{R}^n} u_{x,\xi}\overline{u_{\overline{x},\overline{\xi}}}\,\mathrm{d}t = 1. \tag{1.5}$$

Note that the analytic continuation of u is not always L^2-normalized.

Assumption 1.3 *Under Assumption 1.1, let us denote by* $\mathsf{Hess}\ \mu_1(x_0,\xi_0)$ *the Hessian matrix of* μ_1 *at* (x_0,ξ_0)*. We assume that the spectrum of the operator* $\mathsf{Hess}\ \mu_1(x_0,\xi_0)(\sigma,D_\sigma)$ *is simple.*

Remark 1.4 *Assumption 1.3 is automatically satisfied when* $n=1$*.*

The last assumption is a spectral confinement.

Assumption 1.5 *For* $R\geq 0$*, we let* $\Omega_R=\mathbb{R}^{m+n}\setminus\overline{\mathcal{B}(0,R)}$*. We denote by* $\mathfrak{L}_h^{\mathsf{Dir},\Omega_R}$ *the Dirichlet realization on* Ω_R *of* $(-i\nabla_t+A_2(s,t))^2+(-ih\nabla_s+A_1(s,t))^2$*. We assume that there exist* $R_0\geq 0$*,* $h_0>0$ *and* $\mu_0^*>\mu_0$ *such that for all* $h\in(0,h_0)$*, the first eigenvalue of* $\mathfrak{L}_h^{\mathsf{Dir},\Omega_{R_0}}$ *satisfies:*

$$\lambda_1^{\mathsf{Dir},\Omega_{R_0}}(h)\geq\mu_0^*.$$

Remark 1.6 *In particular, due to the monotonicity of the Dirichlet realization with respect to the domain, Assumption 1.5 implies that there exist* $R_0>0$ *and* $h_0>0$ *such that for all* $R\geq R_0$ *and* $h\in(0,h_0)$*:*

$$\lambda_1^{\mathsf{Dir},\Omega_R}(h)\geq\lambda_1^{\mathsf{Dir},\Omega_{R_0}}(h)\geq\mu_0^*.$$

By using the Persson's theorem (see [41]), we have the following proposition.

Proposition 1.7 *Under Assumption 1.5, there exists* $h_0>0$ *such that for all* $h\in(0,h_J)$*:*

$$\inf\mathsf{sp}_{\mathsf{ess}}(\mathfrak{L}_h)\geq\mu_0^*.$$

Theorem 1.8 *Let us assume Assumptions 1.1, 1.3 and 1.5. For all* $n\geq 1$*, there exists* $h_0>0$ *such that for all* $h\in(0,h_0)$ *the* n*-th eigenvalue of* $\mathfrak{L}_h$ *exists and satisfies*

$$\lambda_n(h)=\lambda_{n,0}+\lambda_{n,1}h+o(h),$$

where $\lambda_{n,0}=\mu_0$ *and* $\lambda_{n,1}$ *is the* n*-th eigenvalue of* $\frac{1}{2}\mathsf{Hess}\ \mu_1(x_0,\xi_0)(\sigma,D_\sigma)$*.*

Remark 1.9 *In fact using the double scale construction developed in the proof of the previous theorem, it is possible to get a complete asymptotic expansion of the following type*

$$\lambda_n(h)\underset{h\to 0}{\sim}\sum_{j\geq 0}\mathfrak{l}_{n,j}h^{j/2},$$

where $\mathfrak{l}_{n,0}=\mu_0$*,* $\mathfrak{l}_{n,1}=0$ *and* $\mathfrak{l}_{n,2}=\lambda_{n,1}$*. This double scale construction can be continued at any order by using the same kind of double scale procedure which can be found in [45, 44, 16] (see also [18] more in the Grushin spirit or [9] in an electric case). Anyway as we will see in Theorem 1.10 and thanks to a WKB construction, we can get a complete asymptotic expansion of* $\lambda_n(h)$ *in powers of* h*.*

Strategy of the proof of Theorem 1.8. The proof of Theorem 1.8 is divided into two main steps. The first step is to construct quasimodes as formal series expansions and to apply the spectral theorem. In order to succeed we will establish Feynman-Hellmann formulas with multiple parameters which are consequences of the perturbation theory of Kato. The second step which is slightly more difficult is to get an equivalent of the spectral splitting between the eigenvalues. For that purpose, we will follow the strategy of [46] by using a partial coherent states decomposition with respect to the semiclassical variables s and use it to establish polynomial estimates (in the spirit of [44] and also [22]) in the phase space satisfied by the eigenfunctions. Then the Feshbach-Grushin type reduction is used to rigorously reduce the dimension and get the spectral splitting. Proof of Theorem 1.8 is the aim of Section 2.

1.2.2 Magnetic WKB expansions: simple well case

We provide now WKB expansions of the lowest eigenpairs in a pure magnetic case. We reduce here our study to the case when $A_2 = 0$ for reasons motivated in Remark 1.12. We therefore focus now on operators of the form

$$\mathfrak{L}_h = D_t^2 + (hD_s + A_1(s,t))^2. \tag{1.6}$$

Let us state one of the most important results of this paper.

Theorem 1.10 *We assume $A_2 = 0$. Under Assumptions 1.1, 1.3 and 1.5, there exist a function $\Phi = \Phi(s)$ defined in a neighborhood $\mathcal{V}$ of x_0 with $\mathrm{Re}\,\mathsf{Hess}\,\Phi(x_0) > 0$ and, for any $n \geq 1$, a sequence of real numbers $(\lambda_{n,j})_{j\geq 0}$ such that*

$$\lambda_n(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \lambda_{n,j} h^j,$$

in the sense of formal series, with $\lambda_{n,0} = \mu_0$. Besides there exists a formal series of smooth functions on $\mathcal{V} \times \mathbb{R}^n_t$

$$a_n(.;h) \underset{h\to 0}{\sim} \sum_{j\geq 0} a_{n,j} h^j$$

with $a_{n,0} \neq 0$ such that

$$(\mathfrak{L}_h - \lambda_n(h)) \left(a_n(.;h) \mathrm{e}^{-\Phi/h} \right) = \mathcal{O}\left(h^\infty\right) \mathrm{e}^{-\Phi/h}.$$

Furthermore the functions $t \mapsto a_{n,j}(s,t)$ belong to the Schwartz class uniformly in $s \in \mathcal{V}$. In addition, there exists $c_0 > 0$ such that for all $h \in (0, h_0)$

$$\mathcal{B}\Big(\lambda_{n,0} + \lambda_{n,1} h, c_0 h\Big) \cap \mathsf{sp}\left(\mathfrak{L}_h\right) = \{\lambda_n(h)\},$$

and $\lambda_n(h)$ is a simple eigenvalue.

In the previous theorem we used the following definition of formal series of functions.

Notation 1.11 *Let $n \geq 1$. We write $a_n(s,t;h) \underset{h\to 0}{\sim} \sum_{j\geq 0} a_{n,j}(s,t) h^j$ when for all $J \geq 0$ and $\alpha \in \mathbb{N}^{n+m}$, there exist $h_{J,\alpha} > 0$ and $C_{J,\alpha} > 0$ such that for all $h \in (0, h_{J,\alpha})$, we have*

$$\left| \partial^\alpha \Big(a_n(s,t;h) - \sum_{j=0}^{J} a_{n,j}(s,t) h^j \Big) \right| \leq C_{J,\alpha} h^{J+1} \quad \text{locally in } (s,t) \in \mathcal{V} \times \mathbb{R}^n_t.$$

We also write $a = \mathcal{O}(h^\infty)$ when $a \sim 0$. The case of formal series of numbers is similar.

Let us also recall that for any arbitrary sequence of smooth functions a_j one can always find, by a procedure of Borel type, a unique smooth function $a(s,t;h)$ (called a realization) (up to $\mathcal{O}(h^\infty)$) such that $a(s,t;h) \underset{h\to 0}{\sim} \sum_{j\geq 0} a_j(s,t) h^j$ (see e.g. [35]). We call this function a a realization.

Remark 1.12 *When A_2 is not zero, it appears that the dimensional reduction is prevented by the oscillations of the eigenfunctions of the model operator $\mathcal{M}_{x,\xi}$. The problem already appears in the case $n = 1$: we can gauge out A_2 at the price to replace A_1 by $A_1 + h\nabla_s \varphi(s,t)$ which is h dependent. As a consequence of our analysis, we can check that the spectrum associated with the potential $(A_1 + h\nabla_s\varphi, 0)$ is shifted by a factor $\mathcal{O}(h)$ compared to the one associated with $(A_1, 0)$. When $n = 1$ we can even prove with our method (and a change of gauge) that the phase Φ in the WKB expansion is (s,t)-dependent.*

Strategy of the proof of Theorem 1.10. Thanks to Theorem 1.8, we have sharp asymptotic expansions of the eigenvalues. In particular, one knows that they become simple in the semiclassical limit. Therefore, to get the (WKB) approximation of the corresponding eigenfunctions, we have just to use an appropriate Ansatz for our quasimodes and to apply the spectral theorem. The new Ansatz considered here will be given by a partial WKB expansion with respect to the variable s. Under some analyticity assumptions, the effective eikonal equation will be solved thanks to the classical stable manifold theorem and analytic extensions of the eigenpairs of some "model" operators. The corresponding effective transport equation will be obtained as the Fredholm condition of an operator valued transport equation jointly with the Feynman-Hellmann formulas. Theorem 1.10 will be proved in Section 3.

1.2.3 Generalized Montgomery operators: towards the magnetic tunnel effect

Let us introduce a family of magnetic Laplacians in dimension two which is related to [27] and the more recent result by Fournais and Persson [20]. For $k \in \mathbb{N} \setminus \{0\}$, we consider the operator on $\mathrm{L}^2(\mathbb{R}^2, \mathsf{ds}\,\mathsf{dt})$:

$$\mathcal{L}_\hbar^{[k],\mathsf{gM}} = \hbar^2 D_{\mathsf{t}}^2 + \left(\hbar D_{\mathsf{s}} - \gamma(\mathsf{s})\frac{\mathsf{t}^{k+1}}{k+1}\right)^2,$$

where γ does not vanish. We call $\lambda_{n,\hbar}^{[k],\mathsf{gM}}$ the n-th eigenvalue (if exists) of this operator. In order to stick to the previous analysis, we start by the following naive but fundamental rescaling

$$\mathsf{s} = s, \qquad \mathsf{t} = \hbar^{\frac{1}{k+2}} t. \tag{1.7}$$

The operator becomes

$$\hbar^{\frac{2k+2}{k+2}}\left(D_t^2 + \left(\hbar^{\frac{1}{k+2}} D_s - \gamma(s)\frac{t^{k+1}}{k+1}\right)^2\right).$$

The investigation is then reduced to the one of

$$\mathfrak{L}_h^{[k]} = D_t^2 + \left(hD_s - \gamma(s)\frac{t^{k+1}}{k+1}\right)^2, \tag{1.8}$$

with $h = \hbar^{\frac{1}{k+2}}$. Let us now introduce one of the generic assumptions, corresponding to the simple well case, under which we will work.

Assumption 1.13 (simple well) *The function γ admits a unique minimum $\gamma_0 > 0$ at $s_0 = 0$ which is non degenerate and $\lim_{s\to\pm\infty} \gamma(s) = \gamma_{\pm\infty}$ with $\gamma_0 < \gamma_{\pm\infty} < +\infty$.*

Under this assumption, the operator $\mathfrak{L}_h^{[k]}$ is a particular case of the previous theory. We will see in Section 4.1 (Proposition 4.1) that operator $\mathfrak{L}_h^{[k]}$ satisfies Assumptions 1.1, 1.3 and 1.5. As a consequence, we could directly apply Theorem 1.10. But, at least in the case when the amplitude of the function γ is small enough, we can prove that the first eigenfunctions are *globally* in the WKB form:

Proposition 1.14 *Suppose Assumption 1.13. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, if $\left|1 - \frac{\gamma_0}{\gamma}\right| \leq \varepsilon$, we can take $x_0 = s_0$, $\mathcal{V} = \mathbb{R}$ in Theorem 1.10.*

As a direct reformulation and using the rescaling (1.7), we get the following result in the original variables

Corollary 1.15 *The n-th eigenvalue of $\mathfrak{L}_\hbar^{[k],\mathsf{gM}}$ and the corresponding WKB solution are given on $\mathbb{R}^2_{\mathsf{s},\mathsf{t}}$ by*

$$\lambda_{n,\hbar}^{[k],\mathsf{gM}} = \hbar^{\frac{2k+2}{k+2}}\lambda_n(\hbar^{\frac{j}{k+2}}) \sim \hbar^{\frac{2k+2}{k+2}}\sum_{j\geq 0}\lambda_{n,j}\hbar^{\frac{j}{k+2}},$$

and

$$u_{n,\hbar}^{[k],\mathsf{gM}}(\mathsf{s},\mathsf{t}) \sim a_n(\mathsf{s},\hbar^{-\frac{1}{k+2}}\mathsf{t};\hbar^{\frac{1}{k+2}})\mathrm{e}^{-\Phi(\mathsf{s})/\hbar^{\frac{1}{k+2}}}$$

where a_n and λ_n are given by Theorem 1.10.

In the perspective of the analysis of the magnetic tunnelling, we will now suppose that γ, instead of having a unique non degenerate minimum, satisfies the following assumption of double well type:

Assumption 1.16 (double well) *The function γ is even and has two non degenerate minima at $s_- < 0$ and $s_+ = -s_- > 0$.*

Proposition 1.17 *We assume Assumption 1.16. Let us fix $\delta \in (0, s_+)$ and let*

$$\mathsf{z}(s) = \chi_{\delta,-}(s)\left|\int_{s_-}^{s}\tilde{\chi}(s')\sqrt{\gamma(s')^{\frac{2}{k+2}}-\gamma_0^{\frac{2}{k+2}}}\,\mathrm{d}s'\right| + \chi_{\delta,+}(s)\left|\int_{s_+}^{s}\tilde{\chi}(s')\sqrt{\gamma(s')^{\frac{2}{k+2}}-\gamma_0^{\frac{2}{k+2}}}\,\mathrm{d}s'\right|,$$

where $0\leq\tilde{\chi}\leq 1$ is a smooth cutoff function whose support contains s_- and s_+ and where $\chi_{\delta,-}$ and $\chi_{\delta,+}$ are smooth cutoff functions such that

$$\chi_{\delta,-}(s) = \begin{cases} 1 & \text{for } s\leq\frac{\delta}{2},\\ 0 & \text{for } s\geq\delta,\end{cases} \qquad \text{and} \qquad \chi_{\delta,+}(s) = \begin{cases} 1 & \text{for } s\geq-\frac{\delta}{2},\\ 0 & \text{for } s\leq-\delta.\end{cases}$$

Let us consider $C_0 > 0$. There exist $\varepsilon_0 > 0$, $C > 0$ and $h_0 > 0$ such that for all eigenpairs (λ,ψ) of $\mathfrak{L}_h^{[k]}$ satisfying $\lambda\leq\mu_0 + C_0 h$ we have, for all $h\in(0,h_0)$,

$$\|\mathrm{e}^{\varepsilon_0\mathsf{z}/h}\psi\| \leq C\|\psi\| \qquad \text{and} \qquad \mathfrak{Q}_h^{[k]}(\mathrm{e}^{\varepsilon_0\mathsf{z}/h}\psi)\leq C\|\psi\|^2.$$

We can now deal with the tunnel effect for the rescaled Montgomery models. For that purpose, let us fix $\delta\in(0,s_+)$ and define the two symmetric model wells. We assume that γ satisfies Assumption 1.16. We consider the Dirichlet realizations on $\mathrm{L}^2((-\infty,s_-+\delta)\times\mathbb{R})$ and $\mathrm{L}^2((s_+-\delta)\times\mathbb{R},+\infty))$ of $D_t^2+\left(hD_s-\gamma(s)\frac{t^{k+1}}{k+1}\right)^2$ respectively denoted by $\mathfrak{H}_{h,-}^{\mathsf{Dir}}$ and $\mathfrak{H}_{h,+}^{\mathsf{Dir}}$. These operators are isospectral by symmetry. We want to compare the spectrum of $\mathfrak{L}_h^{[k]}$ with the one of the direct sum $\mathfrak{H}_h = \mathfrak{H}_{h,-}^{\mathsf{Dir}}\oplus\mathfrak{H}_{h,+}^{\mathsf{Dir}}$.

Theorem 1.18 *Let us consider $C_0 > 0$. There exist $c > 0$, $C > 0$, $h_0 > 0$ such that for all $\mu\in\mathsf{sp}\,(\mathfrak{H}_h)$ and $\lambda\in\mathsf{sp}\left(\mathfrak{L}_h^{[k]}\right)$ such that $\mu,\lambda\leq\mu_0+C_0h$, we have, for all $h\in(0,h_0)$,*

$$\mathsf{range}\left(\mathbb{1}_{[\mu-C\mathrm{e}^{-c/h},\mu+C\mathrm{e}^{-c/h}]}(\mathfrak{L}_h^{[k]})\right) = 2$$

and

$$\mathsf{dist}\,(\lambda,\mathsf{sp}\,(\mathfrak{H}_h))\leq C\mathrm{e}^{-c/h}.$$

The following corollary is a direct consequence of the previous theorem

Corollary 1.19 *In terms of operator $\mathcal{L}_\hbar^{[k],\mathsf{gM}}$ the gap between pairs of eigenvalues is given by*

$$\lambda_{2n,\hbar}^{[k],\mathsf{gM}} - \lambda_{2n-1,\hbar}^{[k],\mathsf{gM}} = \mathcal{O}(\mathrm{e}^{-c/\hbar^{\frac{1}{k+2}}}).$$

for h small enough depending on n.

Strategy of the analysis of $\mathfrak{L}_\hbar^{[k]}$ In the simple well case the study of $\mathfrak{L}_\hbar^{[k]}$ follows the same lines as for Theorem 1.10 jointly with a normal form argument inspired by [45, 16, 48] which permits simultaneously to make the WKB expansions global and to get quasi-optimal estimates of Agmon. Concerning the double case, in order to get the tunneling effect, we can follow the classical procedure based on the spectral theorem and the previous estimates of Agmon.

1.3 WKB constructions: influence of the geometry

This section is devoted to the fully semiclassical magnetic Laplacian $(\hbar D_{\mathsf{x}}+\mathbf{A})^2$ on $\mathrm{L}^2(\mathbb{R}^2)$. We now investigate three kinds of models for which the geometry is more intricate and for which our theorems do not directly apply. Nevertheless, our WKB strategy is robust enough and still effective: we are able to exhibit WKB expansions. Note that all the forthcoming situations have been already studied in the literature but never from the WKB point of view. The geometric perturbations at stake are: vanishing magnetic fields on curves or possibly singular boundary.

1.3.1 Vanishing magnetic fields... or not

We will work under the following assumption on $\mathbf{B}$ as in [21] and [16].

Assumption 1.20 *The zero locus of $\boldsymbol{B}$ is a smooth closed non empty curve Γ:*

$$\Gamma = \{\boldsymbol{B}(\mathsf{x}) = 0\},$$

and $\boldsymbol{B}$ vanishes exactly at the order $k \geq 1$ on Γ. Moreover we assume that the k-th normal derivative of $\boldsymbol{B}$ admits a non degenerate minimum on Γ at x_0.

Remark 1.21 *Here we work in dimension two, but there is no doubt that we could adapt the presentation, modulo a few technicalities, to cover the case of magnetic fields vanishing at a given order on hypersurfaces as in [21].*

In Section 5.1 we will construct, in a neighborhood of x_0 and in normal coordinates, WKB expansions which come within the study of the generalized Montgomery operator $\mathcal{L}_\hbar^{[k],\mathsf{gM}}$. Under the additional assumption that the minimum is uniquely reached at x_0, the splitting between the lowest eigenvalues has been established in [16] (for the case $k = 1$ and the proof is completely similar for $k \geq 2$) so that these WKB expansions are local (in the sense of Theorem 1.10) approximations of the true eigenfunctions.

These considerations can be extended to the case $k = 0$ which can be interpreted as follows. The curve Γ represents the boundary of an enclosed open set Ω carrying a magnetic Neumann condition (with a magnetic field which does not vanish on $\overline{\Omega}$). In other words, we can perform a WKB construction, for the Neumann realization on Ω of

$(\hbar D_{\mathsf{x}} + \mathbf{A})^2$, near each x_0 where $\mathbf{B}_{|\partial\Omega}$ is non degenerately minimal. Assuming moreover that

$$\Theta_0 \min_{\partial\Omega} \mathbf{B} < \min_{\Omega} \mathbf{B},$$

and that $\mathbf{B}_{|\partial\Omega}$ admits a unique and non degenerate minimum, we can get, by using the spectral splitting proved in [45], the local WKB approximation of the lowest eigenfunctions. All the scaling properties, stated in normal coordinates, will be addressed in Section 5.1.

1.3.2 Varying edge

The strategy of this paper can also deal with more singular situations in dimension three. Such a situation is described in the paper [43] where the semiclassical analysis is done when the boundary of the domain contains a varying edge. We propose to perform the WKB constructions for a simplified version of the operator introduced there. We are interested in the operator defined on $\mathrm{L}^2(\mathcal{W}_\alpha, \mathrm{d}s\,\mathrm{d}t\,\mathrm{d}z)$ and with Neumann conditions

$$\mathcal{L}^{\mathsf{e}}_\hbar = \hbar^2 D_t^2 + \hbar^2 D_z^2 + (\hbar D_s - t)^2,$$

where

$$\mathcal{W}_\alpha = \left\{(s,t,z) \in \mathbb{R}^3 : |z| \leq \mathcal{T}(s)t\right\},$$

with $\mathcal{T}(s) = \tan\left(\frac{\alpha(s)}{2}\right)$ and where $\alpha : \mathbb{R} \to \mathbb{R}$ is a smooth function which represents the (varying) opening of the wedge $\mathcal{W}_\alpha$. We will work under the following assumption.

Assumption 1.22 *The function $s \mapsto \alpha(s)$ admits a unique and non degenerate maximum α_0 at $s = 0$.*

In Section 5.2, we will provide local (near the point of the edge giving the maximal aperture) WKB expansions of the lowest eigenfunctions.

1.3.3 Curvature induced magnetic bound states

As we have seen, in many situations the spectral splitting appears in the second term of the asymptotic expansion of the eigenvalues. It turns out that we can also deal with more degenerate situations. The next lines are motivated by the initial paper [26] whose main result is recalled in (1.4). Their fundamental result establishes that a smooth Neumann boundary can trap the lowest eigenfunctions near the points of maximal curvature. These considerations are generalized in [18, Theorem 1.1] where the complete asymptotic expansion of the eigenpairs is proved

$$\lambda^{\mathsf{c}}_{n,\hbar} = \Theta_0\hbar - C_1\kappa_{\max}\hbar^{3/2} + (2n-1)C_1\Theta_0^{1/4}\sqrt{\frac{3k_2}{2}}\hbar^{7/4} + o(\hbar^{7/4}), \tag{1.9}$$

where $k_2 = -\kappa''(0)$. In our paper, as in [18], we will consider the magnetic Neumann Laplacian on a smooth domain Ω such that the algebraic curvature κ satisfies the following assumption.

Assumption 1.23 *The function κ is smooth and admits a unique and non degenerate maximum.*

We will prove that the lowest eigenfunctions are approximated by local WKB expansions which can be made global when for instance $\partial\Omega$ is the graph of a smooth function. In particular we will recover the term $C_1\Theta_0^{1/4}\sqrt{\frac{3k_2}{2}}$ by another method.

1.4 Organization of the paper

The paper is organized as follows. Section 2 is devoted to models with simple magnetic wells and to the proof of Theorem 1.8. Section 3 is concerned with the proof of Theorem 1.10. In Section 4 we establish that the Montgomery operators satisfy the assumption of Theorem 1.10, we prove that the WKB expansions are global (Proposition 1.14) and we give an upper bound of the tunnel effect (proof of Theorem 1.18). These theoretical results are illustrated by numerical simulations. Section 5 deals with the geometrical examples introduced in Section 1.3.

2 Simple magnetic wells

This section is devoted to the proof of Theorem 1.8. In order to perform the investigation we use the following rescaling

$$s = x_0 + h^{1/2}\sigma, \qquad t = \tau, \tag{2.1}$$

and a gauge transform $\mathrm{e}^{i\xi_0\sigma/h^{1/2}}$, so that $\mathfrak{L}_h$ becomes

$$\mathcal{L}_h = (-i\nabla_\tau + A_2(x_0 + h^{1/2}\sigma, \tau))^2 + (\xi_0 - ih^{1/2}\nabla_\sigma + A_1(x_0 + h^{1/2}\sigma, \tau))^2. \tag{2.2}$$

The corresponding quadratic form is denoted by $\mathcal{Q}_h$.

2.1 Formal series and general Feynman-Hellmann formulas

This section is devoted to the proof of the following proposition.

Proposition 2.1 *Let us assume Assumption 1.1. For all $n \geq 1$, there exist $C > 0$ and $h_0 > 0$ such that, for $h \in (0, h_0)$,*

$$\mathsf{dist}\left(\sum_{j=0}^{2} \mathfrak{l}_{n,j} h^{j/2}, \mathsf{sp}(\mathcal{L}_h)\right) \leq Ch^{3/2},$$

where $\mathfrak{l}_{n,0} = \mu_0$, $\mathfrak{l}_{n,1} = 0$ and $\mathfrak{l}_{n,2}$ is the n-th eigenvalue of $\frac{1}{2}\mathsf{Hess}\ \mu_1(x_0, \xi_0)(\sigma, D_\sigma)$.

We will need the so-called Feynman-Hellmann formulas which are obtained by taking the derivative of the eigenvalue equation

$$\mathcal{M}_{x,\xi} u_{x,\xi} = \mu_1(x, \xi) u_{x,\xi},$$

with respect to x_j and ξ_k.

Proposition 2.2 *Let η and θ denote one of the x_j or ξ_k. Then we have*

$$(\mathcal{M}_{x,\xi} - \mu_1(x, \xi))(\partial_\eta u)_{x,\xi} = (\partial_\eta \mu_1(x, \xi) - \partial_\eta \mathcal{M}_{x,\xi}) u_{x,\xi} \tag{2.3}$$

and

$$\begin{aligned}&(\mathcal{M}_{x_0,\xi_0} - \mu_0)(\partial_\eta \partial_\theta u)_{x_0,\xi_0} \\ &= \partial_\eta \partial_\theta \mu_1(x_0, \xi_0) u_{x_0,\xi_0} - \partial_\eta \mathcal{M}_{x_0,\xi_0}(\partial_\theta u)_{x_0,\xi_0} - \partial_\theta \mathcal{M}_{x_0,\xi_0}(\partial_\eta u)_{x_0,\xi_0} - \partial_\eta \partial_\theta \mathcal{M}_{x_0,\xi_0} u_{x_0,\xi_0},\end{aligned} \tag{2.4}$$

where $(\partial_\eta u)_{x_0,\xi_0}$ *denotes* $(\partial_\eta u_{x,\xi})_{|(x,\xi)=(x_0,\xi_0)}$ *and similarly for the other derivatives of* $u_{x,\xi}$. *In a neighborhood of* (x_0,ξ_0) *in* $\mathbb{R}^m \times \mathbb{C}^m$, *we have,*

$$\partial_\eta \mu_1(x,\xi) = \int_{\mathbb{R}^n} \partial_\eta \mathcal{M}_{x,\xi}\, u_{x,\xi}(\tau)\overline{u_{x,\bar{\xi}}(\tau)}\,\mathrm{d}\tau. \tag{2.5}$$

Proof of Proposition 2.1: Since A_1 and A_2 are polynomial, we can write, for some $M \in \mathbb{N}$,

$$\mathcal{L}_h = \sum_{j=0}^{M} h^{j/2}\mathcal{L}_j,$$

with

$$\mathcal{L}_0 = \mathcal{M}_{x_0,\xi_0}, \qquad \mathcal{L}_1 = \sum_{k=1}^{m}(\partial_{x_k}\mathcal{M})_{x_0,\xi_0}\sigma_k + \sum_{k=1}^{m}(\partial_{\xi_k}\mathcal{M})_{x_0,\xi_0}D_{\sigma_k},$$

$$\mathcal{L}_2 = \frac{1}{2}\sum_{k,j=1}^{m}\Big((\partial_{x_j}\partial_{x_k}\mathcal{M})_{x_0,\xi_0}\sigma_j\sigma_k + (\partial_{\xi_j}\partial_{\xi_k}\mathcal{M})_{x_0,\xi_0}D_{\sigma_j}D_{\sigma_k} + (\partial_{\xi_j}\partial_{x_k}\mathcal{M})_{x_0,\xi_0}D_{\sigma_j}\sigma_k$$
$$+(\partial_{x_k}\partial_{\xi_j}\mathcal{M})_{x_0,\xi_0}\sigma_k D_{\sigma_j}\Big).$$

We look for quasimodes in the form

$$\psi = \sum_{j=0}^{2} h^{j/2}\psi_j \qquad \text{and} \qquad \mathfrak{l} = \sum_{j=0}^{2} h^{j/2}\mathfrak{l}_j,$$

so that they solve in the sense of formal series

$$\mathcal{L}_h\psi = \mathfrak{l}\psi + \mathcal{O}(h^{3/2}).$$

Let us now deal with each power of h.

Terms of order h^0. By collecting the terms of order h^0, we get the equation

$$\mathcal{M}_{x_0,\xi_0}\psi_0 = \mathfrak{l}_0\psi_0.$$

This leads to take

$$\mathfrak{l}_0 = \mu_0 \qquad \text{and} \qquad \psi_0(\sigma,\tau) = f_0(\sigma)u_0(\tau),$$

where $u_0 = u_{x_0,\xi_0}$ and f_0 is a function to be determined in the Schwartz class.

Terms of order $h^{1/2}$. By collecting the terms of order $h^{1/2}$, we find

$$(\mathcal{M}_{x_0,\xi_0} - \mu_1(x_0,\xi_0))\psi_1 = (\mathfrak{l}_1 - \mathcal{L}_1)\psi_0.$$

By using (2.3) and the Fredholm alternative (applied for σ fixed) we get $\mathfrak{l}_1 = 0$ and

$$\psi_1(\sigma,\tau) = \sum_{k=1}^{m}(\partial_{x_k}u)_{x_0,\xi_0}(\tau)\,\sigma_k f_0(\sigma) + \sum_{k=1}^{m}(\partial_{\xi_k}u)_{x_0,\xi_0}(\tau)\,D_{\sigma_k}f_0(\sigma) + f_1(\sigma)u_0(\tau), \tag{2.6}$$

where f_1 is a function to be determined in the Schwartz class.

Terms of order h. The equation reads

$$(\mathcal{M}_{x_0,\xi_0} - \mu_1(x_0,\xi_0))\psi_2 = (\mathfrak{l}_2 - \mathcal{L}_2)\psi_0 - \mathcal{L}_1\psi_1.$$

The Fredholm condition gives

$$\langle \mathcal{L}_2\psi_0 + \mathcal{L}_1\psi_1, u_0\rangle_{\mathrm{L}^2(\mathbb{R}^n,\mathrm{d}\tau)} = \mathfrak{l}_2 f_0. \tag{2.7}$$

Let us examine each term which appears when computing the l.h.s. and recall that Proposition 2.2 holds (especially the Fredholm condition of (2.4)).
The coefficient in front of $\sigma_j\sigma_k f_0$ is

$$\langle(\partial_{x_j}\mathcal{M})_{x_0,\xi_0}(\partial_{x_k}u)_{x_0,\xi_0}, u_0\rangle_{\mathrm{L}^2(\mathbb{R}^n,\mathrm{d}\tau)} + \langle(\partial_{x_k}\mathcal{M})_{x_0,\xi_0}(\partial_{x_j}u)_{x_0,\xi_0}, u_0\rangle_{\mathrm{L}^2(\mathbb{R}^n,\mathrm{d}\tau)} + \langle(\partial_{x_j}\partial_{x_k}\mathcal{M})_{x_0,\xi_0}u_0, u_0\rangle_{\mathrm{L}^2(\mathbb{R}^n,\mathrm{d}\tau)} = \partial_{x_j}\partial_{x_k}\mu_1(x_0,\xi_0).$$

The coefficient in front of $D_{\sigma_j}D_{\sigma_k}$ is

$$\langle(\partial_{\xi_j}\mathcal{M})_{x_0,\xi_0}(\partial_{\xi_k}u)_{x_0,\xi_0}, u_0\rangle_{\mathrm{L}^2(\mathbb{R}^n,\mathrm{d}\tau)} + \langle(\partial_{\xi_k}\mathcal{M})_{x_0,\xi_0}(\partial_{\xi_j}u)_{x_0,\xi_0}, u_0\rangle_{\mathrm{L}^2(\mathbb{R}^n,\mathrm{d}\tau)} + \langle(\partial_{\xi_j}\partial_{\xi_k}\mathcal{M})_{x_0,\xi_0}u_0, u_0\rangle_{\mathrm{L}^2(\mathbb{R}^n,\mathrm{d}\tau)} = \partial_{\xi_j}\partial_{\xi_k}\mu_1(x_0,\xi_0).$$

To deal with the coefficient in front of $D_{\sigma_j}\sigma_k + \sigma_k D_{\sigma_j}$, we use the formula

$$\begin{aligned}\langle(\partial_{\xi_j}\mathcal{M})_{x_0,\xi_0}(\partial_{x_k}u)_{x_0,\xi_0}, u_0\rangle_{\mathrm{L}^2(\mathbb{R}^n,\mathrm{d}\tau)} &+ \langle(\partial_{x_k}\mathcal{M})_{x_0,\xi_0}(\partial_{\xi_j}u)_{x_0,\xi_0}, u_0\rangle_{\mathrm{L}^2(\mathbb{R}^n,\mathrm{d}\tau)}\\ + \langle(\partial_{\xi_j}\partial_{x_k}\mathcal{M})_{x_0,\xi_0}u_0, u_0\rangle_{\mathrm{L}^2(\mathbb{R}^n,\mathrm{d}\tau)} &+ \langle(\partial_{\xi_k}\mathcal{M})_{x_0,\xi_0}(\partial_{x_j}u)_{x_0,\xi_0}, u_0\rangle_{\mathrm{L}^2(\mathbb{R}^n,\mathrm{d}\tau)}\\ + \langle(\partial_{x_j}\mathcal{M})_{x_0,\xi_0}(\partial_{\xi_k}u)_{x_0,\xi_0}, u_0\rangle_{\mathrm{L}^2(\mathbb{R}^n,\mathrm{d}\tau)} &+ \langle(\partial_{\xi_k}\partial_{x_j}\mathcal{M})_{x_0,\xi_0}u_0, u_0\rangle_{\mathrm{L}^2(\mathbb{R}^n,\mathrm{d}\tau)}\\ &= \partial_{x_j}\partial_{\xi_k}\mu_1(x_0,\xi_0) + \partial_{x_k}\partial_{\xi_j}\mu_1(x_0,\xi_0).\end{aligned}$$

Therefore the Fredholm condition (2.7) becomes

$$\frac{1}{2}\sum_{j,k=1}^{m}\left(\partial_{x_j}\partial_{x_k}\mu_1\sigma_j\sigma_k + \partial_{\xi_j}\partial_{\xi_k}\mu_1 D_{\sigma_j}D_{\sigma_k} + \partial_{\xi_j}\partial_{x_k}\mu_1 D_{\sigma_j}\sigma_k + \partial_{x_k}\partial_{\xi_j}\mu_1\sigma_k D_{\sigma_j}\right) f_0 = \mathfrak{l}_2 f_0,$$

where for shortness, we write $\partial_\eta\partial_\theta\mu_1 := (\partial_\eta\partial_\theta\mu_1)(x_0,\xi_0)$. In other words, we have

$$\tfrac{1}{2}\mathsf{Hess}\ \mu_1(x_0,\xi_0)(\sigma, D_\sigma)f_0 = \mathfrak{l}_2 f_0.$$

We take $\mathfrak{l}_2 = \mathfrak{l}_{n,2}$ the n-th eigenvalue of $\frac{1}{2}\mathsf{Hess}\ \mu_1(x_0,\xi_0)(\sigma, D_\sigma)$ and we choose f_0 a corresponding normalized eigenfunction. The spectral theorem completes the proof of Proposition 2.1. ∎

Furthers terms As mentionned in the introduction, this double scale construction could be continued at any order by using the same kind of procedure as in [45, 44, 16].
We deduce from Propositions 1.7 and 2.1:

Corollary 2.3 *For all $n \geq 1$ there exist $h_0 > 0$ and $C > 0$ such that for all $h \in (0, h_0)$ the n-th eigenvalue of $\mathfrak{L}_h$ exists and satisfies:*

$$\lambda_n(h) \leq \mu_0 + Ch.$$

2.2 Semiclassical estimates of Agmon-Persson

This section is devoted to the rough localization and microlocalization estimates satisfied by the eigenfunctions and resulting from Assumptions 1.1 and 1.5 and Corollary 2.3.

Proposition 2.4 *Let $C_0 > 0$. There exist $h_0, C, \varepsilon_0 > 0$ such that for all eigenpairs (λ, ψ) of $\mathfrak{L}_h$ with $\lambda \leq \mu_0 + C_0 h$, we have*

$$\left\| e^{\varepsilon_0 |t|} \psi \right\|^2 \leq C \|\psi\|^2, \qquad \mathfrak{Q}_h \left(e^{\varepsilon_0 |t|} \psi \right) \leq C \|\psi\|^2.$$

Proof: The proof is standard but we recall it for completeness. We consider a smooth cutoff function χ_1 supported in a fixed neighborhood of 0 and, for $\ell \geq 1$, we introduce $\chi_\ell(t) = \chi_1(\ell^{-1} t)$. We let $\Phi_\ell(t) = \varepsilon_0 \chi_\ell(t)|t|$ and we write the Agmon identity (see [1, 2])

$$\mathfrak{Q}_h(e^{\Phi_\ell}\psi) = \lambda \|e^{\Phi_\ell}\psi\| + \| \, |\nabla \Phi_\ell| e^{\Phi_\ell}\psi\|^2.$$

There exists $C > 0$ such that for all $\ell \geq 1$ we have

$$\| \, |\nabla \Phi_\ell| e^{\Phi_\ell}\psi\|^2 \leq C \varepsilon_0^2 \|e^{\Phi_\ell}\psi\|^2.$$

We infer that

$$\mathfrak{Q}_h(e^{\Phi_\ell}\psi) \leq (\mu_0 + C_0 h + C\varepsilon_0^2) \|e^{\Phi_\ell}\psi\|^2.$$

For $R > 0$, we introduce a partition of unity $(\chi_{1,R}, \chi_{2,R})$ in t-variables such that

$$\chi_{1,R}^2(t) + \chi_{2,R}^2(t) = 1, \quad |\nabla \chi_{1,R}|^2 + |\nabla \chi_{2,R}|^2 \leq C R^{-2} \quad \text{and} \quad \mathsf{supp}\, \chi_{2,R} \cap \mathcal{B}(0,R) = \emptyset.$$

With the so-called IMS formula (see [13, Chapter 3]), we deduce

$$\mathfrak{Q}_h(\chi_{1,R} e^{\Phi_\ell}\psi) + \mathcal{Q}_h(\chi_{2,R} e^{\Phi_\ell}\psi) - C R^{-2} \|e^{\Phi_\ell}\psi\|^2 \leq (\mu_0 + C_0 h + C\varepsilon_0^2) \|e^{\Phi_\ell}\psi\|^2.$$

Since e^{Φ_ℓ} is bounded on the support of $\chi_{1,R}$, we get the existence of $C, \tilde{C} > 0$ such that for all $\ell \geq 1$ and $h \in (0,1)$:

$$\mathfrak{Q}_h(\chi_{2,R} e^{\Phi_\ell}\psi) - (\mu_0 + C_0 h + C\varepsilon_0^2 + C R^{-2}) \|\chi_{2,R} e^{\Phi_\ell}\psi\|^2 \leq \tilde{C} \|\psi\|^2.$$

By using Assumption 1.5 and Remark 1.6, there exist $R_0 > 0$ and $h_0 > 0$ such that for all $R \geq R_0$ and $h \in (0, h_0)$ we have

$$\mathfrak{Q}_h(\chi_{2,R} e^{\Phi_\ell}\psi) \geq \mu_0^* \|\chi_{2,R} e^{\Phi_\ell}\psi\|^2.$$

We infer the existence of $c > 0$ such that for $h \in (0, h_0)$

$$c \|\chi_{2,R} e^{\Phi_\ell}\psi\|^2 \leq \tilde{C} \|\psi\|^2.$$

Then there exist $C > 0$ and $h_0 > 0$ such that for all $\ell \geq 1$ and $h \in (0, h_0)$

$$\|e^{\Phi_\ell}\psi\|^2 \leq C \|\psi\|^2.$$

It remains to consider the limit $\ell \to +\infty$ and to use the Fatou lemma and the conclusion follows. ∎

Proposition 2.5 *Let $C_0 > 0$. There exist $h_0, C, \varepsilon_0 > 0$ such that for all eigenpairs (λ, ψ) of $\mathfrak{L}_h$ with $\lambda \leq \mu_0 + C_0 h$, we have*

$$\left\| e^{\varepsilon_0 |s|} \psi \right\|^2 \leq C \|\psi\|^2, \qquad \mathfrak{Q}_h \left(e^{\varepsilon_0 |s|} \psi \right) \leq C \|\psi\|^2.$$

Proof: The proof is the same as that of Proposition 2.4 with $\Phi(s) = \varepsilon_0 \chi_\ell(s)|s|$. ∎

We get the following.

Corollary 2.6 *Let $C_0 > 0$ and $k, l, d \in \mathbb{N}$. There exist $h_0, C, \varepsilon_0 > 0$ such that for all eigenpairs (λ, ψ) of $\mathfrak{L}_h$ with $\lambda \leq \mu_0 + C_0 h$ and all $h \in (0, h_0)$, we have*

$$\|t^k s^l \psi\| \leq C\|\psi\|, \qquad \mathfrak{Q}_h(t^k s^l \psi) \leq C\|\psi\|^2,$$
$$\|(-i\nabla_\tau)^d s^l t^k \psi\| \leq C\|\psi\|^2, \qquad \|(-ih\nabla_s)^d s^l t^k \psi\| \leq C\|\psi\|^2.$$

Proof: For $d = 1$, this is an immediate consequence of Propositions 2.4 and 2.5. Taking successive derivatives of the eigenvalue equation $\mathfrak{L}_h \psi = \lambda \psi$ we deduce the result for $d \geq 2$. ∎

For another purpose, we will need the following localization result which is again a consequence of Propositions 2.4 and 2.5.

Proposition 2.7 *Let $k \in \mathbb{N}$. Let $\eta > 0$ and χ a smooth cutoff function defined on $\mathbb{R}_+$ and being zero in a neighborhood of 0. There exists $h_0 > 0$ such that for all eigenpairs (λ, ψ) of $\mathfrak{L}_h$ with $\lambda \leq \mu_0 + C_0 h$ and all $h \in (0, h_0)$, we have*

$$\|\chi(h^\eta |s|)\psi\|_{B^k(\mathbb{R}^{m+n})} \leq O(h^\infty)\|\psi\|, \qquad \|\chi(h^\eta |t|)\psi\|_{B^k(\mathbb{R}^{m+n})} \leq O(h^\infty)\|\psi\|,$$

where $\|\cdot\|_{B^k(\mathbb{R}^{n+m})}$ is the standard norm on

$$B^k(\mathbb{R}^{m+n}) = \{\psi \in \mathrm{L}^2(\mathbb{R}^{m+n}) | y_j^q \partial_{y_l}^p \psi \in \mathrm{L}^2(\mathbb{R}^{n+m}), \forall j, l \in \{1, \ldots, m+n\},\ p+q \leq k\}.$$

By using a rough pseudo-differential calculus jointly with the space localization of Proposition 2.7 and standard elliptic estimates, we get

Proposition 2.8 *Let $k \in \mathbb{N}$. Let $\eta > 0$ and χ a smooth cutoff function being zero in a neighborhood of 0. There exists $h_0 > 0$ such that for all eigenpairs (λ, ψ) of $\mathfrak{L}_h$ such that $\lambda \leq \mu_0 + C_0 h$, we have*

$$\|\chi(h^\eta h D_s)\psi\|_{B^k(\mathbb{R}^{m+n})} \leq O(h^\infty)\|\psi\|, \qquad \|\chi(h^\eta D_t)\psi\|_{B^k(\mathbb{R}^{m+n})} \leq O(h^\infty)\|\psi\|.$$

2.3 Microlocalization and coherent states

In this section we follow the same philosophy as in [46].

2.3.1 Formalism and application

Let us recall the formalism of coherent states (see for instance [17] and [12]). We define

$$g_0(\sigma) = \pi^{-m/4} \mathrm{e}^{-|\sigma|^2/2},$$

and the usual creation and annihilation operators

$$\mathfrak{a}_j = \tfrac{1}{\sqrt{2}}(\sigma_j + \partial_{\sigma_j}), \qquad \mathfrak{a}_j^* = \tfrac{1}{\sqrt{2}}(\sigma_j - \partial_{\sigma_j}),$$

which satisfy the commutator identities

$$[\mathfrak{a}_j, \mathfrak{a}_j^*] = 1, \qquad [\mathfrak{a}_j, \mathfrak{a}_k^*] = 0 \quad \text{if } k \neq j.$$

We notice that

$$\sigma_j = \tfrac{1}{\sqrt{2}}(\mathfrak{a}_j + \mathfrak{a}_j^*), \qquad \partial_{\sigma_j} = \tfrac{1}{\sqrt{2}}(\mathfrak{a}_j - \mathfrak{a}_j^*), \qquad \mathfrak{a}_j\mathfrak{a}_j^* = \tfrac{1}{2}(D_{\sigma_j}^2 + \sigma_j^2 + 1).$$

For $(u,p) \in \mathbb{R}^m \times \mathbb{R}^m$, we introduce the coherent state

$$f_{u,p}(\sigma) = \mathrm{e}^{ip\cdot\sigma} g_0(\sigma - u),$$

and the associated projection, defined for $\psi \in \mathrm{L}^2(\mathbb{R}^m \times \mathbb{R}^n)$ by

$$\Pi_{u,p}\psi = \langle \psi, f_{u,p}\rangle_{\mathrm{L}^2(\mathbb{R}^m,\,\mathrm{d}\sigma)} f_{u,p} = \psi_{u,p} f_{u,p},$$

which satisfies

$$\psi = \int_{\mathbb{R}^{2m}} \Pi_{u,p}\psi \,\mathrm{d}u\,\mathrm{d}p,$$

and the Parseval formula

$$\|\psi\|^2 = \int_{\mathbb{R}^n}\int_{\mathbb{R}^{2m}} |\psi_{u,p}|^2 \,\mathrm{d}u\,\mathrm{d}p\,\mathrm{d}\tau.$$

We recall that

$$\mathfrak{a}_j f_{u,p} = \frac{u_j + ip_j}{\sqrt{2}} f_{u,p}$$

and

$$(\mathfrak{a}_j)^\ell (\mathfrak{a}_k^*)^q \psi = \int_{\mathbb{R}^{2m}} \left(\frac{u_j + ip_j}{\sqrt{2}}\right)^\ell \left(\frac{u_k - ip_k}{\sqrt{2}}\right)^q \Pi_{u,p}\psi \,\mathrm{d}u\,\mathrm{d}p.$$

We recall that (see (2.2))

$$\mathcal{L}_h = \big(-i\nabla_\tau + A_2(x_0 + h^{1/2}\sigma,\tau)\big)^2 + \big(\xi_0 - ih^{1/2}\nabla_\sigma + A_1(x_0 + h^{1/2}\sigma,\tau)\big)^2$$

and

$$\mathcal{L}_h = \mathcal{L}_0 + h^{1/2}\mathcal{L}_1 + h\mathcal{L}_2 + \ldots + h^{M/2}\mathcal{L}_M.$$

If we write the Wick ordered operator, we get

$$\mathcal{L}_h = \underbrace{\mathcal{L}_0 + h^{1/2}\mathcal{L}_1 + h\mathcal{L}_2^{\mathsf{W}} + \ldots + (h^{1/2})^M \mathcal{L}_M^{\mathsf{W}}}_{\mathcal{L}_h^{\mathsf{W}}} + \underbrace{hR_2 + \ldots + (h^{1/2})^M R_M}_{\mathcal{R}_h}, \tag{2.8}$$

where the R_d are the remainders in the Wick ordering and satisfy, for $d \geq 2$,

$$h^{d/2}R_d = h^{d/2}\mathcal{O}_{d-2}(\sigma, D_\sigma), \tag{2.9}$$

where the notation $\mathcal{O}_d(\sigma, D_\sigma)$ stands for a polynomial operator with total degree in (σ, D_σ) less than d. We recall that

$$\mathcal{L}_h^{\mathsf{W}} = \int_{\mathbb{R}^{2m}} \mathcal{M}_{x_0 + h^{1/2}u, \xi_0 + h^{1/2}p} \,\mathrm{d}u\,\mathrm{d}p.$$

Proposition 2.9 *There exist $h_0, C > 0$ such that for all eigenpairs (λ,ψ) of $\mathcal{L}_h$ such that $\lambda \leq \mu_0 + C_0 h$ and all $h \in (0,h_0)$, we have*

$$\mathcal{Q}_h(\psi) \geq \int_{\mathbb{R}^{2m}} Q_{h,u,p}(\psi_{u,p}) \,\mathrm{d}u\,\mathrm{d}p - Ch\|\psi\|^2 \geq (\mu_0 - Ch)\|\psi\|^2, \tag{2.10}$$

where $Q_{h,u,p}$ is the quadratic form associated with the operator $\mathcal{M}_{x_0 + h^{1/2}u, \xi_0 + h^{1/2}p}$.

Proof: We use (2.8). Then the terms of $\mathcal{R}_h$ (see (2.9)) are in the form $hh^{d/2}\sigma^l D_\sigma^q \tau^\alpha D_\tau^\beta$ with $l + q \leq d$ and $\beta = 0, 1$. With Corollary 2.6 and the rescaling (2.1), we have

$$\|h^{d/2}\sigma^l D_\sigma^q \tau^\alpha D_\tau^\beta \psi\| \leq C\|\psi\|$$

and the conclusion follows. ∎

2.3.2 Localization in the phase space

This section is devoted to elliptic regularity properties (both in space and frequency) satisfied by the eigenfunctions. We will use the following generalization of the "IMS" formula the proof of which can be found in [46].

Lemma 2.10 ("Localization" of P^2 with respect to $\mathfrak{A}$) *Let H be a Hilbert space and P and $\mathfrak{A}$ be two unbounded operators defined on a domain $\mathsf{D} \subset \mathsf{H}$. We assume that P is symmetric and that $P(\mathsf{D}) \subset \mathsf{D}$, $\mathfrak{A}(\mathsf{D}) \subset \mathsf{D}$, $\mathfrak{A}^*(\mathsf{D}) \subset \mathsf{D}$. Then, for $\psi \in \mathsf{D}$, we have*

$$\begin{aligned}\mathrm{Re}\,\langle P^2\psi, \mathfrak{A}\mathfrak{A}^*\psi\rangle = \|P(\mathfrak{A}^*\psi)\|^2 - \|[\mathfrak{A}^*, P]\psi\|^2 + \mathrm{Re}\,\langle P\psi, [[P,\mathfrak{A}],\mathfrak{A}^*]\psi\rangle \\ + \mathrm{Re}\,\left(\langle P\psi, \mathfrak{A}^*[P,\mathfrak{A}]\psi\rangle - \overline{\langle P\psi, \mathfrak{A}[P,\mathfrak{A}^*]\psi\rangle}\right). \quad (2.11)\end{aligned}$$

The following lemma is a straightforward consequence of Assumption 1.1.

Lemma 2.11 *Under Assumption 1.1, there exist $\varepsilon_0 > 0$ and $c > 0$ such that*

$$\mu_1(x_0 + x, \xi_0 + \xi) - \mu_1(x_0, \xi_0) \geq c(|x|^2 + |\xi|^2), \qquad \forall (x,\xi) \in \mathcal{B}(\varepsilon_0),$$

and

$$\mu_1(x_0 + x, \xi_0 + \xi) - \mu_1(x_0, \xi_0) \geq c, \qquad \forall (x,\xi) \in \complement\mathcal{B}(\varepsilon_0),$$

where $\mathcal{B}(\varepsilon_0) = \{(x,\xi), |x| + |\xi| \leq \varepsilon_0\}$ and $\complement\mathcal{B}(\varepsilon_0)$ is its complement.

Notation 2.12 *In what follows we will denote by $\tilde{\eta} > 0$ all the quantities which are multiples of $\eta > 0$,i.e. in the form $j\eta$ for $j \in \mathbb{N} \setminus \{0\}$. We recall that $\eta > 0$ can be chosen arbitrarily small.*

Proposition 2.13 *There exist $h_0, C, \varepsilon_0 > 0$ such that for all eigenpairs (λ, ψ) of $\mathcal{L}_h$ with $\lambda \leq \mu_0 + C_0 h$, we have*

$$\|\sigma\psi\|^2 + \|\nabla_\sigma \psi\|^2 \leq C\|\psi\|^2.$$

Proof: We recall that (2.10) holds. We have

$$\mathcal{Q}_h(\psi) = \lambda\|\psi\|^2 \leq (\mu_0 + C_0 h)\|\psi\|^2.$$

We deduce that

$$\int_{\mathbb{R}^{2m}} Q_{h,u,p}(\psi_{u,p}) - \mu_0|\psi_{u,p}|^2 \,\mathrm{d}u\,\mathrm{d}p \leq Ch\|\psi\|^2$$

and thus by the min-max principle

$$\int_{\mathbb{R}^{2m}} \left(\mu_1(x_0 + h^{1/2}u, \xi_0 + h^{1/2}p) - \mu_0\right)|\psi_{u,p}|^2 \,\mathrm{d}u\,\mathrm{d}p \leq Ch\|\psi\|^2.$$

Let $\varepsilon_0 > 0$ be given in Lemma 2.11. We split the integral into two parts and find

$$\int_{\mathcal{B}(h^{-1/2}\varepsilon_0)} (|u|^2 + |p|^2)|\psi_{u,p}|^2 \,\mathrm{d}u\,\mathrm{d}p \leq C\|\psi\|^2, \quad (2.12)$$

$$\int_{\complement\mathcal{B}(h^{-1/2}\varepsilon_0)} |\psi_{u,p}|^2 \,\mathrm{d}u\,\mathrm{d}p \leq Ch\|\psi\|^2. \quad (2.13)$$

The first inequality is not enough to get the conclusion. We also need a control of momenta in the region $\complement\mathcal{B}(h^{-1/2}\varepsilon_0)$. For that purpose, we write:

$$\mathcal{Q}_h(\mathfrak{a}_j^*\psi) = \int_{\mathbb{R}^{2m}} Q_{h,u,p}\left(\frac{u_j - ip_j}{\sqrt{2}}\psi_{u,p}\right)\mathrm{d}u\,\mathrm{d}p + \langle \mathcal{R}_h\mathfrak{a}_j^*\psi, \mathfrak{a}_j^*\psi\rangle. \quad (2.14)$$

Up to lower order terms we must estimate terms in the form:

$$h\langle h^{d/2}\sigma^l D_\sigma^q \tau^\alpha D_\tau^\beta \mathfrak{a}_j^*\psi, \mathfrak{a}_j^*\psi\rangle,$$

with $l+q=d$, $\alpha\in\mathbb{N}$ and $\beta=0,1$. By using the *a priori* estimates of Propositions 2.7 and 2.8, we have

$$\|h^{d/2}\sigma^l D_\sigma^q \tau^\alpha D_\tau^\beta \mathfrak{a}_j^*\psi\| \leq Ch^{-\tilde{\eta}}\|\mathfrak{a}_j^*\psi\|.$$

The remainder is controlled by

$$|\langle \mathcal{R}_h \mathfrak{a}_j^*\psi, \mathfrak{a}_j^*\psi\rangle| \leq Ch^{1-\tilde{\eta}}(\|\nabla_\sigma\psi\|^2 + \|\sigma\psi\|^2).$$

Then we analyze $\mathcal{Q}_h(\mathfrak{a}_j^*\psi)$ by using Lemma 2.10 with $\mathfrak{A}=\mathfrak{a}_j$. We need to estimate the different remainder terms. We notice that

$$\begin{aligned}
\|[\mathfrak{a}_j^*, P_{k,r,h}]\psi\| &\leq Ch^{1/2}\|\psi\|,\\
|\langle P_{k,r,h}\psi, \mathfrak{a}_j^*[P_{k,r,h},\mathfrak{a}_j]\psi\rangle| &\leq \|P_{k,r,h}\psi\|\ \|\mathfrak{a}_j^*[P_{k,r,h},\mathfrak{a}_j]\psi\|,\\
|\langle P_{k,r,h}\psi, \mathfrak{a}_j[P_{k,r,h},\mathfrak{a}_j^*]\psi\rangle| &\leq \|P_{k,r,h}\psi\|\ \|\mathfrak{a}_j[P_{k,r,h},\mathfrak{a}_j^*]\psi\|,\\
|\langle P_{k,r,h}\psi, [[P_{k,r,h},\mathfrak{a}_j],\mathfrak{a}_j^*]\psi\rangle| &\leq \|P_{k,r,h}\psi\|\ \|[[P_{k,r,h},\mathfrak{a}_j],\mathfrak{a}_j^*]\psi\|,
\end{aligned}$$

where $P_{1,r,h}$ denotes the magnetic momentum $h^{1/2}D_{\sigma_r} + A_{1,r}(x_0+h^{1/2}\sigma,\tau)$ and $P_{2,r,h}$ denotes $D_{\tau_r} + A_{2,r}(x_0+h^{1/2}\sigma,\tau)$. We have

$$\|P_{k,r,h}\psi\| \leq C\|\psi\|$$

and

$$\|\mathfrak{a}_j^*[P_{k,r,h},\mathfrak{a}_j]\psi\| \leq Ch^{1/2}\|\mathfrak{a}_j^* Q(h^{1/2}\sigma,\tau)\psi\|,$$

where Q is polynomial. The other terms are bounded in the same way. We apply the estimates of Propositions 2.7 and 2.8 to get

$$\|\mathfrak{a}_j^* Q(h^{1/2}\sigma,\tau)\psi\| \leq Ch^{-\tilde{\eta}}\|\mathfrak{a}_j^*\psi\|.$$

We have

$$\mathcal{Q}_h(\mathfrak{a}_j^*\psi) = \lambda\|\mathfrak{a}_j^*\psi\|^2 + O(h)\|\psi\|^2 + O(h^{\frac{1}{2}-\tilde{\eta}})(\|\nabla_\sigma\psi\|^2+\|\sigma\psi\|^2),$$

so that

$$\mathcal{Q}_h(\mathfrak{a}_j^*\psi) \leq \mu_0\|\mathfrak{a}_j^*\psi\|^2 + Ch\|\psi\|^2 + O(h^{\frac{1}{2}-\tilde{\eta}})(\|\nabla_\sigma\psi\|^2+\|\sigma\psi\|^2).$$

By using (2.14) and splitting again the integral into two parts, it follows

$$\begin{aligned}
\int_{\mathcal{B}(h^{-1/2}\varepsilon_0)} (|u|^2+|p|^2)|(u_j-ip_j)\psi_{u,p}|^2\,\mathrm{d}u\,\mathrm{d}p &\leq C\|\psi\|^2 + Ch^{-\frac{1}{2}-\tilde{\eta}}(\|\nabla_\sigma\psi\|^2+\|\sigma\psi\|^2),\\
\int_{\complement\mathcal{B}(h^{-1/2}\varepsilon_0)} |(u_j-ip_j)\psi_{u,p}|^2\,\mathrm{d}u\,\mathrm{d}p &\leq Ch\|\psi\|^2 + Ch^{\frac{1}{2}-\tilde{\eta}}(\|\nabla_\sigma\psi\|^2+\|\sigma\psi\|^2).
\end{aligned}$$

Combining the last inequality with the first one of (2.12) and the Parseval formula we get the conclusion. ∎

Proposition 2.14 *Let $P\in\mathbb{C}_2[X_1,\dots,X_{2m}]$. There exist $h_0, C, \varepsilon_0 > 0$ such that for all eigenpairs (λ,ψ) of $\mathfrak{L}_h$ with $\lambda\leq\mu_0+C_0h$, we have*

$$\|P(\sigma,D_\sigma)\psi\|^2 \leq Ch^{-\frac{1}{2}-\tilde{\eta}}\|\psi\|^2.$$

Proof: From the proof of Proposition 2.13, we infer

$$\int_{\mathcal{B}(h^{-1/2}\varepsilon_0)} (|u|^2+|p|^2)|(u_j-ip_j)\psi_{u,p}|^2\,\mathrm{d}u\,\mathrm{d}p \;\leq\; Ch^{-\frac{1}{2}-\tilde{\eta}}\|\psi\|^2, \tag{2.15}$$

$$\int_{\complement\mathcal{B}(h^{-1/2}\varepsilon_0)} |(u_j-ip_j)\psi_{u,p}|^2\,\mathrm{d}u\,\mathrm{d}p \;\leq\; Ch^{\frac{1}{2}-\tilde{\eta}}\|\psi\|^2. \tag{2.16}$$

Then we use Lemma 2.10 with $\mathfrak{A}=\mathfrak{a}_j\mathfrak{a}_j$. The worst remainders in (2.11) are

$$\begin{aligned}\|[P_{k,r,h},\mathfrak{a}_j^*\mathfrak{a}_j^*]\psi\|^2 &\leq Ch\|\psi\|^2,\\ |\langle P_{k,r,h}\psi,\mathfrak{a}_j^*\mathfrak{a}_j^*[P_{k,r,h},\mathfrak{a}_j\mathfrak{a}_j]\psi\rangle| &\leq Ch^{\frac{1}{2}-\tilde{\eta}}\|\psi\|\|\mathfrak{a}_j^*\mathfrak{a}_j^*\psi\|.\end{aligned}$$

We infer

$$\mathcal{Q}_h(\mathfrak{a}_j^*\mathfrak{a}_j^*\psi)\leq\mu_0\|\mathfrak{a}_j^*\mathfrak{a}_j^*\psi\|^2+Ch^{\frac{1}{2}-\tilde{\eta}}\|\psi\|^2+Ch^{\frac{1}{2}-\tilde{\eta}}\|\mathfrak{a}_j^*\mathfrak{a}_j^*\psi\|^2.$$

We deduce

$$\int_{\mathcal{B}(h^{-1/2}\varepsilon_0)} (|u|^2+|p|^2)|(u_j-ip_j)^2\psi_{u,p}|^2\,\mathrm{d}u\,\mathrm{d}p\leq Ch^{-\frac{1}{2}-\tilde{\eta}}\|\psi\|^2+Ch^{-\frac{1}{2}-\tilde{\eta}}\|\mathfrak{a}_j^*\mathfrak{a}_j^*\psi\|^2,$$

$$\int_{\complement\mathcal{B}(h^{-1/2}\varepsilon_0)} |(u_j-ip_j)^2\psi_{u,p}|^2\,\mathrm{d}u\,\mathrm{d}p\leq Ch^{\frac{1}{2}-\tilde{\eta}}\|\psi\|^2+Ch^{\frac{1}{2}-\tilde{\eta}}\|\mathfrak{a}_j^*\mathfrak{a}_j^*\psi\|^2. \tag{2.17}$$

Jointly with Proposition 2.13, estimates (2.15) and (2.17) imply that

$$\int_{\mathbb{R}^{2m}} (|u|^4+|p|^4)|\psi_{u,p}|^2\,\mathrm{d}u\,\mathrm{d}p\leq Ch^{-\frac{1}{2}-\tilde{\eta}}\|\psi\|^2.$$

The conclusion follows from the Parseval formula and Proposition 2.13. ∎

2.4 Spectral gap

We introduce the projection

$$\Psi_0=\Pi_0\psi=\langle\psi,u_{x_0,\xi_0}\rangle_{\mathrm{L}^2(\mathbb{R}^n,\,\mathrm{d}\tau)}u_{x_0,\xi_0}$$

and, inspired by (2.6) where f_0 is replaced by $\langle\psi,u_{x_0,\xi_0}\rangle_{\mathrm{L}^2(\mathbb{R}^n,\,\mathrm{d}\tau)}$ and f_1 by 0,

$$\Psi_1=\sum_{j=1}^m(\partial_{x_j}u)_{x_0,\xi_0}\,\sigma_j\langle\psi,u_{x_0,\xi_0}\rangle_{\mathrm{L}^2(\mathbb{R}^n,\,\mathrm{d}\tau)}+\sum_{j=1}^m(\partial_{\xi_j}u)_{x_0,\xi_0}\,D_{\sigma_j}\langle\psi,u_{x_0,\xi_0}\rangle_{\mathrm{L}^2(\mathbb{R}^n,\,\mathrm{d}\tau)}. \tag{2.18}$$

This leads to defined the corrected Feshbach projection

$$\Pi_h\psi=\Psi_0+h^{1/2}\Psi_1 \tag{2.19}$$

and

$$R_h=\psi-\Pi_h\psi$$

Note that the functions Ψ_0 and Ψ_1 will be *a priori* h-dependent. By the L^2-normalization of $u_{x,\xi}$ (when $\xi\in\mathbb{R}^m$), Ψ_1 and R_h are orthogonal (with respect to the τ-variable) to u_0 (and Ψ_0). Furthermore, we have by construction and Proposition 2.2

$$(\mathcal{L}_0-\mu_0)\Psi_1=-\mathcal{L}_1\Psi_0 \tag{2.20}$$

and, by the Fredholm alternative,

$$\langle\mathcal{L}_1\Psi_0,\Psi_0\rangle_{\mathrm{L}^2(\mathbb{R}^n,\,\mathrm{d}\tau)}=0.$$

2.4.1 Approximation results

We can prove a first approximation.

Proposition 2.15 *There exist $h_0, C > 0$ such that for all eigenpairs (λ, ψ) of $\mathcal{L}_h$ with $\lambda \leq \mu_0 + C_0 h$, we have*

$$\|\psi - \Pi_0 \psi\| \leq C h^{1/2-\tilde{\eta}} \|\psi\|.$$

Proof: We can write

$$(\mathcal{L}_0 - \mu_0)\psi = (\lambda - \mu_0)\psi - h^{1/2}\mathcal{L}_1\psi - h\mathcal{L}_2\psi + \ldots - h^{M/2}\mathcal{L}_M\psi.$$

By using the rough microlocalization given in Propositions 2.7 and 2.8 and Proposition 2.14, we infer that for $d \geq 2$

$$h^{d/2}\|\tau^\alpha D_\tau^\beta \sigma^l D_\sigma^q \psi\| \leq C h^{\frac{d}{2} - \frac{d-2}{2} - \frac{1}{4} - \tilde{\eta}} \|\psi\| = C h^{\frac{3}{4} - \tilde{\eta}} \|\psi\|, \tag{2.21}$$

and thanks to Proposition 2.13

$$\|\mathcal{L}_1 \psi\| \leq C h^{-\tilde{\eta}} \|\psi\|,$$

so that

$$\|(\mathcal{L}_0 - \mu_0)\psi\| \leq C h^{\frac{1}{2} - \tilde{\eta}} \|\psi\|,$$

and the conclusion follows. ∎

Corollary 2.16 *There exist $h_0, C > 0$ such that for all eigenpairs (λ, ψ) of $\mathcal{L}_h$ with $\lambda \leq \mu_0 + C_0 h$, we have*

$$\|\sigma(\psi - \Pi_0\psi)\| \leq C h^{\frac{1}{4} - \tilde{\eta}} \|\psi\|, \qquad \|D_\sigma(\psi - \Pi_0\psi)\| \leq C h^{\frac{1}{4} - \tilde{\eta}} \|\psi\|.$$

We can now estimate $\psi - \Pi_h \psi$.

Proposition 2.17 *There exist $h_0, C > 0$ such that for all eigenpairs (λ, ψ) of $\mathcal{L}_h$ with $\lambda \leq \mu_0 + C_0 h$, we have*

$$\|R_h \psi\| = \|\psi - \Pi_h \psi\| \leq C h^{\frac{3}{4} - \tilde{\eta}} \|\psi\|.$$

Proof: Let us write

$$\mathcal{L}_h \psi = \lambda \psi.$$

We have

$$(\mathcal{L}_0 + h^{1/2}\mathcal{L}_1)\psi = (\mu_0 + O(h))\psi - h\mathcal{L}_2\psi - \ldots - h^{M/2}\mathcal{L}_M\psi.$$

Let us notice that, as in (2.21), we have, for $d \geq 2$,

$$h^{d/2}\|\mathcal{L}_d \psi\| \leq C h^{\frac{3}{4} - \tilde{\eta}} \|\psi\|.$$

We get

$$(\mathcal{L}_0 - \mu_0) R_h = -h^{1/2}\mathcal{L}_1(\psi - \Psi_0) + O(h)\psi - h\mathcal{L}_2\psi - \ldots - h^{M/2}\mathcal{L}_M\psi$$

It remains to apply Corollary 2.16 and obtain

$$h^{1/2}\|\mathcal{L}_1(\psi - \Psi_0)\| \leq \tilde{C} h^{\frac{3}{4} - \tilde{\eta}} \|\psi\|.$$

∎

2.4.2 Proof of Theorem 1.8

Let us introduce a subspace of dimension $P \geq 1$. For $j \in \{1, \dots, P\}$ we can consider a L^2-normalized eigenfunction of $\mathcal{L}_h$ denoted by $\psi_{j,h}$ and so that the family $(\psi_{j,h})_{j\in\{1,\dots,P\}}$ is orthogonal. We let

$$\mathfrak{E}_P(h) = \operatorname*{span}_{j\in\{1,\dots,P\}} \psi_{j,h}.$$

Remark 2.18 *We can extend all the local and microlocal estimates as well as our approximations to $\psi \in \mathfrak{E}_P(h)$.*

Let us prove a lower bound for the quadratic form on $\mathfrak{E}_P(h)$:

$$\mathcal{Q}_h(\psi) = \langle \mathcal{L}_0\psi, \psi\rangle + h^{1/2}\langle \mathcal{L}_1\psi, \psi\rangle + h\langle \mathcal{L}_2\psi, \psi\rangle + \ldots + h^{p/2}\langle \mathcal{L}_p\psi, \psi\rangle + \ldots + h^{M/2}\langle \mathcal{L}_M\psi, \psi\rangle.$$

Using Propositions 2.13, 2.14, 2.7 and 2.8, we have, for $d \geq 3$

$$|h^{d/2}\langle \mathcal{L}_d\psi, \psi\rangle| \leq Ch^{\frac{d}{2}-\frac{d-3}{2}-\tilde{\eta}-\frac{1}{4}}\|\psi\|^2 = Ch^{\frac{5}{4}-\tilde{\eta}}\|\psi\|^2.$$

We infer

$$\mathcal{Q}_h(\psi) \geq \langle \mathcal{L}_0\psi, \psi\rangle + h^{1/2}\langle \mathcal{L}_1\psi, \psi\rangle + h\langle \mathcal{L}_2\psi, \psi\rangle - Ch^{\frac{5}{4}-\tilde{\eta}}\|\psi\|^2.$$

It remains to analyze the different terms. We have

$$\langle \mathcal{L}_0\psi, \psi\rangle = \langle \mathcal{L}_0(\Psi_0 + h^{1/2}\Psi_1 + R_h), \Psi_0 + h^{1/2}\Psi_1 + R_h\rangle.$$

The orthogonality (with respect to τ) cancels the terms $\langle \mathcal{L}_0\Psi_1, \Psi_0\rangle$ and $\langle R_h, \Psi_0\rangle$. Moreover, we have, with Propositions 2.7 and 2.8,

$$h^{1/2}|\langle \mathcal{L}_0 R_h, \Psi_1\rangle| \leq h^{1/2-\tilde{\eta}}\|R_h\|\|\Psi_1\|,$$

and we use Proposition 2.13 to get

$$\|\Psi_1\| \leq C\|\psi\|,$$

so that, with Proposition 2.17,

$$\langle \mathcal{L}_0\psi, \psi\rangle = \mu_0\|\Psi_0\|^2 + h\langle \mathcal{L}_0\Psi_1, \Psi_1\rangle + O(h^{\frac{5}{4}-\tilde{\eta}})\|\psi\|^2.$$

We have

$$\langle \mathcal{L}_1\psi, \psi\rangle = \langle \mathcal{L}_1\Psi_0, \Psi_0\rangle + 2h^{1/2}\langle \mathcal{L}_1\Psi_0, \Psi_1\rangle + h\langle \mathcal{L}_1\Psi_1, \Psi_1\rangle + 2\langle \mathcal{L}_1\psi, R_h\rangle.$$

Then, a Feynman-Hellmann formula provides $\langle \mathcal{L}_1\Psi_0, \Psi_0\rangle = 0$. Using again Propositions 2.7, 2.8, 2.13, 2.14 and 2.17, we notice that

$$\langle \mathcal{L}_1\psi, \psi\rangle = 2h^{1/2}\langle \mathcal{L}_1\Psi_0, \Psi_1\rangle + O(h^{\frac{3}{4}-\tilde{\eta}})\|\psi\|^2.$$

We notice

$$\langle \mathcal{L}_2\psi, \psi\rangle = \langle \mathcal{L}_2\Psi_0, \Psi_0\rangle + \langle \mathcal{L}_2(\psi - \Psi_0), \psi\rangle + \langle \mathcal{L}_2\psi, \psi - \Psi_0\rangle.$$

Writing $\psi - \Psi_0 = h^{1/2}\Psi_1 + R_h$, we have the estimate

$$|\langle \mathcal{L}_2(\psi - \Psi_0), \psi\rangle + \langle \mathcal{L}_2\psi, \psi - \Psi_0\rangle| \leq Ch^{-\frac{1}{4}-\tilde{\eta}}h^{\frac{1}{2}-\tilde{\eta}}\|\psi\|^2.$$

We infer

$$\mathcal{Q}_h(\psi) \geq \mu_0\|\Psi_0\|^2 + h\langle \mathcal{L}_0\Psi_1, \Psi_1\rangle + h\langle \mathcal{L}_1\Psi_0, \Psi_1\rangle + h\langle \mathcal{L}_1\Psi_1, \Psi_0\rangle + h\langle \mathcal{L}_2\Psi_0, \Psi_0\rangle - Ch^{\frac{5}{4}-\tilde{\eta}}\|\psi\|^2.$$

Using (2.20), we get

$$h\langle \mathcal{L}_0\Psi_1, \Psi_1\rangle + h\langle \mathcal{L}_1\Psi_0, \Psi_1\rangle = h\mu_0\|\Psi_1\|^2,$$

so that, by orthogonality,

$$\mathcal{Q}_h(\psi) \geq \mu_0\|\Psi_0 + h^{1/2}\Psi_1\|^2 + h\langle \mathcal{L}_1\Psi_1, \Psi_0\rangle + h\langle \mathcal{L}_2\Psi_0, \Psi_0\rangle - Ch^{\frac{5}{4}-\tilde{\eta}}\|\psi\|^2.$$

Since $\langle R_h, \Psi_0\rangle = 0$ we deduce that

$$\|\Psi_0 + h^{1/2}\Psi_1\|^2 = \|\Psi_0 + h^{1/2}\Psi_1 + R_h\|^2 + O(h^{\frac{5}{4}-\tilde{\eta}})\|\psi\|^2.$$

It follows that

$$\mathcal{Q}_h(\psi) - \mu_0\|\psi\|^2 \geq h\langle \mathcal{L}_1\Psi_1, \Psi_0\rangle + h\langle \mathcal{L}_2\Psi_0, \Psi_0\rangle + O(h^{\frac{5}{4}-\tilde{\eta}})\|\psi\|^2,$$

and, since $\mathcal{Q}_h(\psi) \leq \lambda_P(h)\|\psi\|^2$, we have

$$(\lambda_P(h) - \mu_0)\|\psi\|^2 \geq h\langle \mathcal{L}_1\Psi_1, \Psi_0\rangle + h\langle \mathcal{L}_2\Psi_0, \Psi_0\rangle + O(h^{\frac{5}{4}-\tilde{\eta}})\|\psi\|^2.$$

Thus we get

$$(\lambda_P(h) - \mu_0)\|\Psi_0\|^2 \geq h\langle \mathcal{L}_1\Psi_1, \Psi_0\rangle + h\langle \mathcal{L}_2\Psi_0, \Psi_0\rangle + O(h^{\frac{5}{4}-\tilde{\eta}})\|\psi\|^2.$$

We recall that (see (2.7) and below)

$$\begin{aligned}\langle \mathcal{L}_1\Psi_1, \Psi_0\rangle + \langle \mathcal{L}_2\Psi_0, \Psi_0\rangle \\ = \left\langle \tfrac{1}{2}\mathsf{Hess}\ \mu(x_0,\xi_0)(\sigma, D_\sigma)(\langle \psi, u_0\rangle_{\mathrm{L}^2(\mathbb{R}^n,\,\mathrm{d}\tau)}), \langle \psi, u_0\rangle_{\mathrm{L}^2(\mathbb{R}^n,\,\mathrm{d}\tau)}\right\rangle_{\mathrm{L}^2(\mathbb{R}^m,\,\mathrm{d}\sigma)}.\end{aligned}$$

Finally we apply the min-max principle to the P-dimensional space $\langle \mathfrak{E}_P(h), u_0\rangle_{\mathrm{L}^2(\mathbb{R}^n,\,\mathrm{d}\tau)}$ to get the spectral gap. Theorem 1.8 is a consequence of Proposition 2.1.

3 Magnetic WKB constructions

3.1 Stable manifold and eikonal equation

In this section we study the construction of WKB solutions in the general case

$$\mathfrak{L}_h = (hD_s + A_1(s,t))^2 + (D_t + A_2(s,t))^2 \qquad \text{with} \qquad D = -i\nabla.$$

As mentionned in the introduction, for $(x,\xi) \in \mathbb{R}^m \times \mathbb{R}^m$, we are interested in the electromagnetic Laplacian acting on $\mathrm{L}^2(\mathbb{R}^n,\,\mathrm{d}t)$, when looking at the partial semiclassical symbol of $\mathfrak{L}_h$ in variable s

$$\mathcal{M}_{x,\xi} = (D_t + A_2(x,t))^2 + (\xi + A_1(x,t))^2. \tag{3.1}$$

Denoting by $\mu_1(x,\xi)$ its lowest eigenvalue, we would like to replace (in spirit) $\mathfrak{L}_h$ by the m-dimensional pseudo-differential operator (with the Weyl quantization) $\mu_1(x, hD_x)$. In order to complete this program, the main assumption on the operator $\mathcal{M}_{x,\xi}$ in variable t concerns its lowest eigenvalue μ_1 and is stated in Assumption 1.1 in the introduction.

In order to build suitable quasimodes for $\mathfrak{L}_h$, it will be of great use to first study the following eikonal equation

$$\mu_1(x_0 + x, \xi_0 + i\nabla\Phi(x)) = \mu_0, \tag{3.2}$$

with unknown Φ, where we recall that (x_0, ξ_0) is the point where the minimum μ_0 of μ_1, is attained (as a real function of $(x, \xi) \in \mathbb{R}^{2m}$). Although certainly well-known (see e.g. [49]), in particular in the context of Sjöstrand's theory of FIO with complex phases, we recall in the next subsections this construction with elementary tools. In order to simplify the notation, we denote in the following

$$p(x, \xi) = \mu^\sharp(x, \xi) - \mu_0 \qquad \text{with} \qquad \mu^\sharp(x, \xi) = \mu_1(x_0 + x, \xi_0 + \xi).$$

With these notations, we deal with a real analytic symbol p defined at least in a neighborhood of $(0, 0)$ in the complex plane, and such that

$$p(0, 0) = 0, \quad \nabla p(0, 0) = 0, \quad \text{and} \quad \mathsf{Hess}\, p(0, 0) \text{ is (real) positive definite.}$$

The point $\rho_0 = (0, 0)$ is then a so-called doubly characteristic point for p, and the eikonal equation now reads

$$p(x, i\nabla\Phi(x)) = 0.$$

In the next subsection, we recall the quadratic case for solving this equation, and in the following one we study the general case.

3.1.1 The quadratic case

In this subsection we recall basic facts from [49] about the quadratic case and also about symplectic geometry. Recall that $T^*\mathbb{R}^m$ is endowed with the canonical symplectic 2-form which writes $\omega = \sum_j \mathrm{d}\xi_j \wedge \mathrm{d}x_j$ in coordinates, and that this form naturally extends to a symplectic 2-form in $T^*\mathbb{C}^m$ with the same expression

$$\omega\big((x, \xi), (y, \eta)\big) = \xi \cdot y - x \cdot \eta, \qquad (x, \xi), (y, \eta) \in T^*\mathbb{C}^m.$$

Note that

$$\frac{1}{i}\omega(X, \overline{X}) = \frac{1}{i}(\xi \cdot \overline{x} - \overline{\xi} \cdot x) = 2\mathrm{Im}\,(\xi \cdot \overline{x}) \in \mathbb{R}. \tag{3.3}$$

We recall that an endomorphism κ^0 of $T^*\mathbb{R}^m$ is said to be symplectic if $\omega(\kappa^0., \kappa^0.) = \omega(., .)$. In coordinates, we get that ω is represented by a matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Posing $X = (x, \xi) \in T^*\mathbb{C}^m$, we consider here a quadratic form

$$x \longrightarrow p_0(X, X) \quad \text{ on } T^*\mathbb{C}^m,$$

which is real definite positive when restricted to $T^*\mathbb{R}^m$. Note that this implies that

$$p_0(X, \overline{X}) > 0, \qquad \forall X \in T^*\mathbb{C}^m \setminus \{0\}.$$

We first recall that the fundamental matrix F_0 of p_0 may be defined through the symplectic form via the following formula

$$p_0(X, X) = \omega(X, F_0 X), \qquad \forall X \in T^*\mathbb{C}^m.$$

Let us recall some properties of F_0 (see [15, 35]). Let M be the matrix of the quadratic form p_0. We have $F_0 = JM$. Since M is positive, we may consider $M^{1/2}$ and $M^{1/2}F_0M^{-1/2} = M^{1/2}JM^{1/2}$ is a real antisymmetric matrix. Its eigenvalues are purely imaginary and conjugate (the matrices are real). We denote them by $\pm i\lambda_j$ with $\lambda_j > 0$.

Following [15], we introduce

$$\Lambda^0_+ = \bigoplus_j E_{i\lambda_j}, \qquad \Lambda^0_- = \bigoplus_j E_{-i\lambda_j},$$

and note that Λ^0_+ and Λ^0_- are Lagrangian vector spaces of $T^*\mathbb{C}^m$ (just recall that this is in particular linked to the fact that the family $(E_{\pm i\lambda_k})_k$ is ω-orthogonal).

The next step is to show some transversality property.

Lemma 3.1 *We have* $\Lambda^0_+ \cap T^*\mathbb{R}^m = \{0\}$.

Proof: Let us take $X \in \Lambda^0_+ \cap T^*\mathbb{R}^m$. Then by stability of generalized eigenspaces, we also have $F_0X \in \Lambda^0_+$ and we get

$$p_0(X, X) = \omega(X, F_0X) = 0,$$

since Λ^0_+ is isotropic (because lagrangian). On the other hand we know that p_0 is positive definite, and this implies $X = 0$. ∎

We now show that

Lemma 3.2 *For all* $X \in \Lambda^0_+ \setminus \{0\}$, *we have* $\frac{1}{i}\omega(X, \overline{X}) > 0$.

Proof: This is done by a perturbation argument. We denote $p_1(X, X) = \xi^2 + x^2$ the harmonic oscillator (here $X = (x, \xi) \in T^*\mathbb{C}^m$), and we introduce

$$p_t = (1 - t)p_0 + tp_1, \qquad \forall t \in [0, 1].$$

From direct computations we already know completely the description of the eigenvalues and Lagrangian subspaces for p_1 and its fundamental matrix F_t. The eigenvalues are of the form $\pm i$, and the lagrangian subspaces are given by

$$\Lambda^1_+ = \{\xi = ix\}, \qquad \Lambda^1_- = \{\xi = -ix\}.$$

In particular we have from (3.3) that for all $X = (x, ix) \in \Lambda^1_+ \setminus \{0\}$,

$$\frac{1}{i}\omega(X, \overline{X}) = \frac{1}{i}\omega((x, ix), \overline{(x, ix)}) = 2\mathrm{Im}\,(ix \cdot \overline{x}) = 2|x|^2 > 0.$$

Now we can consider

$$\Lambda^t_+ = \bigoplus_j E_{i\lambda^t_j},$$

where $i\lambda^t_j$ are the eigenvalues of the fundamental matrix of p_t, which satisfy the same properties as the ones of p_0. The family Λ^t_+ is a continuous family of subspaces, defined e.g. by

$$\Lambda^t_+ = \mathsf{range}\frac{1}{2i\pi}\int_\gamma (z - F_t)^{-1}dz,$$

where γ is made of a segment $[-R, R]$ and a semi-circle $\mathcal{C}(0, R) \cap \{\mathrm{Im}\, z > 0\}$. For R sufficiently large, the open semi-disc contains all eigenvalues $i\lambda^t_j$, since $\lambda^t_j > 0$ for all $t \in [0, 1]$.

Now we want to show that for all $X \in \Lambda_+^t \setminus \{0\}$ we have $\frac{1}{i}\omega(X, \overline{X}) > 0$, and already know that this is the case for $t = 1$. Suppose by contradiction that there is $t_0 \in [0, 1]$ and $X_0 \in \Lambda_+^{t_0} \setminus \{0\}$ such that

$$\frac{1}{i}\omega(X_0, \overline{X_0}) = 0$$

and that this t_0 is the largest in $[0, 1]$ for which this happens. In particular, by definition of t_0 and continuity this implies that $(X, Y) \longrightarrow \frac{1}{i}\omega(X, \overline{Y})$ is a positive semi-definite hermitian form $\Lambda_+^{t_0}$. Then we can use the following version of the Cauchy-Schwarz inequality,

$$\forall X, Y \in \Lambda_+^{t_0}, \qquad \left|\frac{1}{i}\omega(X, \overline{Y})\right|^2 \leq \frac{1}{i}\omega(X, \overline{X})\frac{1}{i}\omega(Y, \overline{Y}).$$

Applying this to $Y = X_0$, we have for all $X \in \Lambda_+^{t_0}$

$$\omega(X, \overline{X_0}) = 0,$$

which implies $\overline{X_0} \in (\Lambda_+^{t_0})^{\perp_\omega} = \Lambda_+^{t_0}$, where the last equality is due to the fact that $\Lambda_+^{t_0}$ is Lagrangian. As a consequence we get that

$$\operatorname{Re} X_0 = \frac{1}{2}(X_0 + \overline{X_0}) \in \Lambda_+^{t_0} \cap T^*R^m, \qquad \operatorname{Im} X_0 = \frac{1}{2i}(X_0 - \overline{X_0}) \in \Lambda_+^{t_0} \cap T^*R^m.$$

From Lemma 3.1, we get that $X_0 = 0$, which gives a contradiction. We therefore can conclude that for all $t \in [0, 1]$, we have

$$X \in \Lambda_+^t \setminus \{0\}, \qquad \frac{1}{i}\omega(X, \overline{X}) > 0$$

and taking $t = 0$ gives the lemma. ∎

We want now to show that Λ_+^0 is a graph. For this we show that

Lemma 3.3 $\Lambda_+^0 \cap \{x = 0\} = \{0\}$.

Proof: Suppose by contradiction that there exists $0 \neq (0, \xi_0) \in \Lambda_+^0$. Then one has directly by computation

$$\frac{1}{i}\omega((0, \xi_0), (0, \overline{\xi_0})) = 0.$$

On the other hand we already have shown that $\frac{1}{i}\omega((0, \xi_0), (0, \overline{\xi_0})) > 0$ since $\xi_0 \neq 0$. This gives the contradiction. ∎

We are therefore able to prove the following lemma.

Lemma 3.4 *There exists a matrix $B \in M_m(\mathbb{C})$ with ${}^tB = B$ and $\operatorname{Im} B > 0$ and such that $\Lambda_+^0 = \{(x, Bx),\ x \in \mathbb{C}^m\}$.*

Proof: From the previous transversality lemma, we already know that there exists a matrix B such that $\Lambda_+^0 = \{(x, Bx),\ x \in \mathbb{C}^m\}$. We have to check that B satisfies the required properties. We first note that since Λ_+^0 is lagrangian, we have for all $x \in \mathbb{C}^*$,

$$0 = \omega((x, Bx), (y, By)) = Bx \cdot y - By \cdot x = x({}^tB - B)y.$$

This implies ${}^tB = B$. On the other hand, we have also for all $x \in \mathbb{C}^m \setminus \{0\}$,

$$0 < \tfrac{1}{i}\omega((x, Bx), (\overline{x}, \overline{Bx})) = \tfrac{1}{i}\left(Bx \cdot \overline{x} - \overline{B}\overline{x} \cdot x\right) = \tfrac{1}{i}(B - B^*)x \cdot \overline{x} = 2\operatorname{Im} Bx \cdot \overline{x}. \qquad (3.4)$$

This implies $\operatorname{Im} B > 0$ which is the desired result. ∎

As a consequence, we get and denoting $\Phi_0(x) = \frac{1}{2}Bx.x$. We have obtained

Proposition 3.5 *There exists a quadratic homogeneous polynomial* Φ_0 *with* $\operatorname{Re}\mathsf{Hess}\,\Phi_0 > 0$ *such that*

$$\Lambda_+^0 = \{\xi = i\nabla\Phi_0(x)\}\,.$$

Furthermore this function satisfies the eikonal equation

$$p_0(x, i\nabla\Phi_0(x)) = 0, \qquad \forall x \in \mathbb{R}^n.$$

Proof: For the first part, we only have to pose $\Phi_0(x) = \frac{1}{2i}Bx.x$. Note that $\nabla\Phi_0(x) = \frac{1}{i}Bx$ since ${}^tB = B$, and use the positivity result of the previous lemma. For the eikonal equation, we check that for $X = (x, Bx) \in \Lambda_+^0$, we have

$$p_0(x, i\nabla\Phi_0(x)) = p_0(X, X) = \omega(X, FX) = 0,$$

since X and $FX \in \Lambda_+^0$, and Λ_+^0 is Lagrangian. This concludes the proof. ∎

3.1.2 Solving the eikonal equation in the general case

In order to stick to the standard theory (see e.g. [15]), we introduce

$$q(x, \xi) = -p(x, i\xi)$$

and the eikonal equation reads then

$$q(x, \nabla\Phi(x)) = 0. \tag{3.5}$$

We focus from now on (3.5).

Now we note that contrary to the particular case of the Schrödinger operator, the symbol q is not real, so that we cannot expect to get a real phase Φ. Anyway the classical construction in $\mathbb{C}^{2m}$ of the phase remains true. We give it shortly in detail now.

We can look at the H_q flow in a small neighborhood of $(0, 0)$.

$$H_q(x, \xi) = \frac{\partial q}{\partial \xi}(x, \xi)\partial_x - \frac{\partial q}{\partial x}(x, \xi)\partial_\xi.$$

Since the point $\rho_0 = (0, 0)$ is the doubly characteristic point, we look at the linearization of the H_q flow at ρ_0, which writes

$$H_q(x, \xi) = F_q(x, \xi).(\partial_x, \partial_\xi) + \mathcal{O}\left((x, \xi)^3\right)(\partial_x, \partial_\xi),$$

where F_q is the fundamental matrix, with expression

$$F_q(\rho_0) = \begin{pmatrix} \frac{\partial^2 q}{\partial x \partial \xi}(\rho_0) & \frac{\partial^2 q}{\partial \xi^2}(\rho_0) \\ -\frac{\partial^2 q}{\partial x^2}(\rho_0) & -\frac{\partial^2 q}{\partial x \partial \xi}(\rho_0) \end{pmatrix},$$

since the p is real, the hessian of p is of course real, and of the form

$$M_p = \mathsf{Hess}\ p(\rho_0) = \begin{pmatrix} A & B \\ B & C \end{pmatrix},$$

with A, B and C real symmetric matrices. We therefore get that

$$M_q = \begin{pmatrix} -A & -iB \\ -iB & C \end{pmatrix}.$$

We also know that since ρ_0 is a non degenerate minimum of p, the Hessian of p at ρ_0 is positive definite. We directly check that $F_q = JM_q$ so that

$$F_q = \begin{pmatrix} -iB & A \\ C & iB \end{pmatrix}.$$

Now using that $F_p = JM_p$, we also have

$$F_p = F_0 = \begin{pmatrix} B & C \\ -A & -B \end{pmatrix},$$

so that

$$F_q = i \begin{pmatrix} I & 0 \\ 0 & -iI \end{pmatrix} F_p \begin{pmatrix} I & 0 \\ 0 & iI \end{pmatrix}. \tag{3.6}$$

From this transformation and the previous subsection, we get directly the following lemma where the notations are the same as previously.

Lemma 3.6 *The matrix F_q is antisymmetric with respect to ω. The eigenvalues are of the form $(-\lambda_j, +\lambda_j)$, $j \in \mathbb{N}^m$, defined in the previous subsection, where the $\lambda_j > 0$ are counted with multiplicity. In addition, for q_0 the quadratic approximation of q at 0, we have*

$$q_0(x, \nabla\Phi_0(x)) = 0,$$

that is to say Φ_0 solves the quadratic eikonal equation for the quadratic approximation.

Here we recall that $\Lambda_+^0 = \{(x, Bx)\}$ was the lagrangian subspace associated with eigenvalues $i\lambda_j$, $\lambda_j > 0$. The transformation

$$(x, \xi) \longrightarrow (x, -i\xi)$$

and (3.6) give directly that $\Lambda_+^{0,q} = \{(x, -iBx)\}$ is also a lagrangian subspace associated to eigenvalues of negative real part $\{-\lambda_j\}$. We can therefore have an other interpretation of the set $\Lambda_+^{0,q}$: For this we study the linearized flow at ρ_0, which is given by

$$Z'(t) = F_q(Z).\nabla Z.$$

This is clear that $\Lambda_+^{0,q}$ and $\Lambda_-^{0,q}$ are respectively the unstable and stable manifolds associated to the vector field $F_q(Z).\nabla$. By this we mean that

$$\forall Z_0 \in \Lambda_\mp^{0,q}, \qquad \lim_{t \longrightarrow \pm\infty} Z(t) = \rho_0.$$

As already noted, these two spaces $\Lambda_\pm^{0,q}$ are lagrangian, and we will show this in more generality for the stable and unstable manifolds associated to the vector field H_q and its characteristics $X(t)$, define by

$$X'(t) = H_q(X(t)).$$

Knowing the spectrum of the linearization of H_q, we now just have to apply the stable manifold theorem, with value in $\mathbb{C}$ and we directly get that there exists one unstable holomorphic manifold Λ_+^q and one stable holomorphic manifold Λ_-^q, for which we have

$$T\big|_{\rho_0, \mathbb{C}^{2m}} \Lambda_\pm = \Lambda_\pm^0. \tag{3.7}$$

We also know that $\Lambda_\pm$ are m dimensional, since $\Lambda_\pm^0$ are. Our next result is the following

Lemma 3.7 *The manifolds $\Lambda^q_\pm$ are transverse to $\{x=0\}$ and $\{\xi=0\}$.*

Proof: The first result is a direct consequence of Lemma 3.3 for p_0 and (3.7). ∎

A consequence of the previous result is the following lemma.

Lemma 3.8 *There exists an holomorphic function Φ such that $\Lambda^q_+ = \{(x,\xi),\ \xi = \nabla\Phi(x)\}$. In addition Φ solves the eikonal equation $q(x,\nabla\Phi(x))=0$ and $\Lambda^q_-\ \{(x,\xi),\ \xi=-\nabla\Phi(\overline{x})\}$, and $\mathrm{Re}\,\mathsf{Hess}\,\Phi(\rho_0)$ is positive definite.*

Proof: The existence of Φ is a direct consequence of the fact that $\Lambda^q_+ \cap \{x=0\} = \{0\}$. Now for x in a neighborhood of ρ_0 in $\mathbb{C}^m$, and looking at the characteristic $X(t)$ of H_q beginning at $X_0 = (x, \nabla\Phi(x))$, we get that

$$q(x,\nabla\Phi(x)) = q(X_0) = \lim_{t\to-\infty} q(X(t)) = q(\rho_0) = 0,$$

since q is constant along the characterstics of H_q. The phase Φ therefore solves the eikonal equation. For the description of Λ^q_-, we first check that for all x in a neighboorhood of ρ_0 in $\mathbb{C}^m$, $\overline{x}$ also belongs to a neighborhood of ρ_0 and therefore $q(\overline{x},\nabla\Phi(\overline{x}))=0$. Taking the complex conjugate and using that p is real analytic gives

$$0 = \overline{q(\overline{x},\nabla\Phi(\overline{x}))} = \overline{p(\overline{x}, i\nabla\Phi(\overline{x}))} = p(x, -i\overline{\nabla\Phi(\overline{x})}) = q(x,-\overline{\nabla\Phi(\overline{x})})$$

which gives that q is zero on $\left\{(x,\xi),\ \xi=-\overline{\nabla\Phi(\overline{x})}\right\}$. Now this holomorphic manifold is of dimension m, is clearly lagrangian (of the form $\{\xi=\psi'(x)\}$ for some ψ), and we only have to check that it coincides with Λ_-. For this, it is sufficient to check that $(x,-\overline{\nabla\Phi(\overline{x})}) \in \Lambda_-$. Now if we look at the solution of $X'(t) = H_q(X(t))$ with initial condition $X_0 = (x,-\overline{\nabla\Phi(\overline{x})})$ then we easily get that $\lim_{t\to+\infty} X(t) = \rho_0$. From dimensional considerations, we get that $\left\{\xi=-\overline{\nabla\Phi(\overline{x})}\right\} = \Lambda^q_-$.

Now for the last point, we first recall that at the linearization level, we have

$$\Lambda^{0,q}_+ = \{\xi = \nabla\Phi_0(x)\}$$

where $\nabla\Phi_0(x) = -iBx$ is exactly the linear part of the expansion of $\nabla\Phi(x)$ at ρ_0. From the previous subsection, we get that $\mathrm{Re}\,\mathsf{Hess}\,\Phi(\rho_0)$ is a real positive definite quadratic form. ∎

3.2 WKB expansions

In this section, we go back to (3.1) with the coordinates (x,ξ) there and we assume that $A_2 = 0$ (note that this was not the case in the previous subsection). Before starting our magnetic WKB analysis we center again the operator $\mathfrak{L}_h$ at x_0 and perform a change of gauge in order to center the phase at ξ_0. This means that we rather consider

$$\mathfrak{L}^\natural_h = D_t^2 + (hD_s + A^\natural)^2, \quad A^\natural(s,t) = \xi_0 + A_1(x_0+s,t).$$

In order to lighten the notation, we introduce

$$\mathcal{M}^\natural_{s,\xi} = \mathcal{M}_{s+x_0,\xi+\xi_0}, \quad u^\natural_{s,\xi} = u_{s+x_0,\xi+\xi_0}, \quad \mu^\natural(s,\xi) = \mu_1(s+x_0,\xi+\xi_0).$$

The assumption $A_2=0$ implies the fundamental property:

$$\left(\mathcal{M}^\natural_{s,\xi}\right)^* = \mathcal{M}^\natural_{s,\bar\xi}$$

so that

$$\overline{u^\natural_{s,\xi}} = u^\natural_{s,\bar{\xi}}. \tag{3.8}$$

We conjugate $\mathfrak{L}^\natural_h$ via a weight function $\Phi = \Phi(s)$ and define

$$\begin{aligned}\mathfrak{L}^\natural_\Phi &= \mathrm{e}^{\Phi(s)/h}\mathfrak{L}^\natural_h \mathrm{e}^{-\Phi(s)/h} \\ &= D_t^2 + (hD_s + i\nabla\Phi + A^\natural)^2 \\ &= \mathfrak{L}^\natural_0 + h\mathfrak{L}^\natural_1 + h^2\mathfrak{L}^\natural_2,\end{aligned}$$

with

$$\mathfrak{L}^\natural_0 = D_t^2 + (i\nabla\Phi + A^\natural)^2 = \mathcal{M}^\natural_{s,i\nabla\Phi(s)}$$
$$\mathfrak{L}^\natural_1 = D_s \cdot (i\nabla\Phi + A^\natural) + (i\nabla\Phi + A^\natural)\cdot D_s = \tfrac{1}{2}\left(D_s \cdot (\nabla_\xi \mathcal{M}^\natural)_{s,i\nabla\Phi(s)} + (\nabla_\xi \mathcal{M}^\natural)_{s,i\nabla\Phi(s)} \cdot D_s\right)$$
$$\mathfrak{L}^\natural_2 = D_s^2\Phi,$$

where we have used

$$(\nabla_\xi \mathcal{M}^\natural)_{s,i\nabla\Phi(s)} = 2(i\nabla\Phi(s) + A^\natural). \tag{3.9}$$

We now look for a formal solution on the form

$$\lambda \sim \sum_{j\geq 0}\lambda_j h^j, \qquad a \sim \sum_{j\geq 0} a_j h^j$$

such that $\mathfrak{L}^\natural_\Phi a = \lambda a$. We cancel each power of h step by step.

3.2.1 Term in h^0: Solving the operator valued eikonal equation

We have to find (λ_0, a_0) such that

$$\mathfrak{L}^\natural_0 a_0 = \lambda_0 a_0.$$

According to Theorem 1.8, we must choose

$$\lambda_0 = \mu_0,$$

Thus we have to find a_0 such that

$$\mathfrak{L}^\natural_0 a_0 = \mu_0 a_0, \tag{3.10}$$

that is to say

$$\mathcal{M}^\natural_{s,i\nabla\Phi(s)} a_0 = \mu_0 a_0.$$

We can choose a_0 on the form

$$a_0(s,t) = u^\natural_{s,i\nabla\Phi(s)}(t) b_0(s),$$

with b_0 to determine and Φ is the solution constructed in Section 3.1 of the following eikonal equation

$$\mu^\natural(s, i\nabla_s\Phi) = \mu_0,$$

so that a_0 solves (3.10).

3.2.2 Term in h^1: Solving the operator valued transport equation

We must now to solve the second equation, which is also the first transport equation,

$$(\mathfrak{L}_1^\natural - \lambda_1)a_0 = -(\mathfrak{L}_0^\natural - \mu_0)a_1$$

Pointwise in s, the Fredholm compatibility condition writes

$$(\lambda_1 - \mathfrak{L}_1^\natural)a_0 \in (\mathrm{Ker}(\mathfrak{L}_0^\natural - \mu_0)^*)^\perp = (\mathrm{Ker}(\mathfrak{L}_0^{\natural\,*} - \mu_0))^\perp$$

Note that this last space is closed and that we can write

$$\mathfrak{L}_0^\natural u^\natural_{s,i\nabla\Phi(s)}(t) = \mu_0 u^\natural_{s,i\nabla\Phi(s)}(t), \qquad \mathfrak{L}_0^{\natural\,*} u^\natural_{s,-i\nabla\overline{\Phi}(s)}(t) = \mu_0 u^\natural_{s,-i\nabla\overline{\Phi}(s)}(t).$$

Thus we have $(\mathrm{Ker}(\mathfrak{L}_0^{\natural\,*} - \mu_0))^\perp = \mathsf{span}(u_{s,-i\nabla\overline{\Phi}(s)})$. The compatibility condition is equivalent to

$$\lambda_1 \left\langle u^\natural_{s,i\nabla\Phi(s)} b_0(s), u^\natural_{s,-i\nabla\overline{\Phi}(s)} \right\rangle_{\mathrm{L}^2(\mathbb{R}^m,\,\mathrm{d}t)} = \left\langle \mathfrak{L}_1^\natural u^\natural_{s,i\nabla\Phi(s)} b_0(s), u^\natural_{s,-i\nabla\overline{\Phi}(s)} \right\rangle_{\mathrm{L}^2(\mathbb{R}^m,\,\mathrm{d}t)}, \ \forall s \in \mathbb{R}^m.$$

By (1.5), we have

$$\int_{\mathbb{R}^n} u^\natural_{s,i\nabla\Phi(s)}(t)\overline{u^\natural_{s,-i\nabla\overline{\Phi}(s)}(t)}dt = 1, \qquad \forall s \in \mathbb{R}^m. \tag{3.11}$$

Let us rewrite the first term:

$$\lambda_1 \left\langle u^\natural_{s,i\nabla\Phi(s)} b_0(s), u^\natural_{s,-i\nabla\overline{\Phi}(s)} \right\rangle_{\mathrm{L}^2(\mathbb{R}^m,\,\mathrm{d}t)} \quad = \quad \lambda_1 b_0(s). \tag{3.12}$$

Let us deal with the second term. Differentiating the relation $\mathcal{M}^\natural_{x,\xi} u^\natural_{x,\xi} = \mu_1^\natural(x,\xi) u^\natural_{x,\xi}$ with respect to (the complex variable) ξ_k leads to (see Proposition 2.2)

$$(\mathcal{M}^\natural_{x,\xi} - \mu_1^\natural(x,\xi))\nabla_\xi u^\natural_{x,\xi} = (\nabla_\xi \mu_1^\natural(x,\xi) - \nabla_\xi \mathcal{M}^\natural_{x,\xi}) u^\natural_{x,\xi}.$$

The Fredholm condition writes

$$\langle (\nabla_\xi \mathcal{M}^\natural_{x,\xi} - \nabla_\xi \mu_1^\natural(x,\xi)) u^\natural_{x,\xi}, u^\natural_{x,\overline{\xi}} \rangle_{\mathrm{L}^2(\mathbb{R}^n,\,\mathrm{d}t)} = 0.$$

Consequently, taking $\xi = i\nabla\Phi$ and using (3.8), we get

$$\nabla_\xi \mu_1^\natural(s, i\nabla\Phi(s)) = \int_{\mathbb{R}^n} (\nabla_\xi \mathcal{M}^\natural)_{s,i\nabla\Phi(s)} \ u^\natural_{s,i\nabla\Phi(s)}(t) u^\natural_{s,i\nabla\Phi(s)}(t)\,\mathrm{d}t. \tag{3.13}$$

Multiplying by b_0 and differentiating with respect to s, we infer

$$\begin{aligned}
D_s \cdot (b_0(s)\nabla_\xi \mu_1^\natural(s, i\nabla\Phi(s))) &= \int_{\mathbb{R}^n} D_s \cdot \Big((\nabla_\xi \mathcal{M}^\natural)_{s,i\nabla\Phi(s)} b_0(s) u^\natural_{s,i\nabla\Phi(s)}(t) \Big) \, u^\natural_{s,i\nabla\Phi(s)}(t)\,\mathrm{d}t \\
&\quad + \int_{\mathbb{R}^n} (\nabla_\xi \mathcal{M}^\natural)_{s,i\nabla\Phi(s)} b_0(s) u^\natural_{s,i\nabla\Phi(s)}(t) \cdot D_s u^\natural_{s,i\nabla\Phi(s)}(t)\,\mathrm{d}t \\
&= \int_{\mathbb{R}^n} D_s \cdot \Big((\nabla_\xi \mathcal{M}^\natural)_{s,i\nabla\Phi(s)} b_0(s) u^\natural_{s,i\nabla\Phi(s)}(t) \Big) \, u^\natural_{s,i\nabla\Phi(s)}(t)\,\mathrm{d}t \\
&\quad + \int_{\mathbb{R}^n} (\nabla_\xi \mathcal{M}^\natural)_{s,i\nabla\Phi(s)} u^\natural_{s,i\nabla\Phi(s)}(t) \cdot D_s \big(b_0(s) u^\natural_{s,i\nabla\Phi(s)}(t) \big)\,\mathrm{d}t \\
&\quad - D_s b_0(s) \cdot \int_{\mathbb{R}^n} (\nabla_\xi \mathcal{M}^\natural)_{s,i\nabla\Phi(s)} u^\natural_{s,i\nabla\Phi(s)}(t) u^\natural_{s,i\nabla\Phi(s)}(t)\,\mathrm{d}t,
\end{aligned}$$

where we have used $b_0 D_s u = D_s(b_0 u) - u D_s b_0$. We recognize the expression of $\mathfrak{L}_1^\natural$ and using (3.13), this yields

$$\langle \mathfrak{L}_1^\natural u^\natural_{s,i\nabla\Phi(s)} b_0(s), u^\natural_{s,-i\nabla\overline{\Phi}(s)} \rangle_{\mathrm{L}^2(\mathbb{R}^m,\mathrm{d}t)} \\ = \tfrac{1}{2}\Big(\nabla_\xi \mu^\natural(s, i\nabla\Phi(s)) \cdot D_s b_0(s) + D_s \cdot \Big(\nabla_\xi \mu^\natural(s, i\nabla\Phi(s)) b_0(s) \Big) \Big).$$

We get the effective transport equation

$$\tfrac{1}{2}\left(\nabla_\xi \mu^\natural \cdot D_s + D_s \cdot \nabla_\xi \mu^\natural \right) b_0 = \lambda_1 b_0.$$

where $\mu^\natural$ stands for $\mu^\natural(s, i\nabla\Phi(s)$ for short. In the following we denote

$$\mathscr{T} = \tfrac{1}{2}\left(\nabla_\xi \mu^\natural \cdot D_s + D_s \cdot \nabla_\xi \mu^\natural \right),$$

and we therefore have to solve the transport equation

$$\mathscr{T} b_0 = \lambda_1 b_0.$$

We recall now some notations from the previous subsection. We posed there

$$q(s,\xi) = -\mu^\natural(s, i\xi)$$

so that with this notation

$$\mathscr{T} = \tfrac{1}{2}\left(\nabla_\xi q(s, \nabla\Phi(s)) \cdot \nabla_s + \nabla_s \cdot \nabla_\xi q(s, \nabla\Phi(s)) \right). \tag{3.14}$$

Now from the analysis in the previous section, we see that apart from the term without derivative

$$\mathsf{v}(s) = \tfrac{1}{2i}\nabla_s \cdot (\nabla_\xi \mu^\natural) = \tfrac{1}{2}\nabla_s \cdot (\nabla_\xi q(s, \nabla\Phi(s)),$$

the transport $\mathscr{T}$ is exactly the projection on $T\mathbb{R}^m_s$ of the hamiltonian of $q(s,\xi)$ at the point $(s, \nabla\Phi(s)) \in \Lambda_+$,

$$\mathscr{T} = \pi_s(H_q(s, \nabla\Phi(s))),$$

which reads in suitable coordinates

$$\mathscr{T} = \sum_j \mu_j y_j \nabla_{y_j} + \mathcal{O}(y^2)\nabla_{y_j} + \mathsf{v}(s),$$

where $(y,\eta) = (\kappa(s), {}^t\,\mathrm{d}\kappa(s)^{-1}\xi)$ is the corresponding symplectic change of variable (note that for correctly defining κ we used here that $\Lambda_+ \cap \{x = 0\} = \{0\}$). In particular, we get that $\mathscr{T}$ is an expansive vector field since Λ_+^q is the outgoing manifold associated to q at the doubly charactersitic point $(0,0)$. Using again this change of coordinates, we also directly get that

$$\mathsf{v}(s) = \sum_j \tfrac{1}{2}\mu_j + \mathcal{O}(s).$$

We are reduced to the study of a standard transport equation (see e.g. [15]) and we can at least formally solve the equation $\mathscr{T} b_0 = \lambda_1 b_0$ in the space of formal series first provided

$$\lambda_1 \in \Big\{ \sum_j \mu_j(\tfrac{1}{2} + \alpha_j), \quad \alpha \in \mathbb{N}^m \Big\}.$$

These values are exactly the eigenvalues of $\frac{1}{2}\mathsf{Hess}\ \mu_1(x_0,\xi_0)(\sigma, D_\sigma)$, with in particular $\frac{1}{2}\sum_j \mu_j$ for the smallest one and recall that they were supposed to be simple by Assumption 1.3. This is of course coherent with the statement of Theorem 1.8. Following again e.g. [15], we can also solve this first transport equation in $\mathcal{C}^\infty$ in a neighborhood of 0.

Let us now take $\alpha \in \mathbb{N}^m$ fixed. We shall now look at the complete first transport equation:

$$(\mathfrak{L}_1^\natural - \lambda_1)a_0 = -(\mathfrak{L}_0^\natural - \mu_0)a_1$$

where a_0 is now given by

$$a_0(s,t) = u^\natural_{s,i\nabla\Phi(s)}(t)b_0(s) \tag{3.15}$$

We look at a function a_1 not necessary given in a tensor form, but with an orthogonal decomposition of the form

$$a_1 = u^\natural_{s,i\nabla\Phi(s)}(t)b_1(s) + \tilde{a}_1(s,t) \tag{3.16}$$

with

$$\tilde{a}_1 \in (\mathrm{Ker}(\mathfrak{L}_0^\natural - \mu_0))^\perp$$

from the transport equation, we see that we can directly find

$$\tilde{a}_1 = -(\mathfrak{L}_0^\natural - \mu_0)^{-1}(\mathfrak{L}_1^\natural - \lambda_1)a_0,$$

where by operator $(\mathfrak{L}_0^\natural - \mu_0)^{-1}$ we mean the inverse of operator

$$(\mathfrak{L}_0^\natural - \mu_0) : \mathsf{Dom}(\mathfrak{L}_0^\natural - \mu_0) \cap (\mathrm{Ker}(\mathfrak{L}_0^\natural - \mu_0))^\perp \longrightarrow (\mathrm{Ker}(\mathfrak{L}_0^\natural - \mu_0))^\perp$$

where $\mathsf{Dom}(\mathfrak{L}_0^\natural - \mu_0)$ is the domain in $\mathrm{L}^2(\mathbb{R}^m, \mathrm{d}t)$ of operator $\mathfrak{L}_0^\natural - \mu_0$, pointwisely defined in s. Note that this is possible to apply this operator to $a_0(s,.)$ because of the properties of $u^\natural$, and that we indeed get that $\tilde{a}_1$ is smooth by elliptic regularity.

To summarize, at this point, and whatever $b_1(s)$ is, we have been able to find a function

$$a^{[1]}(s,t) = a_0(s,t) + ha_1(s,t)$$

such that

$$(\mathfrak{L}_h^\natural - \mu_0 - h\lambda_1)a_1 \mathrm{e}^{-\Phi/h} = r^{[1]}\mathrm{e}^{-\Phi/h}$$

where for all $\beta \in \mathbb{N}^m$

$$|\partial^\beta_{s,t} r^{[1]}(s,t)| = \mathcal{O}(h^2), \qquad \text{locally in } (s,t).$$

3.2.3 Full asymptotic expansion

The first step is to find the function $b_1(s)$ built in the previous subsection. For this we look at the next transport equation, which reads

$$(\mathfrak{L}_1^\natural - \lambda_1)a_1 = -(\mathfrak{L}_0^\natural - \mu_0)a_2 - \mathfrak{L}_2^\natural a_0 + \lambda_2 a_0$$

We look at the natural compatibility Fredholm condition which gives, pointwisely in s,

$$(\mathfrak{L}_1^\natural - \lambda_1)a_1 + (\mathfrak{L}_2^\natural - \lambda_2)a_0 \in (\mathrm{Ker}(\mathfrak{L}_0^\natural - \mu_0)^*)^\perp = \mathsf{span}(u_{s,-i\nabla\overline{\Phi}(s)})$$

This compatibility condition is then equivalent to

$$\begin{aligned}
&\left\langle (\mathfrak{L}_1^\natural - \lambda_1) a_1(s), u^\natural_{s,-i\nabla\overline{\Phi}(s)} \right\rangle_{\mathrm{L}^2(\mathbb{R}^m,\,\mathrm{d}t)} \\
&= -\left\langle \mathfrak{L}_2^\natural a_0, u^\natural_{s,-i\nabla\overline{\Phi}(s)} \right\rangle_{\mathrm{L}^2(\mathbb{R}^m,\,\mathrm{d}t)} + \lambda_2 \left\langle a_0, u^\natural_{s,-i\nabla\overline{\Phi}(s)} \right\rangle_{\mathrm{L}^2(\mathbb{R}^m,\,\mathrm{d}t)} \\
&= -\left\langle \mathfrak{L}_2^\natural a_0, u^\natural_{s,-i\nabla\overline{\Phi}(s)} \right\rangle_{\mathrm{L}^2(\mathbb{R}^m,\,\mathrm{d}t)} + \lambda_2 b_0(s), \qquad \forall s \in \mathbb{R}^m
\end{aligned}$$

where we used (3.12). Now Using the splitting (3.16) and the expression for a_0 in 3.15 we get that for all $s \in \mathbb{R}^m$,

$$\begin{aligned}
&\left\langle (\mathfrak{L}_1^\natural - \lambda_1) u^\natural_{s,i\nabla\Phi(s)} b_1(s), u^\natural_{s,-i\nabla\overline{\Phi}(s)} \right\rangle_{\mathrm{L}^2(\mathbb{R}^m,\,\mathrm{d}t)} \\
&= -\left\langle \mathfrak{L}_2^\natural a_0, u^\natural_{s,-i\nabla\overline{\Phi}(s)} \right\rangle_{\mathrm{L}^2(\mathbb{R}^m,\,\mathrm{d}t)} + \lambda_2 \left\langle a_0, u^\natural_{s,-i\nabla\overline{\Phi}(s)} \right\rangle_{\mathrm{L}^2(\mathbb{R}^m,\,\mathrm{d}t)} \\
&\qquad\qquad + \left\langle (\mathfrak{L}_1^\natural - \lambda_1)\tilde{a}_1(s), u^\natural_{s,-i\nabla\overline{\Phi}(s)} \right\rangle_{\mathrm{L}^2(\mathbb{R}^m,\,\mathrm{d}t)} \\
&= -\left\langle \mathfrak{L}_2^\natural a_0, u^\natural_{s,-i\nabla\overline{\Phi}(s)} \right\rangle_{\mathrm{L}^2(\mathbb{R}^m,\,\mathrm{d}t)} + \lambda_2 b_0(s) + \left\langle (\mathfrak{L}_1^\natural - \lambda_1)\tilde{a}_1(s), u^\natural_{s,-i\nabla\overline{\Phi}(s)} \right\rangle_{\mathrm{L}^2(\mathbb{R}^m,\,\mathrm{d}t)}
\end{aligned}$$

Using the definition of the reduced transport introduced in (3.14), this equation reads

$$\mathscr{T} b_1(s) = \lambda_1 b_1(s) + \lambda_2 b_0(s) + R_1(s),$$

where λ_2 has to be defined and R_1 is already completely known and defined by

$$R_1(s) = -\left\langle \mathfrak{L}_2^\natural a_0, u^\natural_{s,-i\nabla\overline{\Phi}(s)} \right\rangle_{\mathrm{L}^2(\mathbb{R}^m,\,\mathrm{d}t)} + \left\langle (\mathfrak{L}_1^\natural - \lambda_1)\tilde{a}_1(s), u^\natural_{s,-i\nabla\overline{\Phi}(s)} \right\rangle_{\mathrm{L}^2(\mathbb{R}^m,\,\mathrm{d}t)}$$

Using the classical theory of formal series (and using that that both b_1 and b_0 have at first non zero term in their Taylor expansion, we see that it completely determines the value of λ_2. Now, exactly as we did for b_0 in the previous section, we can also solve, first in formal series, and then in $\mathcal{C}^\infty$ this transport equation on b_1.

We can look for the next function a_2 of the form

$$a_2 = u^\natural_{s,i\nabla\Phi(s)}(t) b_2(s) + \tilde{a}_2(s,t) \tag{3.17}$$

with

$$\tilde{a}_2 \in (\mathrm{Ker}(\mathfrak{L}_0^\natural - \mu_0))^\perp$$

from the transport equation, we see that we can directly find

$$\tilde{a}_2 = -(\mathfrak{L}_0^\natural - \mu_0)^{-1}(\mathfrak{L}_1^\natural - \lambda_{1,\alpha_0}) a_1.$$

To summarize, at this point, and whatever $b_2(s)$ is, we have been able to find a function

$$a^{[2]}(s,t) = a_0(s,t) + h a_1(s,t) + h^2 a_2(s,t)$$

such that

$$(\mathfrak{L}_h^\natural - \mu_0 - h\lambda_1 - h^2\lambda_2) a^{[2]} \mathrm{e}^{-\Phi/h} = r^{[2]} \mathrm{e}^{-\Phi/h}$$

where, for all $\beta \in \mathbb{N}^m$,

$$|\partial^\beta_{s,t} r^{[2]}| = \mathcal{O}(h^3), \qquad \text{locally in } (s,t).$$

The same procedure can be followed at any order (we only saw the complete construction of a_0 and a_1), and continuing the previous construction, we see that we are able to find a full family of functions a_j and λ_j such that the formal series

$$a \sim \sum_{j\geq 0} a_j h^j \qquad \lambda \sim \sum_{j\geq 0} \lambda_j h^j,$$

formally solves the equation

$$(\mathfrak{L}_h^\flat - \lambda) a \mathrm{e}^{-\Phi/h} = \mathcal{O}(h^\infty)\mathrm{e}^{-\Phi/h}.$$

Using the simplicity and the localization of each eigenvalue induced by Theorem 1.8, we complete the proof of Theorem 1.10.

4 Generalized Montgomery operators

We focus in this section on the operator

$$\mathfrak{L}_h^{[k]} = D_t^2 + \left(hD_s - \gamma(s)\frac{t^{k+1}}{k+1}\right)^2$$

obtained in (1.8) after the rescaling described there. We denote by $\lambda_n^{[k]}(h)$ its n-th eigenvalue if it exists.

4.1 Verifying assumptions

The aim of this section is to prove that operator $\mathfrak{L}_h^{[k]}$ fulfills the assumptions mentioned in Section 1.2.

Proposition 4.1 *For $k \in \mathbb{N}\setminus\{0\}$ and under Assumption 1.13, the operator $\mathfrak{L}_h^{[k]}$ satisfies Assumptions 1.1, 1.3 and 1.5. Moreover we can choose*

$$\mu_0^* \in \left(\mu_0, (\min(\gamma_{-\infty}, \gamma_{+\infty}))^{\frac{2}{k+2}} \min \nu_1^{[k]}\right),$$

where $\nu_1^{[k]}(\zeta)$ denotes the first eigenvalue of the generalized Montgomery operator:

$$D_\tau^2 + \left(\zeta - \frac{\tau^{k+1}}{k+1}\right)^2. \tag{4.1}$$

Notation 4.2 *Let $u_\zeta^{[k]}$ be a L^2-normalized eigenfunction depending analytically on ζ and associated with $\nu_1^{[k]}(\zeta)$.*

4.1.1 Uniqueness and non-degeneracy

The symbol of $\mathfrak{L}_h^{[k]}$ with respect to s is

$$\mathcal{M}_{x,\xi}^{[k]} = D_t^2 + \left(\xi - \gamma(x)\frac{t^{k+1}}{k+1}\right)^2.$$

The lowest eigenvalue of $\mathcal{M}_{x,\xi}^{[k]}$, denoted by $\mu_1^{[k]}(x,\xi)$, satisfies:

$$\mu_1^{[k]}(x,\xi) = (\gamma(x))^{\frac{2}{k+2}} \nu_1^{[k]}\left((\gamma(x))^{-\frac{1}{k+2}}\xi\right).$$

It is proved in [20, Theorem 1.3] that $\zeta \mapsto \nu_1^{[k]}(\zeta)$ admits a unique and non degenerate minimum. Therefore Assumption 1.1 is satisfied. Note that Assumption 1.3 is satisfied since $n = 1$ according to Remark 1.4.

Notation 4.3 *We denote by $\zeta_0^{[k]}$ the point ζ where the minimum of $\nu_1^{[k]}$ is reached.*

4.1.2 Confinement

This verification is a little more subtle. It is based on a normal form procedure. Let us denote by $\mathfrak{Q}_h^{[k]}$ the quadratic form associated with $\mathfrak{L}_h^{[k]}$. For $\psi \in \mathsf{Dom}(\mathfrak{Q}_h^{[k]})$, we have

$$\mathfrak{Q}_h^{[k]}(\psi) = \int_{\mathbb{R}^2} |D_t\psi|^2 + \left| \left(hD_s - \gamma(s)\frac{t^{k+1}}{k+1} \right) \psi \right|^2 \mathrm{d}t\,\mathrm{d}s.$$

We would like to prove a lower bound for $\mathfrak{Q}_h^{[k]}(\psi)$ when $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ is supported away from the square $[-R_0, R_0]^2$.

Magnetic confinement for the variable t. If ψ is supported in $\{(s,t) \in \mathbb{R}^2 : |t| > R\}$, we can use the standard inequality (see [13, Chapter 6])

$$\mathfrak{Q}_h^{[k]}(\psi) \geq \int_{\mathbb{R}^2} \gamma(s)|t|^k|\psi|^2 \,\mathrm{d}s\,\mathrm{d}t \geq R^k\gamma_0\|\psi\|^2. \tag{4.2}$$

Spectral confinement for the variable s. If ψ is supported in $\{(s,t) \in \mathbb{R}^2 : |t| < R, \quad |s| \geq R\}$, we use the canonical transformation associated with the change of variables

$$t = (\gamma(\sigma))^{-\frac{1}{k+2}}\tau, \quad s = \sigma, \tag{4.3}$$

we deduce that $\mathfrak{L}_h^{[k]}$ is unitarily equivalent to the operator on $\mathrm{L}^2(\,\mathrm{d}\sigma\,\mathrm{d}\tau)$

$$\mathfrak{L}_h^{[k],\flat} = \gamma(\sigma)^{\frac{2}{k+2}}D_\tau^2 + \left(hD_\sigma - \gamma(\sigma)^{\frac{1}{k+2}}\frac{\tau^{k+1}}{k+1} + \frac{h}{2(k+2)}\frac{\gamma'(\sigma)}{\gamma(\sigma)}(\tau D_\tau + D_\tau\tau) \right)^2.$$

Let us denote by $\psi^\flat$ the function ψ transported by the canonical transformation. In the terms of the quadratic from, we have

$$\mathfrak{Q}_h^{[k]}(\psi) = \mathfrak{Q}_h^{[k],\flat}(\psi^\flat).$$

Let us notice that $\psi^\flat$ is supported in $\left\{(\sigma,\tau) \in \mathbb{R}^2 : |\sigma| \geq R, \quad |\tau| \leq R\gamma(\sigma)^{\frac{1}{k+2}}\right\}$. We can write, for all $\varepsilon \in (0,1)$

$$\begin{aligned}\mathfrak{Q}_h^{[k],\flat}(\psi^\flat) \geq \\ (1-\varepsilon)\int_{\mathbb{R}^2} \gamma(\sigma)^{\frac{2}{k+2}} \left(|D_\tau\psi^\flat|^2 + \left| \left(h\gamma(\sigma)^{-\frac{1}{k+2}}D_\sigma - \frac{\tau^{k+1}}{k+1} \right)\psi^\flat \right|^2 \right) \mathrm{d}\sigma\,\mathrm{d}\tau \\ - \frac{\varepsilon^{-1}h^2}{(2(k+2))^2}\int_{\mathbb{R}^2} \left| \frac{\gamma'(\sigma)}{\gamma(\sigma)}(\tau D_\tau + D_\tau\tau)\psi^\flat \right|^2 \mathrm{d}\sigma\,\mathrm{d}\tau.\end{aligned} \tag{4.4}$$

Thanks to Assumption 1.13, there exists $\eta_0 > 0$ and for all $\eta \in (0,\eta_0)$, $R_0 > 0$ such that, on $\{|\sigma| \geq R_0\}$

$$\gamma(\sigma)^{\frac{2}{k+2}} \min_{\zeta\in\mathbb{R}} \nu_1^{[k]}(\zeta) \geq \left\{ (\min(\gamma_{-\infty}, \gamma_{+\infty}))^{\frac{2}{k+2}} - \eta \right\} \min_{\zeta\in\mathbb{R}} \nu_1^{[k]}(\zeta) > \mu_1^{[k]}(x_0,\xi_0). \tag{4.5}$$

Up to choosing R_0 larger, we may also assume that

$$\left(\frac{R_0}{2}\right)^k \gamma_0 \geq \left\{(\min(\gamma_{-\infty},\gamma_{+\infty}))^{\frac{2}{k+2}} - \eta\right\} \min_{\zeta\in\mathbb{R}} \nu_1^{[k]}(\zeta). \tag{4.6}$$

Moreover, we have

$$\int_{\mathbb{R}^2} \left|\frac{\gamma'(\sigma)}{\gamma(\sigma)}(\tau D_\tau + D_\tau \tau)\psi^\flat\right|^2 \mathrm{d}\sigma\,\mathrm{d}\tau \leq C\int_{\mathbb{R}^2} |\psi^\flat|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ + 4\int_{\mathbb{R}^2}\left(\frac{\gamma'(\sigma)}{\gamma(\sigma)}\right)^2 |\tau D_\tau\psi^\flat|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau. \tag{4.7}$$

Using support considerations, we get

$$\int_{\mathbb{R}^2} \left|\frac{\gamma'(\sigma)}{\gamma(\sigma)}(\tau D_\tau + D_\tau \tau)\psi^\flat\right|^2 \mathrm{d}\sigma\,\mathrm{d}\tau \leq C\int_{\mathbb{R}^2} |\psi^\flat|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ + 4R^2\int_{\mathbb{R}^2} \gamma(\sigma)^{\frac{2}{k+2}-2}\gamma'(\sigma)^2 |D_\tau\psi^\flat|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau,$$

so that

$$\int_{\mathbb{R}^2} \left|\frac{\gamma'(\sigma)}{\gamma(\sigma)}(\tau D_\tau + D_\tau \tau)\psi^\flat\right|^2 \mathrm{d}\sigma\,\mathrm{d}\tau \leq C\int_{\mathbb{R}^2} |\psi^\flat|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau + 4CR^2\int_{\mathbb{R}^2} |D_\tau\psi^\flat|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau.$$

We deduce

$$\mathfrak{Q}_h^{[k],\flat}(\psi^\flat) \geq \\ (1-\varepsilon)\left((\min(\gamma_{-\infty},\gamma_{+\infty}))^{\frac{2}{k+2}} - \eta\right)\int_{\mathbb{R}^2}\left(|D_\tau\psi^\flat|^2 + \left|\left(h\gamma(\sigma)^{-\frac{1}{k+2}}D_\sigma - \frac{\tau^{k+1}}{k+1}\right)\psi^\flat\right|^2\right)\mathrm{d}\sigma\,\mathrm{d}\tau \\ - C\varepsilon^{-1}h^2\int_{\mathbb{R}^2}\left|\psi^\flat\right|^2\mathrm{d}\sigma\,\mathrm{d}\tau - 4CR^2\varepsilon^{-1}h^2\int_{\mathbb{R}^2}\left|D_\tau\psi^\flat\right|^2\mathrm{d}\sigma\,\mathrm{d}\tau.$$

We choose $\varepsilon = h$, we infer that

$$\mathfrak{Q}_h^{[k],\flat}(\psi^\flat) \geq \left\{(1-h)\left((\min(\gamma_{-\infty},\gamma_{+\infty}))^{\frac{2}{k+2}} - \eta\right) - 4CR^2h\right\}\mathfrak{q}_h^{[k],\flat}(\psi^\flat) \\ - Ch\int_{\mathbb{R}^2}\left|\psi^\flat\right|^2\mathrm{d}\sigma\,\mathrm{d}\tau, \tag{4.8}$$

where

$$\mathfrak{q}_h^{[k],\flat}(\psi^\flat) = \int_{\mathbb{R}^2}\left(|D_\tau\psi^\flat|^2 + \left|\left(h\gamma(\sigma)^{-\frac{1}{k+2}}D_\sigma - \frac{\tau^{k+1}}{k+1}\right)\psi^\flat\right|^2\right)\mathrm{d}\sigma\,\mathrm{d}\tau.$$

We have

$$\mathfrak{q}_h^{[k],\flat}(\psi^\flat) = \\ \int_{\mathbb{R}^2}\left(|D_\tau\psi^\flat|^2 + \left|\left(h\Xi(\sigma,D_\sigma) - \frac{\tau^{k+1}}{k+1} - \frac{ih}{2k+4}\gamma'(\sigma)\gamma(\sigma)^{-\frac{k+3}{k+2}}\right)\psi^\flat\right|^2\right)\mathrm{d}\sigma\,\mathrm{d}\tau, \tag{4.9}$$

with $\Xi(\sigma,D_\sigma)=\gamma(\sigma)^{-\frac{1}{2k+4}}D_\sigma\gamma(\sigma)^{-\frac{1}{2k+4}}$. We get the lower bound:

$$\mathfrak{q}_h^{[k],\flat}(\psi^\flat)\geq\int_{\mathbb{R}^2}\left(|D_\tau\psi^\flat|^2+\left|\left(h\Xi(\sigma,D_\sigma)-\frac{\tau^{k+1}}{k+1}\right)\psi^\flat\right|^2\right)\mathrm{d}\sigma\,\mathrm{d}\tau$$
$$+\frac{2h}{2k+4}\mathrm{Re}\int_{\mathbb{R}^2}i\gamma'(\sigma)\gamma(\sigma)^{-\frac{k+3}{k+2}}\left(h\Xi(\sigma,D_\sigma)-\frac{\tau^{k+1}}{k+1}\right)\psi^\flat\,\overline{\psi^\flat}\,\mathrm{d}\sigma\,\mathrm{d}\tau.$$

This becomes

$$\mathfrak{q}_h^{[k],\flat}(\psi^\flat)\geq\int_{\mathbb{R}^2}\left(|D_\tau\psi^\flat|^2+\left|\left(h\Xi(\sigma,D_\sigma)-\frac{\tau^{k+1}}{k+1}\right)\psi^\flat\right|^2\right)\mathrm{d}\sigma\,\mathrm{d}\tau$$
$$+\frac{2h^2}{2k+4}\mathrm{Re}\int_{\mathbb{R}^2}i\gamma'(\sigma)\gamma(\sigma)^{-\frac{k+3}{k+2}}\Xi(\sigma,D_\sigma)\psi^\flat\,\overline{\psi^\flat}\,\mathrm{d}\sigma\,\mathrm{d}\tau.$$

By using in particular that $2\mathrm{Re}\left(\partial_\sigma\psi^\flat\,\overline{\psi^\flat}\right)=\partial_\sigma|\psi^\flat|^2$ and by integrating by parts, we infer (since the derivatives of γ are bounded):

$$\left|\frac{2h^2}{2k+4}\mathrm{Re}\int_{\mathbb{R}^2}i\gamma'(\sigma)\gamma(\sigma)^{-\frac{k+3}{k+2}}\Xi(\sigma,D_\sigma)\psi^\flat\,\overline{\psi^\flat}\,\mathrm{d}\sigma\,\mathrm{d}\tau\right|\leq Ch^2\int_{\mathbb{R}^2}|\psi^\flat|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau.$$

By using the functional calculus, we get

$$\mathfrak{q}_h^{[k],\flat}(\psi^\flat)\geq\left(\min_{\zeta\in\mathbb{R}}\nu_1^{[k]}(\zeta)-Ch^2\right)\int_{\mathbb{R}^2}|\psi^\flat|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau.\tag{4.10}$$

Fixing $R=R_0$ and η_0 defined in (4.5) and (4.6) and combining (4.8) and (4.10), we infer the existence of $h_0>0$ and $C>0$ such that that for $h\in(0,h_0)$ and all $\eta\in(0,\eta_0)$

$$\mathfrak{Q}_h^{[k],\flat}(\psi^\flat)\geq\left((\min(\gamma_{-\infty},\gamma_{+\infty}))^{\frac{2}{k+2}}\min_{\zeta\in\mathbb{R}}\nu_1^{[k]}(\zeta)-\eta-Ch\right)\int_{\mathbb{R}^2}|\psi^\flat|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau$$

or equivalently

$$\mathfrak{Q}_h^{[k]}(\psi)\geq\left((\min(\gamma_{-\infty},\gamma_{+\infty}))^{\frac{2}{k+2}}\min_{\zeta\in\mathbb{R}}\nu_1^{[k]}(\zeta)-\eta-Ch\right)\int_{\mathbb{R}^2}|\psi|^2\,\mathrm{d}s\,\mathrm{d}t,$$

for all $\psi\in\mathcal{C}_0^\infty(\mathbb{R}^2)$ supported in $\{(s,t)\in\mathbb{R}^2:|t|<R_0,\quad|s|\geq R_0\}$.

Gluing the lower bounds. Using (4.2) (with $R=\frac{R_0}{2}$) and (4.6), we deduce

$$\mathfrak{Q}_h^{[k]}(\psi)\geq\left((\min(\gamma_{-\infty},\gamma_{+\infty}))^{\frac{2}{k+2}}\min_{\zeta\in\mathbb{R}}\nu_1^{[k]}(\zeta)-\eta-Ch\right)\int_{\mathbb{R}^2}|\psi|^2\,\mathrm{d}s\,\mathrm{d}t,$$

for all $\psi\in\mathcal{C}_0^\infty(\mathbb{R}^2)$ supported in $\{(s,t)\in\mathbb{R}^2:|t|<R_0,\quad|s|\geq R_0\}$ or in $\{(s,t)\in\mathbb{R}^2:|t|>\frac{R_0}{2}\}$. We now use a standard partition of unity with respect to t such that

$$\chi_{1,R_0}^2+\chi_{2,R_0}^2=1,\qquad\chi_{1,R_0}=\begin{cases}1\text{ for }|t|\leq\frac{R_0}{2},\\0\text{ for }|t|\geq R_0,\end{cases}\qquad\text{and }\left(\chi_{1,R_0}'\right)^2+\left(\chi_{2,R_0}'\right)^2\leq CR_0^{-2}.$$

The "IMS" formula provides, for all $\psi\in\mathcal{C}_0^\infty\left(\mathbb{R}^2\setminus[-R_0,R_0]^2\right)$:

$$\mathfrak{Q}_h^{[k]}(\psi)\geq\mathfrak{Q}_h^{[k]}(\chi_{1,R_0}\psi)+\mathfrak{Q}_h^{[k]}(\chi_{2,R_0}\psi)-CR_0^{-2}\|\psi\|^2.$$

By supports considerations we have

$$\mathsf{supp}\,(\chi_{1,R_0}\psi)\subset\{(s,t)\in\mathbb{R}^2:|t|<R_0,\ |s|\geq R_0\},\quad \mathsf{supp}\,(\chi_{2,R_0}\psi)\subset\{(s,t)\in\mathbb{R}^2:|t|\geq\tfrac{R_0}{2}\},$$

so that, for all $\psi\in\mathcal{C}_0^\infty\left(\mathbb{R}^2\setminus[-R_0,R_0]^2\right)$, there holds

$$\mathfrak{Q}_h^{[k]}(\psi)\geq\left((\min(\gamma_{-\infty},\gamma_{+\infty}))^{\frac{2}{k+2}}\min_{\zeta\in\mathbb{R}}\nu_1^{[k]}(\zeta)-\eta-Ch-CR_0^{-2}\right)\int_{\mathbb{R}^2}|\psi|^2\,\mathrm{d}s\,\mathrm{d}t.$$

Therefore, for all $\mu_0^*\in\left(\mu_0,(\min(\gamma_{-\infty},\gamma_{+\infty}))^{\frac{2}{k+2}}\min\nu_1^{[k]}\right)$, up to choosing η, h small enough and possibly R_0 larger, we have

$$\mathfrak{Q}_h^{[k]}(\psi)\geq\mu_0^*\int_{\mathbb{R}^2}|\psi|^2\,\mathrm{d}s\,\mathrm{d}t,\quad\forall\psi\in\mathcal{C}_0^\infty\left(\mathbb{R}^2\setminus[-R_0,R_0]^2\right).$$

4.2 Explicit WKB expansions, weak magnetic barrier

4.2.1 Renormalization

The key point to perform the spectral analysis of $\mathfrak{L}_h^{[k]}$ is the normal form procedure introduced in [45, 16, 43], see also [48] which is pervaded by this spirit. Let us explain this basic idea. We use the canonical transformation associated with the change of variables[1]

$$t=(\gamma(\sigma))^{-\frac{1}{k+2}}\tau,\qquad s=\sigma.$$

We deduce that $\mathfrak{L}_h^{[k]}$ is unitarily equivalent to the operator on $\mathrm{L}^2(\,\mathrm{d}\sigma\,\mathrm{d}\tau)$

$$\mathfrak{L}_h^{[k],\flat}=\gamma(\sigma)^{\frac{2}{k+2}}D_\tau^2+\left(hD_\sigma-\gamma(\sigma)^{\frac{1}{k+2}}\frac{\tau^{k+1}}{k+1}+\frac{h}{2(k+2)}\frac{\gamma'(\sigma)}{\gamma(\sigma)}(\tau D_\tau+D_\tau\tau)\right)^2.$$

In order to estimate the low lying eigenvalues, we may write the heuristic approximation

$$\begin{aligned}\mathfrak{L}_h^{[k],\flat}&\approx\gamma(\sigma)^{\frac{2}{k+2}}\left(D_\tau^2+\left(h\gamma(\sigma)^{-\frac{1}{k+2}}D_\sigma-\frac{\tau^{k+1}}{k+1}\right)^2\right)\\&\approx\gamma(\sigma)^{\frac{2}{k+2}}\nu_1^{[k]}\left(h\gamma(\sigma)^{-\frac{1}{k+1}}D_\sigma\right).\end{aligned}$$

Let now make this approximation more rigorous. We may change the gauge

$$\begin{aligned}&\mathrm{e}^{-ig(\sigma)/h}\mathfrak{L}_h^{[k],\flat}\mathrm{e}^{ig(\sigma)/h}\\&=\gamma(\sigma)^{\frac{2}{k+2}}D_\tau^2+\left(hD_\sigma+\zeta_0^{[k]}\gamma(\sigma)^{\frac{1}{k+2}}-\gamma(\sigma)^{\frac{1}{k+2}}\frac{\tau^{k+1}}{k+1}+\frac{h}{2(k+2)}\frac{\gamma'(\sigma)}{\gamma(\sigma)}(\tau D_\tau+D_\tau\tau)\right)^2.\end{aligned}$$

with

$$g(\sigma)=\zeta_0^{[k]}\int_0^\sigma\gamma(\tilde\sigma)^{\frac{1}{k+2}}\,\mathrm{d}\tilde\sigma.$$

For some function $\Phi=\Phi(\sigma)$ to be determined, we consider

$$\mathfrak{L}_h^{[k],\mathsf{wg}}=\mathrm{e}^{\Phi/h}\mathrm{e}^{-ig(\sigma)/h}\mathfrak{L}_h^{[k],\flat}\mathrm{e}^{ig(\sigma)/h}\mathrm{e}^{-\Phi/h}=\mathfrak{L}^{[k],\mathsf{wg},0}+h\mathfrak{L}^{[k],\mathsf{wg},1}+h^2\mathfrak{L}^{[k],\mathsf{wg},2},$$

[1] see for instance [31, Theorem 18.5.9 and below]

with

$$\mathfrak{L}^{[k],\mathsf{wg},0} = \gamma^{\frac{2}{k+2}} \left(D_\tau^2 + \left(V_{\zeta_0^{[k]}} + i\gamma^{-\frac{1}{k+2}} \Phi' \right)^2 \right),$$
$$\mathfrak{L}^{[k],\mathsf{wg},1} = \left(\gamma^{\frac{1}{k+2}} V_{\zeta_0^{[k]}} + i\Phi' \right) D_\sigma + D_\sigma \left(\gamma^{\frac{1}{k+2}} V_{\zeta_0^{[k]}} + i\Phi' \right) + \mathfrak{R}_1(\sigma, \tau; D_\tau),$$
$$\mathfrak{L}^{[k],\mathsf{wg},2} = D_\sigma^2 + \mathfrak{R}_2(\sigma, \tau; D_\sigma, D_\tau),$$

where

$$V_\zeta(\tau) = \zeta - \frac{\tau^{k+1}}{k+1},$$

and where the $\mathfrak{R}_1(\sigma, \tau; D_\tau)$ is of order zero in D_σ and cancels for $\sigma = 0$ whereas $\mathfrak{R}_2(\sigma, \tau; D_\sigma, D_\tau)$ is an operator of order one with respect to the variable σ.

Now, let us solve, as in the previous section, the eigenvalue equation

$$\mathfrak{L}_h^{[k],\mathsf{wg}} a = \lambda a$$

in the sense of formal series in h,

$$a \sim \sum_{j \geq 0} h^j a_j, \qquad \lambda \sim \sum_{j \geq 0} h^j \lambda_j.$$

4.2.2 Solving the operator valued eikonal equation

The first equation is

$$\mathfrak{L}^{[k],\mathsf{wg},0} a_0 = \lambda_0 a_0.$$

We must choose

$$\lambda_0 = \gamma_0^{\frac{2}{k+2}} \nu_1^{[k]}(\zeta_0^{[k]})$$

and we are led to take

$$a_0(\sigma, \tau) = f_0(\sigma) u^{[k]}_{w(\sigma)}(\tau), \qquad \text{with} \quad w(\sigma) = \zeta_0^{[k]} + i\gamma(\sigma)^{-\frac{1}{k+2}} \Phi'(\sigma). \tag{4.11}$$

Recall that $u_\zeta^{[k]}$ denotes a L^2-normalized eigenfunction associated with $\nu_1^{[k]}(\zeta)$ for the generalized Montgomery operator (4.1). Then the equation becomes

$$\gamma(\sigma)^{\frac{2}{k+2}} \nu_1^{[k]} \left(\zeta_0^{[k]} + i\gamma(\sigma)^{-\frac{1}{k+2}} \Phi'(\sigma) \right) = \gamma_0^{\frac{2}{k+2}} \nu_1^{[k]}(\zeta_0^{[k]})$$

and this can be written in the form

$$\nu_1^{[k]} \left(\zeta_0^{[k]} + i\gamma(\sigma)^{-\frac{1}{k+2}} \Phi'(\sigma) \right) - \nu_1^{[k]}(\zeta_0^{[k]}) = \left(\gamma_0^{\frac{2}{k+2}} \gamma(\sigma)^{-\frac{2}{k+2}} - 1 \right) \nu_1^{[k]}(\zeta_0^{[k]}).$$

Therefore we are in the framework of the following elementary lemma.

Lemma 4.4 *For $r > 0$, let us consider a holomorphic function $\nu : D(0, r) \to \mathbb{C}$ such that $\nu(0) = \nu'(0) = 0$ and $\nu''(0) \in \mathbb{R}_+$. Let us also introduce a smooth F defined in a real neighborhood of $\sigma = 0$ such that $\sigma = 0$ is a non degenerate maximum. Then, there exists a neighborhood of $\sigma = 0$ such that the equation*

$$\nu(i\varphi(\sigma)) = F(\sigma) \tag{4.12}$$

admits a smooth solution φ such that $\varphi(0) = 0$ and $\varphi'(0) > 0$.

Proof: We can apply the Morse lemma to deduce that (4.12) is equivalent to

$$\tilde{\nu}(i\varphi(\sigma))^2 = -f(\sigma)^2,$$

where f is a non negative function such that $f'(0) = \sqrt{-\frac{F''(0)}{2}}$ and $F(\sigma) = -f(\sigma)^2$ and $\tilde{\nu}$ is a holomorphic function in a neighborhood of 0 such that $\tilde{\nu}^2 = \nu$ and $\tilde{\nu}'(0) = \sqrt{\frac{\nu''(0)}{2}}$. This provides the equations

$$\tilde{\nu}(i\varphi(\sigma)) = if(\sigma), \quad \text{or} \quad \tilde{\nu}(i\varphi(\sigma)) = -if(\sigma).$$

Since $\tilde{\nu}$ is a local biholomorphism and $f(0) = 0$, we can write the equivalent equations

$$\varphi(\sigma) = -i\tilde{\nu}^{-1}(if(\sigma)), \quad \text{or} \quad \varphi(\sigma) = -i\tilde{\nu}^{-1}(-if(\sigma)).$$

The function $\varphi = -i\tilde{\nu}^{-1}(if)$ satisfies our requirements since $\varphi'(0) = \sqrt{-\frac{F''(0)}{\nu''(0)}}$. ∎

We use the lemma with $F(\sigma) = \left(\gamma_0^{\frac{2}{k+2}}\gamma(\sigma)^{-\frac{2}{k+2}} - 1\right)\nu_1^{[k]}(\zeta_0^{[k]})$ and, for the function φ given by the lemma, we have (up to a translation by $\zeta_0^{[k]}$):

$$\Phi'(\sigma) = \gamma(\sigma)^{\frac{1}{k+2}}\varphi(\sigma)$$

and we take

$$\Phi(\sigma) = \int_0^\sigma \gamma(\tilde{\sigma})^{\frac{1}{k+2}}\varphi(\tilde{\sigma})\,\mathrm{d}\tilde{\sigma},$$

which is defined in a fixed neighborhood of 0 and satisfies $\Phi(0) = \Phi'(0) = 0$ and

$$\Phi''(0) = \gamma_0^{\frac{1}{k+2}}\sqrt{\frac{2}{k+2}\frac{\gamma''(0)\nu_1^{[k]}(\zeta_0^{[k]})}{\left(\nu_1^{[k]}\right)''(\zeta_0^{[k]})\gamma(0)}} > 0. \tag{4.13}$$

Therefore (4.11) is well defined in a neighborhood of $\sigma = 0$.

4.2.3 Solving the transport equation

We can now deal with the operator valued transport equation

$$(\mathfrak{L}^{[k],\mathsf{wg},0} - \lambda_0)a_1 = (\lambda_1 - \mathfrak{L}^{[k],\mathsf{wg},1})a_0.$$

For each σ the Fredholm condition is

$$\left\langle (\lambda_1 - \mathfrak{L}^{[k],\mathsf{wg},1})a_0, \overline{u^{[k]}_{w(\sigma)}} \right\rangle_{\mathrm{L}^2(\mathbb{R},\,\mathrm{d}\tau)} = 0.$$

Using (1.5), (3.8) and a Feynman-Hellmann formula (as in Section 3), we get the transport equation

$$\begin{aligned}\lambda_1 f_0 = \left\langle \mathfrak{L}^{[k],\mathsf{wg},1}a_0, \overline{u^{[k]}_{w(\sigma)}} \right\rangle_{\mathrm{L}^2(\mathbb{R},\,\mathrm{d}\tau)} \\ = \tfrac{1}{2}\left\{\gamma(\sigma)^{\frac{1}{k+2}}\left(\nu_1^{[k]}\right)'(w(\sigma))D_\sigma + D_\sigma\gamma(\sigma)^{\frac{1}{k+2}}\left(\nu_1^{[k]}\right)'(w(\sigma))\right\}f_0 \\ + \left\langle \mathfrak{R}_1 u^{[k]}_{w(\sigma)}, \overline{u^{[k]}_{w(\sigma)}} \right\rangle_{\mathrm{L}^2(\mathbb{R},\,\mathrm{d}\tau)} f_0.\end{aligned}$$

The term

$$\left\langle \mathfrak{R}_1 u^{[k]}_{w(\sigma)}, \overline{u^{[k]}_{w(\sigma)}} \right\rangle_{\mathrm{L}^2(\mathbb{R},\,\mathrm{d}\tau)}$$

is just a smooth function which cancels in $\sigma = 0$ so that we have just to consider the linearization of the first part of the equation. The linearized operator is

$$\frac{1}{2}\left(\nu_1^{[k]}\right)''(\zeta_0^{[k]})\Phi''(0)(\sigma\partial_\sigma + \partial_\sigma\sigma).$$

The eigenvalues of this operator in the corresponding weighted space are

$$\left\{\left(\nu_1^{[k]}\right)''(\zeta_0^{[k]})\Phi''(0)\left(j+\tfrac{1}{2}\right),\quad j\in\mathbb{N}\right\}. \tag{4.14}$$

Let us notice that

$$\left(\nu_1^{[k]}\right)''(\zeta_0^{[k]})\Phi''(0) = \gamma_0^{\frac{1}{k+2}}\sqrt{\frac{2}{k+2}\frac{\gamma''(0)\nu_1^{[k]}(\zeta_0^{[k]})\left(\nu_1^{[k]}\right)''(\zeta_0^{[k]})}{\gamma(0)}}.$$

4.3 Estimates of Agmon in the normal form spirit

4.3.1 Weighted semiclassical elliptic estimates

Proposition 4.5 *Let us assume Assumption 1.5. Let $C_0 > 0$ and $\mathsf{z} : \mathbb{R}^m \to \mathbb{R}_+$ be a Lipschitzian function. There exist $\varepsilon_0 > 0$, $\varepsilon_1 > 0$, $h_0 > 0$ and $C > 0$ such that for all eigenpairs (λ, ψ) of $\mathfrak{L}_h$ satisfying $\lambda \le \mu_0 + C_0 h$ we have, for all $p \ge 1$:*

$$\left\|\mathrm{e}^{\varepsilon_1|t|+\varepsilon_0 h^{-1}\chi_p\mathsf{z}}\psi\right\|^2 \le C\|\mathrm{e}^{\varepsilon_0 h^{-1}\chi_p\mathsf{z}}\psi\|^2,$$

$$\mathfrak{Q}_h\left(\mathrm{e}^{\varepsilon_1|t|+\varepsilon_0 h^{-1}\chi_p\mathsf{z}}\psi\right) \le C\|\mathrm{e}^{\varepsilon_0 h^{-1}\chi_p\mathsf{z}}\psi\|^2,$$

where $\chi_p(s) = \chi_1(p^{-1}s)$, with $0 \le \chi_1 \le 1$ a smooth cutoff function supported near 0.

Proof: Let us consider an eigenpair (λ, ψ) of $\mathfrak{L}_h$ such that $\lambda \le \mu_0 + C_0 h$. We have the Agmon type formula

$$\mathfrak{Q}_h\left(\mathrm{e}^{\Phi}\psi\right) = \lambda\|\mathrm{e}^{\Phi}\psi\|^2 + \|\partial_t\Phi\mathrm{e}^{\Phi}\psi\|^2 + \|h\partial_s\Phi\mathrm{e}^{\Phi}\psi\|^2,$$

where Φ is bounded and Lipschitzian. We look for Φ in the form

$$\Phi(s,t) = \Phi_p(s,t) = \varepsilon_1|t| + \varepsilon_0 h^{-1}\chi_p(s)\mathsf{z}(s).$$

We get

$$\mathfrak{Q}_h\left(\mathrm{e}^{\Phi_p}\psi\right) \le (\mu_0 + C_0 h + \varepsilon_1^2)\|\mathrm{e}^{\Phi_p}\psi\|^2 + 2\varepsilon_0^2\|\mathsf{z}'\chi_p\mathrm{e}^{\Phi_p}\psi\|^2 + 2\varepsilon_0^2\|\mathsf{z}\chi_p'\mathrm{e}^{\Phi_p}\psi\|^2.$$

Since z is Lipschitzian, there exists $K \ge 0$ such that for all $s \in \mathbb{R}^m$

$$|\mathsf{z}(s)| \le |c(0)| + K|s|.$$

Therefore there exists $\tilde{C} > 0$ such that for all $p \ge 1$, $\varepsilon_0 > 0$ and $\varepsilon_1 > 0$ we have

$$\mathfrak{Q}_h\left(\mathrm{e}^{\Phi_p}\psi\right) \le (\mu_0 + C_0 h + \varepsilon_1^2 + \tilde{C}\varepsilon_0^2)\|\mathrm{e}^{\Phi_p}\psi\|^2,$$

where we have used that $s \mapsto \chi_p'(s)\mathsf{z}(s)$ is uniformly bounded with respect to p. We introduce the partition of the unity

$$\chi_{1,R}^2(t) + \chi_{2,R}^2(t) = 1,$$

where $\chi_{2,R}$ is a supported in $\{|t| \geq R\}$. The partition is assumed to be such that there exists $C > 0$ such that for all $R > 0$

$$\chi_{1,R}'^2 + \chi_{1,R}'^2 \leq CR^{-2}.$$

The so-called "IMS" formula implies that

$$\mathfrak{Q}_h(\chi_{1,R}\mathrm{e}^{\Phi_p}\psi) + \mathfrak{Q}_h(\chi_{2,R}\mathrm{e}^{\Phi_p}\psi) - CR^{-2}\|\mathrm{e}^{\Phi_p}\psi\|^2 \leq (\mu_0 + C_0 h + \varepsilon_1^2 + C\varepsilon_0^2)\|\mathrm{e}^{\Phi_p}\psi\|^2.$$

We choose R large enough, $\varepsilon_0 > 0$, $\varepsilon_1 > 0$, $h_0 > 0$ small enough such that we have for $h \in (0, h_0)$ and $p \geq 1$

$$\mathfrak{Q}_h(\chi_{2,R}\mathrm{e}^{\Phi_p}\psi) \geq \mu_0^*\|\chi_{2,R}\mathrm{e}^{\Phi_p}\psi\|^2$$

and

$$C_0 h + \varepsilon_1^2 + \tilde{C}\varepsilon_0^2 + CR^{-2} < \frac{\mu_0^* - \mu_0}{2}.$$

For these choices of R, ε_0, ε_1 and h_0, we find $\hat{c}, \hat{C} > 0$ such that for all $h \in (0, h_0)$ and $p \geq 1$, we have

$$\hat{c}\|\chi_{2,R}\mathrm{e}^{\Phi_p}\psi\|^2 \leq \hat{C}\|\mathrm{e}^{\varepsilon_0\chi_p h^{-1}\mathsf{z}}\psi\|^2,$$

and the conclusion easily follows. ∎

4.3.2 Estimates of Agmon

Proposition 4.6 *Let us assume Assumption 1.13. We let*

$$\mathsf{z}(s) = \left| \int_{s_0}^{s} \chi(s')\sqrt{\gamma(s')^{\frac{2}{k+2}} - \gamma_0^{\frac{2}{k+2}}}\,\mathrm{d}s' \right|,$$

with $0 \leq \chi \leq 1$ a smooth cutoff function whose support contains s_0. Let us consider $C_0 > 0$. There exist $\varepsilon_0 > 0$, $C > 0$ and $h_0 > 0$ such that for all eigenpairs (λ, ψ) of $\mathfrak{L}_h^{[k]}$ satisfying $\lambda \leq \mu_0 + C_0 h$ we have

$$\|\mathrm{e}^{\varepsilon_0 h^{-1}\mathsf{z}}\psi\| \leq C\|\psi\|$$

and

$$\mathfrak{Q}_h^{[k]}(\mathrm{e}^{\varepsilon_0 h^{-1}\mathsf{z}}\psi) \leq C\|\psi\|^2.$$

Proof: Let us consider an eigenpair (λ, ψ) of $\mathfrak{L}_h^{[k]}$ such that $\lambda \leq \mu_0 + C_0 h$. We first use the Agmon estimate

$$\mathfrak{Q}_h^{[k]}(\mathrm{e}^{\varepsilon_0\chi_p h^{-1}\mathsf{z}}\psi) \leq \lambda\|\mathrm{e}^{\varepsilon_0\chi_p h^{-1}\mathsf{z}}\psi\|^2 + \tilde{C}\varepsilon_0^2\|\mathsf{z}'\mathrm{e}^{\varepsilon_0\chi_p h^{-1}\mathsf{z}}\psi\|^2 + \tilde{C}\varepsilon_0^2\|\chi_p'\mathsf{z}\mathrm{e}^{\varepsilon_0\chi_p h^{-1}\mathsf{z}}\psi\|^2. \quad (4.15)$$

In order to simplify the notation, we consider the weighted ψ:

$$\psi^{\mathsf{wg}} = \mathrm{e}^{\varepsilon_0\chi_p h^{-1}\mathsf{z}}\psi.$$

Now, we shall establish a very fine lower bound of $\mathfrak{Q}_h^{[k]}(\psi^{\mathsf{wg}})$. For that purpose we use the normal form already introduced in Section 4.1.2 and associated with the change of variables

(4.3). If we denote by $\psi^{\mathsf{wg},\flat}$ the function ψ^{wg} transported by the canonical transform, we get

$$\mathfrak{Q}_h^{[k]}(\psi^{\mathsf{wg}}) = \mathfrak{Q}_h^{[k],\flat}(\psi^{\mathsf{wg},\flat}).$$

Using again (4.4) with $\varepsilon = h$, we deduce with (4.7)

$$\begin{aligned}\mathfrak{Q}_h^{[k],\flat}(\psi^{\mathsf{wg},\flat}) \geq \\ (1-h)\int_{\mathbb{R}^2}\gamma(\sigma)^{\frac{2}{k+2}}\left(|D_\tau\psi^{\mathsf{wg},\flat}|^2 + \left|\left(h\gamma(\sigma)^{-\frac{1}{k+2}}D_\sigma - \frac{\tau^{k+1}}{k+1}\right)\psi^{\mathsf{wg},\flat}\right|^2\right)\mathrm{d}\sigma\,\mathrm{d}\tau \\ - Ch\int_{\mathbb{R}^2}\left|\tau D_\tau\psi^{\mathsf{wg},\flat}\right|^2\mathrm{d}\sigma\,\mathrm{d}\tau - Ch\|\psi^{\mathsf{wg},\flat}\|^2. \quad (4.16)\end{aligned}$$

But we have

$$\int_{\mathbb{R}^2}\left|\tau D_\tau\psi^{\mathsf{wg},\flat}\right|^2\mathrm{d}\sigma\,\mathrm{d}\tau = \int_{\mathbb{R}^2}|tD_t\psi^{\mathsf{wg}}|^2\,\mathrm{d}s\,\mathrm{d}t$$

and we apply Proposition 4.5 to get

$$\int_{\mathbb{R}^2}|tD_t\psi^{\mathsf{wg}}|^2\,\mathrm{d}s\,\mathrm{d}t \leq C\|\psi^{\mathsf{wg},\flat}\|^2.$$

With (4.16), we infer

$$\begin{aligned}\mathfrak{Q}_h^{[k],\flat}(\psi^{\mathsf{wg},\flat}) \geq \\ (1-h)\int_{\mathbb{R}^2}\gamma(\sigma)^{\frac{2}{k+2}}\left(|D_\tau\psi^{\mathsf{wg},\flat}|^2 + \left|\left(h\gamma(\sigma)^{-\frac{1}{k+2}}D_\sigma - \frac{\tau^{k+1}}{k+1}\right)\psi^{\mathsf{wg},\flat}\right|^2\right)\mathrm{d}\sigma\,\mathrm{d}\tau \\ - Ch\|\psi^{\mathsf{wg},\flat}\|^2. \quad (4.17)\end{aligned}$$

We can write

$$\begin{aligned}&\int_{\mathbb{R}^2}\gamma(\sigma)^{\frac{2}{k+2}}\left(|D_\tau\psi^{\mathsf{wg},\flat}|^2 + \left|\left(h\gamma(\sigma)^{-\frac{1}{k+2}}D_\sigma - \frac{\tau^{k+1}}{k+1}\right)\psi^{\mathsf{wg},\flat}\right|^2\right)\mathrm{d}\sigma\,\mathrm{d}\tau \\ &= \int_{\mathbb{R}^2}\left(|D_\tau\phi^{\mathsf{wg},\flat}|^2 + \left|\left(hD_\sigma\gamma(\sigma)^{-\frac{1}{k+2}} - \frac{\tau^{k+1}}{k+1}\right)\phi^{\mathsf{wg},\flat}\right|^2\right)\mathrm{d}\sigma\,\mathrm{d}\tau \\ &= \int_{\mathbb{R}^2}\left(|D_\tau\phi^{\mathsf{wg},\flat}|^2 + \left|\left(h\Xi(\sigma,D_\sigma) - \frac{\tau^{k+1}}{k+1} + \frac{ih}{2k+4}\gamma'(\sigma)\gamma(\sigma)^{-\frac{k+3}{k+2}}\right)\phi^{\mathsf{wg},\flat}\right|^2\right)\mathrm{d}\sigma\,\mathrm{d}\tau,\end{aligned}$$

with

$$\phi^{\mathsf{wg},\flat}(\sigma,\tau) = \gamma(\sigma)^{\frac{1}{k+2}}\psi^{\mathsf{wg},\flat}(\sigma,\tau).$$

We are reduced to the same analysis as after (4.9). We deduce that

$$\mathfrak{Q}_h^{[k],\flat}(\psi^{\mathsf{wg},\flat}) \geq (1-h)\mathfrak{q}_h^{[k],\flat}(\gamma^{\frac{1}{k+2}}\psi^{\mathsf{wg},\flat}) - Ch\|\psi^{\mathsf{wg},\flat}\|^2.$$

We may use the functional calculus for the self-adjoint operator $\gamma^{-\frac{1}{2k+4}}D_\sigma\gamma^{-\frac{1}{2k+4}}$ and it follows that

$$\mathfrak{Q}_h^{[k],\flat}(\psi^{\mathsf{wg},\flat}) \geq (1-h)\min_{\zeta\in\mathbb{R}}\nu_1^{[k]}(\zeta)\|\gamma^{\frac{1}{k+2}}\psi^{\mathsf{wg},\flat}\|^2 - Ch\|\psi^{\mathsf{wg},\flat}\|^2.$$

We now come back in the variables (s,t) and we have proved

$$\min_{\zeta\in\mathbb{R}} \nu_1^{[k]}(\zeta)\|\gamma^{\frac{1}{k+2}}\psi^{\mathsf{wg}}\|^2 - Ch\|\psi^{\mathsf{wg}}\|^2 \leq \lambda\|\psi^{\mathsf{wg}}\|^2 + \tilde{C}\varepsilon_0^2\|\mathsf{z}'\psi^{\mathsf{wg}}\|^2 + \tilde{C}\varepsilon_0^2\|\chi_p'\mathsf{z}\psi^{\mathsf{wg}}\|^2.$$

We deduce that

$$\min_{\zeta\in\mathbb{R}} \nu_1^{[k]}(\zeta)\int(\gamma^{\frac{2}{k+2}} - \gamma_0^{\frac{2}{k+2}})|\psi^{\mathsf{wg}}|^2\,\mathrm{d}s\,\mathrm{d}t - \tilde{C}h\|\psi^{\mathsf{wg}}\|^2 \leq \tilde{C}\varepsilon_0^2\|\mathsf{z}'\psi^{\mathsf{wg}}\|^2 + \tilde{C}\varepsilon_0^2\|\chi_p'\mathsf{z}\psi^{\mathsf{wg}}\|^2. \quad (4.18)$$

We infer that there exist $c>0$, $\tilde{C}>0$, $h_0>0$ and $\varepsilon_0>0$ such that for all $h\in(0,h_0)$ and $p\geq 1$ we have:

$$c\int(\gamma^{\frac{2}{k+2}} - \gamma_0^{\frac{2}{k+2}})|\psi^{\mathsf{wg}}|^2\,\mathrm{d}s\,\mathrm{d}t - \tilde{C}h\|\psi^{\mathsf{wg}}\|^2 \leq \tilde{C}\varepsilon_0^2\|\chi_p'\mathsf{z}\psi^{\mathsf{wg}}\|^2. \quad (4.19)$$

We deduce by standard arguments that there exist $\hat{C}>0$, $h_0>0$ and $\varepsilon_0>0$ such that for all $h\in(0,h_0)$ and $p\geq 1$ we have:

$$\|\psi^{\mathsf{wg}}\|^2 \leq \hat{C}\|\psi\|^2 + h^{-1}\hat{C}\varepsilon_0^2\|\chi_p'\mathsf{z}\psi^{\mathsf{wg}}\|^2.$$

Since c is bounded and $|\chi_p'|\leq Cp^{-1}$, it remains to take the $\liminf_{p\to+\infty}$ and to apply the Fatou lemma. ∎

In fact, when γ admits two non degenerate minima, we also have Agmon estimates. Let us now prove Proposition 1.17

Proof: The proof is essentially the same as for Proposition 4.6. The inequality (4.19) becomes in this case

$$c\int(\gamma^{\frac{2}{k+2}} - \gamma_0^{\frac{2}{k+2}})|\psi^{\mathsf{wg}}|^2\,\mathrm{d}s\,\mathrm{d}t - \tilde{C}h\|\psi^{\mathsf{wg}}\|^2 \leq \tilde{C}\varepsilon_0^2\|\chi_p'\mathsf{z}\psi^{\mathsf{wg}}\|^2 + C\varepsilon_0^2\|f_-\psi^{\mathsf{wg}}\|^2 + C\varepsilon_0^2\|f_+\psi^{\mathsf{wg}}\|^2,$$

with

$$f_-(s) = \chi_{-,d}'(s)\left|\int_{s_-}^s \tilde{\chi}(s')\sqrt{\gamma(s')^{\frac{2}{k+2}} - \gamma_0^{\frac{2}{k+2}}}\,\mathrm{d}s'\right|,$$

$$f_+(s) = \chi_{+,d}'(s)\left|\int_{s_-}^s \tilde{\chi}(s')\sqrt{\gamma(s')^{\frac{2}{k+2}} - \gamma_0^{\frac{2}{k+2}}}\,\mathrm{d}s'\right|.$$

We can deal with the last two terms by using the positivity of $\gamma^{\frac{2}{k+2}} - \gamma_0^{\frac{2}{k+2}}$ away from the minima and the conclusion standardly follows. ∎

4.4 Tunnelling estimates

Let us first state an elementary result.

Proposition 4.7 *The asymptotic expansions (modulo h^∞) of the n-th eigenvalue of $\mathfrak{H}_{h,-}^{\mathsf{Dir}}$ are the same as for $\mathfrak{L}_h^{[k]}$. In particular, the spectral gap between two consecutive eigenvalues is of order h. Moreover the eigenfunctions of $\mathfrak{H}_{h,-}^{\mathsf{Dir}}$ satisfy the same kind of Agmon estimates as in Proposition 4.6.*

Proof: The construction of quasimodes is the same as for Proposition 2.1. We have just to add a suitable cutoff function and to use the exponential decay of the explicit quasimodes. In order to estimate the spectral gap between the lowest eigenvalues of $\mathfrak{H}_{h,-}^{\mathsf{Dir}}$ we have just to notice that $\mathfrak{H}_{h,-}^{\mathsf{Dir}}$ is bounded from below by an operator on $\mathrm{L}^2(\mathbb{R}^2)$ in the form $D_t^2 + \left(hD_s - \gamma(s)\frac{t^{k+1}}{k+1}\right)^2$ where γ admits a unique minimum at s_-. This lower bound is enough to deduce the estimates of Agmon. ∎

Let us now prove that the low lying eigenvalues of $\mathfrak{L}_h^{[k]}$ are exponentially close to the eigenvalues of $\mathfrak{H}_h$.

Proposition 4.8 *Let us consider $C_0 > 0$. There exist $c > 0$, $C > 0$, $h_0 > 0$ such that for all $\mu \in \mathsf{sp}\left(\mathfrak{L}_h^{[k]}\right)$ such that $\mu \leq \mu_0 + C_0 h$, we have, for all $h \in (0, h_0)$,*

$$\mathsf{dist}\left(\mu, \mathsf{sp}\left(\mathfrak{H}_h\right)\right) \leq C\mathrm{e}^{-c/h}.$$

Proof: The proof is based on the introduction of suitable quasimodes for the operator $\mathfrak{H}_h$ and on the estimates of Agmon satisfied by the eigenfunctions of $\mathfrak{L}_h^{[k]}$. Let us consider an eigenpair (λ, ψ) of $\mathfrak{L}_h^{[k]}$ such that $\lambda \leq \mu_0 + C_0 h$. One knows that ψ satisfies the estimates of Proposition 1.17. In particular, we deduce that

$$\mathfrak{H}_{h,-}^{\mathsf{Dir}}(\chi_{-,d'}\psi) = \lambda(\chi_{-,d'}\psi) + O(\mathrm{e}^{-c/h})\|\psi\|^2, \quad \mathfrak{H}_{h,+}^{\mathsf{Dir}}(\chi_{+,d'}\psi) = \lambda(\chi_{+,d'}\psi) + O(\mathrm{e}^{-c/h})\|\psi\|^2.$$

The spectral theorem provides the conclusion. ∎

In fact, exponentially close to each eigenvalue of $\mathfrak{H}_h$ there are at least two eigenvalues of $\mathfrak{L}_h$.

Proposition 4.9 *Let us consider $C_0 > 0$. There exist $c > 0$, $C > 0$, $h_0 > 0$ such that for all $\mu \in \mathsf{sp}\left(\mathfrak{H}_h\right)$ such that $\mu \leq \mu_0 + C_0 h$, we have, for all $h \in (0, h_0)$,*

$$\mathsf{range}\left(\mathbb{1}_{[\mu - C\mathrm{e}^{-c/h}, \mu + C\mathrm{e}^{-c/h}]}(\mathfrak{L}_h^{[k]})\right) \geq 2.$$

Proof: Let us consider $\mu \in \mathsf{sp}\left(\mathfrak{H}_h\right)$ such that $\mu \leq \mu_0 + C_0 h$. We can find a corresponding normalized eigenfunction (φ_l, φ_r) of $\mathfrak{H}_h$. The estimates of Agmon imply that

$$\left(\mathfrak{L}_h^{[k]} - \mu\right)(\chi_{-,d'}\varphi_-) = O(\mathrm{e}^{-c/h}), \quad \left(\mathfrak{L}_h^{[k]} - \mu\right)(\chi_{+,d'}\varphi_+) = O(\mathrm{e}^{-c/h})$$

and where $\chi_{-,d'}\varphi_-$ and $\chi_{+,d'}\varphi_+$ are orthogonal. The proof follows again from the spectral theorem. ∎

We have now all the elements for our tunnelling result. It remains to prove that, in Proposition 4.9, the range of the spectral projection is exactly 2. By contradiction, if this range were at least 3, then we could consider three eigenfunctions $(u_{j,h}^{[k]})$ of $\mathfrak{L}_h^{[k]}$ mutually orthogonal associated with an eigenvalue λ such that there exists $\mu \in \mathsf{sp}\left(\mathfrak{H}_h\right)$ satisfying $|\mu - \lambda| \leq C\mathrm{e}^{-c/h}$. Then one could apply the same argument as in the proof of Proposition 4.8 with $\psi = \psi_j$. The spectral theorem would imply that the multiplicity of μ is at least 3 but this is impossible since the multiplicity of the lowest eigenvalues of $\mathfrak{H}_h$ is 2.

4.5 Numerical simulations

4.5.1 Method

Let us now deal with numerical simulations for the model operator $\mathfrak{L}_h^{[k]}$ on $\mathbb{R}^2$ if $k = 1$ and on the half-plane $\mathbb{R}^2_+$ if $k = 0$. In each case, we propose simulations for the simple and double well models. We analyze the convergence of the eigenvalues as $h \to 0$ and give the first eigenfunctions for small h. This illustrates the localization of the modulus and the behavior of the phase.

To approximate the eigenpairs $(\lambda_n^{[k]}(h), u_{n,h}^{[k]})$ of the operator $\mathfrak{L}_h^{[k]}$ on Ω_k (with $\Omega_0 = \mathbb{R}^2_+$ and $\Omega_1 = \mathbb{R}^2$), we use the Finite Element Library MÉLINA++ [33]. Since the domain Ω is unbounded, we use an artificial domain $\mathcal{R}_{k,a,b} = \Omega_k \cap (-a, a) \times (-b, b)$. We compute the eigenpairs $(\lambda_n^{[k]}(h, a, b), u_{n,h}^{[k]}(a, b))$ on $\mathcal{R}_{k,a,b}$ and impose Dirichlet conditions on the artificial boundaries $\{|x| = a\} \cup \{|y| = b\}$. We use quadrangular elements and polynomial approximation $\mathbb{Q}_p$ and a mesh $n_x \times n_y$. By this way, we obtain upper bounds for $\lambda_n^{[k]}(h)$ and we know that

$$\lambda_n^{[k]}(h, a, b) \to \lambda_n^{[k]}(h) \quad \text{as} \quad \min(a, b) \to +\infty.$$

We compute the eigenpairs for several sets of parameters (a, b) with several combinations of degree of approximation p and of size of the meshes $n_x \times n_y$ until convergence is found. We normalize the numerical eigenfunction so that $\|u_{n,h}^{[k]}(a, b))\|_\infty = 1$.

4.5.2 Simple well models

For the numerical simulations, we take

$$\gamma(s) = \min(1 + 4s^2, M), \qquad s \in \mathbb{R},$$

where $M \gg 1$ is a fixed positive real. Such a function γ satisfies Assumption 1.13. Parameters used for the numerical simulations are given in Tables 1. Among the computations, we take, for each value of h, the smallest numerical eigenvalues for the different choices of a, b, p, n_x, n_y.

a	n_x	b	n_y	p	$1/h$
1	5	20	20	16	$1:1:100,\ 110:10:1000$
1	5	40	20	14	$10:10:1000$
1.5	5	30	20	14	$1:1:100$
2	5	40	20	14	$1:1:100$

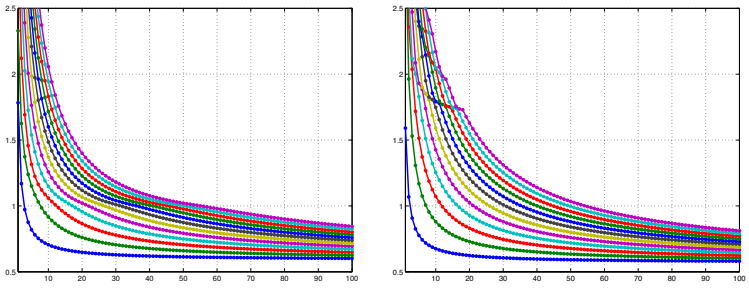
Table 1: Simple well: Parameters for the numerical simulations for $\mathfrak{L}_h^{[k]}$, $k = 0, 1$.

In Figure 1, we analyze the convergence of the eigenvalues $\lambda_n^{[k]}(h)$ as $h \to 0$ for $1 \le n \le 12$. Figure 1(a) gives an approximation of the first twelve eigenvalues $\lambda_n^{[k]}(h)$, $1/h \in [1, 100]$ and corroborates the convergence

$$\lambda_n^{[k]}(h) \to \underline{\nu}^{[k]} \qquad \text{as } h \to 0.$$

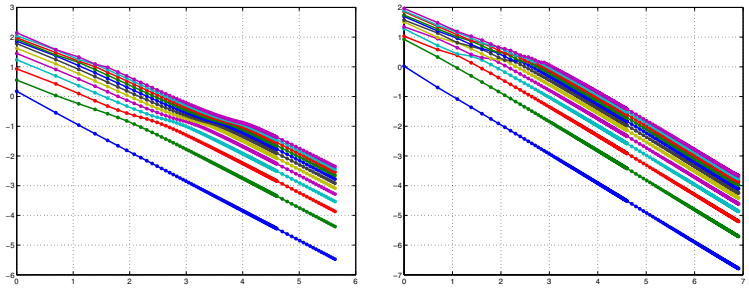
Using [10, 6], we know that

$$\underline{\nu}^{[1]} := \nu_1^{[1]}(\zeta_0^{[1]}) \simeq 0.5698, \qquad \underline{\nu}^{[0]} := \nu_1^{[0]}(\zeta_0^{[0]}) = \Theta_0 \simeq 0.5901.$$



$k = 0$ $k = 1$

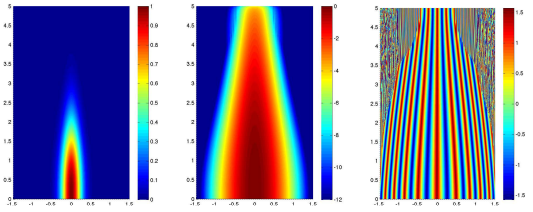
(a) $\lambda_n^{[k]}(h)$ vs. $1/h$, $\frac{1}{h} \in \{1, \ldots, 100\}$.



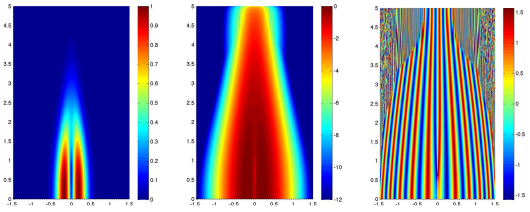
$k = 0$ $k = 1$

(b) $\ln\left(\lambda_n^{[k]}(h) - \underline{\nu}^{[k]}\right)$ vs. $\ln \frac{1}{h}$, $\frac{1}{h} \in \{1, \ldots, 1000\}$.

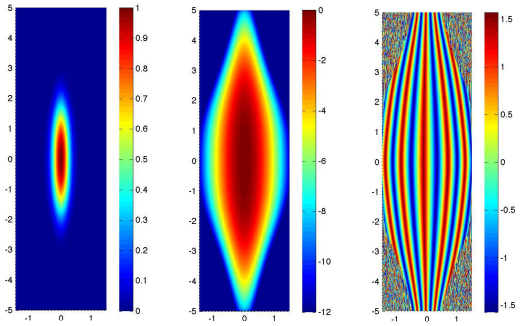
Figure 1: Convergence of the eigenvalues $\lambda_n^{[k]}(h)$.



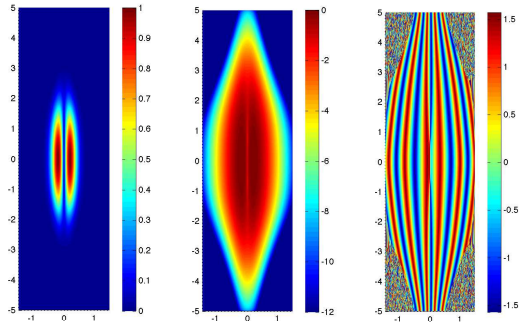
(a) First eigenfunction $u_{1,h}^{[0]}$, $h = \frac{1}{15}$



(b) Second eigenfunction $u_{2,h}^{[0]}$, $h = \frac{1}{15}$



(c) First eigenfunction $u_{1,h}^{[1]}$, $h = \frac{1}{20}$



(d) Second eigenfunction $u_{2,h}^{[1]}$, $h = \frac{1}{20}$

Figure 2: Moduli, $\log_{10}$(moduli) and phases of the first two eigenvectors, $u_{n,h}^{[k]}$.

To catch the following term in the expansion of the eigenvalues, we plot in Figure 1(b)

$$\ln \frac{1}{h} \mapsto \ln \frac{\lambda_n^{[k]}(h) - \underline{\nu}^{[k]}}{h}.$$

We observe linear convergence: the slope r illustrates the behavior

$$\lambda_n^{[k]}(h) \simeq \underline{\nu}^{[k]} + C_n h^r + o(h^r).$$

In Figure 2, we give the approximation of the first two eigenfunctions $u_{n,h}^{[k]}$ for $h = 1/20$ if $k = 1$ and $h = 1/15$ if $k = 0$. We draw the modulus, le logarithm of the modulus and the phase.

4.5.3 Double well models

Let us now consider the double well model ant take

$$\gamma(s) = \min(1 + (s^2 - 1)^2, M), \qquad s \in \mathbb{R},$$

where $M \gg 1$ is a fixed positive real. Such a function γ satisfies Assumption 1.16. Parameters used for the numerical simulations are given in Tables 2.

a	n_x	b	n_y	p	$1/h$
2	5	40	20	12	$1 : \frac{1}{10} : 500$
2	5	40	20	16	$10 : 1 : 1000$
3	5	60	20	14	$10 : 1 : 300$
4	5	40	20	14	$10 : 1 : 300$
5	5	40	20	14	$10 : 1 : 200$
10	10	50	50	10	$10 : 1 : 200$

Table 2: Double well: Parameters for the numerical simulations for $\mathfrak{L}_h^{[k]}$, $k = 0, 1$.

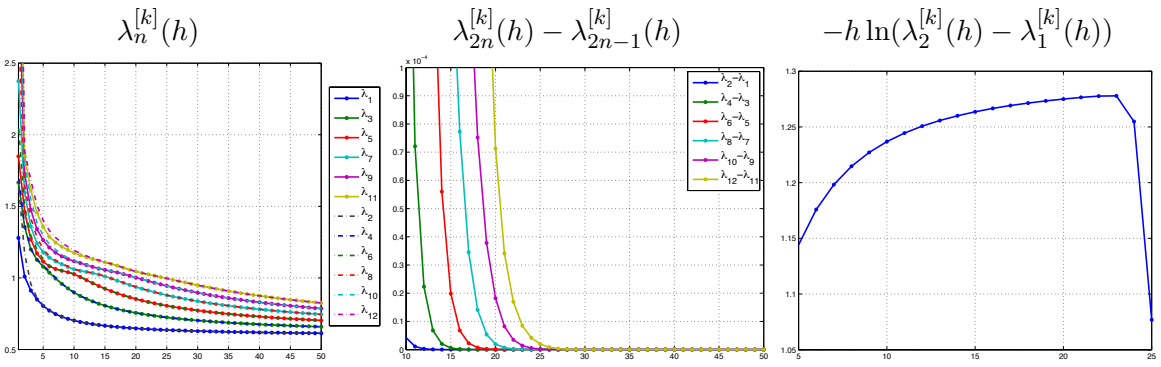
Figure 3 illustrates Corollary 1.15, Theorem 1.18 and Remark 1.19 in the scale h instead of $\hbar$. The first line concerns the low eigenvalues $\lambda_n^{[k]}(h)$ with $k = 0$ (on the half-plane) and the second line with $k = 1$. The first column illustrates the convergence

$$\lambda_n^{[k]}(h) \to \underline{\nu}^{[k]} \qquad \text{as } h \to 0.$$

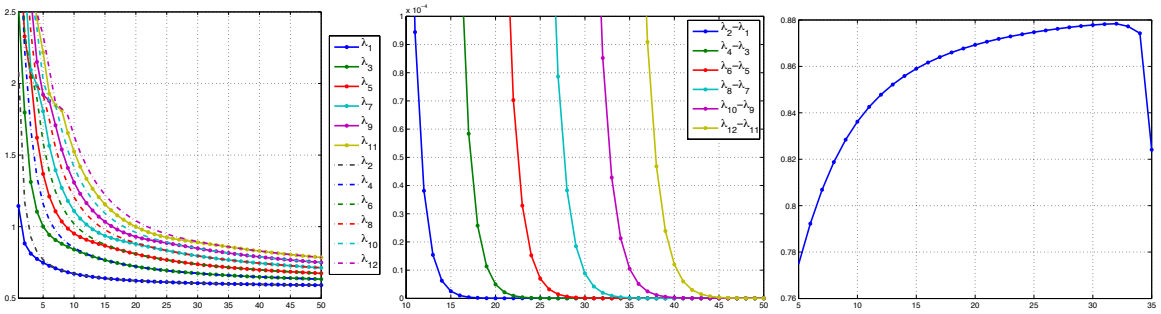
The second column represents the splitting $\lambda_{2n}^{[k]}(h) - \lambda_{2n-1}^{[k]}(h)$ according to $1/h \in \{1, \ldots, 50\}$. We recover the exponential decay of Remark 1.19. This decay is faster when $k = 0$. In the last column, we aim at catching the exponential decay rate and we plot

$$1/h \mapsto -h \ln \Big(\lambda_2^{[k]}(h) - \lambda_1^{[k]}(h) \Big).$$

Let us discuss this last column a little more. We observe a break of the curve when $1/h$ becomes too large ($1/h \geq 24$ for $k = 0$ and $1/h \geq 33$ for $k = 1$). For smaller h, the gap between the first two eigenvalues is very small $\lambda_2^{[k]}(h) - \lambda_1^{[k]}(h) < 3\ 10^{-12}$, which is the accuracy of the computations. So the gap is no more significant when h becomes too small: the error due to the computations and the splitting is at the same order $\simeq 10^{-12}$. We try here to catch the scales: a polynomial scale for the convergence of the eigenvalues as $h \to 0$ and an exponential scale for the splitting. So the range of h to have the two

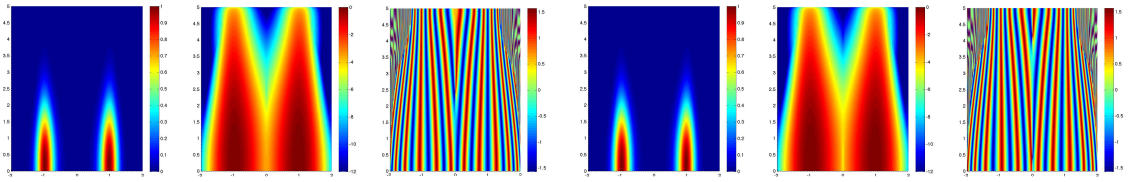


(a) $k = 0$



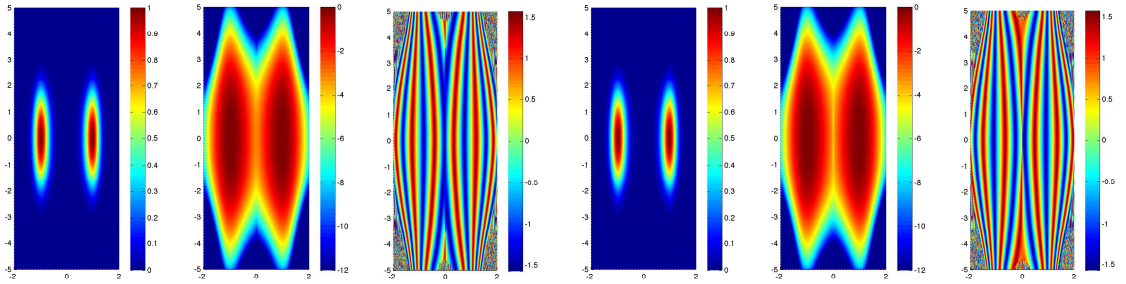
(b) $k = 1$

Figure 3: First eigenvalues $\lambda_n^{[k]}(h)$ vs. $1/h$, $k = 0, 1$.



(a) First eigenvector $h = \frac{1}{15}$

(b) Second eigenvector $h = \frac{1}{15}$



(c) First eigenvector $h = \frac{1}{20}$

(d) Second eigenvector $h = \frac{1}{20}$

Figure 4: Modules and phases of the first two eigenvectors, $h = \frac{1}{20}$.

convergences is small. The third column in Figure 3 gives an estimate for the constant

$c = c_k$ in Remark 1.19 such that

$$\lambda_2^{[k]}(h) - \lambda_1^{[k]}(h) = \mathcal{O}(\mathrm{e}^{-c_k/h}) \qquad \text{with} \quad 1.25 \leq c_0 \leq 1.3, \quad 0.86 \leq c_1 \leq 0.9.$$

In Figure 4, we give the first two eigenvectors (modulus, logarithm of the modulus and phase) of $\mathfrak{L}_h^{[k]}$ for $h = 1/15$ if $k = 0$ and $h = 1/20$ if $k = 1$. To compute them, we use a $\mathbb{Q}_{10}$ approximation and $\mathcal{R}_{k,2,5}$ as artificial domain for computations. We observe the change of symmetry between the first two eigenmodes: the first eigenvector satisfies Neumann condition along the symmetry axis $x = 0$ whereas the second one is antisymmetric as it can be seen on the phase or on the logarithm of the modulus. If we take too small values for h ($h \leq 1/24$ if $k = 0$ and $h \leq 1/33$ if $k = 1$), then the accuracy of our computations is no more sufficient to catch the tunneling effect and the modulus the first two eigenfunctions is no more symmetric: The first eigenvector is essentially localized in one well whereas the second eigenfunction is localized in the other well.

5 Geometric models: an application of the strategy

In this section, we use the same notation as previously but add an exponent $\sharp = \mathsf{vf}, \mathsf{e}, \mathsf{c}$ to distinguish the three geometrical operators studied now.

5.1 Vanishing magnetic fields

This section is concerned with the result announced in Section 1.3.1.

5.1.1 Description of the operator in curvilinear coordinates

If $k \geq 0$ is an integer, we let $\Omega_k = \mathbb{R}$ if $k \geq 1$ and $\Omega_k = \mathbb{R}_+$ if $k = 0$. By using the standard tubular coordinates near Γ (see [19, Appendix F]), we are reduced to analyze the following operator, depending on the integer $k \geq 0$ and acting on $\mathrm{L}^2(\mathbb{R} \times \Omega_k, m(s,t)\,\mathrm{d}s\,\mathrm{d}t)$ and with Neumann condition on $t = 0$ if $k = 0$:

$$\begin{aligned}\mathcal{L}_\hbar^{\mathsf{vf},[k]} = (1 - t\kappa(s))^{-1}\hbar D_t(1 - t\kappa(s))hD_t \\ + (1 - t\kappa(s))^{-1}(\hbar D_s - A^{\mathsf{vf},[k]}(s,t))(1 - t\kappa(s))^{-1}(\hbar D_s - A^{\mathsf{vf},[k]}(s,t)),\end{aligned}$$

with again $m(s,t) = 1 - t\kappa(s)$ and

$$A^{\mathsf{vf},[k]}(s,t) = \int_0^t (1 - t'\kappa(s))B^{\mathsf{vf},[k]}(s,t')\,\mathrm{d}t'$$

and where $B^{\mathsf{vf},[k]}$ is a magnetic field which satisfies

$$B^{\mathsf{vf},[k]}(s,t) = \gamma(s)t^k + \delta(s)t^{k+1} + O(t^{k+2})$$

so that

$$A^{\mathsf{vf},[k]}(s,t) = \gamma(s)\frac{t^{k+1}}{k+1} + \tilde{\delta}(s)\frac{t^{k+2}}{k+2} + O(t^{k+3})$$

where

$$\tilde{\delta}(s) = \delta(s) - \gamma(s)\kappa(s).$$

The function κ can be interpreted as the curvature function of the zero line of the magnetic field (if $k \geq 1$) or of the boundary (if $k = 0$). We will work under the following assumption.

Assumption 5.1 *The functions κ and B are smooth and γ admits a non degenerate minimum at $s = 0$.*

Let us perform an appropriate rescaling

$$h = \hbar^{\frac{1}{k+2}}, \qquad s = \sigma, \qquad t = h\tau.$$

We denote by $\mathfrak{L}_h^{\mathsf{vf},[k]}$ the rescaled operator divided by h^{2k+2} and we have

$$\begin{aligned}\mathfrak{L}_h^{\mathsf{vf},[k]} &= (1-h\tau\kappa(\sigma))^{-1}D_\tau(1-h\tau\kappa(\sigma))D_\tau \\ &\quad + (1-h\tau\kappa(\sigma))^{-1}(hD_\sigma - A_h^{\mathsf{vf},[k]}(\sigma,\tau))(1-h\tau\kappa(\sigma))^{-1}(hD_\sigma - A_h^{\mathsf{vf},[k]}(\sigma,\tau)).\end{aligned}$$

with

$$A_h^{\mathsf{vf},[k]}(\sigma,\tau) = h^{-(k+1)}A^{\mathsf{vf},[k]}(\sigma,h\tau).$$

Theorem 5.2 *Under Assumption 5.1, there exist a function $\Phi = \Phi(\sigma)$ defined in a neighborhood $\mathcal{V}$ of $(0,0)$ with $\operatorname{Re}\mathsf{Hess}\,\Phi(0) > 0$ and, for any $n \geq 1$, a sequence of real numbers $(\lambda_{n,j}^{\mathsf{vf},[k]})_{j\geq 0}$ such that*

$$\lambda_n^{\mathsf{vf},[k]}(h) \underset{h\to 0}{\sim} \sum_{j\geq 0}\lambda_{n,j}^{\mathsf{vf},[k]}h^j,$$

in the sense of formal series. Besides there exists a formal series of smooth functions on $\mathcal{V}$

$$a_n^{\mathsf{vf},[k]}(.;h) \underset{h\to 0}{\sim} \sum_{j\geq 0} a_{n,j}^{\mathsf{vf},[k]}h^j,$$

with $a_{n,0}^{\mathsf{vf},[k]}(0,0) \neq 0$ such that

$$\left(\mathfrak{L}_h^{\mathsf{vf},[k]} - \lambda_n^{\mathsf{vf},[k]}(h)\right)\left(a_n(.;h)\mathrm{e}^{-\Phi/h}\right) = \mathcal{O}\left(h^\infty\right)\mathrm{e}^{-\Phi/h}.$$

We also have that $\lambda_{n,0}^{\mathsf{vf},[k]} = \gamma(0)^{\frac{2}{k+2}}\nu_1^{[k]}(\zeta_0^{[k]})$ and that $\lambda_{n,1}^{\mathsf{vf},[k]}$ is the n-th eigenvalue of the operator

$$\tfrac{1}{2}\mathsf{Hess}\;\mu_1^{[k]}(0,\zeta_0^{[k]})(\sigma,D_\sigma) + R^{\mathsf{vf},[k]}(0), \tag{5.1}$$

with

$$\begin{aligned}R^{\mathsf{vf},[k]}(0) = 2\gamma(0)\left(\delta(0) + \frac{\kappa(0)\gamma(0)}{k+1}\right)\int_{\Omega_k}\frac{\tau^{2k+3}}{(k+1)(k+2)}u_{0,i\Phi'(0)}^2(\tau)\,\mathrm{d}\tau \\ + \kappa(0)\int_{\Omega_k}\partial_\tau u_{0,i\Phi'(0)}(\tau)u_{0,i\Phi'(0)}(\tau)\,\mathrm{d}\tau.\end{aligned} \tag{5.2}$$

The main term in the Ansatz is

$$a_{n,0}^{\mathsf{vf},[k]} = f_{n,0}^{\mathsf{vf},[k]}(\sigma)u_{\sigma,i\Phi'(\sigma)}^{\mathsf{vf},[k]}(\tau),$$

where $f_{n,0}^{\mathsf{vf},[k]}(\sigma)$ is the n-th normalized eigenfunction of (5.1). Moreover, for all $n \geq 1$, there exist $h_0 > 0$, $c > 0$ such that for all $h \in (0,h_0)$, we have

$$\mathcal{B}\big(\lambda_{n,0}^{\mathsf{vf},[k]} + \lambda_{n,1}^{\mathsf{vf},[k]}h, ch\big) \cap \mathsf{sp}\left(\mathfrak{L}_h^{\mathsf{vf},[k]}\right) = \{\lambda_n^{\mathsf{vf},[k]}(h)\},$$

and $\lambda_n^{\mathsf{vf},[k]}(h)$ is a simple eigenvalue.

5.1.2 WKB expansion

Let us now prove Theorem 5.2. We have the expansion (in powers of h):

$$A_h^{\mathsf{vf},[k]}(\sigma,\tau) = \gamma(\sigma)\frac{\tau^{k+1}}{k+1} + \tilde{\delta}(\sigma)h\frac{\tau^{k+2}}{k+2} + O(h^2\tau^{k+3}).$$

We can write the following formal series expansion:

$$\mathrm{e}^{\Phi(\sigma)/h}\mathfrak{L}_h^{\mathsf{vf},[k]}\mathrm{e}^{-\Phi(\sigma)/h} \sim \sum_{j\geq 0} h^j \mathfrak{L}_j,$$

where we have

$$\mathfrak{L}_0 = D_\tau^2 + \left(i\Phi'(\sigma) - \gamma(\sigma)\frac{\tau^{k+1}}{k+1}\right)^2$$

and

$$\mathfrak{L}_1 = -\frac{\tau^{k+1}}{k+1}(D_\sigma\gamma(\sigma)+\gamma(\sigma)D_\sigma)+2\gamma(\sigma)\tilde{\delta}(\sigma)\frac{\tau^{2k+3}}{(k+2)(k+1)}+\kappa(\sigma)\partial_\tau+2\tau\kappa(\sigma)\left(\gamma(\sigma)\frac{\tau^{k+1}}{k+1}\right)^2.$$

Our Ansätze are again in the form

$$a \sim \sum_{j\geq 0} h^j a_j, \qquad \lambda \sim \sum_{j\geq 0} h^j \lambda_j.$$

The first equation is given by

$$\mathfrak{L}_0 a_0 = \lambda_0 a_0.$$

This leads to the choice

$$a_0(\sigma,\tau) = f_0(\sigma)u^{[k]}_{\sigma,i\Phi'(\sigma)}(\tau)$$

and Φ must be such that

$$\mu_1^{[k]}(\sigma, i\Phi'(\sigma)) = \lambda_0$$

and $\lambda_0 = \gamma(0)^{\frac{2}{k+2}}\nu_1^{[k]}(\zeta_0^{[k]})$. The next equation to solve is

$$(\mathfrak{L}_0 - \mu_0)a_1 = (\lambda_1 - \mathfrak{L}_1)a_0$$

and the associated Fredholm condition is given by

$$\left\langle \mathfrak{L}_1 a_0, u^{[k]}_{\sigma,-i\overline{\Phi}'(\sigma)}\right\rangle_{\mathrm{L}^2(\Omega_k,\,\mathrm{d}\tau)} = \lambda_1 f_0(\sigma).$$

We get the transport equation

$$\left(D_\sigma\partial_\zeta\mu_1(\sigma,i\Phi'(\sigma)) + \partial_\zeta\mu_1(\sigma,i\Phi'(\sigma))D_\sigma\right)f_0 + R^{\mathsf{vf}}(\sigma)f_0 = \lambda_1 f_0,$$

where R^{vf} is a smooth function. Considering the linearized equation near $\sigma = 0$, we are led to choose λ_1 in the set

$$\mathsf{sp}\left(\tfrac{1}{2}\mathsf{Hess}\ \mu_1^{[k]}(0,\zeta_0^{[k]})(\sigma,D_\sigma)\right) + R^{\mathsf{vf}}(0).$$

We recognize the set which appears in [45, $\gamma_{n,2}$ in Theorem 1.3] (for $k=0$), [16, θ_2^n in Theorem 1.6] (for $k=1$) and in the conjecture of [21, A for our $R^{\mathsf{vf}}(0)$ in (4.5)] (for $k\geq 1$).

5.2 Along a varying edge in dimension three

Let us now deal with the situation described in Section 1.3.2. Let us recall that the bottom of the spectrum of the magnetic Neumann Laplacian on the wedge $\mathcal{W}_\alpha$ with constant aperture α with a magnetic field normal to the symmetry plane is a non increasing function with respect to α (see [42]). The wedge is so that $\mathcal{W}_\alpha = \mathbb{R} \times \mathcal{S}_\alpha$ where $\mathcal{S}_\alpha$ is the angular sector in $\mathbb{R}^2$.

5.2.1 Framework

We will need the Neumann realization of the operator defined on $\mathrm{L}^2(\mathcal{S}_{\alpha_0}, \mathrm{d}t\,\mathrm{d}z)$ by

$$\mathcal{M}^{\mathsf{e}}_{s,\eta} = D_t^2 + \mathcal{T}(s)^{-2}\mathcal{T}(0)^2 D_z^2 + (\eta - t)^2,$$

whose form domain is

$$\mathsf{Dom}(\mathcal{Q}^{\mathsf{e}}_{s,\eta}) = \left\{\psi \in \mathrm{L}^2(\mathcal{S}_{\alpha_0}) : D_t\psi \in \mathrm{L}^2(\mathcal{S}_{\alpha_0}), D_z\psi \in \mathrm{L}^2(\mathcal{S}_{\alpha_0}), t\psi \in \mathrm{L}^2(\mathcal{S}_{\alpha_0})\right\}$$

and with operator domain

$$\mathsf{Dom}(\mathcal{M}^{\mathsf{e}}_{s,\eta}) = \left\{\psi \in \mathsf{Dom}(\mathcal{Q}^{\mathsf{e}}_{s,\eta}) : \mathcal{M}^{\mathsf{e}}_{s,\eta}\psi \in \mathrm{L}^2(\mathcal{S}_{\alpha_0}), \mathfrak{T}(s)\psi = 0\right\},$$

where

$$\mathfrak{T}(s) = -\mathsf{sgn}(z)D_t + \mathcal{T}(s)^{-2}\mathcal{T}(0)D_z.$$

The lowest eigenvalue of $\mathcal{M}^{\mathsf{e}}_{s,\eta}$ is denoted by $\mu^{\mathsf{e}}_1(s,\eta)$. As in [43], we will also investigate the consequences of the following conjecture (see [43, Remark 1.8]).

Conjecture 5.3 *For all $\alpha_0 \in (0,\pi)$, the function $\eta \mapsto \mu^{\mathsf{e}}_1(0,\eta)$ admits a unique critical point η_0 which is a non degenerate minimum.*

Proposition 5.4 *Under Assumption 1.22 and if Conjecture 5.3 is true, the function μ^{e}_1 admits a local non degenerate minimum at $(0,\eta_0)$. Moreover the Hessian at $(0,\eta_0)$ is given by*

$$4\kappa\mathcal{T}(0)^{-1}\|D_z u_{0,\eta_0}\|s^2 + \left(\partial_\eta^2\mu^{\mathsf{e}}_1\right)_{0,\eta_0}\eta^2. \tag{5.3}$$

Proof: The proof follows from the perturbation theory. We have the eigenvalue equation

$$\mathcal{M}^{\mathsf{e}}_{s,\eta}u_{s,\eta} = \mu^{\mathsf{e}}_1(s,\eta)u_{s,\eta}, \qquad \mathfrak{T}(s)u_{s,\eta} = 0.$$

We notice that $\mathfrak{T}'(0) = 0$ and $\mathfrak{T}''(0) = 4\kappa\mathcal{T}(0)^{-2}D_z$. Let us analyze the derivative with respect to s. We have

$$\left(\mathcal{M}^{\mathsf{e}}_{0,\eta_0} - \mu^{\mathsf{e}}_1(0,\eta_0)\right)\left(\partial_s u\right)_{0,\eta_0} = \left(\partial_s\mu^{\mathsf{e}}_1\right)_{0,\eta_0} u_{0,\eta_0},$$

with $\mathfrak{T}(0)\left(\partial_s u\right)_{0,\eta_0} = 0$. This implies that $\left(\partial_s\mu^{\mathsf{e}}_1\right)_{0,\eta_0} = 0$ by the Fredholm condition and that $\left(\partial_s u\right)_{0,\eta_0}$ is proportional to u_{0,η_0}. Therefore $(0,\eta_0)$ is a critical point of μ^{e}_1. Let us now consider the derivative with respect to s and η. We get:

$$\left(\mathcal{M}^{\mathsf{e}}_{0,\eta_0} - \mu^{\mathsf{e}}_1(0,\eta_0)\right)\left(\partial_s\partial_\eta u\right)_{0,\eta_0} = \left(\partial_s\partial_\eta\mu^{\mathsf{e}}_1\right)_{0,\eta_0} u_{0,\eta_0} - \partial_\eta\mathcal{M}^{\mathsf{e}}_{0,\eta_0}\left(\partial_s u\right)_{0,\eta_0}$$

and $\mathfrak{T}(0)\left(\partial_s\partial_\eta u\right)_{0,\eta_0} = 0$. The Fredholm condition and a Feynman-Hellmann formula give that $\left(\partial_s\partial_\eta\mu^{\mathsf{e}}_1\right)_{0,\eta_0} = 0$. We shall now analyze the second order derivative with respect to s:

$$\left(\mathcal{M}^{\mathsf{e}}_{0,\eta_0} - \mu^{\mathsf{e}}_1(0,\eta_0)\right)\left(\partial_s^2 u\right)_{0,\eta_0} = \left(\partial_s^2\mu^{\mathsf{e}}_1\right)_{0,\eta_0} u_{0,\eta_0} - \partial_s^2\mathcal{M}^{\mathsf{e}}_{0,\eta_0}u_{0,\eta_0},$$

with boundary condition $\mathfrak{T}(0)\left(\partial_s^2 u\right)_{0,\eta_0} = -\mathfrak{T}''(0)u_{0,\eta_0}$. We have $\partial_s^2\mathcal{M}^{\mathsf{e}}_{0,\eta_0} = 4\kappa\mathcal{T}(0)^{-1}D_z^2$. With the Fredholm condition, we get $\left(\partial_s^2\mu^{\mathsf{e}}_1\right)_{0,\eta_0} = 4\kappa\mathcal{T}(0)^{-1}\|D_z u_{0,\eta_0}\|^2$. ∎

The function (5.3) is the symbol of the effective harmonic oscillator introduced in [43]. We will see that our WKB analysis succeeds as soon as we work near a local and non degenerate minimum of μ_1^{e}. The goal of Assumption 1.22 and Conjecture 5.3 is to provide sufficient conditions to have such a critical point as well as to get the spectral splitting as in [43].

5.2.2 Normal form

We introduce the change of variables

$$\check{s} = s, \qquad \check{t} = t, \qquad \check{z} = \mathcal{T}(s)^{-1}\mathcal{T}(0)z$$

and we let

$$\check{\nabla}_\hbar = \begin{pmatrix} \hbar D_{\check{s}} \\ \hbar D_{\check{t}} \\ \hbar \mathcal{T}(\check{s})^{-1}\mathcal{T}(0) D_{\check{z}} \end{pmatrix} + \begin{pmatrix} -\check{t} - \hbar \frac{\mathcal{T}'}{2\mathcal{T}}(\check{z}D_{\check{z}} + D_{\check{z}}\check{z}) \\ 0 \\ 0 \end{pmatrix}.$$

The operator $\mathfrak{L}_\hbar^{\mathsf{e}}$ is unitarily equivalent to the operator $\check{\mathfrak{L}}_\hbar^{\mathsf{e}}$ on $\mathrm{L}^2(\mathcal{W}_{\alpha_0}, \mathrm{d}\check{s}\,\mathrm{d}\check{t}\,\mathrm{d}\check{z})$ defined by

$$\check{\mathcal{L}}_\hbar^{\mathsf{e}} = \left(\hbar D_{\check{s}} - \check{t} - \hbar \frac{\mathcal{T}'}{2\mathcal{T}}(\check{z}D_{\check{z}} + D_{\check{z}}\check{z})\right)^2 + \hbar^2 D_{\check{t}}^2 + \hbar^2 \mathcal{T}(\check{s})^{-2}\mathcal{T}(0)^2 D_{\check{z}}^2.$$

The boundary condition becomes, on $\partial\mathcal{W}_{\alpha_0}$,

$$\check{\nabla}_\hbar \check{\psi} \cdot \check{\mathbf{n}} = 0, \qquad \text{with} \qquad \check{\mathbf{n}} = \begin{pmatrix} -\mathcal{T}'(\check{s})\check{t} \\ -\mathcal{T}(\check{s}) \\ \pm 1 \end{pmatrix}. \tag{5.4}$$

We now perform the scaling which preserves $\mathcal{W}_{\alpha_0}$:

$$h = \hbar^{1/2}, \qquad \check{s} = \sigma, \qquad \check{t} = h^{1/2}\tau, \qquad \check{z} = h^{1/2}z.$$

The operator $\hbar^{-1}\check{\mathcal{L}}_\hbar^{\mathsf{e}}$ becomes $\mathfrak{L}_h^{\mathsf{e}}$:

$$\mathfrak{L}_h^{\mathsf{e}} = \left(hD_\sigma - \tau - h\frac{\mathcal{T}'}{2\mathcal{T}}(zD_z + D_z z)\right)^2 + D_\tau^2 + \mathcal{T}(\sigma)^{-2}\mathcal{T}(0)^2 D_z^2.$$

Now, the boundary condition is, on $\partial\mathcal{W}_{\alpha_0}$,

$$\widehat{\nabla}_h \psi \cdot \hat{\mathbf{n}} = 0 \qquad \text{with} \qquad \hat{\mathbf{n}} = \mathbf{n}_0 + h^{1/2}\mathbf{n}_1 \tag{5.5}$$

where

$$\mathbf{n}_0 = \begin{pmatrix} 0 \\ -\mathcal{T}(\sigma) \\ \pm 1 \end{pmatrix}, \qquad \mathbf{n}_1 = \begin{pmatrix} -\mathcal{T}'(\sigma)\tau \\ 0 \\ 0 \end{pmatrix}$$

and

$$\widehat{\nabla}_h = \begin{pmatrix} hD_\sigma \\ D_\tau \\ \mathcal{T}(\sigma)^{-1}\mathcal{T}(0)D_z \end{pmatrix} + \begin{pmatrix} -\tau - h\frac{\mathcal{T}'}{2\mathcal{T}}(zD_z + D_z z) \\ 0 \\ 0 \end{pmatrix}.$$

Theorem 5.5 *Under Assumption 1.22 and Conjecture 5.3, there exist a function $\Phi = \Phi(\sigma)$ defined in a neighborhood $\mathcal{V}$ of $(0,0)$ such that $\mathrm{Re}\,\mathsf{Hess}\,\Phi(0) > 0$ on $\mathcal{V}$ and sequence of real numbers $(\lambda_{n,j}^{\mathsf{e}})$ such that*

$$\lambda_n^{\mathsf{e}}(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \lambda_{n,j}^{\mathsf{e}} h^j.$$

in the sense of formal series. Besides there exists a formal series of smooth functions $(a^{\mathsf{e}}_{n,j}(s,t))$ *defined for* $(s,t)\in\mathcal{V}$,

$$a^{\mathsf{e}}_n \underset{h\to 0}{\sim} \sum_{j\geq 0} a^{\mathsf{e}}_{n,j} h^j,$$

such that

$$(\mathfrak{L}^{\mathsf{e}}_h - \lambda^{\mathsf{e}}_n(h))\left(a^{\mathsf{e}}_n \mathrm{e}^{-\Phi/h}\right) = \mathcal{O}\left(h^\infty\right)\mathrm{e}^{-\Phi/h}.$$

We also have that $\lambda^{\mathsf{e}}_{n,0} = \mu^{\mathsf{e}}_1(0,\eta_0)$ *and that* $\lambda^{\mathsf{e}}_{n,1}$ *is the* n*-th eigenvalue of the operator*

$$\tfrac{1}{2}\mathsf{Hess}\ \mu^{\mathsf{e}}_1(0,\eta_0)(\sigma,D_\sigma). \tag{5.6}$$

The main term in the Ansatz is in the form $a^{\mathsf{e}}_{n,0}(\sigma,\tau,z) = f^{\mathsf{e}}_{n,0}(\sigma)u^{\mathsf{e}}_{\sigma,i\Phi'(\sigma)}(\tau,z)$. *Moreover, for all* $n\geq 1$, *there exist* $h_0>0$, $c>0$ *such that for all* $h\in(0,h_0)$, *we have*

$$\mathcal{B}(\lambda^{\mathsf{e}}_{n,0}+\lambda^{\mathsf{e}}_{n,1}h, ch)\cap \mathsf{sp}\left(\mathfrak{L}^{\mathsf{e}}_h\right) = \{\lambda^{\mathsf{e}}_n(h)\},$$

and $\lambda^{\mathsf{e}}_n(h)$ *is a simple eigenvalue.*

5.2.3 WKB expansion for the normal form

Let us consider the conjugate operator

$$\mathrm{e}^{\Phi(\sigma)/h}\mathfrak{L}^{\mathsf{e}}_h\mathrm{e}^{-\Phi(\sigma)/h} = \left(hD_\sigma + i\Phi'(\sigma) - \tau - h\frac{\mathcal{T}'(\sigma)}{2\mathcal{T}(\sigma)}(zD_z + D_z z)\right)^2 + D^2_\tau + \frac{\mathcal{T}(0)^2}{\mathcal{T}(\sigma)^2}D^2_z,$$

with the corresponding boundary conditions. We can write the formal power series expansion:

$$\mathrm{e}^{\Phi(\sigma)/h}\mathfrak{L}^{\mathsf{e}}_h\mathrm{e}^{-\Phi(\sigma)/h} \sim \sum_{j\geq 0} h^j\mathfrak{L}_j$$

with

$$\mathfrak{L}_0 = D^2_\tau + \mathcal{T}(\sigma)^{-2}\mathcal{T}(0)^2D^2_z + (i\Phi'(\sigma)-\tau)^2,$$

$$\mathfrak{L}_1 = (i\Phi'(\sigma)-\tau)D_\sigma + D_\sigma(i\Phi'(\sigma)-\tau) - i\Phi'(\sigma)\frac{\mathcal{T}'(\sigma)}{\mathcal{T}(\sigma)}(zD_z + D_z z).$$

Our Ansätze are in the form:

$$a\sim\sum_{j\geq 0}h^j a_j, \qquad \lambda\sim\sum_{j\geq 0}h^j\lambda_j.$$

The first equation is given by

$$\mathfrak{L}_0 a_0 = \lambda_0 a_0,$$

with boundary condition (which is in fact a Neumann condition)

$$\begin{pmatrix} -\tau \\ D_\tau \\ \mathcal{T}(\sigma)^{-1}\mathcal{T}(0)D_z\end{pmatrix} a_0\cdot\mathbf{n}_0 = 0.$$

We take

$$a_0(\sigma,\tau) = f_0(\sigma)u^{\mathsf{e}}_{\sigma,i\Phi'(\sigma)}(\tau)$$

and $\lambda_0 = \mu_1^{\mathsf{e}}(0, \eta_0)$. The equation becomes

$$\mu_1^{\mathsf{e}}(\sigma, i\Phi'(\sigma)) = \lambda_0.$$

The second equation is

$$(\mathfrak{L}_0 - \lambda_0)a_1 = (\lambda_1 - \mathfrak{L}_1)a_0$$

with boundary condition

$$\begin{pmatrix} -\tau \\ D_\tau \\ \mathcal{T}(\sigma)^{-1}\mathcal{T}(0)D_{\mathcal{Z}} \end{pmatrix} a_0 \cdot \mathbf{n}_1 + \begin{pmatrix} -\tau \\ D_\tau \\ \mathcal{T}(\sigma)^{-1}\mathcal{T}(0)D_{\mathcal{Z}} \end{pmatrix} a_1 \cdot \mathbf{n}_0 = 0.$$

The Fredholm condition is

$$\left\langle \mathfrak{L}_1 a_0, u^{\mathsf{e}}_{\sigma, -i\overline{\Phi}'(\sigma)} \right\rangle_{\mathrm{L}^2(\mathcal{S}_{\alpha_0}, \mathrm{d}\tau\, \mathrm{d}\mathcal{Z})} = \lambda_1 f_0(\sigma)$$

and can be rewritten in the form

$$\left\{ \frac{1}{2} \left(\partial_\eta \mu_1^{\mathsf{e}}(\sigma, i\Phi'(\sigma)) D_\sigma + D_\sigma \mu_1^{\mathsf{e}}(\sigma, i\Phi'(\sigma)) \right) + R^{\mathsf{e}}(\sigma) \right\} f_0 = \lambda_1 f_0,$$

where the smooth function $\sigma \mapsto R^{\mathsf{e}}(\sigma)$ vanishes at $\sigma = 0$ since $\mathcal{T}'(0) = 0$. The conclusion follows by iteration.

5.3 Curvature induced magnetic bound states

This section is devoted to the analysis of the result announced in Section 1.3.3.

5.3.1 A higher order degeneracy

Let us consider the following Neumann realization on $\mathrm{L}^2(\mathbb{R}^2_+, m(s,t)\,\mathrm{d}s\,\mathrm{d}t)$,

$$\begin{aligned} \mathcal{L}^{\mathsf{c}}_\hbar &= m(s,t)^{-1}\hbar D_t m(s,t)\hbar D_t \\ &\quad + m(s,t)^{-1}\left(\hbar D_s + \zeta_0 \hbar^{\frac{1}{2}} - t + \kappa(s)\frac{t^2}{2}\right) m(s,t)^{-1}\left(\hbar D_s + \zeta_0 \hbar^{\frac{1}{2}} - t + \kappa(s)\frac{t^2}{2}\right), \end{aligned} \tag{5.7}$$

where $m(s,t) = 1 - t\kappa(s)$. Thanks to the rescaling

$$h = \hbar^{1/2}, \qquad t = h\tau, \qquad s = \sigma,$$

and after division by h^2 the operator $\mathcal{L}^{\mathsf{c}}_\hbar$ becomes

$$\begin{aligned} \mathfrak{L}^{\mathsf{c}}_h &= m(\sigma, h\tau)^{-1} D_\tau m(\sigma, h\tau) D_\tau \\ &\quad + m(\sigma, h\tau)^{-1}\left(hD_\sigma + \zeta_0 - \tau + h\kappa(\sigma)\frac{\tau^2}{2}\right) m(\sigma, h\tau)^{-1}\left(hD_\sigma + \zeta_0 - \tau + h\kappa(\sigma)\frac{\tau^2}{2}\right), \end{aligned} \tag{5.8}$$

on the space $\mathrm{L}^2(m(\sigma, h\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau)$.

Theorem 5.6 *Under Assumption 1.22, there exist a function*

$$\Phi = \Phi(\sigma) = \left(\frac{2C_1}{\nu''(\zeta_0)}\right)^{1/2} \left| \int_0^\sigma (\kappa(0) - \kappa(s))^{1/2} \, \mathrm{d}s \right|$$

defined in a neighborhood $\mathcal{V}$ of $(0,0)$ such that $\operatorname{Re} \Phi''(0) > 0$, and a sequence of real numbers $(\lambda^{\mathsf{c}}_{n,j})$ such that

$$\lambda^{\mathsf{c}}_n(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \lambda^{\mathsf{c}}_{n,j} h^{\frac{j}{2}}.$$

Besides there exists a formal series of smooth functions on $\mathcal{V}$,

$$a^{\mathsf{c}}_n \underset{h\to 0}{\sim} \sum_{j\geq 0} a^{\mathsf{c}}_{n,j} h^{\frac{j}{2}}$$

such that

$$(\mathfrak{L}^{\mathsf{c}}_h - \lambda^{\mathsf{c}}_n(h)) \left(a^{\mathsf{c}}_n \mathrm{e}^{-\Phi/h^{\frac{1}{2}}} \right) = \mathcal{O}\left(h^\infty\right) \mathrm{e}^{-\Phi/h^{\frac{1}{2}}}.$$

We also have that $\lambda^{\mathsf{c}}_{n,0} = \Theta_0$, $\lambda^{\mathsf{c}}_{n,1} = 0$, $\lambda^{\mathsf{c}}_{n,1} = -C_1 \kappa_{max}$ and $\lambda^{\mathsf{c}}_{n,3} = (2n-1) C_1 \Theta_0^{1/4} \sqrt{\frac{3k_2}{2}}$. The main term in the Ansatz is in the form

$$a^{\mathsf{c}}_{n,0} = f^{\mathsf{c}}_{n,0}(\sigma) u_{\zeta_0}(\tau).$$

Moreover, for all $n \geq 1$, there exist $h_0 > 0$, $c > 0$ such that for all $h \in (0, h_0)$, we have

$$\mathcal{B}\left(\lambda^{\mathsf{c}}_{n,0} + \lambda^{\mathsf{c}}_{n,2} h + \lambda^{\mathsf{c}}_{n,3} h^{\frac{3}{2}}, c h^{\frac{3}{2}}\right) \cap \mathsf{sp}\left(\mathfrak{L}^{\mathsf{c}}_h\right) = \{\lambda^{\mathsf{c}}_n(h)\},$$

and $\lambda^{\mathsf{c}}_n(h)$ is a simple eigenvalue.

Remark 5.7 *In particular, Theorem 5.6 proves that there are no odd powers of $\hbar^{\frac{1}{8}}$ in the expansion of the eigenvalues (as in [18, Theorem 1.1]).*

5.3.2 WKB expansion

Let us introduce a phase function $\Phi = \Phi(\sigma)$ defined in a neighborhood of $\sigma = 0$ the unique and non degenerate maximum of the curvature κ. We consider the conjugate operator

$$\mathfrak{L}^{\mathsf{c},\mathsf{wg}}_h = \mathrm{e}^{\Phi(\sigma)/h^{\frac{1}{2}}} \mathfrak{L}^{\mathsf{c}}_h \mathrm{e}^{-\Phi(\sigma)/h^{\frac{1}{2}}}.$$

As usual, we look for

$$a \sim \sum_{j\geq 0} h^{\frac{j}{2}} a_j, \qquad \lambda \sim \sum_{j\geq 0} \lambda_j h^{\frac{j}{2}}$$

such that, in the sense of formal series we have

$$\mathfrak{L}^{\mathsf{c},\mathsf{wg}}_h a \sim \lambda a.$$

We may write

$$\mathfrak{L}^{\mathsf{c},\mathsf{wg}}_h \sim \mathfrak{L}_0 + h^{\frac{1}{2}} \mathfrak{L}_1 + h \mathfrak{L}_2 + h^{\frac{3}{2}} \mathfrak{L}_3 + \dots,$$

where

$$\mathfrak{L}_0 = D_\tau^2 + (\zeta_0 - \tau)^2,$$
$$\mathfrak{L}_1 = 2(\zeta_0 - \tau) i\Phi'(\sigma),$$
$$\mathfrak{L}_2 = \kappa(\sigma)\partial_\tau + 2\left(D_\sigma + \kappa(\sigma)\frac{\tau^2}{2}\right)(\zeta_0 - \tau) - \Phi'(\sigma)^2 + 2\kappa(\sigma)(\zeta_0 - \tau)^2\tau,$$
$$\mathfrak{L}_3 = \left(D_\sigma + \kappa(\sigma)\frac{\tau^2}{2}\right)(i\Phi'(\sigma)) + (i\Phi'(\sigma))\left(D_\sigma + \kappa(\sigma)\frac{\tau^2}{2}\right) + 4i\Phi'(\sigma)\tau\kappa(\sigma)(\zeta_0 - \tau).$$

Let us now solve the formal system. The first equation is

$$\mathfrak{L}_0 a_0 = \lambda_0 a_0$$

and leads to take

$$\lambda_0 = \Theta_0, \qquad a_0(\sigma, \tau) = f_0(\sigma) u_{\zeta_0}(\tau),$$

where f_0 has to be determined. The second equation is

$$(\mathfrak{L}_0 - \lambda_0)a_1 = (\lambda_1 - \mathfrak{L}_1)a_0 = (\lambda_1 - 2(\zeta_0 - \tau))u_{\zeta_0}(\tau) i\Phi'(\sigma) f_0$$

and, due to the Fredholm alternative, we must take $\lambda_1 = 0$ and

$$a_1(\sigma, \tau) = i\Phi'(\sigma) f_0(\sigma) \left(\partial_\zeta u\right)_{\zeta_0}(\tau) + f_1(\sigma) u_{\zeta_0}(\tau),$$

where f_1 is to be determined in a next step. Then the third equation is

$$(\mathfrak{L}_0 - \lambda_0)a_2 = (\lambda_2 - \mathfrak{L}_2)a_0 - \mathfrak{L}_1 a_1.$$

Let us explicitly write the r.h.s. It equals

$$\lambda_2 u_{\zeta_0} f_0 + \Phi'^2(u_{\zeta_0} + 2(\zeta_0 - \tau)(\partial_\zeta u)_{\zeta_0}) f_0 - 2(\zeta_0 - \tau)u_{\zeta_0}(i\Phi' f_1 - i\partial_\sigma f_0) \\ + \kappa(\sigma) f_0(\partial_\tau u_{\zeta_0} - 2(\zeta_0 - \tau)^2 \tau u_{\zeta_0} - \tau^2(\zeta_0 - \tau)u_{\zeta_0}).$$

Therefore the equation becomes

$$(\mathfrak{L}_0 - \lambda_0)\tilde{a}_2 = \lambda_2 u_{\zeta_0} f_0 + \frac{\nu''(\zeta_0)}{2}\Phi'^2 u_{\zeta_0} f_0 + \kappa(\sigma) f_0(-\partial_\tau u_{\zeta_0} - 2(\zeta_0 - \tau)^2\tau u_{\zeta_0} - \tau^2(\zeta_0 - \tau)u_{\zeta_0}),$$

where

$$\tilde{a}_2 = a_2 - v_{\zeta_0}(i\Phi' f_1 - i\partial_\sigma f_0) + \tfrac{1}{2}(\partial_\zeta^2 u)_{\zeta_0}\Phi'^2 f_0.$$

Let us now use the Fredholm alternative (with respect to τ). We will need the following lemma the proof of which relies on Feynman-Hellmann formulas (like in Proposition 2.2) and on [18, p. 19] (for the last one).

Lemma 5.8 *We have:*

$$\int_{\mathbb{R}_+} (\zeta_0 - \tau) u_{\zeta_0}^2(\tau)\,\mathrm{d}\tau = 0, \qquad \int_{\mathbb{R}_+} (\partial_\zeta u)_{\zeta_0}(\tau) u_{\zeta_0}(\tau)\,\mathrm{d}\tau = 0,$$
$$2\int_{\mathbb{R}_+} (\zeta_0 - \tau)(\partial_\zeta u)_{\zeta_0}(\tau) u_{\zeta_0}(\tau)\,\mathrm{d}\tau = \frac{\nu''(\zeta_0)}{2} - 1,$$
$$\int_{\mathbb{R}_+} 2\tau(\zeta_0 - \tau)^2 + \tau^2(\zeta_0 - \tau)u_{\zeta_0}^2 + u_{\zeta_0}\partial_\tau u_{\zeta_0}\,\mathrm{d}\tau = -C_1.$$

We get the equation

$$\lambda_2 + \frac{\nu''(\zeta_0)}{2}\Phi'^2(\sigma) + C_1\kappa(\sigma) = 0, \qquad C_1 = \frac{u_{\zeta_0}^2(0)}{3}.$$

This eikonal equation is the one of a pure electric problem in dimension one whose potential is given by the curvature. Thus we take

$$\lambda_2 = -C_1\kappa(0),$$

and

$$\Phi(\sigma) = \left(\frac{2C_1}{\nu''(\zeta_0)}\right)^{1/2} \left| \int_0^\sigma (\kappa(0) - \kappa(s))^{1/2}\, \mathrm{d}s \right|.$$

In particular we have:

$$\Phi''(0) = \left(\frac{k_2 C_1}{\nu''(\zeta_0)}\right)^{1/2},$$

where $k_2 = -\kappa''(0) > 0$.
This leads to take

$$a_2 = f_0\hat{a}_2 + (\partial_\zeta u)_{\zeta_0}(i\Phi' f_1 - i\partial_\sigma f_0) - \tfrac{1}{2}(\partial_\eta^2 u)_{\zeta_0}\Phi'^2 f_0 + f_2 u_{\zeta_0},$$

where $\hat{a}_2$ is the unique solution, orthogonal to u_{ζ_0} for all σ, of

$$(\mathfrak{L}_0 - \nu_0)\hat{a}_2 = \nu_2 u_{\zeta_0} + \frac{\nu''(\zeta_0)}{2}\Phi'^2 u_{\zeta_0} + \kappa(\sigma)\left(-\partial_\tau u_{\zeta_0} - 2(\zeta_0 - \tau)^2\tau u_{\zeta_0} - \tau^2(\zeta_0 - \tau)u_{\zeta_0}\right),$$

and f_2 has to be determined.
Finally we must solve the fourth equation given by

$$(\mathfrak{L}_0 - \lambda_0)a_3 = (\lambda_3 - \mathfrak{L}_3)a_0 + (\lambda_2 - \mathfrak{L}_2)a_1 - \mathfrak{L}_1 a_2.$$

The Fredholm condition provides the following equation in the variable σ:

$$\langle \mathfrak{L}_3 a_0 + (\mathfrak{L}_2 - \lambda_2)a_1 + \mathfrak{L}_1 a_2, u_{\zeta_0}\rangle_{\mathrm{L}^2(\mathbb{R}_+,\, \mathrm{d}\tau)} = \lambda_3 f_0.$$

Using the previous steps of the construction, it is not very difficult to see that this equation does not involve f_1 and f_2 (due to the choice of Φ and λ_2 and Feynman-Hellmann formulas). Using the same formulas, we may write it in the form

$$\frac{\nu''(\zeta_0)}{2}\left(\Phi'(\sigma)\partial_\sigma + \partial_\sigma\Phi'(\sigma)\right) f_0 + F(\sigma)f_0 = \lambda_3 f_0, \tag{5.9}$$

where F is a smooth function which vanishes at $\sigma = 0$. Therefore the linearized equation at $\sigma = 0$ is given by

$$\Phi''(0)\frac{\nu''(\zeta_0)}{2}\left(\sigma\partial_\sigma + \partial_\sigma\sigma\right) f_0 = \lambda_3 f_0.$$

We recall that

$$\frac{\nu''(\zeta_0)}{2} = 3C_1\Theta_0^{1/2}$$

so that the linearized equation becomes

$$C_1\Theta_0^{1/4}\sqrt{\frac{3k_2}{2}}\left(\sigma\partial_\sigma + \partial_\sigma\sigma\right) f_0 = \lambda_3 f_0.$$

We have to choose λ_3 in the spectrum of this transport equation, which is given by the set

$$\left\{ (2n-1)C_1\Theta_0^{1/4}\sqrt{\frac{3k_2}{2}}, \quad n \geq 1 \right\}.$$

If λ_3 belongs to this set, we may solve locally the transport equation (5.9) and thus find f_0. This procedure can be continued at any order.

5.3.3 Numerical estimates of the magnetic camel

For the numerical computations, we consider the magnetic potential $\mathbf{A} = (-x_2, 0)$ and we denote by $(\lambda_n^{\mathsf{c}}(h), u_{n,h}^{\mathsf{c}})$ the n-th eigenpairs of the magnetic Laplacian $-(ih\partial_{x_1}+x_2)^2-h^2\partial_{x_2}^2$ on Ω.

Camel with one bump Let us first consider the case where Ω is an unbounded domain with a unique point with maximum curvature. We consider

$$\Omega = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^-, |x_1| < 4x_2^2\}.$$

For the numerical computations, we proceed as explained in Section 4.5.1: we bound the domain and impose Dirichlet condition on the artificial boundary. Let us define the truncated domain

$$\Omega_H = \{(x_1, x_2) \in \Omega, x_2 > -H\}.$$

We consider triangular elements of degree $\mathbb{P}_6$. For the numerical computations, we take $H = 2.5, 3, 4$ and a mesh with approximately 3000, 3600, 4800 triangular elements and $1/h \in \{1 : 0.1 : 1000\}$.

Figure 6 illustrate the asymptotic expansion (1.4) for the first eigenvalue:

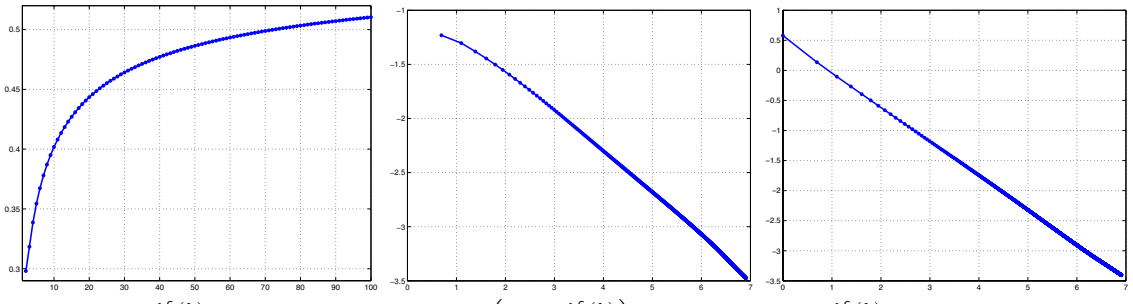
$$\frac{\lambda_1(h)^{\mathsf{c}}}{h} = \Theta_0 - C_1\kappa_{max}h^{1/2} + o(h^{1/2}), \qquad \text{with } C_1 = \frac{u_{\zeta_0}^2(0)}{3}. \tag{5.10}$$

In our example, we have $\kappa_{max} = 8$. Using [6], we have

$$\Theta_0 \simeq 0.59010 \qquad \text{and} \qquad C_1 \simeq 0.873043.$$

Figure 5(a) gives the first term Θ_0 of the expansion. Note that the convergence is quite slow because only in $\mathcal{O}(h^{1/2})$. In Figures 5(b)–5(c), we aim at recovering numerically the power appearing in the expansion. For this, we plot, according to $\ln \frac{1}{h}$ the quantities

$$\ln\left(\Theta_0 - \frac{\lambda_1^{\mathsf{c}}(h)}{h}\right) \qquad \text{and} \qquad \ln \frac{\lambda_1^{\mathsf{c}}(h)}{h} - \Theta_0 + C_1\kappa_{max}h^{1/2}.$$



(a) $\frac{\lambda_1^{\mathsf{c}}(h)}{h}$ vs. $\frac{1}{h}$ (b) $\ln\left(\Theta_0 - \frac{\lambda_1^{\mathsf{c}}(h)}{h}\right)$ vs. $\ln \frac{1}{h}$ (c) $\ln \frac{\lambda_1^{\mathsf{c}}(h)}{h} - \Theta_0 + C_1\kappa_{max}h^{1/2}$ vs. $\ln \frac{1}{h}$

Figure 5: Convergence of the eigenvalues $\lambda_1^{\mathsf{c}}(h)$.

In Figure 8, is represented the modulus, the logarithm of the modulus and the phase of the first eigenfunction for $h = 1/20$.

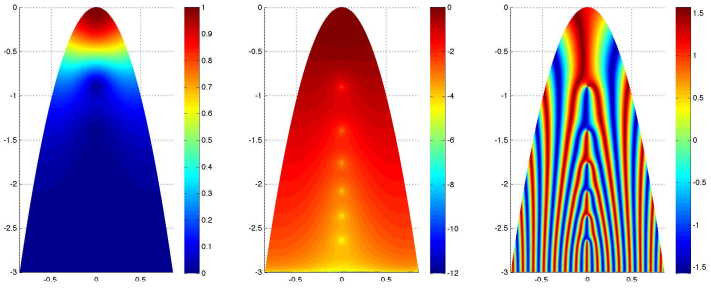


Figure 6: Modulu, logarithm of the modulus and phase of the first eigenvector, $h = \frac{1}{20}$.

Camel with two bumps Let us now deal with the case of a double well on the geometry. For this, we consider

$$\Omega = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^-, |x_1| < (1 - x_2^2)^2\}.$$

Let us look at the behavior of the first two eigenpairs. Figure 7(a) illustrates the convergence of the first two eigenvalues $\lambda_n^{\mathsf{c}}(h)$ to Θ_0 as $h \to 0$. We represent

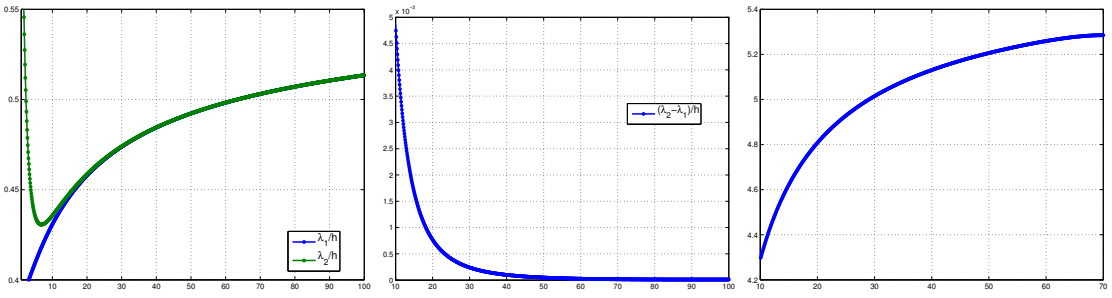
$$\frac{1}{h} \mapsto \frac{\lambda_n^{\mathsf{c}}(h)}{h}, \qquad n = 1, 2.$$

To analyze the splitting between the first two eigenvalues, we plot in Figures 7(b)–7(c), according to $\ln 1/h$

$$\frac{\lambda_2^{\mathsf{c}}(h) - \lambda_1^{\mathsf{c}}(h)}{h} \qquad \text{and} \qquad -h^{1/4} \ln \frac{\lambda_2^{\mathsf{c}}(h) - \lambda_1^{\mathsf{c}}(h)}{h}.$$

For the last figure, we take $h \geq 1/70$ otherwise the splitting computed numerically is of the same order than the accuracy of our computation and the numerics is more relevant when $h < 1/70$. These computations suggest that

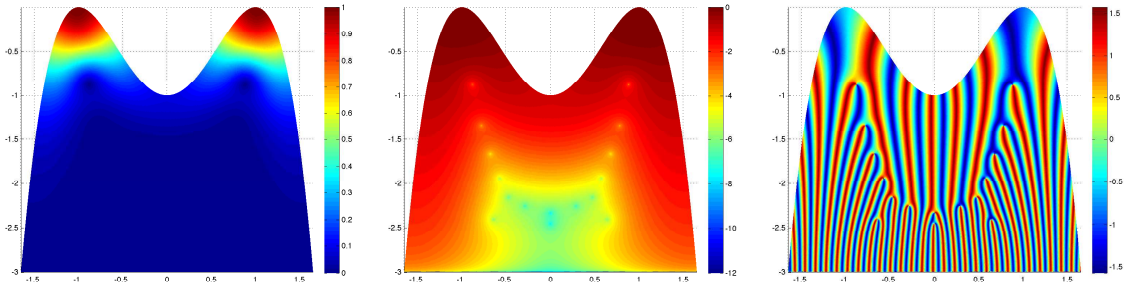
$$\frac{\lambda_2^{\mathsf{c}}(h) - \lambda_1^{\mathsf{c}}(h)}{h} = \mathcal{O}(\mathrm{e}^{-C/h^{1/4}}) \qquad \text{with} \quad 5.2 \leq C \leq 5.4.$$



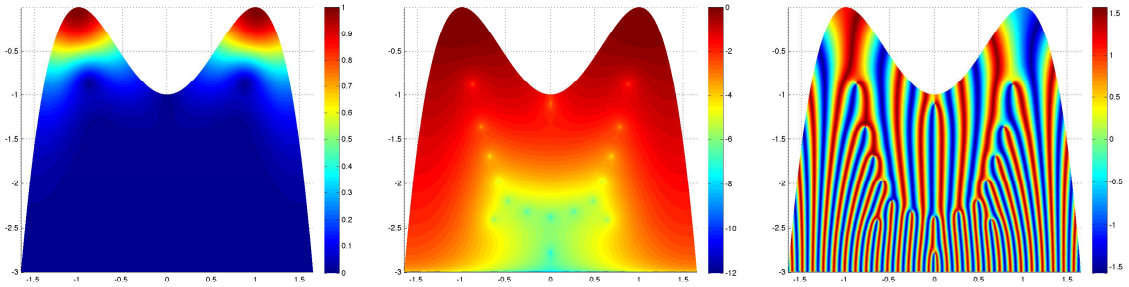
(a) $\frac{\lambda_n^{\mathsf{c}}(h)}{h}$, $n = 1, 2$ vs. $\frac{1}{h}$ (b) $\frac{\lambda_2^{\mathsf{c}}(h) - \lambda_1^{\mathsf{c}}(h)}{h}$ vs. $\frac{1}{h}$ (c) $-h^{1/4} \ln \frac{\lambda_2(h) - \lambda_1(h)}{h}$ vs. $\frac{1}{h}$

Figure 7: Convergence for $\lambda_n^{\mathsf{c}}(h)$.

Figure 8 gives the modulus, logarithm of the modulus and the phase of the first two eigenvectors for $h = 1/20$.



(a) First eigenvector (modulus, log-modulus, phase)



(b) Second eigenvector (modulus, log-modulus, phase)

Figure 8: Modulus and phases of the first two eigenvectors, $h = \frac{1}{20}$.

6 Perspectives

Let us finally provide some perspectives. As we have seen in Section 4.3.2, even in explicit situations, the optimal estimates of Agmon are still an open problem. If these estimates are improved, one will obtain an accurate asymptotics of the splitting between the low-lying eigenvalues. As well as the globalness of the WKB constructions, this problem is related to the spectral properties of the analytic extensions of the model operators eigenpairs (for instance the generalized Montgomery operators). Furthermore in the case of curvature induced magnetic bound states, we have proved, at the WKB expansion level, that the effective operator is purely electric so that we can think that the optimal estimates of Agmon are accessible. Numerically, this paper was concerned with one symmetry (camel with two bumps) and we have observed that the lowest eigenvalues seemed to be simple. With more symmetries, we expect multiplicity (see Figure 9). Moreover, in more singular geometrical situations (see [8]), the WKB structure of the eigenfunctions is not clear at all since there is no obvious dimensional reduction (for example, the case of polygonal domains is based on models on angular sectors).

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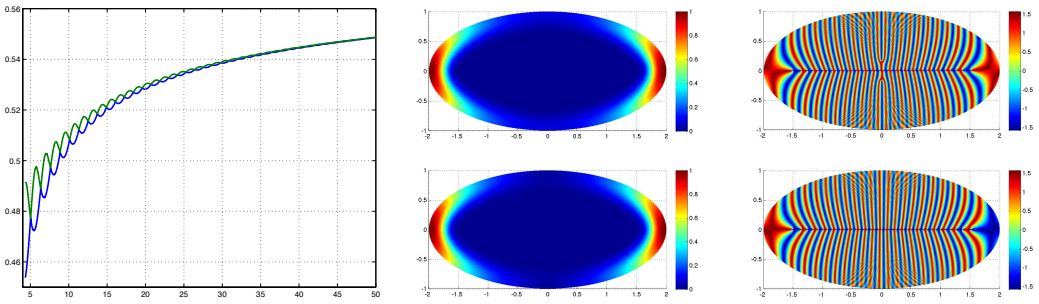

(a) $\frac{\lambda_n(h)}{h}$ vs $\frac{1}{h}$. (b) First two eigenvectors

Figure 9: Eigenpairs on the ellipse, $h = \frac{1}{50}$, $\mathbf{A} = (-x_2, 0)$.

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