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Veronese transform, and Castelnuovo-Mumford regularity of modules

MARCEL MORALES

Université de Grenoble I, Institut Fourier, UMR 5582, B.P.74,
38402 Saint-Martin D'Hères Cedex,
and IUFM de Lyon, 5 rue Anselme,
69317 Lyon Cedex (FRANCE)

NGUYEN THI DUNG

Thai Nguyen University of Agriculture and Forestry,
Thai Nguyen,
Vietnam,

Abstract ¹

Veronese rings, Segre embeddings or more generally Segre-Veronese embeddings are very important rings in Algebraic Geometry. In this paper we present an original, elementary way to compute the Hilbert-Poincaré series of these rings, as a consequence we compute their Castelnuovo-Mumford regularity, and also the leading term of the h -vector. Moreover, we can compute the Castelnuovo-Mumford regularity of the n -Veronese Module of any finitely generated Cohen-Macaulay graded module.

1 Introduction

Veronese rings, Segre embeddings or more generally Segre-Veronese embeddings are very important rings in Algebraic Geometry. It is well known that these rings are arithmetically Cohen-Macaulay, hence their Hilbert-Poincaré series can be written $P_R(t) = \frac{Q_R(t)}{(1-t)^{\dim R}}$, where $Q_R(t)$ is a polynomial on t with $Q_R(1) \neq 0$ having positive integer coefficients, the sequence of the coefficients of $Q_R(t)$ is also called the h -vector of R . The degree of $Q_R(t)$ is the Castelnuovo-Mumford regularity (c.f.[7][Chapter 4]), and the leading term of $Q_R(t)$ is the highest graded Betti number of R . By using very original and elementary methods we are able to compute the leading term of $Q_R(t)$. Our results allows to compute the Castelnuovo-Mumford regularity of the n Veronese Module of any finitely generated Cohen-Macaulay graded module, and the rings called of Veronese type.

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Note that this result can be proved easily by using local cohomology, but our purpose is to give a very elementary proof.

Our main results improves partially [1] and [5].

Theorem. *Let consider the Segre-Veronese ring $R_{\underline{b},n}$, $\dim R_{\underline{b},n} = b_1 + \dots + b_m + 1$. Let $P_{R_{\underline{b},n}}(t) = \frac{Q_{R_{\underline{b},n}}}{(1-t)^{\dim R_{\underline{b},n}}}$ be the Hilbert-Poincaré series of $R_{\underline{b},n}$, with $Q_{R_{\underline{b},n}} = h_0 + h_1 t + \dots + h_{r_{\underline{b},n}} t^{r_{\underline{b},n}}$, where $r_{\underline{b},n}$ is the Castelnuovo-Mumford regularity of $R_{\underline{b},n}$. We set $\alpha_{\underline{b},n} = \dim R_{\underline{b},n} - r_{\underline{b},n}$. After a permutation of b_1, \dots, b_m , we can assume that, for all $i = 1, \dots, m$, $\lceil \frac{b_i+1}{n_i} \rceil > \frac{b_i}{n_i}$. Then*

$$\alpha_{\underline{b},n} = \lceil \frac{b_1 + 1}{n_1} \rceil \quad ,$$

and the highest Betti number of $R_{\underline{b},n}$ is

$$\beta_{r_{\underline{b},n}} = h_{r_{\underline{b},n}} = \binom{n_1 \alpha_{\underline{b},n} - 1}{b_1} \dots \binom{n_m \alpha_{\underline{b},n} - 1}{b_m}$$

Theorem. *Fix integers $d, n \in \mathbb{N}^*$, $\tau \in \mathbb{Z}$. Let $(a_l)_{l \in \mathbb{Z}}$ be a sequence of complex numbers, such that $a_l = 0$ for $l \ll 0$, set :*

$$f(t) = \sum_{l \in \mathbb{Z}} a_l t^l, \quad f^{<n, \tau>}(t) = \sum_{l \in \mathbb{Z}} a_{nl+\tau} t^l.$$

If $f(t) = \frac{h(t)}{(1-t)^d}$ with $h(t) \in \mathbb{C}[t, t^{-1}]$ then $f^{<n, \tau>}(t) = \frac{h^{<n>}(t)}{(1-t)^d}$ with $h^{<n>}(t) \in \mathbb{C}[t, t^{-1}]$ such that:

- $\deg h^{<n, \tau>}(t) \leq d - \lceil \frac{d - \deg h(t) + \tau}{n} \rceil$,
- *If all the coefficients of $h(t)$ are positive real numbers then*
 $\deg h^{<n, \tau>}(t) = d - \lceil \frac{d - \deg h(t) + \tau}{n} \rceil$,
- *If $\deg h(t) = d$ then $\deg h^{<n>}(t) = d$.*

2 Preliminaries on toric rings and Hilbert-Poincaré series

Let $S = K[x_0, \dots, x_s, x_0^{-1}, \dots, x_s^{-1}]$ be a Laurent polynomial ring over a field K on a finite set of variables. For any finite set \mathcal{M} of monomials in S , let $K[\mathcal{M}] \subset S$ be the subring of S generated by the set

\mathcal{M} . It is the toric ring defined by the semigroup generated by \mathcal{M} . In what follows we consider the special case where $S = K[x_0, \dots, x_s]$ is a polynomial ring over the field K and all the monomials in \mathcal{M} are of the same degree.

Example 2.1. Let $S = K[x_0, \dots, x_b] \oplus_{l \in \mathbb{N}} S_l$, and $\mathcal{M} = \{x_0^{\alpha_0} \dots x_b^{\alpha_b} \mid \alpha_0 + \dots + \alpha_b = n\}$. So that

$$R_{b,n} = K[\mathcal{M}] = \oplus_{l \in \mathbb{N}} S_{nl}.$$

This toric ring is known as the n -Veronese embedding of S .

Example 2.2. More generally, let X_1, \dots, X_m , m sets of independent disjoint variables, with $\text{Card}(X_i) = b_i + 1$. Let $S_i = K[X_i]$ for $i = 1, \dots, m$, $S = K[X_1 \cup X_2 \cup X_m]$, and $\mathcal{M} = \{x_1 x_2 \dots x_m \mid x_i \in X_i\}$. So that

$$R_{b_1, \dots, b_m} = K[\mathcal{M}] = \oplus_{l \in \mathbb{N}} (S_1)_l \otimes \dots \otimes (S_m)_l.$$

This toric ring is known as the Segre embedding of the m polynomial rings S_1, \dots, S_m .

Example 2.3. Let X_1, \dots, X_m , sets of independent disjoint variables such that $X_i = \{x_{i,0}, \dots, x_{i,b_i}\}$, $S_i = K[X_i]$ for $i = 1, \dots, m$, and $n_1, \dots, n_m \in \mathbb{N}$, Let $S = K[X_1 \cup X_2 \cup X_m]$, and

$$\mathcal{M} = \{\underline{x}_1^{\alpha_1} \dots \underline{x}_m^{\alpha_m} \mid |\alpha_i| = n_i\},$$

where $\alpha_i = (\alpha_{i,0}, \dots, \alpha_{i,b_i})$, $\underline{x}_i^{\alpha_i} = x_{i,0}^{\alpha_{i,0}} \dots x_{i,b_i}^{\alpha_{i,b_i}}$, and $|\alpha_i| = \alpha_{i,0} + \dots + \alpha_{i,b_i}$. The Segre-Veronese embedding:

$$R_{\underline{b}, \underline{n}} = K[\mathcal{M}] = \oplus_{l \in \mathbb{N}} (S_1)_{n_1 l} \otimes \dots \otimes (S_m)_{n_m l},$$

where $\underline{b} = (b_1, \dots, b_m)$, $\underline{n} = (n_1, \dots, n_m)$.

Let $S = K[x_0, \dots, x_s]$ be a polynomial ring over the field K , graded by the standard graduation, that is $\deg x_i = 1$, for all i . Let $R := S/I$, where $I \subset S$ is a graded ideal, let $M = \oplus_{l \in \mathbb{Z}} M_l$ be a finitely generated graded R -module, hence M is also a S -module. the Hilbert-function of M is defined by $H_M(l) = \dim_K M_l$, for all $l \in \mathbb{Z}$, and the Hilbert-Poincaré series of M :

$$P_M(t) = \sum_{l \in \mathbb{Z}} H_M(l) t^l.$$

It is well known that

$$P_M(t) = \frac{Q_M(t)}{(1-t)^{\dim M}} ,$$

where $Q_M(t)$ is a Laurent polynomial on t, t^{-1} with $Q_M(1) \neq 0$. Moreover if M is a Cohen-Macaulay S -module, all the coefficients of $Q_M(t)$ are natural integers, and the Castelnuovo-Mumford regularity of M is the degree of $Q_M(t)$. For more details on Hilbert-Poincaré series see [10], [4][Chapter 4], [7][Chapter 4].

Theorem 2.4. (*Hilbert's Theorem*) *let $M = \bigoplus_{l \in \mathbb{Z}} M_l$ be a finitely generated graded S -module. There exists a polynomial with integer coefficients $\Phi_{H_M}(l)$ such that $H_M(l) = \Phi_{H_M}(l)$, for l large enough. Moreover the leading term of $\Phi_{H_M}(l)$ can be written as : $\frac{\deg(M)}{d!} l^d$, where $d+1$ is the dimension of M and $\deg(M)$ the degree or multiplicity of M .*

Remark 2.5. *The postulation number of the Hilbert function is the biggest integer l such that $H_M(l) \neq \Phi_{H_M}(l)$. It is well known, ([10], [4][Chapter 4]), that the postulation number equals the degree of the rational fraction defining the Poincaré series.*

Remark 2.6. *We recall that binomial coefficients can be defined in a more general setting than natural numbers, indeed for $k \in \mathbb{N}$, binomial coefficients are polynomial functions in the variable n . More precisely:*

- (1) *If $k = 0$ then let $\binom{n}{0} = 1$, for all $n \in \mathbb{C}$.*
- (2) *If $k > 0$ then let $\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!}$, for all $n \in \mathbb{C}$.*

Note that for all $n \in \mathbb{C}$, $\binom{n}{k} = (-1)^k \binom{k-n-1}{k}$ and if $n \in \mathbb{N}, n < k$, then $\binom{n}{k} = 0$.

Example 2.7. *Let $S = K[x_0, \dots, x_b]$, be a polynomial ring. Then*

$$H_S(l) = \begin{cases} \binom{l+b}{b} & \text{if } l \geq 0 \\ 0 & \text{if } l < 0 \end{cases} , \quad P_S(t) = \frac{1}{(1-t)^{b+1}} .$$

Note that in fact $\forall l \geq -b$, $H_S(l) = \binom{l+b}{b}$ and $0 = H_S(-b-1) \neq \binom{-b-1+b}{b} = (-1)^b$, so the postulation number of S is $-(b+1)$.

Example 2.8. Let $S = K[x_0, \dots, x_b]$, $\mathcal{M} = \{x_0^{\alpha_0} \dots x_b^{\alpha_b} \mid \alpha_0 + \dots + \alpha_b = n\}$, and $R_{b,n} = K[\mathcal{M}]$ the n -Veronese embedding. Then

$$H_{R_{b,n}}(l) = H_S(nl) = \begin{cases} \binom{nl+b}{b} & \text{if } l \geq 0 \\ 0 & \text{if } l < 0 \end{cases}.$$

Note that $\binom{nl+b}{b} = \frac{(nl+1)(nl+2)\dots(nl+b)}{b!}$ is a polynomial on l with leading term $\frac{n^b l^b}{b!}$, so that $\deg(R_{b,n}) = n^b$, $\dim R_{b,n} = b + 1$. Note also that $\forall l > -\lceil \frac{b+1}{n} \rceil$, $H_{R_{b,n}}(l) = \binom{nl+b}{b}$ and $0 = H_{R_{b,n}}(-\lceil \frac{b+1}{n} \rceil) \neq \binom{-\lceil \frac{b+1}{n} \rceil n + b}{b} = (-1)^b \binom{\lceil \frac{b+1}{n} \rceil n - 1}{b}$, so the postulation number of $R_{b,n}$ is $-\lceil \frac{b+1}{n} \rceil$. More generally the postulation number of $R_{b,n}[\tau]$ is $\lceil \frac{b+1+\tau}{n} \rceil$.

3 Veronese of generating series

In a recent paper [2], Brenti and Walker prove that taking the n -Veronese transform of the h polynomial is a linear function, in this section we improve this result giving a elementary proof of the fact that taking the shifted n -Veronese transform of the h polynomial is a linear function on h .

Let recall the following fact:

Theorem 3.1. Let $(a_l)_{l \in \mathbb{Z}}$ be a sequence of complex numbers, such that $a_l = 0$ for $l \ll 0$, set $f(t) = \sum_{l \in \mathbb{Z}} a_l t^l$, TFAE:

- There exists $h(t) \in \mathbb{C}[t, t^{-1}]$ and a natural integer d such that $f(t) = \frac{h(t)}{(1-t)^d}$.
- There exists $\Phi(t) \in \mathbb{C}[t, t^{-1}]$ of degree $d - 1$ with leading coefficient $e_0/(d - 1)!$, such that $\Phi(l) = a_l$ for l large enough.

Moreover $h(1) = e_0$.

Definition 3.2. Fix integers $d, n \in \mathbb{N}^*$, $\tau \in \mathbb{Z}$. Let $(a_l)_{l \in \mathbb{Z}}$ be a sequence of complex numbers, such that $a_l = 0$ for $l \ll 0$, set :

$$f(t) = \sum_{l \in \mathbb{Z}} a_l t^l, \quad f^{<n, \tau>}(t) = \sum_{l \in \mathbb{Z}} a_{nl+\tau} t^l.$$

By the Theorem 3.1 if $f(t) = \frac{h(t)}{(1-t)^d}$ with $h(t) \in \mathbb{C}[t, t^{-1}]$ then $f^{<n, \tau>}(t) = \frac{h^{<n, \tau>}(t)}{(1-t)^d}$ with $h^{<n, \tau>}(t) \in \mathbb{C}[t, t^{-1}]$.

Let us introduce some notations. To any non zero polynomial $h(t) = h_\sigma t^\sigma + \dots + h_0 + h_1 t + \dots + h_s t^s \in \mathbb{C}[t, t^{-1}]$ we associate the h -vector $\vec{h} = (\dots, 0, h_\sigma, \dots, h_s, 0, \dots)$, and we set $\deg \vec{h} = \deg h(t)$. For $j \in \mathbb{Z}$, let $\vec{\varepsilon}_j$ be the h -vector of the polynomial t^j . Let denote by $[t^k]h(t)$ the coefficient of t^k in the polynomial $h(t)$. For any $i, j \in \mathbb{Z}$ define $\mathcal{D}_{i,j}$ by

$$\mathcal{D}_{i,j} = [t^{in-j}] \left(\frac{(1-t^n)^d}{(1-t)^d} \right) = [t^{in-j}] ((1+t+\dots+t^{n-1})^d).$$

Note that

$$\mathcal{D}_{i,j} = \text{Card}\{(x_1, \dots, x_d) \in \mathbb{N}^d \mid \forall l, x_l \leq n-1; x_1 + \dots + x_d = in - j\}.$$

Finally let $\mathcal{D}[\sigma, \tau]$ be the infinite square matrix $\mathcal{D}[\sigma, \tau] = (\mathcal{D}_{i+\sigma, j+\tau})$. For $\sigma = \tau = 0$ we write \mathcal{D} instead $\mathcal{D}[0, 0]$. We can give some properties of the numbers $\mathcal{D}_{i,j}$.

Lemma 3.3. *Let $i, j, k \in \mathbb{Z}$, we have:*

- $\mathcal{D}_{i,j} = 0$ if either $in - j < 0$ or $in - j > d(n-1)$.
- For any i, j , $\mathcal{D}_{i,j} = \mathcal{D}_{d-i, d-j}$. That is \mathcal{D} is symmetrical around the point $(d/2, d/2)$.
- For $0 \leq k \leq n-1$, $\mathcal{D}_{d, d+k} = \binom{k+d-1}{d-1}$.
- $\mathcal{D}_{1,0} = \binom{n+d-1}{d-1} - d$, and for $1 \leq k \leq n$, $\mathcal{D}_{1,k} = \binom{n-k+d-1}{d-1}$.
- For any integers q, k , $\mathcal{D}_{d+q, nq+k} = \mathcal{D}_{d,k}$.
- For any i , let $d-i = nq - k$ with $q = \lceil \frac{d-i}{n} \rceil$, $0 \leq k < n$, then

$$\mathcal{D}_{d-\lceil \frac{d-i}{n} \rceil, i} = \binom{k+d-1}{d-1} = \binom{n\lceil \frac{d-i}{n} \rceil + i - 1}{d-1}.$$

Proof. The first claim is trivial. In order to prove the other claims, let remark that the map $(x_1, \dots, x_d) \mapsto (y_1, \dots, y_d)$, where $y_l = (n-1) - x_l$ for $l = 1, \dots, d$, establishes a bijection between

$$\{(x_1, \dots, x_d) \in \mathbb{N}^d \mid x_l \leq n-1 \text{ for } l = 1, \dots, d; x_1 + \dots + x_d = in - j\}$$

and

$$\{(y_1, \dots, y_d) \in \mathbb{N}^d \mid y_l \leq n-1 \text{ for } l = 1, \dots, d; y_1 + \dots + y_d = (d-i)n - (d-j)\}.$$

The third claim follows from the second claim, because if $0 \leq k \leq n - 1$, then the sets

$$\{(x_1, \dots, x_d) \in \mathbb{N}^d \mid x_l \leq n-1 \text{ for } l = 1, \dots, d; \ x_1 + \dots + x_d = dn - d - k\}$$

and

$$\{(y_1, \dots, y_d) \in \mathbb{N}^d \mid y_1 + \dots + y_d = k\}$$

are in bijection.

The fourth claim follows trivially from the previous items.

The fifth claim follows from the equality: $(d + q)n - (nq + k) = dn - k$.

Finally the sixth claim follows from the third claim, since, if $d - i = nq - k$ with $0 \leq k < n$, then $(d - q)n - i = dn - (d + k)$, hence $\mathcal{D}_{d-q,i} = \mathcal{D}_{d,d+k}$, and $n \lceil \frac{d-i}{n} \rceil + i - 1 = k + d - 1$. \square

Remark 3.4. *With the notations introduced in 3.2, it is clear that $f^{<n, kn+\tau>}(t) = t^{-k} f^{<n, \tau>}(t)$, which implies $h^{<n, kn+\tau>}(t) = t^{-k} h^{<n, \tau>}(t)$ for any integer numbers k, τ .*

The following Theorem improves and gives a simpler proof of [2, Theorem 1.1]:

Theorem 3.5. *Fix integers $d, n \in \mathbb{N}^*, \tau \in \mathbb{Z}$. Let $(a_l)_{l \in \mathbb{Z}}$ be a sequence of complex numbers, such that $a_l = 0$ for $l \ll 0$, set :*

$$f(t) = \sum_{l \in \mathbb{Z}} a_l t^l = \frac{h(t)}{(1-t)^d},$$

$$f^{<n, \tau>}(t) = \sum_{l \in \mathbb{Z}} a_{nl+\tau} t^l = \frac{h^{<n, \tau>}(t)}{(1-t)^d},$$

where $h(t), h^{<n, \tau>}(t) \in \mathbb{C}[t, t^{-1}]$. Then

$$\text{for any } \tau \in \mathbb{Z}, \overrightarrow{h^{<n, kn+\tau>}} = \mathcal{D}[-k, -\tau] \vec{h}.$$

Proof. Because the Remark 3.4 we have to compute $h^{<n, \tau>}(t)$ only for $0 \leq \tau \leq n - 1$. The following formula is clear:

$$f^{<n, 0>}(t^n) + t f^{<n, 1>}(t^n) + \dots + t^{n-1} f^{<n, n-1>}(t^n) = f(t),$$

hence

$$\frac{h^{<n, 0>}(t^n) + t h^{<n, 1>}(t^n) + \dots + t^{n-1} h^{<n, n-1>}(t^n)}{(1-t^n)^d} = \frac{h(t)}{(1-t)^d},$$

and

$$h^{<n,0>}(t^n) + th^{<n,1>}(t^n) + \dots + t^{n-1}h^{<n,n-1>}(t^n) = h(t) \frac{(1-t^n)^d}{(1-t)^d},$$

$t^\tau h^{<n,\tau>}(t^n)$ equals the sum of all the terms $A_\beta t^\beta$ of $h(t) \frac{(1-t^n)^d}{(1-t)^d}$ with $\beta \equiv \tau \pmod n$. In particular $h^{<n,\tau>}(t)$ is a linear function of $h(t)$. So it is enough to compute $h^{<n,\tau>}(t)$ for the canonical basis $\{\varepsilon_j := t^j, j \in \mathbb{Z}\}$ of $\mathbb{C}[t, t^{-1}]$. We have

$$[t^i](h^{<n,\tau>}(t)) = [t^{ni+\tau}](h(t) \frac{(1-t^n)^d}{(1-t)^d}),$$

hence

$$\forall j \in \mathbb{Z}; [t^i](\varepsilon_j^{<n,\tau>}(t)) = [t^{ni+\tau}](t^j) \frac{(1-t^n)^d}{(1-t)^d} = [t^{ni+\tau-j}](\frac{(1-t^n)^d}{(1-t)^d})$$

which proves our statement. \square

Corollary 3.6. *Fix an integer $d \in \mathbb{N}^*$. For $j \in \mathbb{Z}$, let $\vec{\varepsilon}_j$ be the h -vector of the polynomial t^j . Then for any $n \in \mathbb{N}^*$, we have $\deg \varepsilon_j^{<n>} = d - \lceil \frac{d-j}{n} \rceil$. Moreover the leading coefficient of $\varepsilon_j^{<n>}$ is $\binom{n \lceil \frac{d-j}{n} \rceil + j - 1}{d-1}$.*

Proof. Let remark that the set of $t^j, j \in \mathbb{Z}$ is the canonical basis of $\mathbb{C}[t, t^{-1}]$. We have by Theorem 3.5 that $\mathcal{D}\vec{\varepsilon}_j = \vec{\varepsilon}_j^{<n>}$, hence $\varepsilon_j^{<n>}$ is the j column vector of \mathcal{D} . By the Example 2.8, we have that $\deg \varepsilon_j^{<n>} = d - \lceil \frac{d-j}{n} \rceil$.

The last claim follows from the Lemma 3.3. Indeed for any $j \in \mathbb{Z}$, we have $\mathcal{D}_{d-\lceil \frac{d-j}{n} \rceil, j} = \binom{n \lceil \frac{d-j}{n} \rceil + j - 1}{d-1}$. This proves that the leading coefficient of $\varepsilon_j^{<n>}$ is $\binom{n \lceil \frac{d-j}{n} \rceil + j - 1}{d-1}$. \square

Example 3.7. *Let $d = 2$ and $n \in \mathbb{N}^*$, we can describe the matrix \mathcal{D}*

ij	$-(n+1)$	\dots	-1	0	1	2	3	\dots	n	$n+1$	$n+2$	\dots	$2n$
-1	2	\dots	0	0	0	0	0	\dots	0	0	0	\dots	0
0	$n-2$	\dots	2	1	0	0	0	\dots	0	0	0	\dots	0
1	0	\dots	$n-2$	$n-1$	n	$n-1$	$n-2$	\dots	1	0	0	\dots	0
2	0	\dots	0	0	0	1	2	\dots	$n-1$	n	$n-1$	\dots	1
3	0	\dots	0	0	0	0	0	\dots	0	0	1	\dots	$n-1$

Theorem 3.8. Fix integers $d, n \in \mathbb{N}^*, \tau \in \mathbb{Z}$. Let $(a_l)_{l \in \mathbb{Z}}$ be a sequence of complex numbers, such that $a_l = 0$ for $l \ll 0$, set :

$$f(t) = \sum_{l \in \mathbb{Z}} a_l t^l, \quad f^{<n, \tau>}(t) = \sum_{l \in \mathbb{Z}} a_{nl+\tau} t^l.$$

If $f(t) = \frac{h(t)}{(1-t)^d}$ with $h(t) \in \mathbb{C}[t, t^{-1}]$ then $f^{<n, \tau>}(t) = \frac{h^{<n>}(t)}{(1-t)^d}$ with $h^{<n>}(t) \in \mathbb{C}[t, t^{-1}]$ such that:

- $\deg h^{<n, \tau>}(t) \leq d - \lceil \frac{d - \deg h(t) + \tau}{n} \rceil$,
- If all the coefficients of $h(t)$ are positive real numbers then $\deg h^{<n, \tau>}(t) = d - \lceil \frac{d - \deg h(t) + \tau}{n} \rceil$,
- If $\deg h(t) = d$ then $\deg h^{<n>}(t) = d$.

Proof. Let $f(t) = \sum_{l \in \mathbb{Z}} a_l t^l = \frac{h(t)}{(1-t)^d}$, where $h(t) \in \mathbb{C}[t, t^{-1}]$ $h(t) = \gamma_\sigma t^\sigma + \dots + \gamma_s t^s$ with $\deg h(t) = s, \gamma_s \neq 0$. It follows that $\vec{h} = \sum_{l=\sigma}^s \gamma_l \vec{\varepsilon}_l$. We multiply this relation on the left by $\mathcal{D}[-\tau]$, so Theorem 3.5 implies $\vec{h}^{<n, \tau>} = \sum_{l=\sigma}^s \gamma_l \vec{\varepsilon}_{l-\tau}^{<n>}$. Since $\deg \vec{\varepsilon}_{\sigma-\tau}^{<n>} \leq \deg \vec{\varepsilon}_{\sigma-\tau+1}^{<n>} \leq \dots \leq \deg \vec{\varepsilon}_{s-\tau}^{<n>}$, we have, $\deg \vec{h}^{<n, \tau>} \leq \deg \vec{\varepsilon}_{s-\tau}^{<n>}$. It is clear that if all the coefficients of $h(t)$ are positive real numbers then $\deg \vec{h}^{<n, \tau>} = \deg \vec{\varepsilon}_{s-\tau}^{<n>}$.

In the special case $s = d$, we have seen that for $0 \leq l \leq d-1$ and any $n \in \mathbb{N}^*$, $\deg \vec{\varepsilon}_l^{<n>} = d - \lceil \frac{d-l}{n} \rceil \leq d-1$, and $\deg \vec{\varepsilon}_d^{<n>} = d$, which implies $\deg \vec{h}^{<n>} = d$. \square

Theorem 3.9. Let $n \in \mathbb{N}^*$, S be a standard graded polynomial ring, $M = \bigoplus_{l \in \mathbb{Z}} M_l$ be a finitely generated Cohen-Macaulay graded S -module of dimension $d \geq 1$, and $M^{<n>} = \bigoplus_{l \in \mathbb{Z}} M_{nl}$. Let $\frac{Q(t)}{(1-t)^d}$ be the Hilbert-Poincaré series of M , where $Q(t) = \gamma_\sigma t^\sigma + \dots + \gamma_s t^s \in \mathbb{C}[t, t^{-1}]$ is the h -polynomial of M , with $\text{reg}(M) = \deg Q(t) = s$. Then

- $\text{reg } M^{<n>} = d - \lceil \frac{d - \text{reg } M}{n} \rceil$. Moreover by taking the sum over all index l such that $\lceil \frac{d-l}{n} \rceil = \lceil \frac{d - \text{reg } M}{n} \rceil$, we will get the leading coefficient of $Q^{<n>}(t)$:

$$\sum_{l \mid \lceil \frac{d-l}{n} \rceil = \lceil \frac{d - \text{reg } M}{n} \rceil} \gamma_l \binom{n \lceil \frac{d-l}{n} \rceil + l - 1}{d-1}.$$

- If $\text{reg } M \leq d - 1$ and $n \geq d$ then $\text{reg } M^{<n>} = d - 1$, and the leading coefficient of $Q^{<n>}(t)$ is

$$\sum_{l=0}^{d-1} \gamma_l \binom{n \lceil \frac{d-l}{n} \rceil + l - 1}{d-1}.$$

- If $n > \text{reg } M \geq d$ then $\text{reg } M^{<n>} = d$, and the leading coefficient of $Q^{<n>}(t)$ is

$$\sum_{l=d}^{\text{reg } M} \gamma_l \binom{l-1}{d-1}.$$

Proof. We have $\vec{Q} = \sum_{l=\sigma}^s \gamma_l \vec{\varepsilon}_l$. We multiply this relation on the left by \mathcal{D} , so Theorem 3.5 implies that for any $n \in \mathbb{N}^*$, $\overrightarrow{Q^{<n>}} = \sum_{l=\sigma}^s \gamma_l \overrightarrow{\varepsilon_l^{<n>}}$. Since $\gamma_l \geq 0$ for all l , $\gamma_s > 0$, and $\deg \overrightarrow{\varepsilon_\sigma^{<n>}} \leq \deg \overrightarrow{\varepsilon_{\sigma+1}^{<n>}} \leq \dots \leq \deg \overrightarrow{\varepsilon_s^{<n>}}$, we have, $\deg \overrightarrow{Q^{<n>}} = \deg \overrightarrow{\varepsilon_s^{<n>}} = d - \lceil \frac{d - \text{reg } M}{n} \rceil$, this number is $\text{reg}(M^{<n>})$ since $M^{<n>}$ is a Cohen-Macaulay S -module. The computation of the leading coefficient of $Q^{<n>}(t)$ is immediate from Lemma 3.6. \square

4 h -vector of the Segre-Veronese embedding.

The next Theorem improves partially [1] and [5].

Theorem 4.1. *Let consider the Segre-Veronese ring $R_{\underline{b},n}$, $\dim R_{\underline{b},n} = b_1 + \dots + b_m + 1$. Let $P_{R_{\underline{b},n}}(t) = \frac{Q_{R_{\underline{b},n}}(t)}{(1-t)^{\dim R_{\underline{b},n}}}$ be the Hilbert-Poincaré series of $R_{\underline{b},n}$, with $Q_{R_{\underline{b},n}}(t) = h_0 + h_1 t + \dots + h_{r_{\underline{b},n}} t^{r_{\underline{b},n}}$, where $r_{\underline{b},n} = \deg Q_{R_{\underline{b},n}}(t)$ is the Castelnuovo-Mumford regularity of $R_{\underline{b},n}$. We set $\alpha_{\underline{b},n} = \dim R_{\underline{b},n} - r_{\underline{b},n}$. After a permutation of b_1, \dots, b_m , we can assume that $\lceil \frac{b_1+1}{n_1} \rceil > \frac{b_i}{n_i} \forall i$, then*

$$\alpha_{\underline{b},n} = \lceil \frac{b_1+1}{n_1} \rceil \quad r_{\underline{b},n} = (b_1 + \dots + b_m + 1) - \lceil \frac{b_1+1}{n_1} \rceil,$$

and the highest Betti number of $R_{\underline{b},n}$ is

$$\beta_{r_{\underline{b},n}} = h_{r_{\underline{b},n}} = \binom{n_1 \alpha_{\underline{b},n} - 1}{b_1} \dots \binom{n_m \alpha_{\underline{b},n} - 1}{b_m}$$

Proof. The proof is by double induction on m and b_m . The case $m = 1$ is given by the Example 2.8 and Corollary 3.6, so we can assume $m \geq 2$. We have that $\lceil \frac{b_1+1}{n_1} \rceil > \frac{b_m}{n_m} > \frac{b_m-1}{n_m}$, so by induction hypothesis the theorem is true for $R_{\underline{b}-\epsilon_m, \underline{n}}$, where $\underline{b}-\epsilon_m = (b_1, \dots, b_{m-1}, b_m-1)$. On the other hand the Hilbert function of $R_{\underline{b}, \underline{n}}$ is $H_{R_{\underline{b}, \underline{n}}}(l) = \binom{n_1 l + b_1}{b_1} \dots \binom{n_m l + b_m}{b_m}$, so

$$H_{R_{\underline{b}, \underline{n}}}(l) = \left(1 + \frac{n_m}{b_m} l\right) H_{R_{\underline{b}-\epsilon_m, \underline{n}}}(l). \quad (1)$$

Let $P_{R_{\underline{b}-\epsilon_m, \underline{n}}}(t) = \frac{Q_{R_{\underline{b}-\epsilon_m, \underline{n}}}(t)}{(1-t)^{b_1+\dots+b_m}}$ be the Hilbert-Poincaré series of $R_{\underline{b}-\epsilon_m, \underline{n}}$, where $Q_{R_{\underline{b}-\epsilon_m, \underline{n}}}(t) = h_0 + h_1 t + \dots + h_{r_{\underline{b}-\epsilon_m, \underline{n}}} t^{r_{\underline{b}-\epsilon_m, \underline{n}}}$, with $h_{r_{\underline{b}-\epsilon_m, \underline{n}}} \neq 0$. In order to avoid any confusion we also set : $P_{R_{\underline{b}, \underline{n}}}(t) = \frac{Q_{R_{\underline{b}, \underline{n}}}(t)}{(1-t)^{b_1+\dots+b_m+1}}$ be the Hilbert-Poincaré series of $R_{\underline{b}, \underline{n}}$, where

$$Q_{R_{\underline{b}, \underline{n}}}(t) = \hat{h}_0 + \dots + \hat{h}_{r_{\underline{b}, \underline{n}}} t^{r_{\underline{b}, \underline{n}}},$$

with $\hat{h}_{r_{\underline{b}, \underline{n}}} \neq 0$.

Let $\beta = \frac{n_m}{b_m}$, by simple calculations from (1) we get:

$$P_{R_{\underline{b}, \underline{n}}}(t) = P_{R_{\underline{b}-\epsilon_m, \underline{n}}}(t) + \beta t P'_{R_{\underline{b}-\epsilon_m, \underline{n}}}(t). \quad (2)$$

Hence $\dim R_{\underline{b}, \underline{n}} = \dim R_{\underline{b}-\epsilon_m, \underline{n}} + 1$, and

$$Q_{R_{\underline{b}, \underline{n}}}(t) = Q_{R_{\underline{b}-\epsilon_m, \underline{n}}}(t) + t[Q_{R_{\underline{b}-\epsilon_m, \underline{n}}}(t)(\beta R_{\underline{b}-\epsilon_m, \underline{n}} - 1) + \beta Q'_{R_{\underline{b}-\epsilon_m, \underline{n}}}(t) - \beta t Q'_{R_{\underline{b}-\epsilon_m, \underline{n}}}(t)],$$

note that $Q_{R_{\underline{b}, \underline{n}}}(1) = \beta \dim R_{\underline{b}-\epsilon_m, \underline{n}} Q_{R_{\underline{b}-\epsilon_m, \underline{n}}}(1) \neq 0$.

In particular we have $r_{\underline{b}, \underline{n}} \leq r_{\underline{b}-\epsilon_m, \underline{n}} + 1$ and for all $k = 0, \dots, r_{\underline{b}-\epsilon_m, \underline{n}} + 1$ we have

$$\hat{h}_k = h_{k-1} \left(\frac{n_m}{b_m} \dim R_{\underline{b}-\epsilon_m, \underline{n}} - (k-1) \frac{n_m}{b_m} - 1 \right) + h_k \left(k \frac{n_m}{b_m} + 1 \right). \quad (3)$$

By induction hypothesis we have $\alpha_{\underline{b}-\epsilon_m, \underline{n}} = \lceil \frac{b_1+1}{n_1} \rceil \neq \frac{b_m}{n_m}$, so we put $k = r_{\underline{b}-\epsilon_m, \underline{n}} + 1$ in equality (3), and we get:

$$\hat{h}_{r_{\underline{b}-\epsilon_m, \underline{n}}+1} = h_{r_{\underline{b}-\epsilon_m, \underline{n}}} \left(\frac{n_m \alpha_{\underline{b}-\epsilon_m, \underline{n}} - b_m}{b_m} \right) \neq 0.$$

Hence $\hat{h}_{r_{\underline{b}-\epsilon_m, \underline{n}}+1}$ is the leading coefficient of $Q_{R_{\underline{b}, \underline{n}}}$ and $r_{\underline{b}, \underline{n}} = r_{\underline{b}-\epsilon_m, \underline{n}} + 1$ and $\alpha_{\underline{b}, \underline{n}} = \alpha_{\underline{b}-\epsilon_m, \underline{n}} = \lceil \frac{b_1+1}{n_1} \rceil$. By induction hypothesis

$$h_{r_{\underline{b}-\epsilon_m, \underline{n}}} = \binom{n_1 \alpha_{\underline{b}, \underline{n}} - 1}{b_1} \dots \binom{n_{m-1} \alpha_{\underline{b}, \underline{n}} - 1}{b_{m-1}} \binom{n_m \alpha_{\underline{b}, \underline{n}} - 1}{b_m}$$

so that

$$\begin{aligned}\hat{h}_{r_{b,n}} &= \binom{n_1\alpha_{b,n} - 1}{b_1} \cdots \binom{n_{m-1}\alpha_{b,n} - 1}{b_{m-1}} \binom{n_m\alpha_{b,n} - 1}{b_m} \left(\frac{n_m\alpha_{b,n} - b_m}{b_m}\right) \\ &= \binom{n_1\alpha_{b,n} - 1}{b_1} \cdots \binom{n_{m-1}\alpha_{b,n} - 1}{b_{m-1}} \binom{n_m\alpha_{b,n} - 1}{b_m}.\end{aligned}$$

□

5 Rings of Veronese type

Let $b, n \in \mathbb{N}^*$, $\underline{a} = (a_0, \dots, a_b) \in \mathbb{N}^{b+1}$ such that $1 \leq a_i \leq n$, $a_0 + \dots + a_b > n$, and $\mathcal{M}_{b,n,\underline{a}}$ be the set of monomials of the polynomial ring $K[x_0, \dots, x_b]$:

$$\mathcal{M}_{b,n,\underline{a}} = \{x_0^{\alpha_0} \cdots x_b^{\alpha_b} \mid \alpha_0 + \dots + \alpha_b = n, \alpha_i \leq a_i, \forall i = 0, \dots, b\}.$$

Let denote by $R_{b,n,\underline{a}}$ the toric subring of $K[x_0, \dots, x_b]$ generated by $\mathcal{M}_{b,n,\underline{a}}$. It is well known that $R_{b,n,\underline{a}}$ is a Cohen-Macaulay ring. Let \mathcal{S} be the collection of subsets of $\{0, \dots, b\}$ such that : $S \in \mathcal{S}$ if and only if $S \subset \{0, \dots, b\}$, and $\Sigma S := \sum_{i \in S} a_i < n$.

Theorem 5.1. ([9]) *With the above notations the Hilbert-function of $R_{b,n,\underline{a}}$ is*

$$\forall l \geq 0; \quad H_{b,n,\underline{a}}(l) = \sum_{S \in \mathcal{S}} (-1)^{|S|} \binom{l(n - \Sigma S) - |S| + b}{b}$$

We have $\dim(R_{b,n,\underline{a}}) = b + 1$, and its degree or multiplicity is

$$\deg(R_{b,n,\underline{a}}) = \sum_{S \in \mathcal{S}} (-1)^{|S|} (n - \Sigma S)^b.$$

Our aim is to study the Hilbert-Poincaré series of $R_{b,n,\underline{a}}$:

$$P_{R_{b,n,\underline{a}}} = \sum_{S \in \mathcal{S}} (-1)^{|S|} \sum_{l \geq 0} \binom{l(n - \Sigma S) - |S| + b}{b} t^l.$$

The following Corollary follows immediately from 3.6.

Corollary 5.2. *For any $S \in \mathcal{S}$ and $k \in \mathbb{N}^*$, we have:*

$$\sum_{l \geq 0} \binom{kl - |S| + b}{b} t^l = \frac{Q_{S,k}(t)}{(1-t)^{b+1}},$$

where $Q_{S,k}(t)$ is a polynomial with $Q_{S,k}(1) \neq 0$, with leading term

$$\binom{k\alpha_{S,k} + |S| - 1}{b} t^{b+1-\alpha_{S,k}},$$

with $\alpha_{S,k} = \lceil \frac{b+1-|S|}{k} \rceil$.

The following theorem is immediate from 5.1 and Corollary 5.2. It improves the description of the Hilbert Poincaré series given in [9].

Theorem 5.3. *With the above notations, let \mathcal{S} be the collection of subsets of $\{0, \dots, b\}$ such that $S \in \mathcal{S}$ if and only if $S \subset \{0, \dots, b\}$, and $\Sigma S := \sum_{i \in S} a_i < n$. Then we can write the Hilbert-Poincaré series of $R_{b,n,\underline{a}}$:*

$$P_{R_{b,n,\underline{a}}} = \frac{Q_{b,n,\underline{a}}(t)}{(1-t)^{b+1}},$$

with $Q_{b,n,\underline{a}}(t) = \sum_{S \in \mathcal{S}} (-1)^{|S|} Q_{S,n}(t)$, where $Q_{S,n}(t)$ is a polynomial with $Q_{S,n}(1) \neq 0$, with leading term

$$\binom{(n - \Sigma S)\alpha_{S,n-\Sigma S} + |S| - 1}{b} t^{b+1-\alpha_{S,n-\Sigma S}},$$

where $\alpha_{S,n-\Sigma S} = \lceil \frac{b+1-|S|}{n-\Sigma S} \rceil$.

Part one of the following corollary improves [9][Cor. 2.12].

Corollary 5.4. *With the above notations:*

1. $\text{reg}(R_{b,n,\underline{a}}) \leq b + 1 - \lceil \frac{b+1}{n} \rceil$, and the equality is true if and only if

$$\sum_{S \in \mathcal{S}, \alpha_{S,n-\Sigma S} = \lceil \frac{b+1}{n} \rceil} (-1)^{|S|} \binom{(n - \Sigma S)\alpha_{S,n-\Sigma S} + |S| - 1}{b} \neq 0$$

2. If $b + 1 > n^2$ then $\text{reg}(R_{b,n,\underline{a}}) = b + 1 - \lceil \frac{b+1}{n} \rceil$. Moreover the leading term of $Q_{b,n,\underline{a}}(t)$ is $\binom{(n \lceil \frac{b+1}{n} \rceil - 1)}{b} t^{b+1 - \lceil \frac{b+1}{n} \rceil}$,

Proof. 1. It is enough to prove that $\min_{S \in \mathcal{S}} \lceil \frac{b+1-|S|}{n-\Sigma S} \rceil = \lceil \frac{b+1}{n} \rceil$. We consider two cases,

- if $b + 1 < n$ then $\lceil \frac{b+1}{n} \rceil = 1 \leq \lceil \frac{b+1-|S|}{n-\Sigma S} \rceil, \forall S \in \mathcal{S}$.

- If $b + 1 \geq n$, then

$$\frac{b+1}{n} \leq \frac{b+1-|S|}{n-\Sigma S} \Leftrightarrow (b+1)(n-\Sigma S) \leq n(b+1-|S|) \Leftrightarrow (b+1)\Sigma S \geq n|S|,$$

this is true since by hypothesis $\frac{b+1}{n} \geq 1 \geq \frac{|S|}{\Sigma S}$.

2. Let $b + 1 > n^2$ and $S \neq \emptyset$. By definition $\lceil \frac{b+1}{n} \rceil$ is the integer q such that $b + 1 = qn - r$, with $0 \leq r < n$ and $q \geq n + 1$. We have

$$b + 1 - |S| = qn - r - |S| = q(n - \Sigma S) - r - |S| + q\Sigma S,$$

and $q\Sigma S - |S| \geq (n + 1)\Sigma S - |S| \geq n\Sigma S > r$, so that $q\Sigma S - |S| - r > 0$, hence $\lceil \frac{b+1-|S|}{n-\Sigma S} \rceil > q = \lceil \frac{b+1}{n} \rceil$. \square

In general leading terms of the alternating sum can cancel, as we can see in the next example.

Example 5.5. Let consider the ring $R_{4,3,(1,1,1,1,1)}$, the sets S can have 0, 1 or 2 elements, and we have: If $S = \emptyset$ then $\alpha_{\emptyset,3} = \lceil \frac{5}{3} \rceil = 2$, if $|S| = 1$ then $\alpha_{S,3} = \lceil \frac{4}{2} \rceil = 2$, and finally if $|S| = 2$ then $\alpha_{S,2} = \lceil \frac{3}{1} \rceil = 3$. By using Theorem 5.3 we can write

$$P_{R_{4,3,(1,1,1,1,1)}} = \frac{Q_0(t) - 5Q_1(t) + 10Q_2(t)}{(1-t)^5},$$

, with $Q_0(t) = 5t^3 + \dots$; $Q_1(t) = t^3 + \dots$; $Q_2(t) = t^2 + \dots$. Note that in this case $Q_0(t) - 5Q_1(t) + 10Q_2(t) = h_0 + h_1t + h_2t^2$, where $h_0 = 1, h_1 = 5$ and since $h_0 + h_1 + h_2 = \deg(R_{4,3,(1,1,1,1,1)}) = 11$, we get $h_2 = 5$, so that

$$P_{R_{4,3,(1,1,1,1,1)}} = \frac{1 + 5t + 5t^2}{(1-t)^5}.$$

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