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Abstract[1] In a recent preprint, Ilse Fischer and Martina Kubitzke, proved the bilinearity of the Segre transform under some restricted hypothesis, motivated by their results we show in this paper the bilinearity of the Segre transform in general. We apply these results to compute the postulation number of a series. Our second application is motivated by the paper of David A. Cox, and Evgeny Materov (2009), where is computed the Castelnuovo-Mumford regularity of the Segre Veronese embedding, we can extend partially their result and compute the Castelnuovo-Mumford regularity of the Segre product of Cohen-Macaulay modules.

Key words and phrases: Segre-Veronese, Castelnuovo-Mumford regularity, Cohen-Macaulay, postulation number.
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1 Introduction

In this paper, we deal only with formal Laurent series

$$\mathfrak{a} = \sum_{l \geqslant \sigma_a} a_l t^l, \sigma_{\mathfrak{a}} \in \mathbb{Z}, a_l \in \mathbb{C}$$

such that

$$(*) \quad \mathfrak{a} = \frac{h(\mathfrak{a})(t)}{(1-t)^{d_{\mathfrak{a}}}}, \quad \text{for some } d_{\mathfrak{a}} \geqslant 0, h(\mathfrak{a})(t) \in \mathbb{C}[t, t^{-1}].$$

Given two formal Laurent series $\mathfrak{a}, \mathfrak{b}$ satisfying (*), the Segre transform $\mathfrak{a} \underline{\otimes} \mathfrak{b}$ is defined by

$$\mathfrak{a} \underline{\otimes} \mathfrak{b} = \sum_{l \geqslant \sigma} a_l b_l t^l,$$

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where $\sigma = \max\{\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{b}}\}$. In a recent preprint [F-K], the authors proved the bilinearity of the Segre transform under some restricted hypothesis, motivated by this results we show in this paper the bilinearity of the Segre transform in general. We apply these results to compute the postulation number of a Segre product of series satisfying (*). Property (*) is equivalent to the existence of a polynomial $\Phi_{\mathfrak{a}}(t) \in \mathbb{C}[t]$, such that $a_n = \Phi_{\mathfrak{a}}(n)$ for n large enough. The postulation number is the smallest integer $\beta_{\mathfrak{a}}$ such that $a_n = \Phi(n)$ for $n > \beta_{\mathfrak{a}}$. It is well known that $\beta_{\mathfrak{a}} = \deg h(\mathfrak{a})(t) - d_{\mathfrak{a}}$.

Our second application is motivated by the paper [C-M], where is computed the Castelnuovo-Mumford regularity of the Segre Veronese embedding, we can extend partially their result. Our main result is:

Theorem. *Let $S_1, \ldots, S_s$ be graded polynomial rings on disjoints of set of variables. For all $i = 1, \ldots, s$, let M_i be a graded finitely generated S_i-Cohen-Macaulay module. We assume that $M_i = \oplus_{l \geqslant 0} M_{i,l}$ as S_i-module. Let $d_i = \dim M_i, b_i = d_i - 1 \geq 0$, $\alpha_i = d_i - \operatorname{reg}(M_i)$, where $\operatorname{reg}(M_i)$ is the Castelnuovo-Mumford regularity of M_i. If $\operatorname{reg}(M_i) < d_i$, for all $i = 1, \ldots, s$ then*

(1) *$M_1 \underline{\otimes} \ldots \underline{\otimes} M_s$ is a Cohen-Macaulay $S_1 \underline{\otimes} \ldots \underline{\otimes} S_s$-module.*

(2) $\operatorname{reg}(M_1 \underline{\otimes} \ldots \underline{\otimes} M_s) = (b_1 + \ldots + b_s + 1) - \max\{\alpha_1, \ldots \alpha_s\}$.

(3) *For $n_i \in \mathbb{N}$, let $M_i^{<n_i>}$ be the n_i-Veronese transform of M_i, then*

$$\operatorname{reg}(M_1^{<n_1>} \underline{\otimes} \ldots \underline{\otimes} M_s^{<n_s>}) = (b_1 + \ldots + b_s + 1) - \max\{\lceil \frac{\alpha_1}{n_1} \rceil, \ldots, \lceil \frac{\alpha_s}{n_s} \rceil\}.$$

Note that this result can be proved easily by using local cohomology, but our purpose is to give a very elementary proof.

Segre transform of Laurent series

In this paper, we deal only with formal Laurent series

$$\mathfrak{a} = \sum_{l \geqslant \sigma_a} a_l t^l, \sigma_{\mathfrak{a}} \in \mathbb{Z}, a_l \in \mathbb{C}$$

such that

$$(*) \quad \mathfrak{a} = \frac{h(\mathfrak{a})(t)}{(1-t)^{d_{\mathfrak{a}}}}, \quad \text{for some } d_{\mathfrak{a}} \geqslant 0, h(\mathfrak{a})(t) \in \mathbb{C}[t, t^{-1}].$$

We will set $h(\mathfrak{a})(t) = \sum_{n \geqslant \sigma_{\mathfrak{a}}} h_n(\mathfrak{a}) t^n$.

Definition 1.1. *Let $\mathfrak{a}, \mathfrak{b}$ be two formal Laurent series satisfying (*). the Segre transform $\mathfrak{a} \underline{\otimes} \mathfrak{b}$ is defined by*

$$\mathfrak{a} \underline{\otimes} \mathfrak{b} = \sum_{l \geqslant \sigma} a_l b_l t^l,$$

where $\sigma = \max\{\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{b}}\}$.

In all this paper we assume that $\mathfrak{a} \underline{\otimes} \mathfrak{b} \neq 0$.

Lemma 1.2. $\mathfrak{a}\underline{\otimes}\mathfrak{b}$ *satisfies* $(*)$.

Proof. By [M1], property (*) is equivalent to the existence of a polynomial $\Phi_{\mathfrak{a}}(l)$ such that $\Phi_{\mathfrak{a}}(l) = a_l$ for l large enough. Moreover,

$$\begin{cases} d_{\mathfrak{a}} = \deg \Phi_{\mathfrak{a}} + 1 \text{ if } \Phi_{\mathfrak{a}} \text{ is a non zero polynomial} \\ d_{\mathfrak{a}} = 0 \text{ if } \Phi_{\mathfrak{a}} = 0. \end{cases}$$

We have also a polynomial $\Phi_{\mathfrak{b}}(l)$ such that $\Phi_{\mathfrak{b}}(l) = b_l$ for l large enough. Hence $a_l b_l = \Phi_{\mathfrak{a}}(l)\Phi_{\mathfrak{b}}(l)$ is a polynomial for l large enough, and again by [M1], there exist a Laurent polynomial $h(\mathfrak{a}\underline{\otimes}\mathfrak{b})(t)$ such that

$$\mathfrak{a}\underline{\otimes}\mathfrak{b} = \frac{h(\mathfrak{a}\underline{\otimes}\mathfrak{b})(t)}{(1-t)^{d_{\mathfrak{a}\underline{\otimes}\mathfrak{b}}}},$$

where

$$d_{\mathfrak{a}\underline{\otimes}\mathfrak{b}} = \begin{cases} 0 \text{ if either } \ d_{\mathfrak{a}} = 0 \text{ or } d_{\mathfrak{b}} = 0 \\ d_{\mathfrak{a}} + d_{\mathfrak{b}} - 1 \text{ if } d_{\mathfrak{a}}, d_{\mathfrak{b}} \geqslant 1. \end{cases}$$

□

Remark 1.3. *We recall that binomial coefficients can be defined in a more general setting than natural numbers, indeed for* $k \in \mathbb{N}$, *binomial coefficients are polynomial functions in the variable* n. *More precisely:*

(1) *If* $k = 0$ *then let* $\binom{n}{0} = 1$, *for all* $n \in \mathbb{C}$.

(2) *If* $k > 0$ *then let* $\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!}$, *for all* $n \in \mathbb{C}$.

Note that for all $n \in \mathbb{C}$, $\binom{n}{k} = (-1)^k\binom{k-n-1}{k}$ *and if* $n \in \mathbb{N}, n < k$, *then* $\binom{n}{k} = 0$.

2 Segre transform is bilinear

Let recall the following Lemma 1 from [M1].

Lemma 2.1. *Let*

$$\mathfrak{a} = \sum_{l \geqslant \sigma_{\mathfrak{a}}} a_l t^l = \frac{h(\mathfrak{a})(t)}{(1-t)^{d_{\mathfrak{a}}}},$$

with $h(\mathfrak{a})(t) = h_{\sigma_{\mathfrak{a}}} t^{\sigma_{\mathfrak{a}}} + \ldots + h_{r_{\mathfrak{a}}} t^{r_{\mathfrak{a}}}, d_{\mathfrak{a}} \geqslant 0$ *and* $\sigma_{\mathfrak{a}} \leq r_{\mathfrak{a}} \in \mathbb{Z}$. *We will set* $b_{\mathfrak{a}} = d_{\mathfrak{a}} - 1$. *Then for all* $n = \sigma_{\mathfrak{a}}, \ldots, r_{\mathfrak{a}}$ *we have*

$$h_n(\mathfrak{a}) = \sum_{k=0}^{n-\sigma_{\mathfrak{a}}} (-1)^k \binom{d_{\mathfrak{a}}}{k} a_{n-k} = \sum_{k=\sigma_{\mathfrak{a}}}^{n} (-1)^{n-k} \binom{d_{\mathfrak{a}}}{n-k} a_k. \quad (1)$$

On the other hand we have for all $k \geqslant \sigma_{\mathfrak{a}}$

$$a_k = \sum_{i=0}^{k-\sigma_{\mathfrak{a}}} h_{k-i}(\mathfrak{a}) \binom{b_{\mathfrak{a}} + i}{i} = \sum_{i=\sigma_{\mathfrak{a}}}^{k} h_i(\mathfrak{a}) \binom{b_{\mathfrak{a}} + k - i}{k - i} \quad (2)$$

The first claim follows from the equality:

$$(\sum_{l \geqslant \sigma_{\mathfrak{a}}} a_l t^l)((1-t)^{d_{\mathfrak{a}}}) = h(\mathfrak{a})(t).$$

The second claim since:

$$\sum_{l \geqslant \sigma_{\mathfrak{a}}} a_l t^l = \frac{h(\mathfrak{a})(t)}{(1-t)^{d_{\mathfrak{a}}}} = (h(\mathfrak{a})(t))(\sum_{i \geqslant 0} \binom{b_{\mathfrak{a}} + i}{i} t^i)$$

The following two theorems extend [F-K, Theorem 1].

Theorem 2.2. *Let* $\mathfrak{a}, \mathfrak{b}$ *be two formal power series satisfying property (*). If* $d_{\mathfrak{a}} = 0$ *, then for all* $n \in \mathbb{Z}$

$$h_n(\mathfrak{a} \underline{\otimes} \mathfrak{b}) = a_n b_n = h_n(\mathfrak{a}) \sum_{j=\sigma_{\mathfrak{b}}}^{n} h_j(\mathfrak{b}) \binom{b_{\mathfrak{b}} + n - j}{n - j}.$$

Moreover, $\deg h(\mathfrak{a} \underline{\otimes} \mathfrak{b})(t) \leqslant \deg \mathfrak{a}$*. If* $d_{\mathfrak{b}} > 0$ *and* $h_j(\mathfrak{b}) \geqslant 0$*, for all* j *then* $\deg h(\mathfrak{a} \underline{\otimes} \mathfrak{b})(t) = \deg(\mathfrak{a})$.

Proof. Since $d_{\mathfrak{a}} = 0$, $\mathfrak{a}$ is a Laurent polynomial, we have that $h_n(\mathfrak{a}) = a_n$, for all $n \in \mathbb{Z}$ and $a_n = 0$ for $n > \deg(\mathfrak{a})$, which implies that

$$\mathfrak{a} \underline{\otimes} \mathfrak{b} = \sum_{n \leqslant \deg(\mathfrak{a})} a_n b_n t^n.$$

hence $\deg h(\mathfrak{a} \underline{\otimes} \mathfrak{b})(t) \leq \deg(\mathfrak{a})$. Now suppose that $d_{\mathfrak{b}} > 0$ and $h_j(\mathfrak{b}) \geqslant 0$ for all j, by using equation (2) we have,

$$h_n(\mathfrak{a} \underline{\otimes} \mathfrak{b}) = h_n(\mathfrak{a}) \sum_{j=\sigma_{\mathfrak{b}}}^{n} h_j(\mathfrak{b}) \binom{b_{\mathfrak{b}} + n - j}{n - j}.$$

Note that $h_{\sigma_{\mathfrak{b}}} > 0$, $\binom{b_{\mathfrak{b}} + n - \sigma_{\mathfrak{b}}}{n - \sigma_{\mathfrak{b}}} > 0$, for $n \geq \sigma_{\mathfrak{b}}$. The assumption $(\mathfrak{a} \underline{\otimes} \mathfrak{b}) \neq 0$ implies $\sigma_{\mathfrak{b}} \leqslant \deg \mathfrak{a}$, so $\sum_{j=\sigma_{\mathfrak{b}}}^{\deg \mathfrak{a}} h_j(\mathfrak{b}) \binom{b_{\mathfrak{b}} + \deg \mathfrak{a} - j}{\deg \mathfrak{a} - j} > 0$. Hence $h_{\deg \mathfrak{a}}(\mathfrak{a} \underline{\otimes} \mathfrak{b}) > 0$ and the claim is over. □

Remark 2.3. *Let* $\mathfrak{a}, \mathfrak{b}$ *be two formal power Laurent series satisfying property (*), the product* $h_i(\mathfrak{a}) h_j(\mathfrak{b})$ *is null for any* $i < \sigma, j < \sigma$*, where* σ *is any of the numbers* $\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{b}}, \sigma_{(\mathfrak{a} \underline{\otimes} \mathfrak{b})} = \max(\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{b}}), \min(\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{b}})$.

Theorem 2.4. *Let $\mathfrak{a}, \mathfrak{b}$ be any formal power Laurent series satisfying property (*). Let $b_{\mathfrak{a}} = d_{\mathfrak{a}} - 1 \geqslant 0, b_{\mathfrak{b}} = d_{\mathfrak{b}} - 1 \geqslant 0$ and σ be any of the numbers $\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{b}}, \sigma_{(\mathfrak{a}\underline{\otimes}\mathfrak{b})} = \max(\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{b}}), \min(\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{b}})$. Then for any $n \in \mathbb{Z}$*

$$h_n(\mathfrak{a}\underline{\otimes}\mathfrak{b}) = \sum_{i=\sigma}^{\infty}\sum_{j=\sigma}^{\infty} h_i(\mathfrak{a})h_j(\mathfrak{b})\binom{b_{\mathfrak{a}}+j-i}{n-i}\binom{b_{\mathfrak{b}}+i-j}{n-j}.$$

The proof will follow immediately from the Lemma 2.1 and the proof of [F-K, Theorem 1]:

3 Postulation number, Castelnuovo-Mummford regularity

Lemma 3.1. *Let $i, j \in \mathbb{Z}, b_1 := d_1 - 1 \geqslant 0, b_2 := d_2 - 1 \geqslant 0$ and*

$$T^i_{d_1} := \frac{t^i}{(1-t)^{d_1}}; T^j_{d_2} := \frac{t^j}{(1-t)^{d_2}}$$

Then

$$T^i_{d_1}\underline{\otimes}T^j_{d_2} = \frac{\sum_{n=\max(i,j)}^{r_{i,j}}\binom{b_1+j-i}{n-i}\binom{b_2+i-j}{n-j}t^n}{(1-t)^{b_1+b_2+1}}, \quad (3)$$

where :

$$r_{i,j} = \begin{cases} \min(b_1+j, b_2+i) \text{ if } b_2+i-j \geqslant 0 \quad \text{and } b_1+j-i \geqslant 0 \\ \max(b_1+j, b_2+i) \text{ if } b_2+i-j < 0 \quad \text{or } b_1+j-i < 0. \end{cases}$$

Proof. The equality (3) follows from theorem 2.4. We need to check that for $n > r_{i,j}$, we have $\binom{b_1+j-i}{n-i}\binom{b_2+i-j}{n-j} = 0$, and that for $n = r_{i,j}$, we have $\binom{b_1+j-i}{n-i}\binom{b_2+i-j}{n-j} \neq 0$. We have two cases:

1. If $b_1 + j - i \geq 0$ and $b_2 + i - j \geq 0$, then $\binom{b_1+j-i}{n-i} = 0$ if and only if $b_1 + j < n$. Hence $r_{i,j} = \min(b_1 + j, b_2 + i)$.
2. Either $b_1 + j - i < 0$ or $b_2 + i - j < 0$. Suppose for example that $b_1 + j - i < 0$ then $\binom{b_1+j-i}{n-i} \neq 0$, and $\binom{b_2+i-j}{n-j} = 0$ if and only if $b_2 + i < n$, but $b_1 + j < i \leq b_2 + i < n$. Hence $r_{i,j} = \max(b_1 + j, b_2 + i)$.

□

Example 3.2. *Let $\alpha, \beta \in \mathbb{Z}$, we study $T^{\alpha}_{d}\underline{\otimes}T^{\beta}_{1}$. We consider two cases:*

(1) *If $\max(\alpha, \beta) = \alpha$ then*

$$T^{\alpha}_{d}\underline{\otimes}T^{\beta}_{1} = \frac{t^{\alpha}}{(1-t)^{d}}.$$

(2) *If* $\max(\alpha,\beta)=\beta>\alpha$ *then*

$$T_d^{\alpha}\underline{\otimes}T_1^{\beta}=\frac{t^{\alpha}-t^{\alpha}(\sum_{l=0}^{\beta-\alpha-1}\binom{d-1+l}{l}t^l)(1-t)^d}{(1-t)^d}.$$

Note that $\deg(t^{\alpha}-t^{\alpha}(\sum_{l=0}^{\beta-\alpha-1}\binom{d-1+l}{l}t^l)(1-t)^d)=d-1+\beta$.

Proposition 3.3. *Let*

$$\mathfrak{a}=\frac{h(\mathfrak{a})(t)}{(1-t)^{d_{\mathfrak{a}}}};\mathfrak{b}=\frac{h(\mathfrak{b})(t)}{(1-t)^{d_{\mathfrak{b}}}},\quad \textit{where } h(\mathfrak{a})(t),h(\mathfrak{b})(t)\in\mathbb{C}[t,t^{-1}].$$

For any non null Laurent series satisfying property (*), we denote $\sigma_{\mathfrak{a}}=\min_n h_n(\mathfrak{a})\neq 0, r_{\mathfrak{a}}=\deg h(\mathfrak{a})(t)$. *Then*

(1) $r_{\mathfrak{a}\underline{\otimes}\mathfrak{b}}\leqslant\max(b_{\mathfrak{a}}+r_{\mathfrak{b}},b_{\mathfrak{b}}+r_{\mathfrak{a}})$.

(2) If for all $\sigma_{\mathfrak{a}}\leqslant i\leqslant r_{\mathfrak{a}},\sigma_{\mathfrak{b}}\leqslant j\leqslant r_{\mathfrak{b}}$ *such that* $h_i(\mathfrak{a})\neq 0, h_j(\mathfrak{b})\neq 0$ *we have* $b_{\mathfrak{b}}+i-j\geqslant 0$ *and* $b_{\mathfrak{a}}+j-i\geqslant 0$ *then*

$$r_{\mathfrak{a}\underline{\otimes}\mathfrak{b}}\leqslant\min(b_{\mathfrak{a}}+r_{\mathfrak{b}},b_{\mathfrak{b}}+r_{\mathfrak{a}}).$$

Moreover, if for all $i,j,h_i(\mathfrak{a})\geqslant 0,h_j(\mathfrak{b})\geqslant 0$ *then* $r_{\mathfrak{a}\underline{\otimes}\mathfrak{b}}=\min(b_{\mathfrak{a}}+r_{\mathfrak{b}},b_{\mathfrak{b}}+r_{\mathfrak{a}})$.

(3) If $0\leqslant\sigma_{\mathfrak{a}}\leqslant r_{\mathfrak{a}}\leqslant b_{\mathfrak{a}}$ *and* $0\leqslant\sigma_{\mathfrak{b}}\leqslant r_{\mathfrak{b}}\leqslant b_{\mathfrak{b}}$, *then*

$$r_{\mathfrak{a}\underline{\otimes}\mathfrak{b}}\leqslant\min(b_{\mathfrak{a}}+r_{\mathfrak{b}},b_{\mathfrak{b}}+r_{\mathfrak{a}}).$$

Moreover, if for all $i,j,h_i(\mathfrak{a})\geqslant 0,h_j(\mathfrak{b})\geqslant 0$ *then* $r_{\mathfrak{a}\underline{\otimes}\mathfrak{b}}=\min(b_{\mathfrak{a}}+r_{\mathfrak{b}},b_{\mathfrak{b}}+r_{\mathfrak{a}})$.

Proof. (1) Let $n>\max(b_{\mathfrak{a}}+r_{\mathfrak{b}},b_{\mathfrak{b}}+r_{\mathfrak{a}})$, for all $i\leqslant r_{\mathfrak{a}},j\leqslant r_{\mathfrak{b}}$. By the Lemma 3.1, this implies that $A_{i,j,n}=0$, for all $i\leqslant r_{\mathfrak{a}},j\leqslant r_{\mathfrak{b}}$. Hence $h_n(\mathfrak{a}\underline{\otimes}\mathfrak{b})=0$, which implies that $\sigma_(\mathfrak{a}\underline{\otimes}\mathfrak{b})\leqslant\max(b_{\mathfrak{a}}+j,b_{\mathfrak{b}}+i)$.

(2) Since for all $\sigma_{\mathfrak{a}}\leqslant i\leqslant r_{\mathfrak{a}},\sigma_{\mathfrak{b}}\leqslant j\leqslant r_{\mathfrak{b}}$, we have that $b_{\mathfrak{b}}+i-j\geqslant 0$ and $b_{\mathfrak{a}}+j-i\geqslant 0$, by the Lemma 3.1, we have that $r_{i,j}=\min(b_{\mathfrak{a}}+j,b_{\mathfrak{b}}+i)$. This implies for all i,j that

$$A_{i,j,r_{i,j}}\neq 0 \text{ and } A_{i,j,n}=0 \text{ for } n>r_{i,j}.$$

On the other hand

$$\min(b_{\mathfrak{a}}+r_{\mathfrak{b}},b_{\mathfrak{b}}+r_{\mathfrak{a}})\geqslant\min(b_{\mathfrak{a}}+r_{\mathfrak{b}},b_{\mathfrak{b}}+i)\geqslant\min(b_{\mathfrak{a}}+j,b_{\mathfrak{b}}+i),$$

for all $\sigma_{\mathfrak{a}}\leqslant i\leqslant r_{\mathfrak{a}},\sigma_{\mathfrak{b}}\leqslant j\leqslant r_{\mathfrak{b}}$. Hence $A_{i,j,n}=0$ for $n>\min(b_{\mathfrak{a}}+r_{\mathfrak{b}},b_{\mathfrak{b}}+r_{\mathfrak{a}})$. Note that the conditions $b_{\mathfrak{a}}+j-i\geqslant 0,b_{\mathfrak{b}}+i-j\geqslant 0$ implies that $A_{i,j,n}\geqslant 0$ for all n. Hence if for all $i,j,h_i(\mathfrak{a})\geqslant 0,h_j(\mathfrak{b})\geqslant 0$ then $h_n(\mathfrak{a}\underline{\otimes}\mathfrak{b})\geqslant 0$ for all n, and for $m:=\min(b_{\mathfrak{a}}+r_{\mathfrak{b}},b_{\mathfrak{b}}+r_{\mathfrak{a}})$ we have:

$$h_m(\mathfrak{a}\underline{\otimes}\mathfrak{b})=\sum_{i,j|A_{i,j,m}\neq 0}h_i(\mathfrak{a})h_j(\mathfrak{b})A_{i,j,m}>0.$$

(3) If $0\leqslant\sigma_{\mathfrak{a}}\leqslant r_{\mathfrak{a}}\leqslant b_{\mathfrak{a}}$ and $0\leqslant\sigma_{\mathfrak{b}}\leqslant r_{\mathfrak{b}}\leqslant b_{\mathfrak{b}}$ then $b_{\mathfrak{a}}+j\geqslant b_{\mathfrak{a}}\geqslant i$ and $b_{\mathfrak{b}}+i\geqslant b_{\mathfrak{b}}\geqslant j$. Therefore $b_{\mathfrak{a}}+j-i\geqslant 0$ and $b_{\mathfrak{b}}+i-j\geqslant 0$. Hence, the claim follows from the claim 2. □

Remark 3.4. *The bounds obtained are sharp.*

Lemma 3.5. *The following statements are equivalent:*

(1) For all $\sigma_{\mathfrak{a}} \leqslant i \leqslant r_{\mathfrak{a}}$ *and* $\sigma_{\mathfrak{b}} \leqslant j \leqslant r_{\mathfrak{b}}$*, we have*

$$b_{\mathfrak{b}} + i - j \geqslant 0 \text{ and } b_{\mathfrak{a}} + j - i \geqslant 0.$$

(2) $b_{\mathfrak{b}} + \sigma_{\mathfrak{a}} - r_{\mathfrak{b}} \geqslant 0$ *and* $b_{\mathfrak{a}} + \sigma_{\mathfrak{b}} - r_{\mathfrak{a}} \geqslant 0$. $(**)$

Proof. (2) $\Rightarrow$ (1). Take $i = \sigma_{\mathfrak{a}}, j = r_{\mathfrak{b}}$ in the first inequality and $j = \sigma_{\mathfrak{b}}, i = r_{\mathfrak{a}}$ in the second.

(1) $\Rightarrow$ (2). Let $\sigma_{\mathfrak{a}} \leqslant i \leqslant r_{\mathfrak{a}}, \sigma_{\mathfrak{b}} \leqslant j \leqslant r_{\mathfrak{b}}$, then

$$i + b_{\mathfrak{b}} - j \geqslant i + b_{\mathfrak{b}} - r_{\mathfrak{b}} \geqslant \sigma_{\mathfrak{a}} + b_{\mathfrak{b}} - r_{\mathfrak{b}} \geqslant 0 \text{ and } \; j + b_{\mathfrak{a}} - i \geqslant i + b_{\mathfrak{a}} - r_{\mathfrak{a}} \geqslant \sigma_{\mathfrak{b}} + b_{\mathfrak{a}} - r_{\mathfrak{a}} \geqslant 0.$$

□

Remark 3.6. *Suppose that* $M_{\mathfrak{a}}, M_{\mathfrak{b}}$ *are Cohen-Macaulay modules of dimensions* $d_{\mathfrak{a}} = b_{\mathfrak{a}}+1 \geq 2$, $d_{\mathfrak{b}} = b_{\mathfrak{b}} + 1 \geq 2$, *with Hilbert-Poincaré series* $\mathfrak{a}, \mathfrak{b}$. *Then the conditions* $b_{\mathfrak{b}} + \sigma_{\mathfrak{a}} - r_{\mathfrak{b}} \geqslant 0$ *and* $b_{\mathfrak{a}} + \sigma_{\mathfrak{b}} - r_{\mathfrak{a}} \geqslant 0$ *are equivalent to say that* $M_{\mathfrak{a}} \underline{\otimes} M_{\mathfrak{b}}$ *is a Cohen-Macaulay module by [G-W][Proposition (4.2.5)].*

Proposition 3.7. *Let consider* $\mathfrak{a}_1, \ldots, \mathfrak{a}_s$ *Laurent formal series satisfying (*)*

$$\mathfrak{a}_i = \frac{h(\mathfrak{a}_i)(t)}{(1-t)^{d_i}}, h(\mathfrak{a}_i)(t) \in \mathbb{C}[t, t^{-1}].$$

We set $r_i = \deg h(\mathfrak{a}_i)(t), \alpha_i = d_i - r_i, b_i = d_i - 1 \geqslant 0$ *and*

$$\mathfrak{a}_1 \underline{\otimes} \ldots \underline{\otimes} \mathfrak{a}_s = \frac{h(\mathfrak{a}_1 \underline{\otimes} \ldots \underline{\otimes} \mathfrak{a}_s)(t)}{(1-t)^{b_1 + \ldots + b_s + 1}}.$$

Then

(1) $\deg(h(\mathfrak{a}_1 \underline{\otimes} \ldots \underline{\otimes} \mathfrak{a}_s)) \leqslant (b_1 + \ldots + b_s + 1) - \min(\alpha_1, \ldots, \alpha_s)$.

(2) If the condition (**) of Lemma 3.5 is fulfilled for

$$\{\mathfrak{a}_1, \mathfrak{a}_2\}, \{\mathfrak{a}_1 \underline{\otimes} \mathfrak{a}_2, \mathfrak{a}_3\}, \ldots, \{\mathfrak{a}_1 \underline{\otimes} \ldots \underline{\otimes} \mathfrak{a}_{s-1}, \mathfrak{a}_s\}$$

and $h_k(\mathfrak{a}_i) \geqslant 0$ *for all* i *and* k*, then*

$$\deg(h(\mathfrak{a}_1 \underline{\otimes} \ldots \underline{\otimes} \mathfrak{a}_s)) = (b_1 + \ldots + b_s + 1) - \max(\alpha_1, \ldots, \alpha_s).$$

(3) If for all $i = 1, ..., s$, $0 \leq \sigma_i r_i < d_i$*, then the condition (**) of Lemma 3.5 is fulfilled for*

$$\{\mathfrak{a}_1, \mathfrak{a}_2\}, \{\mathfrak{a}_1 \underline{\otimes} \mathfrak{a}_2, \mathfrak{a}_3\}, \ldots, \{\mathfrak{a}_1 \underline{\otimes} \ldots \underline{\otimes} \mathfrak{a}_{s-1}, \mathfrak{a}_s\}.$$

Proof. (1) Note that

$$\max(b_1+r_2, b_2+r_1) = \max(b_1+b_2+1-\alpha_2, b_1+b_2+1-\alpha_1) = b_1+b_2+1-\min(\alpha_1,\alpha_2).$$

Now suppose $s \geqslant 3$, we prove the claim by induction. Assume that (1) is true for the case $s-1$:

$$\deg(h(\mathfrak{a}_1 \underline{\otimes} \ldots \underline{\otimes} \mathfrak{a}_{s-1})) \leqslant (b_1+\ldots+b_{s-1}+1) - \min(\alpha_1,\ldots,\alpha_{s-1}).$$

Then, by the Proposition 3.3

$$\begin{aligned}
\deg(h(\mathfrak{a}_1 \underline{\otimes} \ldots \underline{\otimes} \mathfrak{a}_{s-1}) \underline{\otimes} \mathfrak{a}_s)) &\leqslant \max(b_1+\ldots+b_{s-1}+r_s, b_s+\deg(\mathfrak{a}_1 \underline{\otimes} \ldots \underline{\otimes} \mathfrak{a}_{s-1}) \\
&\leqslant \max(b_1+\ldots+b_{s-1}+b_s+1-\alpha_s, b_1+\ldots+b_s+1-\min(\alpha_1,\ldots,\alpha_{s-1}) \\
&= b_1+\ldots+b_s+1+\max(-\alpha_s, -\min(\alpha_1,\ldots,\alpha_{s-1})) \\
&= b_1+\ldots+b_s+1-\min(\alpha_s, \min(\alpha_1,\ldots,\alpha_{s-1})) \\
&= b_1+\ldots+b_s+1-\min(\alpha_s, \min(\alpha_1,\ldots,\alpha_s))
\end{aligned}$$

(2) Since condition (**) is fulfilled, we can apply Proposition 3.3 and we get:

$$\begin{aligned}
\deg(\mathfrak{a}_1 \underline{\otimes} \mathfrak{a}_2) &\leqslant \min(b_1+r_2, b_2+r_1) = \min(b_1+b_2+1-\alpha_2, b_2+b_1+1-\alpha_1) \\
&= b_1+b_2+1+\min(-\alpha_1,-\alpha_2) = b_1+b_2+1-\max(\alpha_1,\alpha_2)
\end{aligned}$$

and we have equality if $h_k(\mathfrak{a}_i) \geqslant 0$, for all i,k. By induction hypothesis we assume that

$$\deg(\mathfrak{a}_1 \underline{\otimes} \ldots \underline{\otimes} \mathfrak{a}_{s-1}) \leqslant (b_1+\ldots+b_{s-1}+1-\max(\alpha_1,\ldots,\alpha_{s-1}) \quad (4)$$

and we have equality if $h_k(\mathfrak{a}_i) \geqslant 0$, for all i,k. Moreover by Proposition 3.3,the coefficients $h_k(\mathfrak{a}_1 \underline{\otimes} \ldots \underline{\otimes} \mathfrak{a}_{s-1})$ are $\geqslant 0$. On the other hand, condition (**) is fulfilled, so we can apply Proposition 3.3, we have:

$$\deg(\mathfrak{a}_1 \underline{\otimes} \ldots \underline{\otimes} \mathfrak{a}_{s-1}) \underline{\otimes} \mathfrak{a}_s)) \leqslant \min(b_1+\ldots+b_{s-1}+r_s, b_s+\deg(\mathfrak{a}_1 \underline{\otimes} \ldots \underline{\otimes} \mathfrak{a}_{s-1}) \quad (5),$$

where we have the equality if $h_k(\mathfrak{a}_s) \geqslant 0$ and $h_k(\mathfrak{a}_1 \underline{\otimes} \ldots \underline{\otimes} \mathfrak{a}_{s-1}) \geqslant 0$, which is true since by hypothesis $h_k(\mathfrak{a}_i) \geqslant 0$ for all i,k.

Using (4) in (5) we get

$$\begin{aligned}
\deg(h(\mathfrak{a}_1 \underline{\otimes} \ldots \underline{\otimes} \mathfrak{a}_s)) \leqslant& \\
&\min(b_1+\ldots+b_{s-1}+1-\alpha_s, b_s+b_1+\ldots+b_{s-1}+1-\max(\alpha_1,\alpha_{s-1})) \\
&= b_1+\ldots+b_s+1+\min(-\alpha_s, -\max(\alpha_1,\ldots,\alpha_{s-1})) \\
&= b_1+\ldots+b_s+1-\max(\alpha_s, \max(\alpha_1,\ldots,\alpha_{s-1})) \\
&= b_1+\ldots+b_s+1-\max(\alpha_s, \min(\alpha_1,\ldots,\alpha_s))
\end{aligned}$$

and we have the equality if $h_k(\mathfrak{a}_i) \geqslant 0$ for all i,k.

(3) The proof is immediate from Proposition 3.3. □

4 $h-$ vector of the Segre product of s power series

The proof of the following theorem is direct from 2.4 by using induction.

Theorem 4.1. *With the notations of Proposition 3.7.*

For $\sigma_s \leq i_s \leq b_1 + ... + b_s - \min\{\alpha_1, ..., \alpha_s\}$ *we have*

$$h_{i_s}(\mathfrak{a}_1 \underline{\otimes} ... \underline{\otimes} \mathfrak{a}_s) = \sum_{(i_1, i_2, ..., i_{s-1}, l_2, ..., l_s) \in \Delta} h_{i_1}(\mathfrak{a}_1) h_{l_2}(\mathfrak{a}_2) ... h_{l_s}(\mathfrak{a}_s) A_{i_1, l_2, i_2} ... A_{i_{s-1}, l_s, i_s}$$

where

$$\forall k = 2, ..., s;\ A_{i_{k-1}, l_k, i_k} = \binom{b_1 + ... + b_{k-1} + l_k - i_{k-1}}{i_k - i_{k-1}} \binom{b_k + i_{k-1} - l_k}{i_k - l_k},$$

and Δ *is defined by : for any* $\tau = 2, ..., s,$

$$\sigma_\tau \leq l_\tau \leq b_\tau + 1 - \alpha_\tau, \quad \sigma_{\tau-1} \leq i_{\tau-1} \leq \min\{b_1 + ... + b_{\tau-1} - \min\{\alpha_1, ..., \alpha_{\tau-1}\}, i_\tau\}.$$

There is an important corollary that will be used in [M1] to prove the conjecture by Simon Newcomb:

Theorem 4.2. *For* $j = 1, ..., n$, *let* S_j *be a polynomial ring over a field* K *in* $b_j + 1$ *variables,* $\mathfrak{a}_j$ *the Hilbert-Poincare series of* S_j, *that is* $\mathfrak{a}_j = \dfrac{1}{(1-t)^{b_j+1}}$ *then:*

For $k = 0, ..., b_1 + ... + b_n - \max\{b_1, ..., b_n\}$, *we have*

$$A([\mathbf{b}], k) = \sum_{(i_2, ..., i_{n-1}) \in \Delta} A_{i_2} A_{i_2, i_3} A_{i_3, i_4} ... A_{i_{n-1}, i_n},$$

where

$$i_n := k; A_{i_2} = \binom{b_1}{i_2} \binom{b_2}{i_2};\ \forall s = 2, ..., n-1,\ A_{i_s, i_{s+1}} = \binom{b_1 + ... + b_s - i_s}{i_{s+1} - i_s} \binom{b_{s+1} + i_s}{i_{s+1}} \geq 0$$

and Δ *is defined by : for any* $\tau = 2, ..., n-1$

$$0 \leq i_\tau \leq \min\{b_1 + ... + b_\tau - \max\{b_1, ..., b_\tau\}, i_{\tau+1}\}$$

Applying Proposition 3.7 to modules we get the following Theorem (note that this theorem can be proved easily by using [G-W], but our purpose is to prove it by using only elementary tools):

Theorem 4.3. *Let* $S_1, \ldots, S_s$ *be graded polynomial rings on disjoints sets of variables. For all* $i = 1, \ldots, s$, *let* M_i *be a graded finitely generated* S_i*-Cohen-Macaulay module. We assume that* $M_i = \oplus_{l \geqslant 0} M_{i,l}$ *as* S_i*-module. Let* $d_i = \dim M_i, b_i = d_i - 1 \geq 0$, $\alpha_i = d_i - \mathrm{reg}(M_i)$, *where*

$\operatorname{reg}(M_i)$ *is the Castelnuovo-Mumford regularity of* M_i. *If* $\operatorname{reg}(M_i) < d_i$, *for all* $i = 1, \ldots, s$ *then*

(1) $M_1 \underline{\otimes} \ldots \underline{\otimes} M_s$ *is a Cohen-Macaulay* $S_1 \underline{\otimes} \ldots \underline{\otimes} S_s$*-module.*

(2) $\operatorname{reg}(M_1 \underline{\otimes} \ldots \underline{\otimes} M_s) = (b_1 + \ldots + b_s + 1) - \max\{\alpha_1, \ldots \alpha_s\}$.

(3) *For* $n_i \in \mathbb{N}$, *let* $M_i^{<n_i>}$ *be the* n_i*-Veronese transform of* M_i, *then*

$$\operatorname{reg}(M_1^{<n_1>} \underline{\otimes} \ldots \underline{\otimes} M_s^{<n_s>}) = (b_1 + \ldots + b_s + 1) - \max\{\lceil \frac{\alpha_1}{n_1} \rceil, \ldots, \lceil \frac{\alpha_s}{n_s} \rceil\}.$$

Proof. We have that for all $i = 1, ..., s$, $0 \leq \sigma(M_i), \operatorname{reg}(M_i) < d_i$, then statement (3) of Proposition 3.7 implies that is a Cohen-Macaulay module by the Remark 3.6 and [G-W][Proposition (4.2.5)]. The second claim follows immediately from Proposition 3.7.

The third claim follows from the second and the fact that for all $i = 1, ..., s$, $\operatorname{reg}(M_i^{<n_i>}) = d_i - \lceil \frac{\alpha_i}{n_i} \rceil$, proved in[MD][Theorem 4.7]. □

If one of the modules has dimension 0, then we get the following Corollary of Theorem 2.2.

Theorem 4.4. *Let* $S_1, \ldots, S_s$ *be graded polynomial rings on disjoints of set of variables. For all* $i = 1, \ldots, s$, *let* M_i *be a graded finitely generated* S_i*-Cohen-Macaulay module of dimension* $d_i = \dim M_i$. *We assume that* $M_i = \oplus_{l \in \mathbb{Z}} M_{i,l}$ *as* S_i*-module, and there is an index* k *such that* $d_k = \dim M_k = 0$. *Then* $M_1 \underline{\otimes} \ldots \underline{\otimes} M_s$ *is a 0-dimensional Cohen-Macaulay* $S_1 \underline{\otimes} \ldots \underline{\otimes} S_s$*-module and* $\operatorname{reg}(M_1 \underline{\otimes} \ldots \underline{\otimes} M_s) = \min_{k | \dim M_k = 0} \operatorname{reg}(M_k)$.

To end this section we exhibit two large classes of ideals that satisfy the hypothesis of Theorem 4.3.

(I) Let $\mathbb{N}\mathcal{A}$ be a finite generated normal semigroup homogeneous, then by [St][13.14], we have that $\operatorname{reg}(K[\mathbb{N}\mathcal{A}]) < \dim(K[\mathbb{N}\mathcal{A}])$. Hence the toric ring $K[\mathbb{N}\mathcal{A}]$ satisfy the hypothesis of Theorem 4.3.

(II) Let Δ be a simplicial complex, and $K[\Delta]$ be the Stanley-Reisner ring associated to Δ. If $K[\Delta]$ is a Cohen-Macaulay ring, then by the main theorem of Reisner $\operatorname{reg}(K[\Delta]) < \dim(K[\Delta])$ if and only if Δ is acyclic.

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