Stability Analysis and Stabilization of Systems with Input Backlash

Sophie Tarbouriech, Isabelle Queinnec, Christophe Prieur

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***Abstract*—This paper deals with the stability analysis and stabilization of linear systems with backlash in the input. Uniform ultimate boundedness stability and stabilization problems are tackled allowing to characterize suitable regions of the state space in which the closed-loop trajectories can be captured. In the state feedback control design, computational oriented solutions are derived to solve suboptimal convex optimization problems able to give a constructive solution.**

***Index Terms*—Stability analysis, stabilization, backlash, uniform ultimate boundedness stability.**

I. Introduction

Backlash operators are nonlinear elements with memory. They are usual phenomena in many physical systems, such as electrical inductors, piezo-actuators, gear trains and mechanical friction systems (see e.g. the survey [18] on such memory elements and also on other operators as Krasnolsel'skii-Pokrovskii hysteresis). Other closely related infinite dimensional operators are also considered in the literature such as the Preisach models [3], the Coleman-Hodgdon operator [7] and the counterclockwise input-output dynamics [1]. For many applications, such nonlinear operators affect the inputs of control systems (see e.g. [17], [19]). The negligence of such nonlinearities during the control design or the stability analysis can lead to an important degradation of the closed-loop performance or even to the loss of stability (see, in particular, [19]).

Here we focus on the stability analysis and the control design for a linear system with a backlash operator in the input. This induces to study an infinite dimensional system, since the backlash operator is a memory-based relation. Several approaches have been developed in the context of this type of non-smooth nonlinearity: see, for example [8] and references therein. Moreover, some solutions consisting in applying inverse nonlinearity have been proposed [25], [26]. However, although there is a large literature dealing with the stability analysis and the estimation of input/output properties (see e.g. [12], [20] for recent results), there are not many papers suggesting design methods for nonlinear systems with memory operators, except [13] where a dissipativity property is used, and [23] where the dynamics of the differential equation modeling a one-dimensional hysteresis is controlled.

In the present paper, sufficient conditions are provided to characterize a compact set for the nonlinear system as a finite-time attractor for all admissible initial conditions. In this set, the system trajectories are uniformly ultimately bounded. The method developed does not impose any assumption on the system matrix at $s = 0$, where s is the Laplace variable, as required in [21] and [22]. Furthermore, the approach pursued does not need to consider that the static gain of the system has to be either null or nonsingular contrary to [10] and [11]. More precisely, generalized sector conditions using properties of the backlash operator and Lyapunov arguments allow us to analyze the stability, i.e., we exhibit, in our first main result (see Theorem 2 below), some conditions to prove that a compact set is a finite-time attractor, for all admissible initial conditions. These conditions are given in terms of Linear Matrix Inequalities (LMI), which can be solved using standard numerical algorithms, within a time of computation which is polynomial with respect to the dimension of the data. Even if the spirit of the approach proposed is quite similar to the SISO statement in Corollary 16 in [12], it allows to deal with MIMO systems. Moreover, while both [12] and our paper address the characterization of the finite-time attractor, our approach, differently from [12], uses the knowledge of the Lyapunov function. In this sense, our approach can be considered as complementary with the method proposed in [12]. Furthermore, at the opposite of conditions developed in [24], we do not need to use the time-derivative version of the system. An implicit objective in the characterization of the finite-time attractor is to make it as small as possible. Then, the second contribution of the paper (see Theorem 3) resides in proposing a technique to design both the state feedback gain not only stabilizing the linear part of the system but also allowing to address the minimization of the associated finite-time attractor. In this sense, the conditions obtained are complementary to those resulting for the stability analysis. This design result is again based on Lyapunov techniques. The computational issues and several examples illustrating the interests and the drawbacks of the conditions are also presented.

Notation. For two vectors x, y of $\Re^n$, the notation $x \succeq y$ means that $x_{(i)} - y_{(i)} \geq 0, \forall i = 1, \ldots, n$. $\mathbf{1}$ and $\mathbf{0}$ denote the identity matrix and the null matrix of appropriate dimensions, respectively. $x \in \Re^n_+$ means that $x \succeq 0$. The Euclidian norm is denoted $\|\cdot\|$. A' and $trace(A)$ denote the transpose and the trace of A, respectively. $He\{A\} = A + A'$. $diag(A; B)$ denotes the diagonal matrix which diagonal blocks are formed by squared matrices A and B. For two symmetric matrices, A and B, $A > B$ means that $A - B$ is positive definite. In partitioned symmetric matrices, the symbol $\star$ stands for symmetric blocks. $\lambda_{max}(A)$ (respectively, $\lambda_{min}(A)$) denotes the maximal (respectively, minimal) eigenvalue of the matrix A.

II. Problem formulation

The class of systems under consideration is described by:

$$\begin{array}{rcl} \dot{x} & = & Ax + B\Phi[w] \\ w & = & Kx \end{array} \qquad (1)$$

where $x \in \Re^n$ is the state and $w \in \Re^m$ is the input of the nonlinearity Φ. A, B and K are matrices of appropriate dimensions. The pair (A, B) is supposed to be controllable. K is the state feedback control gain to be designed or supposed known. Φ is a componentwise backlash operator (see, for example, [18], [21], [25], [3]). We denote the set of continuous, piecewise differentiable functions $f\colon [0, +\infty) \to \Re^m$ by $\mathcal{C}^1_{pw}([0, +\infty); \Re^m)$, that is the set of continuous functions w being, for some unbounded sequence $(t_j)_{j=0}^{\infty}$ in $[0, +\infty)$ with $t_0 = 0$, continuously differentiable on (t_{j-1}, t_j) for all $j \in \mathbb{N}$. Given the vector ρ in $\Re^m_+$ and $L = diag(\ell_{(i)})$, with positive values $\ell_{(i)}$, $i = 1, \ldots, m$, the operator Φ is defined as follows, for all $f \in \mathcal{C}^1_{pw}([0, +\infty); \Re^m)$, for all $j \in \mathbb{N}$, for all $t \in (t_{j-1}, t_j)$ and for all $i \in \{1, \ldots, m\}$:

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$$(\overbrace{\dot{\Phi}[f]}(t))_{(i)} = \begin{cases} \ell_{(i)}\dot{f}_{(i)}(t) \text{ if } \dot{f}_{(i)}(t) \geq 0 \\ \quad \text{and } (\Phi[f](t))_{(i)} = \ell_{(i)}(f_{(i)}(t) - \rho_{(i)}) \\ \ell_{(i)}\dot{f}_{(i)}(t) \text{ if } \dot{f}_{(i)}(t) \leq 0 \\ \quad \text{and } (\Phi[f](t))_{(i)} = \ell_{(i)}(f_{(i)}(t) + \rho_{(i)}) \\ 0 \text{ otherwise} \end{cases} \quad (2)$$

where $0 = t_0 < t_1 < \ldots$ is a partition of $[0, +\infty)$ such that f is continuously differentiable on each of the intervals (t_{j-1}, t_j), $j \in \mathbb{N}$. Thus, Φ is a time-invariant nonlinearity with slope restriction, as in [22]. Note however that it is a memory-based operator, since to compute it, we need to have information about the past values of w (this is not the case in [22]). Such an operator can be depicted in Figure 1.

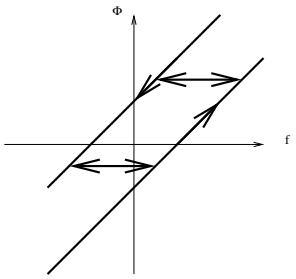


Fig. 1. An illustration of a backlash operator.

Since it is possible to stack several backlash operators into one backlash operator (with the dimension equal to the sum of all dimensions), without loss of generality, it is suitable to assume that only one backlash is present in (1).

Throughout the paper, we define the set of admissible initial conditions ($w(t = 0)$), from which we want to guarantee the stability, as follows

$$L(w(0) + \rho) \succeq \Phi[w](0) \succeq L(w(0) - \rho) \quad (3)$$

According to [19], [8], that means that the nonlinearity Φ is active. Then, with (2), one gets

$$L(w(t) + \rho) \succeq \Phi[w](t) \succeq L(w(t) - \rho), \forall t \geq 0 \quad (4)$$

In a neighborhood of the origin, system (1) operates in open loop and the presence of the backlash $\Phi[w](t)$ may induce the existence of multiple equilibrium points or a limit cycle. Thus, one aims at characterizing compact sets $\mathcal{S}_0$ that are globally attractive in finite-time, or equivalently, such that, for any initial condition $x(0)$ belonging to $\Re^n \backslash \mathcal{S}_0$, the resulting trajectory of the closed-loop system (1) is uniformly ultimately bounded in the set $\mathcal{S}_0$ (see [15, Section 4.8]). This is stated in the following problem:

Problem 1 (Stability analysis): Given a gain K such that $A + BLK$ is Hurwitz, characterize the region of the state space in which the solutions of system (1) are uniformly ultimately bounded when initialized as in (3).

Imposing $A + BLK$ Hurwitz is classical since it corresponds to system (1) when $\Phi[w]$ is neglected and replaced by Lw. Let us emphasize that, unlike what is generally considered in the literature, in the current paper, the Hurwitz assumption on matrix A has been relaxed, i.e. the matrix A can be exponentially unstable.

The second problem we intend to solve is complementary to Problem 1. Actually, we want to make the domain $\mathcal{S}_0$ as small as possible, through the design of the state feedback gain K, as explicated below.

Problem 2 (Synthesis problem): Determine the gain K such that $A+BLK$ is Hurwitz and which minimizes the size of the associated region of the state space in which the solutions of the system (1) are uniformly ultimately bounded when initialized as in (3).

III. Main results

A. Preliminary results

For conciseness, throughout the paper, we denote $\dot{\Phi}$ instead of $\overbrace{\dot{\Phi}[w]}$, and Φ instead of $\Phi[w]$. Recall that the nonlinear operator Φ verifies the properties stated in Lemma 3.1 in [24]. Let us define the nonlinearity Ψ

$$\Psi = \Phi - Lw = \Phi - LKx \quad (5)$$

Hence, from Lemma 3.1 in [24] and (4), we can formulate the following properties with respect to Ψ.

Lemma 1: For any diagonal positive definite matrices N_1, N_2, N_3 in $\Re^{m\times m}$, with $N_3 \geq \mathbf{1}$, we have, for all $w \in \mathcal{C}^1_{pw}([0, +\infty); \Re^m)$, for all $t \in (t_{i-1}, t_i)$

$$(\dot{\Psi} + L\dot{w})' N_1 \Psi \leq 0, \quad (6)$$

$$(\dot{\Psi} + L\dot{w})' N_2 (\dot{\Psi} + (\mathbf{1} - N_3)L\dot{w}) \leq 0, \quad (7)$$

$$-L\rho \preceq \Psi \preceq L\rho \quad (8)$$

where $0 = t_0 < t_1 < \ldots$ is a partition of $[0, +\infty)$ such that w is continuously differentiable on each of the intervals (t_{j-1}, t_j), $j \in \mathbb{N}$.

Remark 1: Property (6) can be seen as a property of passivity with respect to the output $\dot{\Phi}$ and input $N_1 Lw$, and with the storage function $\frac{1}{2}\Phi' N_1 \Phi$ [15].

B. Stability analysis

The following result relative to Problem 1 can be stated.

Theorem 2: Given K such that $A_0 = A + BLK$ is Hurwitz. A positive scalar τ_1 being given, if there exist a symmetric positive definite matrix $P \in \Re^{n\times n}$, three diagonal positive definite matrices $N_1 \in \Re^{m\times m}$, $N_2 \in \Re^{m\times m}$ and $T_3 \in \Re^{m\times m}$ satisfying the following LMI conditions

$$\begin{pmatrix} He\{A_0' P\} + \tau_1 P & \star & \star \\ B'P - N_1 LKA_0 & -T_3 - He\{N_1 LKB\} & \star \\ -N_2 LKA_0 & -N_1 - N_2 LKB & -2N_2 \end{pmatrix} < \mathbf{0} \quad (9)$$

$$\rho' L T_3 L \rho - \tau_1 \leq 0 \quad (10)$$

then, for any initial admissible conditions $(x(0), \Psi(0))$, the resulting trajectories of the closed-loop system (1) are uniformly ultimately bounded in the set $\mathcal{S}_0(P)$ defined as follows:

$$\mathcal{S}_0(P) = \{x \in \Re^n; x'Px \leq 1\} \quad (11)$$

Proof. Consider a quadratic Lyapunov function candidate V defined by $V(x) = x'Px$, $P = P' > \mathbf{0}$, for all x in $\Re^n$. We want to verify that there exists a class $\mathcal{K}$ function α such that $\dot{V}(x) \leq -\alpha(V(x))$, for all x such that $x'Px \geq 1$ (i.e. for any $x \in \Re^n \backslash \mathcal{S}_0$), and for all nonlinearities Ψ satisfying Lemma 1 (i.e. satisfying relations (6), (7), (8)). By using the S-procedure, it corresponds to verify that $\mathcal{L} < 0$, where

$$\begin{aligned} \mathcal{L} = \dot{V}(x) - \tau_1(1 - x'Px) - \sum_{i=1}^{m} \tau_{3i}(\Psi'_{(i)}\Psi_{(i)} - \rho^2_{(i)}\ell^2_{(i)}) \\ -2(\dot{\Psi} + L\dot{w})' N_1 \Psi - 2(\dot{\Psi} + L\dot{w})' N_2 (\dot{\Psi} + (\mathbf{1} - N_3)L\dot{w}) \end{aligned} \quad (12)$$

with τ_1 a positive scalar and $T_3 = diag(\tau_{3i})$ a positive diagonal matrix. Choosing $N_3 = \mathbf{1}$ and noting that $\dot{w} = KA_0 x + KB\Psi$ and $\dot{V}(x) = x'(A'P + PA)x + 2x'PB\Phi$, or equivalently, from the

definition of Ψ in (5), $\dot{V}(x) = x'(A_0'P + PA_0)x + 2x'PB\Psi$, it follows that $\mathcal{L} = \mathcal{L}_0 + \rho' LT_3 L\rho - \tau_1$ with

$$\mathcal{L}_0 = \begin{pmatrix} x \\ \Psi \\ \dot{\Psi} \end{pmatrix}' \mathcal{M}_0 \begin{pmatrix} x \\ \Psi \\ \dot{\Psi} \end{pmatrix}$$

where $\mathcal{M}_0$ is the left-hand-side matrix in (9). The satisfaction of relations (9) and (10) implies both $\mathcal{L}_0 < 0$ and $\rho' LT_3 L\rho - \tau_1 \leq 0$, and then $\mathcal{L} < 0$. Therefore, there exists $\varepsilon > 0$, such that $\mathcal{L} \leq -\varepsilon \|(x' \ \Psi' \ \dot{\Psi}')'\|^2 \leq -\varepsilon x'x$. Hence, since by definition one gets $\dot{V}(x) \leq \dot{V}(x) - \tau_1(1 - x'Px) \leq \mathcal{L}$ one can also verify $\dot{V}(x) - \tau_1(1 - x'Px) \leq -\varepsilon x'x$, for any $x \in \Re^n \backslash \mathcal{S}_0$, or equivalently,

$$\dot{V}(x) \leq -\varepsilon x'x \ , \ \forall x \text{ such that } x'Px \geq 1 \ . \tag{13}$$

Consider now a solution of (1) starting from an admissible initial condition at $t_0 = 0$ such that $x(t_0)'Px(t_0) \geq 1$. According to (13), there exists a time $T \geq t_0 + (x(t_0)'Px(t_0) - 1)\lambda_{max}(P)/\epsilon$ such that $x(t) \in \mathcal{S}_0(P)$, $\forall t \geq T$. Furthermore, $\mathcal{S}_0(P)$ is an invariant set for system (1). Hence, in accordance with [15], it follows that the trajectories are uniformly ultimately bounded in $\mathcal{S}_0(P)$ for any $x(0)$. That concludes the proof of Theorem 2. ■

Theorem 2 ensures the finite-time convergence of the solutions of system (1) inside $\mathcal{S}_0(P)$ for any initial admissible condition. Hence, the set $\mathcal{S}_0(P)$ contains the possible equilibrium points or limit cycles potentially induced by the backlash operator. The equilibrium points of system (1), when they exist, satisfy: $\dot{x}_e = \mathbf{0} = (A + BLK)x_e + B\Psi_e$, or equivalently, since A_0 is Hurwitz, $x_e = -A_0^{-1}B\Psi_e$, $x_e'Px_e \leq 1$ and $L\rho \succeq \Psi_e \succeq -L\rho$.

C. Control design

To address Problem 2 and to remove the products between A_0 and K appearing in condition (9), the control design conditions are developed by using the Finsler lemma [2]. Such an approach is stated in the following result.

Theorem 3: A positive scalar τ_1 being given, if there exist a symmetric positive definite matrix $P \in \Re^{n\times n}$, five matrices $Q_1 \in \Re^{n\times n}$, $Q_2 \in \Re^{n\times n}$, $F_3 \in \Re^{m\times n}$, $F_4 \in \Re^{m\times n}$ and $K \in \Re^{m\times n}$, three diagonal positive definite matrices $N_1 \in \Re^{m\times m}$, $N_2 \in \Re^{m\times m}$ and $T_3 \in \Re^{m\times m}$ satisfying the matrix condition (14) (see top of the next page) and:

$$\rho' LT_3 L\rho - \tau_1 \leq 0 \tag{15}$$

then the state feedback gain K is such that $A + BLK$ is Hurwitz and, for any initial admissible conditions $(x(0), \Psi(0))$, the resulting trajectories of the closed-loop system (1) are uniformly ultimately bounded in the set $\mathcal{S}_0(P)$ as defined in (11).

Proof. Consider the quadratic Lyapunov function defined by $V(x) = x'Px$, $P = P' > \mathbf{0}$, for all x in $\Re^n$. The time-derivative of $V(x)$ along the trajectories of system (1) reads $\dot{V}(x) = \dot{x}'Px + x'P\dot{x}$. Moreover, by noting $\xi = \begin{pmatrix} x' & \dot{x}' & \Psi' & \dot{\Psi}' \end{pmatrix}'$, system (1) can be also written as follows:

$$\begin{pmatrix} A + BLK & -\mathbf{1} & B & \mathbf{0} \end{pmatrix} \xi = \mathbf{0} \tag{16}$$

We then want to verify that there exists a $\mathcal{K}$ function α such that $\dot{V}(x) \leq -\alpha(V(x))$, for all x such that $x'Px \geq 1$, for all nonlinearity Ψ satisfying Lemma 1 and for x, Ψ satisfying (16). In other words, by using the S-procedure, we can express $\mathcal{L}$ defined in (12) as $\mathcal{L} = \xi'\mathcal{L}_0\xi - \tau_1 + \rho' LT_3 L\rho$, i.e., we want to verify:

$$\begin{array}{l} \xi'\mathcal{L}_0\xi = \xi' \begin{pmatrix} \tau_1 P & P & \mathbf{0} & \mathbf{0} \\ P & \mathbf{0} & -K'LN_1 & -K'LN_2 \\ \mathbf{0} & -N_1LK & -T_3 & -N_1 \\ \mathbf{0} & -N_2LK & -N_1 & -2N_2 \end{pmatrix} \xi < 0 \\ \quad \text{and } -\tau_1 + \rho' LT_3 L\rho \leq 0 \\ \text{for } \xi \text{ such that } \begin{pmatrix} A + BLK & -\mathbf{1} & B & 0 \end{pmatrix} \xi = \mathbf{0} \end{array} \tag{17}$$

with τ_1 a positive scalar and T_3 a positive diagonal matrix. By using the Finsler lemma, the satisfaction of (17) consists of finding some multipliers $F_1 \in \Re^{n\times n}$, $F_2 \in \Re^{n\times n}$, $F_3 \in \Re^{m\times n}$, $F_4 \in \Re^{m\times n}$ such that $\mathcal{L}_0 + He\{\mathcal{F} \begin{pmatrix} A + BLK & -\mathbf{1} & B & \mathbf{0} \end{pmatrix}\} < \mathbf{0}$, with $\mathcal{F} = \begin{pmatrix} F_1' & F_2' & F_3' & F_4' \end{pmatrix}'$, which reads:

$$\begin{pmatrix} He\{F_1A + F_1BLK\} + \tau_1 P & \star & & \\ P + F_2A + F_2BLK - F_1' & -F_2 - F_2' & & \\ F_3A + F_3BLK + B'F_1' & -N_1LK + B'F_2' - F_3 & \star & \star \\ F_4A + F_4BLK & -F_4 - N_2LK & \star & \star \\ & & He\{F_3B\} - T_3 & \star \\ & & -N_1 + F_4B & -2N_2 \end{pmatrix} < \mathbf{0}$$

A necessary condition for this inequality to be strict is the non-singularity of matrix F_2. Hence, by considering that F_1 is also nonsingular and by multiplying the previous inequality at left by $diag(Q_1; Q_2; \mathbf{1}; \mathbf{1})$ and at right by $diag(Q_1'; Q_2'; \mathbf{1}; \mathbf{1})$ with $Q_1 = F_1^{-1}$ and $Q_2 = F_2^{-1}$, one obtains relation (14). The satisfaction of relations (14) and (15) implies the satisfaction of condition (17). Furthermore, due to the block $(1,1)$ of the matrix in relation (14), the satisfaction of relation (14) means that the gain K is such that $A + BLK$ is Hurwitz. Similarly to the proof of Theorem 2, by definition, $\mathcal{S}_0(P)$ is an invariant set for the trajectories of system (1). Hence, in accordance with [15], it follows that the trajectories are uniformly ultimately bounded in $\mathcal{S}_0(P)$ for any $x(0)$. That concludes the proof. ■

D. Discussion on systems with backlash in the output

The solution to the analysis problem (Theorem 2) may be directly extended to the case of backlash in the output, considering the following system:

$$\begin{cases} \dot{x} = Ax + B_1u \\ y = Cx + D\Phi[w] \end{cases} \text{ with } \begin{cases} w = Ex \\ u = Ky \end{cases} \tag{18}$$

which can also be written as:

$$\dot{x} = (A + B_1K(C + DLE))x + B_1KD\psi \tag{19}$$

with $\psi = \Phi - Lw$ defined as in equation (5). The static output feedback gain K is supposed chosen such that $(A + B_1K(C + DLE))$ is Hurwitz. Then, by denoting $A_0 = (A + B_1K(C + DLE))$ and $B = B_1KD$, Theorem 2 directly applies. It also applies to the case of a dynamic output feedback with backlash in the measured output. On the other hand, the extension of Theorem 3 to the design problem with backlash in the sensor is not direct.

IV. Numerical implementation

A. Computational issues

As far as τ_1 is fixed, conditions stated in Theorem 2 are linear in the decision variables. The problem may then be solved as an optimization problem to evaluate the smallest set $\mathcal{S}_0(P)$, typically described by its volume, proportional to $\sqrt{\det(P^{-1})}$ [2], in which the trajectories of the closed-loop system are uniformly ultimately

$$\begin{pmatrix} He\{AQ_1' + BLKQ_1'\} + \tau_1 Q_1 P Q_1' & \star & \star & \star \\ Q_2 P Q_1' + AQ_1' + BLKQ_1' - Q_2 & -Q_2 - Q_2' & \star & \star \\ F_3 AQ_1' + F_3 BLKQ_1' + B' & -N_1 LKQ_2' + B' - F_3 Q_2' & F_3 B + B' F_3 - T_3 & \star \\ F_4 AQ_1' + F_4 BLKQ_1' & -F_4 Q_2' - N_2 LKQ_2' & -N_1 + F_4 B & -2N_2 \end{pmatrix} < \mathbf{0} \quad (14)$$

bounded. In the numerical examples which follow, thanks to the use of Yalmip [16] to build the LMI optimization problem, we use the convex minimization problem to solve Problem 1:

$$\begin{array}{l} \min \ -geomean(P) \\ \text{under conditions (9), (10)} \end{array} \quad (20)$$

where $geomean(P)$ is the geometric mean of the eigenvalues of the matrix P.

To solve the design problem expressed through Theorem 3, we exploit the fact that matrices Q_1, Q_2, F_3 and F_4 are Finsler multipliers which may be considered as degrees of freedom in the design conditions. Actually, one selects $Q_2 = Q_1$, $F_3 = \mathbf{0}$, $F_4 = \mathbf{0}$, and introduces the variables $Y = KQ_1'$, $R = Q_1 P Q_1'$, $S_1 = N_1^{-1}$ and $S_2 = N_2^{-1}$. With this change of variables and by pre- and post-mulitplying the matrix in condition (14) by $diag(\mathbf{1}; \mathbf{1}; S_1; S_2)$, condition (14) is then updated as follows:

$$\begin{pmatrix} He\{AQ_1' + BLY\} + \tau_1 R & \star & \star & \star \\ R + AQ_1' + BLY - Q_1 & -Q_1 - Q_1' & \star & \star \\ B' & LY + S_1 B' & -S_1 T_3 S_1 & \star \\ \mathbf{0} & -LY & -S_2 & -2S_2 \end{pmatrix} < \mathbf{0} \quad (21)$$

Then the control design Problem 2 is solved as the optimization problem:

$$\begin{array}{l} \min\limits_{R,Y,Q_1,T_3,S_2} \alpha \\ \text{under conditions (15), (21),} \begin{pmatrix} R & Q_1 \\ Q_1' & \alpha P_g^{-1} \end{pmatrix} > \mathbf{0} \end{array} \quad (22)$$

where

- P_g is selected as the solution to the analysis problem (20) or as directions of interest for the minimization of the set $\mathcal{S}_0(P) \subseteq \{x \in \Re^n,\ x' P_g x \leq \alpha\}$, with $P = (Q_1)^{-1} R (Q_1')^{-1}$, in which the trajectories of the closed-loop system are uniformly ultimately bounded;
- τ_1 is selected, for example on a grid;
- S_1 is selected as the inverse of the solution N_1 to the analysis problem. A grid search may alternatively be performed to select S_1.

Additional constraints may be added to set the performance of the closed-loop system, such as a limit on the size of the gain, or a pole-placement requirement [6]. This can be done by prescribing additionally the following LMI condition with extra variable R_{pp}, issued from the use of Finsler Lemma, which allows to place the poles of $(A + BLK)$ in the disk centered in σ with radius r:

$$\begin{pmatrix} -R_{pp} & Q_1 \frac{(A' - \sigma \mathbf{1})}{r} + Y' \frac{L' B'}{r} \\ \frac{(A - \sigma \mathbf{1})}{r} Q_1' + \frac{BL}{r} Y & R_{pp} - Q_1 - Q_1' \end{pmatrix} > \mathbf{0} \quad (23)$$

B. Illustrative Examples

1) Example 1: The first example is of academic nature and illustrates the conservatism of the approach. Consider the unstable system (1) defined by the following data:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ; \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} ; \quad \begin{array}{c} K = K_{an} = \begin{pmatrix} -2 & -3 \end{pmatrix} \\ L = 1 \ ; \ \rho = 0.5 \end{array}$$

Considering $\tau_1 = 0.9$ and applying Theorem 2, the optimization problem (20) gives the solution to the stability analysis problem

$$P_{an} = \begin{pmatrix} 15.9510 & 6.8038 \\ 6.8038 & 15.1183 \end{pmatrix} ; \quad \begin{array}{l} N_{1an} = 5.3033\ 10^{-5} \\ N_{2an} = 1.3525\ 10^{-4} \end{array}$$

for which the indicator of the volume is $v(P_{an}) = \sqrt{\det(P_{an}^{-1})} = 0.0716$. The influence of the parameter τ_1 on the size of $\mathcal{S}_0(P)$ is illustrated in Figure 2 (the problem becomes unfeasible for $\tau_1 \geq 2 = 2|\lambda_{max}(A_0)|$). An iterative search on τ_1 allows to quickly select a convenient value for this parameter.

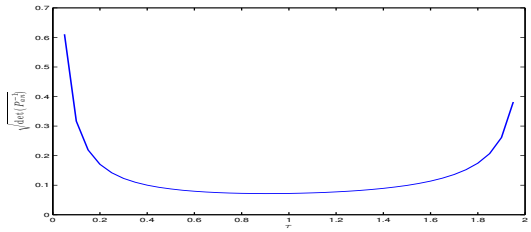


Fig. 2. Example 1 - Analysis problem. Influence of the parameter τ_1 on the size of $\mathcal{S}_0(P)$.

For the synthesis problem, let us consider the optimization problem (22). τ_1 is selected unchanged (equal to 0.9), $P_g = P_{an}$, $S_1 = N_{1an}^{-1}$. An additional pole-placement requirement (23) is used with $\sigma = -6$, $r = 5$. A solution to the optimization problem (22) is given by

$$K_{syn} = \begin{pmatrix} -10.3327 & -11.2293 \end{pmatrix}$$

It is recommended to proceed to the analysis of this new control gain K_{syn}, which results, for τ_1 unchanged, to the solution $v(P_{synan}) = \sqrt{\det(P_{synan}^{-1})} = 0.0057$, with

$$P_{synan} = \begin{pmatrix} 422.7379 & 33.4682 \\ 33.4682 & 74.3510 \end{pmatrix}$$

The phase portraits of some trajectories of the system in closed loop with the control gain K_{an} (solid line) or K_{syn} (dashed line), issued from the initial states $x(0)$, equal to $\begin{pmatrix} 0 & 1 \end{pmatrix}'$ and $\begin{pmatrix} 1 & 1 \end{pmatrix}'$, are plotted in Figure 3 ($\Phi[w](0) = LKx(0)$ which satisfies (3)). They show that the trajectories of the closed-loop system with K_{an} converge to a limit cycle included in the ellipsoid $\mathcal{S}_0(P_{an})$. The trajectories of the closed-loop system with K_{syn} are uniformly ultimately bounded in the set $\mathcal{S}_0(P_{synan})$ included in the set $\mathcal{S}_0(P_{an})$. They may converge either to a limit cycle or to some equilibrium point (trajectory issued from $\begin{pmatrix} 1 & 1 \end{pmatrix}'$), which are included in $\mathcal{S}_0(P_{synan})$.

To evaluate the influence of the choice of S_1 on the solution to the design problem, a grid search in the sets defined by $S_1 = \{10^3, 10^4, 10^5, 10^6, 10^7\}$, $\tau_1 = \{0.1, 0.2, \cdots, 1.8, 1.9\}$ has been performed. It gives as the best solution, after an additional analysis step $v(P_{best}) = \sqrt{\det(P_{best}^{-1})} = 0.0033$ (to be compared to the value 0.0057 obtained with *a priori* given values for τ_1 and $S_1 = N_{an}^{-1}$ solutions to the analysis problem).

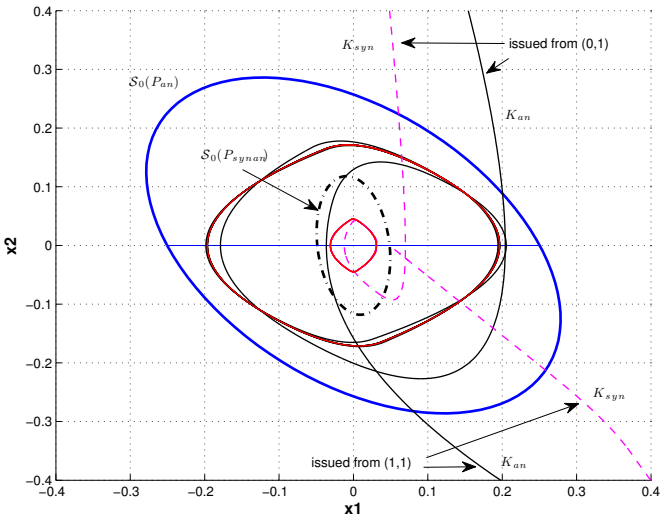


Fig. 3. Example 1. Ellipsoidal sets $\mathcal{S}_0(P_{an})$ and $\mathcal{S}_0(P_{synan})$. Trajectories issued from $(0\ 1)'$ and $(1\ 1)'$ for the system closed with control gains K_{an} (solid line) and K_{syn} (dashed line).

2) Example 2: As a second example, we consider an unstable F-8 aircraft MIMO example borrowed from [4], given by:

$$A_p = \begin{pmatrix} -0.8 & -0.006 & -12 & 0 \\ 0 & -0.014 & -16.64 & -32.2 \\ 1 & -0.0001 & -1.5 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; B_{pu} = \begin{pmatrix} -19 & -3 \\ -0.66 & -0.5 \\ -0.16 & -0.5 \\ 0 & 0 \end{pmatrix}$$

A backlash element interconnects the system with a state-feedback control, which values are given by

$$L = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

$$K = K_{an} = \begin{pmatrix} 0.2672 & 0.0059 & -0.8323 & 0.8089 \\ 0.5950 & -0.1534 & 2.1168 & 0.0525 \end{pmatrix}$$

Considering $\tau_1 = 1.5$ selected from an iterative procedure, the optimization problem (20) gives the solution $v(P_{an}) = \sqrt{\det(P_{an}^{-1})} = 17.7848$, with

$$P_{an} = \begin{pmatrix} 0.1381 & 0.0035 & -0.2112 & 0.2818 \\ 0.0035 & 0.0266 & -0.6522 & 0.4999 \\ -0.2112 & -0.6522 & 16.7322 & -13.0185 \\ 0.2818 & 0.4999 & -13.0185 & 11.5846 \end{pmatrix}$$

For the synthesis case, one considers $P_g = P_{an}$ and a pole-placement requirement with $\sigma = -10$, $r = 8.5$. A grid search in the sets defined by $S_1 = \{10^2\mathbf{1}, 10^3\mathbf{1}, 10^4\mathbf{1}, 10^5\mathbf{1}, 10^6\mathbf{1}\}$, $\tau_1 = \{1.1, 1.2, \cdots, 2.1, 2.2\}$, followed by an additional analysis step, allows to determine the control gain

$$K_{syn} = \begin{pmatrix} 0.9533 & 0.2757 & -6.6111 & 7.9905 \\ 0.1246 & -2.0493 & 37.6241 & -24.2286 \end{pmatrix}$$

with associated set defined from P_{synan} with indication of the volume $v(P_{synan}) = \sqrt{\det(P_{synan}^{-1})} = 0.0831$. The time-evolution of the state x is plotted in Figure 4, issued from the initial state $x(0) = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}'$ and initial nonlinearity $\Phi[w](0) = LKx(0)$. It brings out the reduction of the size of the limit cycle with the gain K_{syn} (plots in dashed line) in accordance with the objective which was actually to reduce the set $\mathcal{S}_0(P)$. Note however that this induces larger amplitude of the system input $\Phi[w]$, as it may also be seen on the form of the backlash operator characteristic in Figure 5.

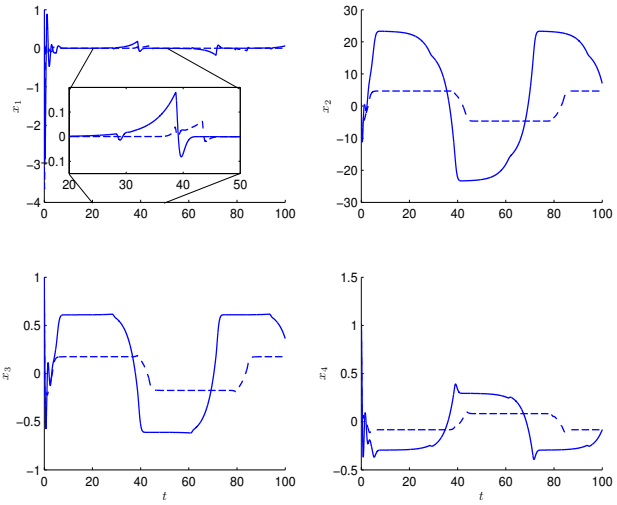


Fig. 4. Example 2 - Time-evolution of the state x. System in closed loop with K_{an} (solid line) or K_{syn} (dashed line) + zoom of the steady-state behavior of x_1 between $t = 20$ and $t = 50$.

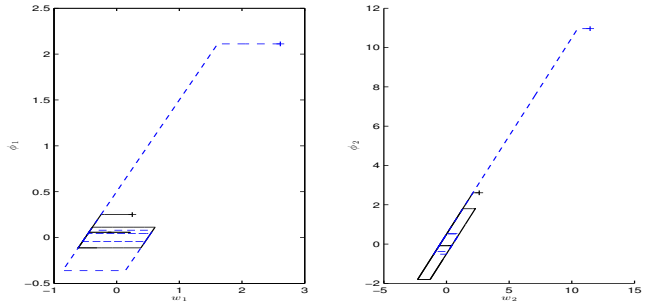


Fig. 5. Example 2 - Backlash characteristic $\Phi[w]$. System in closed loop with K_{an} (solid line) or K_{syn} (dashed line).

To evaluate the degree of conservatism of the sets $\mathcal{S}_0(P_{an})$ and $\mathcal{S}_0(P_{synan})$ in which the trajectories are uniformly ultimately bounded, the time-evolution of $x'P_{an}x$ (solid line, associated to the control gain K_{an}) and $x'P_{synan}x$ (dashed line, associated to the control gain K_{syn}) are plotted in Figure 6. They show that, once the trajectory enters the ellipsoidal set $\mathcal{S}_0(P)$, i.e. $x'Px \leq 1$, $P = P_{an}$ or $P = P_{synan}$, it evolves over the time and comes more or less close to the boundary but never exits.

3) Example 3: A third example is proposed to illustrate how the analysis conditions stated in Theorem 2 may be used in the case of a feedback between the system output and the backlash input involving a dynamic system. This is typically the case when considering a dynamic feedback controller or an actuator device system such as proposed in [9], defined by:

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= c_1x_1 + c_2x_2 + \Phi[w] \end{cases} \quad \text{with} \quad \begin{cases} \dot{x}_A &= c_3x_A + v \\ w &= x_A \end{cases}$$

and a backlash operator with $L = 1$, $\rho = 1.5$. For numerical evaluation, values slightly different from [9] are used to manipulate an unstable open-loop system. We consider the coefficients $c_1 = 1$, $c_2 = -1$, $c_3 = -5$ and a state feedback control $v = -9x_1 - 3x_2$.

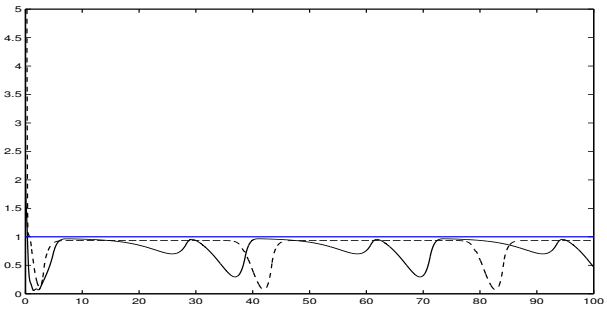


Fig. 6. Example 2 - Time-evolution of $x'P_{an}x$ (solid line) associated to the control gain K_{an} and of $x'P_{synan}x$ (dashed line) associated to the control gain K_{syn}.

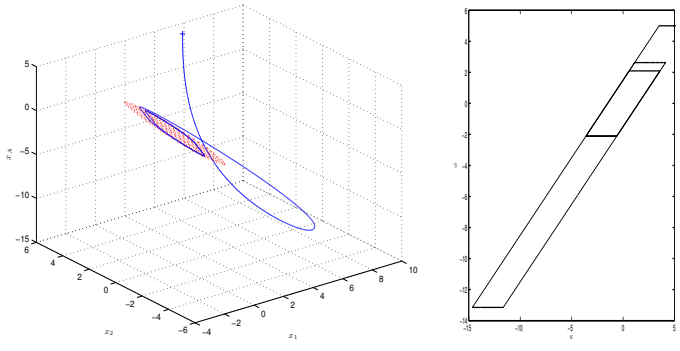
The structure of system (1) is retrieved with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -9 & -3 & -5 \end{pmatrix} \; ; \; B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \; ; \; K = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

Using the optimization procedure (20) and an iterative search of τ_1, one obtains, for $\tau_1 = 0.65$,

$$P_{an} = \begin{pmatrix} 150.9329 & 19.7677 & 82.7974 \\ 19.7677 & 2.9032 & 10.7988 \\ 82.7974 & 10.7988 & 45.4846 \end{pmatrix}$$

The phase portrait and backlash characteristic of the trajectory issued from $\begin{pmatrix} 5 & 5 & 5 \end{pmatrix}'$ are depicted in Figures 7-a and 7-b, respectively.



a b

Fig. 7. Example 3 - a. Ellipsoidal set $\mathcal{S}_0(P_{an})$ and trajectory issued from $(5\ 5\ 5)'$ - b. Backlash characteristic $\Phi[w]$.

V. Concluding remarks

This paper considered systems with backlash operator in the input, aiming at analyzing the stability of such control systems. Uniform ultimate boundedness stability and stabilization problems were tackled allowing to characterize suitable regions of the state space in which the closed-loop trajectories can be captured. Future research directions are open. One idea would be to extend the technique to more general models of nonlinearities as studied, for example, in [5], [14]. Furthermore, as recalled in [12, Page 38], any Prandtl operator may be derived from an infinite weighted sum of backlash operators. It may be fruitful to adapt the present Lyapunov technic to such nonlinear systems. It may also be interesting to consider set-valued nonlinear operators encompassing Prandtl operators (as in [12, Page 37]). Finally, the case of a feedback between the system output and the backlash input involving an actuator device system subject to other nonlinearities (like, for example, saturation) should be considered.

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