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# Veronese transform, and Castelnuovo-Mumford regularity of modules

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## Abstract <sup>1</sup>

Veronese rings, Segre embeddings or more generally Segre-Veronese embeddings are very important rings in Algebraic Geometry. In this paper we present an original, elementary way to compute the Hilbert-Poincaré series of these rings, as a consequence we compute their Castelnuovo-Mumford regularity, and also the leading term of the  $h$ -vector. Moreover, we can compute the Castelnuovo-Mumford regularity of the  $n$ -Veronese Module of any finitely generated Cohen-Macaulay graded module.

## 1 Introduction

Veronese rings, Segre embeddings or more generally Segre-Veronese embeddings are very important rings in Algebraic Geometry. It is well known that these rings are arithmetically Cohen-Macaulay, hence their Hilbert-Poincaré series can be written  $P_R(t) = \frac{Q_R(t)}{(1-t)^{\dim R}}$ , where  $Q_R(t)$  is a polynomial on  $t$  with  $Q_R(1) \neq 0$  having positive integer coefficients, the sequence of the coefficients of  $Q_R(t)$  is also called the  $h$ -vector of  $R$ . The degree of  $Q_R(t)$  is the Castelnuovo-Mumford regularity (c.f.[7][Chapter 4]), and the leading term of  $Q_R(t)$  is the highest graded Betti number of  $R$ . By using very original and elementary methods we are able to compute the leading term of  $Q_R(t)$ . Our results allows to compute the Castelnuovo-Mumford regularity of the  $n$  Veronese Module of any finitely generated Cohen-Macaulay graded module, and the rings called of Veronese type.

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Note that this result can be proved easily by using local cohomology, but our purpose is to give a very elementary proof.

Our main results improves partially [1] and [5].

**Theorem.** *Let consider the Segre-Veronese ring  $R_{\underline{b},n}$ ,  $\dim R_{\underline{b},n} = b_1 + \dots + b_m + 1$ . Let  $P_{R_{\underline{b},n}}(t) = \frac{Q_{R_{\underline{b},n}}}{(1-t)^{\dim R_{\underline{b},n}}}$  be the Hilbert-Poincaré series of  $R_{\underline{b},n}$ , with  $Q_{R_{\underline{b},n}} = h_0 + h_1 t + \dots + h_{r_{\underline{b},n}} t^{r_{\underline{b},n}}$ , where  $r_{\underline{b},n}$  is the Castelnuovo-Mumford regularity of  $R_{\underline{b},n}$ . We set  $\alpha_{\underline{b},n} = \dim R_{\underline{b},n} - r_{\underline{b},n}$ . After a permutation of  $b_1, \dots, b_m$ , we can assume that, for all  $i = 1, \dots, m$ ,  $\lceil \frac{b_i+1}{n_i} \rceil > \frac{b_i}{n_i}$ . Then*

$$\alpha_{\underline{b},n} = \lceil \frac{b_1 + 1}{n_1} \rceil \quad ,$$

and the highest Betti number of  $R_{\underline{b},n}$  is

$$\beta_{r_{\underline{b},n}} = h_{r_{\underline{b},n}} = \binom{n_1 \alpha_{\underline{b},n} - 1}{b_1} \dots \binom{n_m \alpha_{\underline{b},n} - 1}{b_m}$$

**Theorem.** *Fix integers  $d, n \in \mathbb{N}^*$ ,  $\tau \in \mathbb{Z}$ . Let  $(a_l)_{l \in \mathbb{Z}}$  be a sequence of complex numbers, such that  $a_l = 0$  for  $l \ll 0$ , set :*

$$f(t) = \sum_{l \in \mathbb{Z}} a_l t^l, \quad f^{<n, \tau>}(t) = \sum_{l \in \mathbb{Z}} a_{nl + \tau} t^l.$$

*If  $f(t) = \frac{h(t)}{(1-t)^d}$  with  $h(t) \in \mathbb{C}[t, t^{-1}]$  then  $f^{<n, \tau>}(t) = \frac{h^{<n>}(t)}{(1-t)^d}$  with  $h^{<n>}(t) \in \mathbb{C}[t, t^{-1}]$  such that:*

- $\deg h^{<n, \tau>}(t) \leq d - \lceil \frac{d - \deg h(t) + \tau}{n} \rceil$ ,
- *If all the coefficients of  $h(t)$  are positive real numbers then*  
 $\deg h^{<n, \tau>}(t) = d - \lceil \frac{d - \deg h(t) + \tau}{n} \rceil$ ,
- *If  $\deg h(t) = d$  then  $\deg h^{<n>}(t) = d$ .*

## 2 Preliminaries on toric rings and Hilbert-Poincaré series

Let  $S = K[x_0, \dots, x_s, x_0^{-1}, \dots, x_s^{-1}]$  be a Laurent polynomial ring over a field  $K$  on a finite set of variables. For any finite set  $\mathcal{M}$  of monomials in  $S$ , let  $K[\mathcal{M}] \subset S$  be the subring of  $S$  generated by the set

$\mathcal{M}$ . It is the toric ring defined by the semigroup generated by  $\mathcal{M}$ . In what follows we consider the special case where  $S = K[x_0, \dots, x_s]$  is a polynomial ring over the field  $K$  and all the monomials in  $\mathcal{M}$  are of the same degree.

**Example 2.1.** Let  $S = K[x_0, \dots, x_b] \oplus_{l \in \mathbb{N}} S_l$ , and  $\mathcal{M} = \{x_0^{\alpha_0} \dots x_b^{\alpha_b} \mid \alpha_0 + \dots + \alpha_b = n\}$ . So that

$$R_{b,n} = K[\mathcal{M}] = \oplus_{l \in \mathbb{N}} S_{nl}.$$

This toric ring is known as the  $n$ -Veronese embedding of  $S$ .

**Example 2.2.** More generally, let  $X_1, \dots, X_m$ ,  $m$  sets of independent disjoint variables, with  $\text{Card}(X_i) = b_i + 1$ . Let  $S_i = K[X_i]$  for  $i = 1, \dots, m$ ,  $S = K[X_1 \cup X_2 \cup X_m]$ , and  $\mathcal{M} = \{x_1 x_2 \dots x_m \mid x_i \in X_i\}$ . So that

$$R_{b_1, \dots, b_m} = K[\mathcal{M}] = \oplus_{l \in \mathbb{N}} (S_1)_l \otimes \dots \otimes (S_m)_l.$$

This toric ring is known as the Segre embedding of the  $m$  polynomial rings  $S_1, \dots, S_m$ .

**Example 2.3.** Let  $X_1, \dots, X_m$ , sets of independent disjoint variables such that  $X_i = \{x_{i,0}, \dots, x_{i,b_i}\}$ ,  $S_i = K[X_i]$  for  $i = 1, \dots, m$ , and  $n_1, \dots, n_m \in \mathbb{N}$ , Let  $S = K[X_1 \cup X_2 \cup X_m]$ , and

$$\mathcal{M} = \{\underline{x}_1^{\alpha_1} \dots \underline{x}_m^{\alpha_m} \mid |\alpha_i| = n_i\},$$

where  $\alpha_i = (\alpha_{i,0}, \dots, \alpha_{i,b_i})$ ,  $\underline{x}_i^{\alpha_i} = x_{i,0}^{\alpha_{i,0}} \dots x_{i,b_i}^{\alpha_{i,b_i}}$ , and  $|\alpha_i| = \alpha_{i,0} + \dots + \alpha_{i,b_i}$ . The Segre-Veronese embedding:

$$R_{\underline{b}, \underline{n}} = K[\mathcal{M}] = \oplus_{l \in \mathbb{N}} (S_1)_{n_1 l} \otimes \dots \otimes (S_m)_{n_m l},$$

where  $\underline{b} = (b_1, \dots, b_m)$ ,  $\underline{n} = (n_1, \dots, n_m)$ .

Let  $S = K[x_0, \dots, x_s]$  be a polynomial ring over the field  $K$ , graded by the standard graduation, that is  $\deg x_i = 1$ , for all  $i$ . Let  $R := S/I$ , where  $I \subset S$  is a graded ideal, let  $M = \oplus_{l \in \mathbb{Z}} M_l$  be a finitely generated graded  $R$ -module, hence  $M$  is also a  $S$ -module. the Hilbert-function of  $M$  is defined by  $H_M(l) = \dim_K M_l$ , for all  $l \in \mathbb{Z}$ , and the Hilbert-Poincaré series of  $M$ :

$$P_M(t) = \sum_{l \in \mathbb{Z}} H_M(l) t^l.$$

It is well known that

$$P_M(t) = \frac{Q_M(t)}{(1-t)^{\dim M}} \quad ,$$

where  $Q_M(t)$  is a Laurent polynomial on  $t, t^{-1}$  with  $Q_M(1) \neq 0$ . Moreover if  $M$  is a Cohen-Macaulay  $S$ -module, all the coefficients of  $Q_M(t)$  are natural integers, and the Castelnuovo-Mumford regularity of  $M$  is the degree of  $Q_M(t)$ . For more details on Hilbert-Poincaré series see [10], [4][Chapter 4], [7][Chapter 4].

**Theorem 2.4.** (*Hilbert's Theorem*) *let  $M = \bigoplus_{l \in \mathbb{Z}} M_l$  be a finitely generated graded  $S$ -module. There exists a polynomial with integer coefficients  $\Phi_{H_M}(l)$  such that  $H_M(l) = \Phi_{H_M}(l)$ , for  $l$  large enough. Moreover the leading term of  $\Phi_{H_M}(l)$  can be written as :  $\frac{\deg(M)}{d!} l^d$ , where  $d+1$  is the dimension of  $M$  and  $\deg(M)$  the degree or multiplicity of  $M$ .*

**Remark 2.5.** *The postulation number of the Hilbert function is the biggest integer  $l$  such that  $H_M(l) \neq \Phi_{H_M}(l)$ . It is well known, ([10], [4][Chapter 4]), that the postulation number equals the degree of the rational fraction defining the Poincaré series.*

**Remark 2.6.** *We recall that binomial coefficients can be defined in a more general setting than natural numbers, indeed for  $k \in \mathbb{N}$ , binomial coefficients are polynomial functions in the variable  $n$ . More precisely:*

- (1) *If  $k = 0$  then let  $\binom{n}{0} = 1$ , for all  $n \in \mathbb{C}$ .*
- (2) *If  $k > 0$  then let  $\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!}$ , for all  $n \in \mathbb{C}$ .*

*Note that for all  $n \in \mathbb{C}$ ,  $\binom{n}{k} = (-1)^k \binom{k-n-1}{k}$  and if  $n \in \mathbb{N}, n < k$ , then  $\binom{n}{k} = 0$ .*

**Example 2.7.** *Let  $S = K[x_0, \dots, x_b]$ , be a polynomial ring. Then*

$$H_S(l) = \begin{cases} \binom{l+b}{b} & \text{if } l \geq 0 \\ 0 & \text{if } l < 0 \end{cases} \quad , \quad P_S(t) = \frac{1}{(1-t)^{b+1}}.$$

*Note that in fact  $\forall l \geq -b$ ,  $H_S(l) = \binom{l+b}{b}$  and  $0 = H_S(-b-1) \neq \binom{-b-1+b}{b} = (-1)^b$ , so the postulation number of  $S$  is  $-(b+1)$ .*

**Example 2.8.** Let  $S = K[x_0, \dots, x_b]$ ,  $\mathcal{M} = \{x_0^{\alpha_0} \dots x_b^{\alpha_b} \mid \alpha_0 + \dots + \alpha_b = n\}$ , and  $R_{b,n} = K[\mathcal{M}]$  the  $n$ -Veronese embedding. Then

$$H_{R_{b,n}}(l) = H_S(nl) = \begin{cases} \binom{nl+b}{b} & \text{if } l \geq 0 \\ 0 & \text{if } l < 0 \end{cases}.$$

Note that  $\binom{nl+b}{b} = \frac{(nl+1)(nl+2)\dots(nl+b)}{b!}$  is a polynomial on  $l$  with leading term  $\frac{n^b l^b}{b!}$ , so that  $\deg(R_{b,n}) = n^b$ ,  $\dim R_{b,n} = b + 1$ . Note also that  $\forall l > -\lceil \frac{b+1}{n} \rceil$ ,  $H_{R_{b,n}}(l) = \binom{nl+b}{b}$  and  $0 = H_{R_{b,n}}(-\lceil \frac{b+1}{n} \rceil) \neq \binom{-\lceil \frac{b+1}{n} \rceil n + b}{b} = (-1)^b \binom{\lceil \frac{b+1}{n} \rceil n - 1}{b}$ , so the postulation number of  $R_{b,n}$  is  $-\lceil \frac{b+1}{n} \rceil$ . More generally the postulation number of  $R_{b,n}[\tau]$  is  $\lceil \frac{b+1+\tau}{n} \rceil$ .

### 3 Veronese of generating series

In a recent paper [2], Brenti and Walker prove that taking the  $n$ -Veronese transform of the  $h$  polynomial is a linear function, in this section we improve this result giving a elementary proof of the fact that taking the shifted  $n$ -Veronese transform of the  $h$  polynomial is a linear function on  $h$ .

Let recall the following fact:

**Theorem 3.1.** Let  $(a_l)_{l \in \mathbb{Z}}$  be a sequence of complex numbers, such that  $a_l = 0$  for  $l \ll 0$ , set  $f(t) = \sum_{l \in \mathbb{Z}} a_l t^l$ , TFAE:

- There exists  $h(t) \in \mathbb{C}[t, t^{-1}]$  and a natural integer  $d$  such that  $f(t) = \frac{h(t)}{(1-t)^d}$ .
- There exists  $\Phi(t) \in \mathbb{C}[t, t^{-1}]$  of degree  $d - 1$  with leading coefficient  $e_0/(d - 1)!$ , such that  $\Phi(l) = a_l$  for  $l$  large enough.

Moreover  $h(1) = e_0$ .

**Definition 3.2.** Fix integers  $d, n \in \mathbb{N}^*$ ,  $\tau \in \mathbb{Z}$ . Let  $(a_l)_{l \in \mathbb{Z}}$  be a sequence of complex numbers, such that  $a_l = 0$  for  $l \ll 0$ , set :

$$f(t) = \sum_{l \in \mathbb{Z}} a_l t^l, \quad f^{<n, \tau>}(t) = \sum_{l \in \mathbb{Z}} a_{nl+\tau} t^l.$$

By the Theorem 3.1 if  $f(t) = \frac{h(t)}{(1-t)^d}$  with  $h(t) \in \mathbb{C}[t, t^{-1}]$  then  $f^{<n, \tau>}(t) = \frac{h^{<n, \tau>}(t)}{(1-t)^d}$  with  $h^{<n, \tau>}(t) \in \mathbb{C}[t, t^{-1}]$ .

Let us introduce some notations. To any non zero polynomial  $h(t) = h_\sigma t^\sigma + \dots + h_0 + h_1 t + \dots + h_s t^s \in \mathbb{C}[t, t^{-1}]$  we associate the  $h$ -vector  $\vec{h} = (\dots, 0, h_\sigma, \dots, h_s, 0, \dots)$ , and we set  $\deg \vec{h} = \deg h(t)$ . For  $j \in \mathbb{Z}$ , let  $\vec{\varepsilon}_j$  be the  $h$ -vector of the polynomial  $t^j$ . Let denote by  $[t^k]h(t)$  the coefficient of  $t^k$  in the polynomial  $h(t)$ . For any  $i, j \in \mathbb{Z}$  define  $\mathcal{D}_{i,j}$  by

$$\mathcal{D}_{i,j} = [t^{in-j}] \left( \frac{(1-t^n)^d}{(1-t)^d} \right) = [t^{in-j}] ((1+t+\dots+t^{n-1})^d).$$

Note that

$$\mathcal{D}_{i,j} = \text{Card}\{(x_1, \dots, x_d) \in \mathbb{N}^d \mid \forall l, x_l \leq n-1; x_1 + \dots + x_d = in - j\}.$$

Finally let  $\mathcal{D}[\sigma, \tau]$  be the infinite square matrix  $\mathcal{D}[\sigma, \tau] = (\mathcal{D}_{i+\sigma, j+\tau})$ . For  $\sigma = \tau = 0$  we write  $\mathcal{D}$  instead  $\mathcal{D}[0, 0]$ . We can give some properties of the numbers  $\mathcal{D}_{i,j}$ .

**Lemma 3.3.** *Let  $i, j, k \in \mathbb{Z}$ , we have:*

- $\mathcal{D}_{i,j} = 0$  if either  $in - j < 0$  or  $in - j > d(n-1)$ .
- For any  $i, j$ ,  $\mathcal{D}_{i,j} = \mathcal{D}_{d-i, d-j}$ . That is  $\mathcal{D}$  is symmetrical around the point  $(d/2, d/2)$ .
- For  $0 \leq k \leq n-1$ ,  $\mathcal{D}_{d, d+k} = \binom{k+d-1}{d-1}$ .
- $\mathcal{D}_{1,0} = \binom{n+d-1}{d-1} - d$ , and for  $1 \leq k \leq n$ ,  $\mathcal{D}_{1,k} = \binom{n-k+d-1}{d-1}$ .
- For any integers  $q, k$ ,  $\mathcal{D}_{d+q, nq+k} = \mathcal{D}_{d,k}$ .
- For any  $i$ , let  $d-i = nq - k$  with  $q = \lceil \frac{d-i}{n} \rceil$ ,  $0 \leq k < n$ , then

$$\mathcal{D}_{d-\lceil \frac{d-i}{n} \rceil, i} = \binom{k+d-1}{d-1} = \binom{n\lceil \frac{d-i}{n} \rceil + i - 1}{d-1}.$$

*Proof.* The first claim is trivial. In order to prove the other claims, let remark that the map  $(x_1, \dots, x_d) \mapsto (y_1, \dots, y_d)$ , where  $y_l = (n-1) - x_l$  for  $l = 1, \dots, d$ , establishes a bijection between

$$\{(x_1, \dots, x_d) \in \mathbb{N}^d \mid x_l \leq n-1 \text{ for } l = 1, \dots, d; x_1 + \dots + x_d = in - j\}$$

and

$$\{(y_1, \dots, y_d) \in \mathbb{N}^d \mid y_l \leq n-1 \text{ for } l = 1, \dots, d; y_1 + \dots + y_d = (d-i)n - (d-j)\}.$$

The third claim follows from the second claim, because if  $0 \leq k \leq n - 1$ , then the sets

$$\{(x_1, \dots, x_d) \in \mathbb{N}^d \mid x_l \leq n-1 \text{ for } l = 1, \dots, d; \ x_1 + \dots + x_d = dn - d - k\}$$

and

$$\{(y_1, \dots, y_d) \in \mathbb{N}^d \mid y_1 + \dots + y_d = k\}$$

are in bijection.

The fourth claim follows trivially from the previous items.

The fifth claim follows from the equality:  $(d + q)n - (nq + k) = dn - k$ .

Finally the sixth claim follows from the third claim, since, if  $d - i = nq - k$  with  $0 \leq k < n$ , then  $(d - q)n - i = dn - (d + k)$ , hence  $\mathcal{D}_{d-q,i} = \mathcal{D}_{d,d+k}$ , and  $n \lceil \frac{d-i}{n} \rceil + i - 1 = k + d - 1$ . □

**Remark 3.4.** *With the notations introduced in 3.2, it is clear that  $f^{<n, kn+\tau>}(t) = t^{-k} f^{<n, \tau>}(t)$ , which implies  $h^{<n, kn+\tau>}(t) = t^{-k} h^{<n, \tau>}(t)$  for any integer numbers  $k, \tau$ .*

The following Theorem improves and gives a simpler proof of [2, Theorem 1.1]:

**Theorem 3.5.** *Fix integers  $d, n \in \mathbb{N}^*, \tau \in \mathbb{Z}$ . Let  $(a_l)_{l \in \mathbb{Z}}$  be a sequence of complex numbers, such that  $a_l = 0$  for  $l \ll 0$ , set :*

$$f(t) = \sum_{l \in \mathbb{Z}} a_l t^l = \frac{h(t)}{(1-t)^d},$$

$$f^{<n, \tau>}(t) = \sum_{l \in \mathbb{Z}} a_{nl+\tau} t^l = \frac{h^{<n, \tau>}(t)}{(1-t)^d},$$

where  $h(t), h^{<n, \tau>}(t) \in \mathbb{C}[t, t^{-1}]$ . Then

$$\text{for any } \tau \in \mathbb{Z}, \overrightarrow{h^{<n, kn+\tau>}} = \mathcal{D}[-k, -\tau] \vec{h}.$$

*Proof.* Because the Remark 3.4 we have to compute  $h^{<n, \tau>}(t)$  only for  $0 \leq \tau \leq n - 1$ . The following formula is clear:

$$f^{<n, 0>}(t^n) + t f^{<n, 1>}(t^n) + \dots + t^{n-1} f^{<n, n-1>}(t^n) = f(t),$$

hence

$$\frac{h^{<n, 0>}(t^n) + t h^{<n, 1>}(t^n) + \dots + t^{n-1} h^{<n, n-1>}(t^n)}{(1-t^n)^d} = \frac{h(t)}{(1-t)^d},$$



and

$$h^{<n,0>}(t^n) + th^{<n,1>}(t^n) + \dots + t^{n-1}h^{<n,n-1>}(t^n) = h(t) \frac{(1-t^n)^d}{(1-t)^d},$$

$t^\tau h^{<n,\tau>}(t^n)$  equals the sum of all the terms  $A_\beta t^\beta$  of  $h(t) \frac{(1-t^n)^d}{(1-t)^d}$  with  $\beta \equiv \tau \pmod n$ . In particular  $h^{<n,\tau>}(t)$  is a linear function of  $h(t)$ . So it is enough to compute  $h^{<n,\tau>}(t)$  for the canonical basis  $\{\varepsilon_j := t^j, j \in \mathbb{Z}\}$  of  $\mathbb{C}[t, t^{-1}]$ . We have

$$[t^i](h^{<n,\tau>}(t)) = [t^{ni+\tau}](h(t) \frac{(1-t^n)^d}{(1-t)^d}),$$

hence

$$\forall j \in \mathbb{Z}; [t^i](\varepsilon_j^{<n,\tau>}(t)) = [t^{ni+\tau}](t^j) \frac{(1-t^n)^d}{(1-t)^d} = [t^{ni+\tau-j}](\frac{(1-t^n)^d}{(1-t)^d})$$

which proves our statement.  $\square$

**Corollary 3.6.** *Fix an integer  $d \in \mathbb{N}^*$ . For  $j \in \mathbb{Z}$ , let  $\vec{\varepsilon}_j$  be the  $h$ -vector of the polynomial  $t^j$ . Then for any  $n \in \mathbb{N}^*$ , we have  $\deg \varepsilon_j^{<n>} = d - \lceil \frac{d-j}{n} \rceil$ . Moreover the leading coefficient of  $\varepsilon_j^{<n>}$  is  $\binom{n \lceil \frac{d-j}{n} \rceil + j - 1}{d-1}$ .*

*Proof.* Let remark that the set of  $t^j, j \in \mathbb{Z}$  is the canonical basis of  $\mathbb{C}[t, t^{-1}]$ . We have by Theorem 3.5 that  $\mathcal{D}\vec{\varepsilon}_j = \vec{\varepsilon}_j^{<n>}$ , hence  $\varepsilon_j^{<n>}$  is the  $j$  column vector of  $\mathcal{D}$ . By the Example 2.8, we have that  $\deg \varepsilon_j^{<n>} = d - \lceil \frac{d-j}{n} \rceil$ .

The last claim follows from the Lemma 3.3. Indeed for any  $j \in \mathbb{Z}$ , we have  $\mathcal{D}_{d-\lceil \frac{d-j}{n} \rceil, j} = \binom{n \lceil \frac{d-j}{n} \rceil + j - 1}{d-1}$ . This proves that the leading coefficient of  $\varepsilon_j^{<n>}$  is  $\binom{n \lceil \frac{d-j}{n} \rceil + j - 1}{d-1}$ .  $\square$

**Example 3.7.** *Let  $d = 2$  and  $n \in \mathbb{N}^*$ , we can describe the matrix  $\mathcal{D}$*

$ij$	$-(n+1)$	$\dots$	$-1$	$0$	$1$	$2$	$3$	$\dots$	$n$	$n+1$	$n+2$	$\dots$	$2n$
$-1$	$2$	$\dots$	$0$	$0$	$0$	$0$	$0$	$\dots$	$0$	$0$	$0$	$\dots$	$0$
$0$	$n-2$	$\dots$	$2$	$1$	$0$	$0$	$0$	$\dots$	$0$	$0$	$0$	$\dots$	$0$
$1$	$0$	$\dots$	$n-2$	$n-1$	$n$	$n-1$	$n-2$	$\dots$	$1$	$0$	$0$	$\dots$	$0$
$2$	$0$	$\dots$	$0$	$0$	$0$	$1$	$2$	$\dots$	$n-1$	$n$	$n-1$	$\dots$	$1$
$3$	$0$	$\dots$	$0$	$0$	$0$	$0$	$0$	$\dots$	$0$	$0$	$1$	$\dots$	$n-1$

**Theorem 3.8.** Fix integers  $d, n \in \mathbb{N}^*, \tau \in \mathbb{Z}$ . Let  $(a_l)_{l \in \mathbb{Z}}$  be a sequence of complex numbers, such that  $a_l = 0$  for  $l \ll 0$ , set :

$$f(t) = \sum_{l \in \mathbb{Z}} a_l t^l, \quad f^{<n, \tau>}(t) = \sum_{l \in \mathbb{Z}} a_{nl+\tau} t^l.$$

If  $f(t) = \frac{h(t)}{(1-t)^d}$  with  $h(t) \in \mathbb{C}[t, t^{-1}]$  then  $f^{<n, \tau>}(t) = \frac{h^{<n>}(t)}{(1-t)^d}$  with  $h^{<n>}(t) \in \mathbb{C}[t, t^{-1}]$  such that:

- $\deg h^{<n, \tau>}(t) \leq d - \lceil \frac{d - \deg h(t) + \tau}{n} \rceil$ ,
- If all the coefficients of  $h(t)$  are positive real numbers then  $\deg h^{<n, \tau>}(t) = d - \lceil \frac{d - \deg h(t) + \tau}{n} \rceil$ ,
- If  $\deg h(t) = d$  then  $\deg h^{<n>}(t) = d$ .

*Proof.* Let  $f(t) = \sum_{l \in \mathbb{Z}} a_l t^l = \frac{h(t)}{(1-t)^d}$ , where  $h(t) \in \mathbb{C}[t, t^{-1}]$   $h(t) = \gamma_\sigma t^\sigma + \dots + \gamma_s t^s$  with  $\deg h(t) = s, \gamma_s \neq 0$ . It follows that  $\vec{h} = \sum_{l=\sigma}^s \gamma_l \vec{\varepsilon}_l$ . We multiply this relation on the left by  $\mathcal{D}[-\tau]$ , so Theorem 3.5 implies  $\vec{h}^{<n, \tau>} = \sum_{l=\sigma}^s \gamma_l \vec{\varepsilon}_{l-\tau}^{<n>}$ . Since  $\deg \vec{\varepsilon}_{\sigma-\tau}^{<n>} \leq \deg \vec{\varepsilon}_{\sigma-\tau+1}^{<n>} \leq \dots \leq \deg \vec{\varepsilon}_{s-\tau}^{<n>}$ , we have,  $\deg \vec{h}^{<n, \tau>} \leq \deg \vec{\varepsilon}_{s-\tau}^{<n>}$ . It is clear that if all the coefficients of  $h(t)$  are positive real numbers then  $\deg \vec{h}^{<n, \tau>} = \deg \vec{\varepsilon}_{s-\tau}^{<n>}$ .

In the special case  $s = d$ , we have seen that for  $0 \leq l \leq d-1$  and any  $n \in \mathbb{N}^*$ ,  $\deg \vec{\varepsilon}_l^{<n>} = d - \lceil \frac{d-l}{n} \rceil \leq d-1$ , and  $\deg \vec{\varepsilon}_d^{<n>} = d$ , which implies  $\deg \vec{h}^{<n>} = d$ .  $\square$

**Theorem 3.9.** Let  $n \in \mathbb{N}^*$ ,  $S$  be a standard graded polynomial ring,  $M = \bigoplus_{l \in \mathbb{Z}} M_l$  be a finitely generated Cohen-Macaulay graded  $S$ -module of dimension  $d \geq 1$ , and  $M^{<n>} = \bigoplus_{l \in \mathbb{Z}} M_{nl}$ . Let  $\frac{Q(t)}{(1-t)^d}$  be the Hilbert-Poincaré series of  $M$ , where  $Q(t) = \gamma_\sigma t^\sigma + \dots + \gamma_s t^s \in \mathbb{C}[t, t^{-1}]$  is the  $h$ -polynomial of  $M$ , with  $\text{reg}(M) = \deg Q(t) = s$ . Then

- $\text{reg } M^{<n>} = d - \lceil \frac{d - \text{reg } M}{n} \rceil$ . Moreover by taking the sum over all index  $l$  such that  $\lceil \frac{d-l}{n} \rceil = \lceil \frac{d - \text{reg } M}{n} \rceil$ , we will get the leading coefficient of  $Q^{<n>}(t)$ :

$$\sum_{l \mid \lceil \frac{d-l}{n} \rceil = \lceil \frac{d - \text{reg } M}{n} \rceil} \gamma_l \binom{n \lceil \frac{d-l}{n} \rceil + l - 1}{d-1}.$$

- If  $\text{reg } M \leq d - 1$  and  $n \geq d$  then  $\text{reg } M^{<n>} = d - 1$ , and the leading coefficient of  $Q^{<n>}(t)$  is

$$\sum_{l=0}^{d-1} \gamma_l \binom{n \lceil \frac{d-l}{n} \rceil + l - 1}{d-1}.$$

- If  $n > \text{reg } M \geq d$  then  $\text{reg } M^{<n>} = d$ , and the leading coefficient of  $Q^{<n>}(t)$  is

$$\sum_{l=d}^{\text{reg } M} \gamma_l \binom{l-1}{d-1}.$$

*Proof.* We have  $\vec{Q} = \sum_{l=\sigma}^s \gamma_l \vec{\varepsilon}_l$ . We multiply this relation on the left by  $\mathcal{D}$ , so Theorem 3.5 implies that for any  $n \in \mathbb{N}^*$ ,  $\overrightarrow{Q^{<n>}} = \sum_{l=\sigma}^s \gamma_l \overrightarrow{\varepsilon_l^{<n>}}$ . Since  $\gamma_l \geq 0$  for all  $l$ ,  $\gamma_s > 0$ , and  $\deg \overrightarrow{\varepsilon_\sigma^{<n>}} \leq \deg \overrightarrow{\varepsilon_{\sigma+1}^{<n>}} \leq \dots \leq \deg \overrightarrow{\varepsilon_s^{<n>}}$ , we have,  $\deg \overrightarrow{Q^{<n>}} = \deg \overrightarrow{\varepsilon_s^{<n>}} = d - \lceil \frac{d - \text{reg } M}{n} \rceil$ , this number is  $\text{reg}(M^{<n>})$  since  $M^{<n>}$  is a Cohen-Macaulay  $S$ -module. The computation of the leading coefficient of  $Q^{<n>}(t)$  is immediate from Lemma 3.6.  $\square$

#### 4 $h$ -vector of the Segre-Veronese embedding.

The next Theorem improves partially [1] and [5].

**Theorem 4.1.** *Let consider the Segre-Veronese ring  $R_{\underline{b},n}$ ,  $\dim R_{\underline{b},n} = b_1 + \dots + b_m + 1$ . Let  $P_{R_{\underline{b},n}}(t) = \frac{Q_{R_{\underline{b},n}}(t)}{(1-t)^{\dim R_{\underline{b},n}}}$  be the Hilbert-Poincaré series of  $R_{\underline{b},n}$ , with  $Q_{R_{\underline{b},n}}(t) = h_0 + h_1 t + \dots + h_{r_{\underline{b},n}} t^{r_{\underline{b},n}}$ , where  $r_{\underline{b},n} = \deg Q_{R_{\underline{b},n}}(t)$  is the Castelnuovo-Mumford regularity of  $R_{\underline{b},n}$ . We set  $\alpha_{\underline{b},n} = \dim R_{\underline{b},n} - r_{\underline{b},n}$ . After a permutation of  $b_1, \dots, b_m$ , we can assume that  $\lceil \frac{b_1+1}{n_1} \rceil > \frac{b_i}{n_i} \forall i$ , then*

$$\alpha_{\underline{b},n} = \lceil \frac{b_1+1}{n_1} \rceil \quad r_{\underline{b},n} = (b_1 + \dots + b_m + 1) - \lceil \frac{b_1+1}{n_1} \rceil,$$

and the highest Betti number of  $R_{\underline{b},n}$  is

$$\beta_{r_{\underline{b},n}} = h_{r_{\underline{b},n}} = \binom{n_1 \alpha_{\underline{b},n} - 1}{b_1} \dots \binom{n_m \alpha_{\underline{b},n} - 1}{b_m}$$

*Proof.* The proof is by double induction on  $m$  and  $b_m$ . The case  $m = 1$  is given by the Example 2.8 and Corollary 3.6, so we can assume  $m \geq 2$ . We have that  $\lceil \frac{b_1+1}{n_1} \rceil > \frac{b_m}{n_m} > \frac{b_m-1}{n_m}$ , so by induction hypothesis the theorem is true for  $R_{\underline{b}-\epsilon_m, \underline{n}}$ , where  $\underline{b}-\epsilon_m = (b_1, \dots, b_{m-1}, b_m-1)$ . On the other hand the Hilbert function of  $R_{\underline{b}, \underline{n}}$  is  $H_{R_{\underline{b}, \underline{n}}}(l) = \binom{n_1 l + b_1}{b_1} \dots \binom{n_m l + b_m}{b_m}$ , so

$$H_{R_{\underline{b}, \underline{n}}}(l) = \left(1 + \frac{n_m}{b_m} l\right) H_{R_{\underline{b}-\epsilon_m, \underline{n}}}(l). \quad (1)$$

Let  $P_{R_{\underline{b}-\epsilon_m, \underline{n}}}(t) = \frac{Q_{R_{\underline{b}-\epsilon_m, \underline{n}}}(t)}{(1-t)^{b_1+\dots+b_m}}$  be the Hilbert-Poincaré series of  $R_{\underline{b}-\epsilon_m, \underline{n}}$ , where  $Q_{R_{\underline{b}-\epsilon_m, \underline{n}}}(t) = h_0 + h_1 t + \dots + h_{r_{\underline{b}-\epsilon_m, \underline{n}}} t^{r_{\underline{b}-\epsilon_m, \underline{n}}}$ , with  $h_{r_{\underline{b}-\epsilon_m, \underline{n}}} \neq 0$ . In order to avoid any confusion we also set :  $P_{R_{\underline{b}, \underline{n}}}(t) = \frac{Q_{R_{\underline{b}, \underline{n}}}(t)}{(1-t)^{b_1+\dots+b_m+1}}$  be the Hilbert-Poincaré series of  $R_{\underline{b}, \underline{n}}$ , where

$$Q_{R_{\underline{b}, \underline{n}}}(t) = \hat{h}_0 + \dots + \hat{h}_{r_{\underline{b}, \underline{n}}} t^{r_{\underline{b}, \underline{n}}},$$

with  $\hat{h}_{r_{\underline{b}, \underline{n}}} \neq 0$ .

Let  $\beta = \frac{n_m}{b_m}$ , by simple calculations from (1) we get:

$$P_{R_{\underline{b}, \underline{n}}}(t) = P_{R_{\underline{b}-\epsilon_m, \underline{n}}}(t) + \beta t P'_{R_{\underline{b}-\epsilon_m, \underline{n}}}(t). \quad (2)$$

Hence  $\dim R_{\underline{b}, \underline{n}} = \dim R_{\underline{b}-\epsilon_m, \underline{n}} + 1$ , and

$$Q_{R_{\underline{b}, \underline{n}}}(t) = Q_{R_{\underline{b}-\epsilon_m, \underline{n}}}(t) + t[Q_{R_{\underline{b}-\epsilon_m, \underline{n}}}(t)(\beta R_{\underline{b}-\epsilon_m, \underline{n}} - 1) + \beta Q'_{R_{\underline{b}-\epsilon_m, \underline{n}}}(t) - \beta t Q'_{R_{\underline{b}-\epsilon_m, \underline{n}}}(t)],$$

note that  $Q_{R_{\underline{b}, \underline{n}}}(1) = \beta \dim R_{\underline{b}-\epsilon_m, \underline{n}} Q_{R_{\underline{b}-\epsilon_m, \underline{n}}}(1) \neq 0$ .

In particular we have  $r_{\underline{b}, \underline{n}} \leq r_{\underline{b}-\epsilon_m, \underline{n}} + 1$  and for all  $k = 0, \dots, r_{\underline{b}-\epsilon_m, \underline{n}} + 1$  we have

$$\hat{h}_k = h_{k-1} \left( \frac{n_m}{b_m} \dim R_{\underline{b}-\epsilon_m, \underline{n}} - (k-1) \frac{n_m}{b_m} - 1 \right) + h_k \left( k \frac{n_m}{b_m} + 1 \right). \quad (3)$$

By induction hypothesis we have  $\alpha_{\underline{b}-\epsilon_m, \underline{n}} = \lceil \frac{b_1+1}{n_1} \rceil \neq \frac{b_m}{n_m}$ , so we put  $k = r_{\underline{b}-\epsilon_m, \underline{n}} + 1$  in equality (3), and we get:

$$\hat{h}_{r_{\underline{b}-\epsilon_m, \underline{n}}+1} = h_{r_{\underline{b}-\epsilon_m, \underline{n}}} \left( \frac{n_m \alpha_{\underline{b}-\epsilon_m, \underline{n}} - b_m}{b_m} \right) \neq 0.$$

Hence  $\hat{h}_{r_{\underline{b}-\epsilon_m, \underline{n}}+1}$  is the leading coefficient of  $Q_{R_{\underline{b}, \underline{n}}}$  and  $r_{\underline{b}, \underline{n}} = r_{\underline{b}-\epsilon_m, \underline{n}} + 1$  and  $\alpha_{\underline{b}, \underline{n}} = \alpha_{\underline{b}-\epsilon_m, \underline{n}} = \lceil \frac{b_1+1}{n_1} \rceil$ . By induction hypothesis

$$h_{r_{\underline{b}-\epsilon_m, \underline{n}}} = \binom{n_1 \alpha_{\underline{b}, \underline{n}} - 1}{b_1} \dots \binom{n_{m-1} \alpha_{\underline{b}, \underline{n}} - 1}{b_{m-1}} \binom{n_m \alpha_{\underline{b}, \underline{n}} - 1}{b_m}$$

so that

$$\begin{aligned}\hat{h}_{r_{b,n}} &= \binom{n_1\alpha_{b,n} - 1}{b_1} \cdots \binom{n_{m-1}\alpha_{b,n} - 1}{b_{m-1}} \binom{n_m\alpha_{b,n} - 1}{b_m} \left(\frac{n_m\alpha_{b,n} - b_m}{b_m}\right) \\ &= \binom{n_1\alpha_{b,n} - 1}{b_1} \cdots \binom{n_{m-1}\alpha_{b,n} - 1}{b_{m-1}} \binom{n_m\alpha_{b,n} - 1}{b_m}.\end{aligned}$$

□

## 5 Rings of Veronese type

Let  $b, n \in \mathbb{N}^*$ ,  $\underline{a} = (a_0, \dots, a_b) \in \mathbb{N}^{b+1}$  such that  $1 \leq a_i \leq n$ ,  $a_0 + \dots + a_b > n$ , and  $\mathcal{M}_{b,n,\underline{a}}$  be the set of monomials of the polynomial ring  $K[x_0, \dots, x_b]$ :

$$\mathcal{M}_{b,n,\underline{a}} = \{x_0^{\alpha_0} \cdots x_b^{\alpha_b} \mid \alpha_0 + \dots + \alpha_b = n, \alpha_i \leq a_i, \forall i = 0, \dots, b\}.$$

Let denote by  $R_{b,n,\underline{a}}$  the toric subring of  $K[x_0, \dots, x_b]$  generated by  $\mathcal{M}_{b,n,\underline{a}}$ . It is well known that  $R_{b,n,\underline{a}}$  is a Cohen-Macaulay ring. Let  $\mathcal{S}$  be the collection of subsets of  $\{0, \dots, b\}$  such that :  $S \in \mathcal{S}$  if and only if  $S \subset \{0, \dots, b\}$ , and  $\Sigma S := \sum_{i \in S} a_i < n$ .

**Theorem 5.1.** ([9]) *With the above notations the Hilbert-function of  $R_{b,n,\underline{a}}$  is*

$$\forall l \geq 0; \quad H_{b,n,\underline{a}}(l) = \sum_{S \in \mathcal{S}} (-1)^{|S|} \binom{l(n - \Sigma S) - |S| + b}{b}$$

We have  $\dim(R_{b,n,\underline{a}}) = b + 1$ , and its degree or multiplicity is

$$\deg(R_{b,n,\underline{a}}) = \sum_{S \in \mathcal{S}} (-1)^{|S|} (n - \Sigma S)^b.$$

Our aim is to study the Hilbert-Poincaré series of  $R_{b,n,\underline{a}}$ :

$$P_{R_{b,n,\underline{a}}} = \sum_{S \in \mathcal{S}} (-1)^{|S|} \sum_{l \geq 0} \binom{l(n - \Sigma S) - |S| + b}{b} t^l.$$

The following Corollary follows immediately from 3.6.

**Corollary 5.2.** *For any  $S \in \mathcal{S}$  and  $k \in \mathbb{N}^*$ , we have:*

$$\sum_{l \geq 0} \binom{kl - |S| + b}{b} t^l = \frac{Q_{S,k}(t)}{(1-t)^{b+1}},$$

where  $Q_{S,k}(t)$  is a polynomial with  $Q_{S,k}(1) \neq 0$ , with leading term

$$\binom{k\alpha_{S,k} + |S| - 1}{b} t^{b+1-\alpha_{S,k}},$$

with  $\alpha_{S,k} = \lceil \frac{b+1-|S|}{k} \rceil$ .

The following theorem is immediate from 5.1 and Corollary 5.2. It improves the description of the Hilbert Poincaré series given in [9].

**Theorem 5.3.** *With the above notations, let  $\mathcal{S}$  be the collection of subsets of  $\{0, \dots, b\}$  such that  $S \in \mathcal{S}$  if and only if  $S \subset \{0, \dots, b\}$ , and  $\Sigma S := \sum_{i \in S} a_i < n$ . Then we can write the Hilbert-Poincaré series of  $R_{b,n,\underline{a}}$ :*

$$P_{R_{b,n,\underline{a}}} = \frac{Q_{b,n,\underline{a}}(t)}{(1-t)^{b+1}},$$

with  $Q_{b,n,\underline{a}}(t) = \sum_{S \in \mathcal{S}} (-1)^{|S|} Q_{S,n}(t)$ , where  $Q_{S,n}(t)$  is a polynomial with  $Q_{S,n}(1) \neq 0$ , with leading term

$$\binom{(n - \Sigma S)\alpha_{S,n-\Sigma S} + |S| - 1}{b} t^{b+1-\alpha_{S,n-\Sigma S}},$$

where  $\alpha_{S,n-\Sigma S} = \lceil \frac{b+1-|S|}{n-\Sigma S} \rceil$ .

Part one of the following corollary improves [9][Cor. 2.12].

**Corollary 5.4.** *With the above notations:*

1.  $\text{reg}(R_{b,n,\underline{a}}) \leq b + 1 - \lceil \frac{b+1}{n} \rceil$ , and the equality is true if and only if

$$\sum_{S \in \mathcal{S}, \alpha_{S,n-\Sigma S} = \lceil \frac{b+1}{n} \rceil} (-1)^{|S|} \binom{(n - \Sigma S)\alpha_{S,n-\Sigma S} + |S| - 1}{b} \neq 0$$

2. If  $b + 1 > n^2$  then  $\text{reg}(R_{b,n,\underline{a}}) = b + 1 - \lceil \frac{b+1}{n} \rceil$ . Moreover the leading term of  $Q_{b,n,\underline{a}}(t)$  is  $\binom{(n \lceil \frac{b+1}{n} \rceil - 1)}{b} t^{b+1 - \lceil \frac{b+1}{n} \rceil}$ ,

*Proof.* 1. It is enough to prove that  $\min_{S \in \mathcal{S}} \lceil \frac{b+1-|S|}{n-\Sigma S} \rceil = \lceil \frac{b+1}{n} \rceil$ . We consider two cases,

- if  $b + 1 < n$  then  $\lceil \frac{b+1}{n} \rceil = 1 \leq \lceil \frac{b+1-|S|}{n-\Sigma S} \rceil, \forall S \in \mathcal{S}$ .

- If  $b + 1 \geq n$ , then

$$\frac{b+1}{n} \leq \frac{b+1-|S|}{n-\Sigma S} \Leftrightarrow (b+1)(n-\Sigma S) \leq n(b+1-|S|) \Leftrightarrow (b+1)\Sigma S \geq n|S|,$$

this is true since by hypothesis  $\frac{b+1}{n} \geq 1 \geq \frac{|S|}{\Sigma S}$ .

2. Let  $b + 1 > n^2$  and  $S \neq \emptyset$ . By definition  $\lceil \frac{b+1}{n} \rceil$  is the integer  $q$  such that  $b + 1 = qn - r$ , with  $0 \leq r < n$  and  $q \geq n + 1$ . We have

$$b + 1 - |S| = qn - r - |S| = q(n - \Sigma S) - r - |S| + q\Sigma S,$$

and  $q\Sigma S - |S| \geq (n + 1)\Sigma S - |S| \geq n\Sigma S > r$ , so that  $q\Sigma S - |S| - r > 0$ , hence  $\lceil \frac{b+1-|S|}{n-\Sigma S} \rceil > q = \lceil \frac{b+1}{n} \rceil$ .  $\square$

In general leading terms of the alternating sum can cancel, as we can see in the next example.

**Example 5.5.** *Let consider the ring  $R_{4,3,(1,1,1,1,1)}$ , the sets  $S$  can have 0,1 or 2 elements, and we have: If  $S = \emptyset$  then  $\alpha_{\emptyset,3} = \lceil \frac{5}{3} \rceil = 2$ , if  $|S| = 1$  then  $\alpha_{S,3} = \lceil \frac{4}{2} \rceil = 2$ , and finally if  $|S| = 2$  then  $\alpha_{S,2} = \lceil \frac{3}{1} \rceil = 3$ . By using Theorem 5.3 we can write*

$$P_{R_{4,3,(1,1,1,1,1)}} = \frac{Q_0(t) - 5Q_1(t) + 10Q_2(t)}{(1-t)^5},$$

, with  $Q_0(t) = 5t^3 + \dots$ ;  $Q_1(t) = t^3 + \dots$ ;  $Q_2(t) = t^2 + \dots$ . Note that in this case  $Q_0(t) - 5Q_1(t) + 10Q_2(t) = h_0 + h_1t + h_2t^2$ , where  $h_0 = 1, h_1 = 5$  and since  $h_0 + h_1 + h_2 = \deg(R_{4,3,(1,1,1,1,1)}) = 11$ , we get  $h_2 = 5$ , so that

$$P_{R_{4,3,(1,1,1,1,1)}} = \frac{1 + 5t + 5t^2}{(1-t)^5}.$$

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