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# TWO-ISOMETRIES AND de BRANGES-ROVNYAK SPACES

KARIM KELLAY, MOHAMED ZARRABI

ABSTRACT. We characterize the symbols of the de Branges–Rovnyak spaces for which the shift operator is concave or 2–isometry. As applications, we consider wandering  $z$ –invariant subspaces and equality between a de Branges–Rovnyak space and a Dirichlet-type space.

## 1. INTRODUCTION

Let  $\mathbb{D}$  be the open unit disc in the complex plane, and let  $\mathbb{T} = \partial\mathbb{D}$  be the unit circle. Denote  $H^\infty$  the space of bounded analytic function on  $\mathbb{D}$  and  $H^2$  the classical *Hardy space* of analytic functions on  $\mathbb{D}$  having square–summable Taylor coefficients at the origin. Let  $P_+$  be the orthogonal projection of  $L^2(\mathbb{T})$  onto  $H^2$ . For  $\varphi \in L^\infty(\mathbb{T})$ , the Toeplitz operator  $T_\varphi : H^2 \rightarrow H^2$  is given by

$$T_\varphi f := P_+(\varphi f), \quad f \in H^2.$$

Let  $b \in H^\infty$  such that  $\|b\|_\infty \leq 1$ , where  $\|\cdot\|_\infty$  is the supremum norm on  $\mathbb{D}$ . The de Brange–Rovnyak space  $\mathcal{H}(b)$  is the image of  $H^2$  under the operator  $(I - T_b T_{\bar{b}})^{1/2}$ , that is  $\mathcal{H}(b) = (I - T_b T_{\bar{b}})^{1/2}(H^2)$ , endowed with the usual inner product

$$\langle (I - T_b T_{\bar{b}})^{1/2}(f), (I - T_b T_{\bar{b}})^{1/2}(g) \rangle_b = \langle f, g \rangle_{H^2}, \quad f, g \in (\ker(I - T_b T_{\bar{b}})^{1/2})^\perp,$$

where  $\langle \cdot, \cdot \rangle_{H^2}$  is the inner product in  $H^2$  given by

$$\langle f, g \rangle_{H^2} = \frac{1}{2\pi} \int_{\mathbb{T}} f(\zeta) \overline{g(\zeta)} |d\zeta|.$$

It is well known that if  $\|b\|_\infty < 1$  then  $\mathcal{H}(b) = H^2$  and if  $b$  is an inner function, that is  $|b| = 1$  a.e. on  $\mathbb{T}$ , then  $\mathcal{H}(b) = H^2 \ominus bH^2$ , see [10]. We refer to [1, 2, 3, 4] for some recent results on these spaces. We denote by  $S$  the shift operator on  $H^2$ . Note that  $\mathcal{H}(b)$  is invariant by  $S$  if and only if  $b$  is nonextreme in the unit ball of  $H^\infty$  see [10, p. 23]. In this case we investigate here when the shift operator is concave or 2–isometry on the de Brange–Rovnyak spaces. A bounded operator  $T$  on a Hilbert space  $\mathcal{H}$  is called concave if

$$T^{*2}T^2 - 2T^*T + I \leq 0$$

and  $T$  is called 2–*isometry* if

$$T^{*2}T^2 - 2T^*T + I = 0.$$

Note that when

$$b(z) = \frac{c + \gamma z}{1 - \beta z}, \quad z \in \mathbb{D}, \tag{1}$$

where  $c, \gamma, \beta \in \mathbb{C}$  and  $|\beta| < 1$ , then we have by Lemma 6

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$$\|b\|_\infty \leq 1 \iff 2|\beta + \bar{c}\gamma| \leq 1 + |\beta|^2 - |c|^2 - |\gamma|^2$$

and in this case

$$b \text{ is nonextreme} \iff 1 + |\beta|^2 - |c|^2 - |\gamma|^2 > 0.$$

Now we state our main result.

**Theorem 1.** *Suppose that  $b$  is nonextreme in the unit ball of  $H^\infty$ . Then the restriction of the shift operator  $S|_{\mathcal{H}(b)}$  is concave if and only if  $b$  has the form given by (1).*

*Futhermore  $S|_{\mathcal{H}(b)}$  is 2-isometry if and only if  $b$  is given by (1) where  $c, \gamma, \beta \in \mathbb{C}$  are such that  $1 + |\beta|^2 - |c|^2 - |\gamma|^2 = 2|\beta + \bar{c}\gamma|$ .*

Recall that  $\mathcal{H}(b)$  is invariant by  $S|_{\mathcal{H}(b)}$  only in the case where  $b$  is nonextreme. Note also that  $S|_{\mathcal{H}(b)}$  is an isometry if and only if  $b = c$  for some constant with  $|c| < 1$  and in this case  $\mathcal{H}(b) = H^2$ .

The invariant subspaces lattice of the restriction of the shift operator  $S|_{\mathcal{H}(b)}$ , when  $b$  is non extreme, is not known [10, p.35]. Sarason has studied this question ([9]) only for an example, namely  $b(z) = (1 + z)/2$ . By Theorem 1 and Richter wandering subspace Theorem [6, Theorem 1] we have the following

**Corollary 2.** *Let  $b$  has the form given by (1), the closed invariant subspaces of  $S|_{\mathcal{H}(b)}$  are wandering that is*

$$M = \text{Clos}_{\mathcal{H}(b)} \left( \text{Span}\{z^n(M \ominus zM) : n \geq 0\} \right).$$

We do not know for which nonextreme functions  $b$  the  $z$ -closed invariant subspaces of  $\mathcal{H}(b)$  are wandering. Also we do not know for which  $z$ -closed invariant subspaces are singly generated.

N. Chevrot, D. Guillot, T. Ransford in [3] investigate for which  $b$  the de Branges–Rovnyak space  $\mathcal{H}(b)$  coincide with some Dirichlet–type space with equality of norms. As consequence of our main theorem we obtain their Theorem [3, Theorem 3.1], see Corollary 3 below. Let us now introduce the Dirichlet type spaces. If  $\mu$  is a finite positive measure on  $\mathbb{T}$ , the *Dirichlet-type space*  $\mathcal{D}(\mu)$  is the set of analytic functions  $f \in H^2$ , such that

$$\mathcal{D}_\mu(f) := \int_{\mathbb{D}} |f'(z)|^2 P_\mu(z) dA(z) < \infty,$$

where  $dA(z) = dx dy / \pi$  stands for the normalized area measure in  $\mathbb{D}$  and  $P_\mu$  is the Poisson integral of  $\mu$

$$P_\mu(z) := \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} d\mu(\zeta), \quad z \in \mathbb{D}.$$

The space  $\mathcal{D}(\mu)$  is endowed with the norm

$$\|f\|_{\mathcal{D}_\mu}^2 := \|f\|_{H^2}^2 + \mathcal{D}_\mu(f).$$

Note that if  $\mu = 0$ ,  $\mathcal{D}(\mu) = H^2$  and if  $\mu$  is the Lebesgue measure on  $\mathbb{T}$ ,  $P_\mu = 1$  and then  $\mathcal{D}(\mu)$  is the classical *Dirichlet space*, see [5]. These spaces were introduced by Richter [7] by considering the 2-isometries on Hilbert spaces. A bounded operator  $T$  on a Hilbert space  $\mathcal{H}$  is called *analytic* if  $\bigcap_{n \geq 0} T^n \mathcal{H} = \{0\}$ . Note that Richter proved in [7] that for every *cyclic, analytic* and *2-isometry* operator  $T$  on a Hilbert space, there exists a unique finite measure

$\mu$  on  $\mathbb{T}$  such that  $T$  is unitarily equivalent to  $S|\mathcal{D}_\mu$ . We deduce from Theorem 1 the following Corollary

**Corollary 3.** [Chevrot, Guillot, Ransford [3]] *Let  $\mu$  be a finite positive measure on  $\mathbb{T}$  and let  $b$  be in the unit ball of  $H^\infty$ . Then  $\mathcal{H}(b) = \mathcal{D}(\mu)$  with equality of norms, if and only if*

$$\mu = c\delta_\lambda \quad \text{and} \quad b(z) = \frac{\gamma z}{1 - \beta z},$$

where  $\beta \neq 0$ ,  $|\beta| + |\gamma| = 1$ ,  $c = |\gamma|^2/|\beta|$  and  $\lambda = \bar{\beta}/|\beta|$ .

## 2. PROOFS

To give the proofs, we need some additional notations and properties. The space  $\mathcal{H}(b)$  is a reproducing kernel space :

$$f(w) = \langle f, k_w^b \rangle_b, \quad w \in \mathbb{D}, \quad f \in \mathcal{H}(b),$$

where

$$k_w^b(z) := \frac{1 - \overline{b(w)}b(z)}{1 - \overline{w}z}, \quad w, z \in \mathbb{D}.$$

Let  $S^*$  be the backward shift operator on  $H^2$ ,

$$S^*f(z) = \frac{f(z) - f(0)}{z}, \quad f \in H^2.$$

The space  $\mathcal{H}(b)$  is always invariant by  $S^*$  [10, p. 11]. We briefly recall some basic notions about the de Brange–Rovnyak space due to Sarason.

**Proposition 4.** *Let  $b$  be in the unit ball of  $H^\infty$ . The following properties are equivalent :*

- (1)  $b \in \mathcal{H}(b)$
- (2)  $b$  is nonextreme
- (3)  $\log(1 - |b|) \in L^1(\mathbb{T})$
- (4)  $\mathcal{H}(b)$  is invariant by  $S$
- (5) The polynomials are dense in  $\mathcal{H}(b)$ .

*Proof.* see [10, p. 24-25-26]. □

Let  $b$  be a nonextreme function in the unit ball of  $H^\infty$ . Then there exists a unique outer function  $a$  such that  $a(0) > 0$  and

$$|a(\zeta)|^2 + |b(\zeta)|^2 = 1 \quad \text{a.e. } \zeta \in \mathbb{T}.$$

We set  $\mathcal{M}(\bar{a}) = T_{\bar{a}}H^2$ . Since  $b$  is nonextreme

$$f \in \mathcal{H}(b) \iff T_{\bar{b}}f \in \mathcal{M}(\bar{a})$$

and  $\mathcal{M}(\bar{a})$  is dense in  $\mathcal{H}(b)$  (see [10, p. 24–25]). Moreover  $T_{\bar{a}}$  is one-to-one, thus for every  $f \in \mathcal{H}(b)$  there is a unique function  $f^+$  such that

$$T_{\bar{b}}f = T_{\bar{a}}f^+.$$

Note that for  $f, g \in \mathcal{H}(b)$ ,

$$\langle f, g \rangle_b = \langle f, g \rangle_{H^2} + \langle f^+, g^+ \rangle_{H^2}. \quad (2)$$

**Lemma 5.** *Suppose that  $b$  is nonextreme, then*

$$\langle b, S^*b \rangle_b = -\frac{\overline{a'(0)}}{a(0)}.$$

*Proof.* We have  $T_{\bar{b}}b = T_{\bar{a}}((1/\overline{a(0)}) - a)$  and  $T_{\bar{b}}S^*b = -T_{\bar{a}}S^*a$ , hence

$$b^+ = \frac{1}{a(0)} - a, \quad (S^*b)^+ = -S^*a.$$

So by (2)

$$\begin{aligned} \langle b, S^*b \rangle_b &= \langle b, S^*b \rangle_{H^2} + \langle b^+, (S^*b)^+ \rangle_{H^2} \\ &= \langle b, S^*b \rangle_{H^2} + \langle a, S^*a \rangle_{H^2} - \langle 1/\overline{a(0)}, S^*a \rangle_{H^2} \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} \zeta (|b(\zeta)|^2 + |a(\zeta)|^2) |d\zeta| - \frac{\overline{a'(0)}}{a(0)} \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} \zeta d|\zeta| - \frac{\overline{a'(0)}}{a(0)} = -\frac{\overline{a'(0)}}{a(0)}. \end{aligned}$$

□

**Lemma 6.** *Let  $b(z) = (c + \gamma z)/(1 - \beta z)$ ,  $z \in \mathbb{D}$ , where  $c, \gamma, \beta \in \mathbb{C}$ ,  $|\beta| < 1$ . Then*

$$(a) \quad \|b\|_{\infty} \leq 1 \iff 2|\beta + \bar{c}\gamma| \leq 1 + |\beta|^2 - |c|^2 - |\gamma|^2.$$

(b) *If  $\|b\|_{\infty} \leq 1$ , then*

$$b \text{ is nonextreme} \iff 1 + |\beta|^2 - |c|^2 - |\gamma|^2 > 0.$$

*In this case we have*

$$a(z) = \frac{\rho - \sigma z}{1 - \beta z}, \quad z \in \mathbb{D},$$

where  $\rho$  and  $\sigma$  are defined by the following

- (i)  $\rho \geq |\sigma|$ ,
- (ii)  $\rho^2 + |\sigma|^2 = 1 + |\beta|^2 - |c|^2 - |\gamma|^2$  and  $\rho^2|\sigma|^2 = |\beta + \bar{c}\gamma|^2$ ,
- (iii)  $\arg(\sigma) = \arg(\beta + \bar{c}\gamma)$ .

*Proof.* (a) We have

$$\begin{aligned} \|b\|_{\infty} \leq 1 &\iff |c + \gamma\zeta|^2 \leq |1 - \beta\zeta|^2, \quad \forall \zeta \in \mathbb{T}, \\ &\iff |c|^2 + |\gamma|^2 + 2\operatorname{Re}(\bar{c}\gamma\zeta) \leq 1 + |\beta|^2 - 2\operatorname{Re}(\beta\zeta), \quad \forall \zeta \in \mathbb{T}, \\ &\iff 2\operatorname{Re}((\beta + \bar{c}\gamma)\zeta) \leq 1 + |\beta|^2 - |c|^2 - |\gamma|^2, \quad \forall \zeta \in \mathbb{T}, \\ &\iff 2|\beta + \bar{c}\gamma| \leq 1 + |\beta|^2 - |c|^2 - |\gamma|^2. \end{aligned}$$

(b) Assume that  $\|b\|_{\infty} \leq 1$ . So there exist  $\rho$  and  $\sigma$  satisfying (i), (ii) and (iii). In fact  $\rho^2$  and  $|\sigma|^2$  are the solutions of the equation  $x^2 - (1 + |\beta|^2 - |c|^2 - |\gamma|^2)x + |\beta + \bar{c}\gamma|^2 = 0$ . For every  $\zeta \in \mathbb{T}$ ,

$$\begin{aligned} |(1 - \beta\zeta)|^2(1 - |b(\zeta)|^2) &= |1 - \beta\zeta|^2 - |c + \gamma\zeta|^2 \\ &= 1 + |\beta|^2 - |c|^2 - |\gamma|^2 - 2\operatorname{Re}((\beta + \bar{c}\gamma)\zeta) \\ &= \rho^2 + |\sigma|^2 - 2\operatorname{Re}(\rho\sigma\zeta) \\ &= |\rho - \sigma\zeta|^2. \end{aligned}$$

Recall that  $|\beta| < 1$ . Then we have

$$\begin{aligned} \log(1 - |b|^2) \in L^1(\mathbb{T}) &\iff \int_{\mathbb{T}} \log(|1 - \beta\zeta|^2(1 - |b(\zeta)|^2)) |d\zeta| > -\infty \\ &\iff \int_{\mathbb{T}} \log |\rho - \sigma\zeta|^2 |d\zeta| > -\infty. \end{aligned}$$

Since  $\rho \geq |\sigma|$ ,

$$\begin{aligned} \int_{\mathbb{T}} \log |\rho - \sigma\zeta|^2 |d\zeta| > -\infty &\iff \rho > 0 \\ &\iff 1 + |\beta|^2 - |c|^2 - |\gamma|^2 > 0, \end{aligned}$$

which proves the first part of (b). Assume now that  $b$  is nonextreme and let  $a$  be the outer function associated to  $b$ . Since the two outer functions  $(1 - \beta z)a(z)$  and  $\rho - \sigma z$  take positive values at  $z = 0$  and

$$|(1 - \beta\zeta)a(\zeta)|^2 = |(1 - \beta\zeta)|^2(1 - |b(\zeta)|^2) = |\rho - \sigma\zeta|^2, \quad \text{a.e } \zeta \in \mathbb{T},$$

we get  $(1 - \beta z)a(z) = \rho - \sigma z$  on  $\mathbb{D}$ . □

**Remarks.** Let  $b(z) = (c + \gamma z)/(1 - \beta z)$ ,  $z \in \mathbb{D}$ , with  $|\beta| < 1$ .

**1.**  $b(\mathbb{T})$  is the circle of radius  $|\gamma + \beta c|/(1 - |\beta|^2)$  centered at the point  $(c + \gamma\bar{\beta})/(1 - |\beta|^2)$ . So

$$\|b\|_{\infty} = \frac{|c + \gamma\bar{\beta}|}{1 - |\beta|^2} + \frac{|\gamma + \beta c|}{1 - |\beta|^2}.$$

**2.** Suppose  $\|b\|_{\infty} \leq 1$ . It follows from Lemma 6 that  $b$  is extreme if and only if  $1 + |\beta|^2 - |c|^2 - |\gamma|^2 = |\beta + \bar{c}\gamma| = 0$  which is equivalent to  $b(z) = e^{i\theta}(\bar{\beta} - z)/(1 - \beta z)$  for some  $\theta \in \mathbb{R}$ .

For the proof of Corollary 3, we need the following lemma.

**Lemma 7.** Let  $\mu$  be a finite positive measure on  $\mathbb{T}$  and let  $b \in H^{\infty}$  be a nonextreme function in the unit ball of  $H^{\infty}$ . Let  $k_w(z) = 1/(1 - \bar{w}z)$ ,  $w, z \in \mathbb{D}$ . Then

- (i)  $\langle k_z, k_w \rangle_{\mathcal{D}(\mu)} = \left(1 + \bar{z}w \int_{\mathbb{T}} \frac{d\mu(\zeta)}{(1 - \bar{z}\zeta)(1 - w\bar{\zeta})}\right) k_z(w)$ .
- (ii)  $\langle k_z, k_w \rangle_{\mathcal{H}(b)} = \left(1 + \frac{\overline{b(z)b(w)}}{a(z)a(w)}\right) k_z(w)$ .
- (iii)  $\text{Span}\{k_w, w \in \mathbb{D}\}$  is dense in  $\mathcal{D}(\mu)$ .
- (iv)  $\text{Span}\{k_w, w \in \mathbb{D}\}$  is dense in  $\mathcal{H}(b)$ .

*Proof.* By Douglas formula, for every  $f, g \in \mathcal{D}(\mu)$  we have

$$\langle f, g \rangle_{\mathcal{D}(\mu)} = \langle f, g \rangle_{H^2} + \frac{1}{2\pi} \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{(f(\zeta) - f(\zeta'))(\overline{g(\zeta) - g(\zeta')})}{|\zeta - \zeta'|^2} |d\zeta| d\mu(\zeta'),$$

(see [7]). Using now Cauchy formula we get (i). For the proof of (ii) see [10, p. 32].

Now we proof (iii). Let  $f \in \mathcal{D}(\mu)$  and set  $f_r(z) = f(rz)$ ,  $0 < r < 1$ . Note that

$$\|f_{r^2} - f\|_{\mathcal{D}(\mu)} \rightarrow 0, \quad \text{as } r \rightarrow 1-,$$

(see [8, Theorem 5.2] and [10, Corollary]). Since the map  $\zeta \rightarrow f_r(\zeta)k_{r\zeta}$  is continuous from  $\mathbb{T}$  to  $\mathcal{D}(\mu)$  and

$$f_{r^2} = \frac{1}{2\pi} \int_{\mathbb{T}} f(r\zeta)k_{r\zeta} |d\zeta|,$$

we see that  $f_{r,2}$  can be approximated by finite combinations of the functions  $k_w$ . So  $f$  belong to the closure of  $\text{Span}\{k_w, w \in \mathbb{D}\}$ .

To prove (iv), let  $h \in \mathcal{H}(b)$ . We have

$$\langle f, k_w \rangle_b = h(w) + (b(w)/a(w))h^+(w) \quad (3)$$

(see [10, p. 32]).

Suppose that  $h \perp \text{Span}\{k_w : w \in \mathbb{D}\}$ , then by (3)  $h = -(b/a)h^+$ . So

$$(h^+/a) - \bar{a}h^+ = -\bar{b}h. \quad (4)$$

It follows that  $(h^+/a) - \bar{a}h^+ \in L^2(\mathbb{T})$  and then  $h^+/a \in L^2(\mathbb{T})$ . Since  $h^+/a$  belong to Smirnov class,  $h^+/a \in H^2$ . By (4) we get  $P_+\left(\frac{h^+}{a} - \bar{a}h^+\right) = -P_+(\bar{b}h) = -T_{\bar{b}}h$ . Since  $T_{\bar{b}}h = T_{\bar{a}}h^+$ , we obtain  $P_+\left(\frac{h^+}{a} - \bar{a}h^+\right) = -P_+(\bar{a}h^+)$ . Therefore  $P_+(h^+/a) = 0$  and then  $h^+ = 0$ .  $\square$

**Remark.** The polynomials are dense in  $\mathcal{D}(\mu)$  and in  $\mathcal{H}(b)$ , [7, 10]. We can deduce (iii) and (iv) from this fact. Note that for  $f \in \mathcal{H}(b)$  we have not in general  $\|f_r - f\|_{\mathcal{H}(b)} \rightarrow 0$  as  $r \rightarrow 1-$ , see [10, 3].

**Proof of Theorem 1.** We set  $T = S|\mathcal{H}(b)$  and  $X = S^*|\mathcal{H}(b)$ . We have

$$X^*(\cdot) = S(\cdot) - \langle \cdot, S^*b \rangle_b b \quad \text{and} \quad T = X^* + \langle \cdot, S^*b \rangle_b b,$$

see [10, p. 12]. Therefore

$$T^* = X + \langle \cdot, b \rangle_b S^*b = S^*|\mathcal{H}(b) + \langle \cdot, b \rangle_b S^*b.$$

So, for every  $h \in \mathcal{H}(b)$

$$\begin{aligned} T^*T(h) &= T^*(Sh) \\ &= S^*Sh + \langle Sh, b \rangle_b S^*b \\ &= h + \langle h, T^*b \rangle_b S^*b, \end{aligned} \quad (5)$$

and

$$\begin{aligned} T^{*2}T^2(h) &= T^{*2}(S^2h) \\ &= T^*(S^*S^2h + \langle S^2h, b \rangle_b S^*b) \\ &= T^*(Sh + \langle h, T^{*2}b \rangle_b S^*b) \\ &= S^*Sh + \langle h, T^{*2}b \rangle_b S^{*2}b + \left\langle Sh + \langle h, T^{*2}b \rangle_b S^*b, b \right\rangle_b S^*b \\ &= h + \langle h, T^{*2}b \rangle_b S^{*2}b + \left\langle h, T^*b + \langle b, S^*b \rangle_b T^{*2}b \right\rangle_b S^*b. \end{aligned} \quad (6)$$

By (5) and (6), we get

$$(T^{*2}T^2 - 2T^*T + I)(h) = \langle h, T^{*2}b \rangle_b S^{*2}b + \left\langle h, \langle b, S^*b \rangle_b T^{*2}b - T^*b \right\rangle_b S^*b. \quad (7)$$

Moreover, we have

$$T^*b = S^*b + \langle b, b \rangle_b S^*b = (1 + \|b\|_b^2)S^*b \quad (8)$$

and

$$T^{*2}b = (1 + \|b\|_b^2)(S^{*2}b + \langle S^*b, b \rangle_b S^*b). \quad (9)$$

So by (7), (8) and (9),  $T$  is concave if and only if

$$\begin{aligned} \langle (T^{*2}T^2 - 2T^*T + I)h, h \rangle_b &= (1 + \|b\|_b^2) \times \\ &\left[ |\langle h, S^{*2}b \rangle_b|^2 + 2\operatorname{Re} \left( \langle h, S^*b \rangle_b \overline{\langle h, S^{*2}b \rangle_b} \langle b, S^*b \rangle_b \right) - \left( 1 - |\langle b, S^*b \rangle_b|^2 \right) |\langle h, S^*b \rangle_b|^2 \right] \leq 0. \end{aligned} \quad (10)$$

If  $h \in \{S^*b\}^\perp$ , it follows from (10),  $|\langle h, S^{*2}b \rangle_b|^2 \leq 0$  and then  $h \in \{S^{*2}b\}^\perp$ , where  $\{f\}^\perp$  stands for the set of all functions that are orthogonal to  $f$ . So  $\{S^*b\}^\perp \subset \{S^{*2}b\}^\perp$  and then  $S^{*2}b = \beta S^*b$  for some  $\beta \in \mathbb{C}$ . So there exist  $\alpha, \beta \in \mathbb{C}$  such that

$$S^*b(z) = \frac{\alpha}{1 - \beta z}, \quad z \in \mathbb{D}.$$

Note that necessarily  $|\beta| < 1$  if  $S^*b \neq 0$  and we take  $\beta = 0$  if  $S^*b = 0$ . Then there exist  $c, \gamma, \beta \in \mathbb{C}$  such that  $|\beta| < 1$  and

$$b(z) = \frac{c + \gamma z}{1 - \beta z}, \quad z \in \mathbb{D}.$$

We recall that the complex numbers  $c, \gamma$  and  $\beta$  satisfy the conditions found in Lemma 6 corresponding to the facts that  $\|b\|_\infty \leq 1$  and  $b$  nonextreme.

By (9), we have  $T^{*2}b = (1 + \|b\|_b^2) \left( \beta - \frac{a'(0)}{a(0)} \right) S^*b$ . Thus by (10) we get

$$(T^{*2}T^2 - 2T^*T + I)(h) = (1 + \|b\|_b^2) \left\langle h, \left[ \left| \beta - \frac{a'(0)}{a(0)} \right|^2 - 1 \right] S^*b \right\rangle_b S^*b.$$

So  $T$  is concave if and only if  $S^*b = 0$ , that is  $b$  is constant function, or

$$\left| \beta - \frac{a'(0)}{a(0)} \right|^2 - 1 \leq 0. \quad (11)$$

Recall that

$$a(z) = \frac{\rho - \sigma z}{1 - \beta z}$$

where  $\rho$  and  $\sigma$  are given by Lemma 6. Note that  $a(0) = \rho > 0$  and  $a'(0) = -\sigma + \rho\beta$ . So (11) holds if and only if

$$\frac{|\sigma|}{\rho} \leq 1,$$

which is satisfied by Lemma 6.

On the other hand,  $T$  is 2-isometry if and only if  $b$  is constant, or

$$\left| \beta - \frac{a'(0)}{a(0)} \right|^2 - 1 = 0$$

It follows that  $T$  is 2-isometry if and only if  $\frac{|\sigma|}{\rho} = 1$ , which is equivalent to

$$1 + |\beta|^2 - |c|^2 - |\gamma|^2 = 2|\beta + \bar{c}\gamma|.$$

This completes the proof.



**Proof of Corollary 2.** We set  $T = S|\mathcal{H}(b)$ . Note that  $T$  is concave if and only if, for every  $f \in \mathcal{H}(b)$ ,

$$\|T^2 f\|^2 - 2\|Tf\| + \|f\|^2 \leq 0. \quad (12)$$

Let  $M$  be a  $z$ -closed invariant subspace of  $\mathcal{H}(b)$ . It follows from (12) that  $T|M$  is concave. Now the corollary follows from [6, Theorem 1].

**Proof of Corollary 3.** Suppose that  $\mathcal{H}(b) = \mathcal{D}(\mu)$  for some finite positive measure with equality of norms. Then  $b$  is nonextreme since  $\mathcal{H}(b)$  is invariant by  $S$ , see Proposition 4. Since  $S|\mathcal{D}(\mu)$  is 2-isometry,  $T$  is also 2-isometry and by Theorem 1,

$$b(z) = \frac{c + \gamma z}{1 - \beta z}, \quad z \in \mathbb{D},$$

where  $c, \gamma, \beta \in \mathbb{C}$  such that  $|\beta| < 1$  and  $1 + |\beta|^2 - |c|^2 - |\gamma|^2 = 2|\beta + \bar{c}\gamma| > 0$ .

Let  $k_w(z) = 1/(1 - \bar{w}z)$ . Assume now that  $\mathcal{H}(b) = \mathcal{D}(\mu)$  with equality of norms. Then

$$\|k_w\|_{\mathcal{D}(\mu)}^2 = \|k_w\|_b^2, \quad w \in \mathbb{D}. \quad (13)$$

By Lemma 7,

$$\|k_w\|_{\mathcal{D}(\mu)}^2 = \frac{1}{1 - |w|^2} \left( 1 + \frac{|w|^2}{1 - |w|^2} P_\mu(w) \right), \quad w \in \mathbb{D},$$

$$\|k_w\|_b^2 = \frac{1}{1 - |w|^2} \left( 1 + \frac{|b(w)|^2}{|a(w)|^2} \right), \quad w \in \mathbb{D}.$$

Using (13), we obtain  $1 = \|k_0\|_{\mathcal{D}(\mu)}^2 = \|k_0\|_b^2 = 1 + |b(0)|^2/|a(0)|^2$ , so  $b(0) = 0$ . Therefore  $b(z) = \gamma z/(1 - \beta z)$ , with  $\beta \neq 0$  and  $|\beta| + |\gamma| = 1$ . It follows from Lemma 6 that

$$a(z) = \frac{1}{|\beta|^{1/2}} \frac{|\beta| - \beta z}{1 - \beta z}, \quad z \in \mathbb{D}. \quad (14)$$

Again by (13)

$$P_\mu(w) = \frac{|\gamma|^2}{|\beta|} \frac{1 - |w|^2}{|1 - (\beta/|\beta|)w|^2}.$$

So  $\mu = (|\gamma|^2/|\beta|)\delta_\lambda$  where  $\lambda = \bar{\beta}/|\beta|$ .

Conversely, assume that  $b(z) = \gamma z/(1 - \beta z)$  with  $|\beta| + |\gamma| = 1$  and let  $\mu = (|\gamma|^2/|\beta|)\delta_\lambda$  where  $\lambda = \bar{\beta}/|\beta|$ . By (14) and Lemma 7,

$$\langle k_z, k_w \rangle_{\mathcal{D}(\mu)} = \left( 1 + \frac{|\gamma|^2 \bar{z}w}{(|\beta| - \bar{\beta}\bar{z})(|\beta| - \beta w)} \right) k_z(w) = \langle k_z, k_w \rangle_{\mathcal{H}(b)}.$$

Thus  $\langle f, g \rangle_{\mathcal{D}(\mu)} = \langle f, g \rangle_{\mathcal{H}(b)}$  for every  $f, g \in \text{Span}\{k_w, w \in \mathbb{D}\}$ . By (iii) and (iv) of Lemma 7, we have  $\mathcal{H}(b) = \mathcal{D}(\mu)$  with equality of norms.

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