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TWO-ISOMETRIES AND de BRANGES-ROVNYAK SPACES

KARIM KELLAY, MOHAMED ZARRABI

ABSTRACT. We characterize the symbols of the de Branges–Rovnyak spaces for which the shift operator is concave or 2–isometry. As applications, we consider wandering z –invariant subspaces and equality between a de Branges–Rovnyak space and a Dirichlet-type space.

1. INTRODUCTION

Let \mathbb{D} be the open unit disc in the complex plane, and let $\mathbb{T} = \partial\mathbb{D}$ be the unit circle. Denote H^∞ the space of bounded analytic function on \mathbb{D} and H^2 the classical *Hardy space* of analytic functions on \mathbb{D} having square–summable Taylor coefficients at the origin. Let P_+ be the orthogonal projection of $L^2(\mathbb{T})$ onto H^2 . For $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator $T_\varphi : H^2 \rightarrow H^2$ is given by

$$T_\varphi f := P_+(\varphi f), \quad f \in H^2.$$

Let $b \in H^\infty$ such that $\|b\|_\infty \leq 1$, where $\|\cdot\|_\infty$ is the supremum norm on \mathbb{D} . The de Brange–Rovnyak space $\mathcal{H}(b)$ is the image of H^2 under the operator $(I - T_b T_{\bar{b}})^{1/2}$, that is $\mathcal{H}(b) = (I - T_b T_{\bar{b}})^{1/2}(H^2)$, endowed with the usual inner product

$$\langle (I - T_b T_{\bar{b}})^{1/2}(f), (I - T_b T_{\bar{b}})^{1/2}(g) \rangle_b = \langle f, g \rangle_{H^2}, \quad f, g \in (\ker(I - T_b T_{\bar{b}})^{1/2})^\perp,$$

where $\langle \cdot, \cdot \rangle_{H^2}$ is the inner product in H^2 given by

$$\langle f, g \rangle_{H^2} = \frac{1}{2\pi} \int_{\mathbb{T}} f(\zeta) \overline{g(\zeta)} |d\zeta|.$$

It is well known that if $\|b\|_\infty < 1$ then $\mathcal{H}(b) = H^2$ and if b is an inner function, that is $|b| = 1$ a.e. on \mathbb{T} , then $\mathcal{H}(b) = H^2 \ominus bH^2$, see [10]. We refer to [1, 2, 3, 4] for some recent results on these spaces. We denote by S the shift operator on H^2 . Note that $\mathcal{H}(b)$ is invariant by S if and only if b is nonextreme in the unit ball of H^∞ see [10, p. 23]. In this case we investigate here when the shift operator is concave or 2–isometry on the de Brange–Rovnyak spaces. A bounded operator T on a Hilbert space \mathcal{H} is called concave if

$$T^{*2}T^2 - 2T^*T + I \leq 0$$

and T is called 2–*isometry* if

$$T^{*2}T^2 - 2T^*T + I = 0.$$

Note that when

$$b(z) = \frac{c + \gamma z}{1 - \beta z}, \quad z \in \mathbb{D}, \tag{1}$$

where $c, \gamma, \beta \in \mathbb{C}$ and $|\beta| < 1$, then we have by Lemma 6

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$$\|b\|_\infty \leq 1 \iff 2|\beta + \bar{c}\gamma| \leq 1 + |\beta|^2 - |c|^2 - |\gamma|^2$$

and in this case

$$b \text{ is nonextreme} \iff 1 + |\beta|^2 - |c|^2 - |\gamma|^2 > 0.$$

Now we state our main result.

Theorem 1. *Suppose that b is nonextreme in the unit ball of H^∞ . Then the restriction of the shift operator $S|_{\mathcal{H}(b)}$ is concave if and only if b has the form given by (1).*

Futhermore $S|_{\mathcal{H}(b)}$ is 2-isometry if and only if b is given by (1) where $c, \gamma, \beta \in \mathbb{C}$ are such that $1 + |\beta|^2 - |c|^2 - |\gamma|^2 = 2|\beta + \bar{c}\gamma|$.

Recall that $\mathcal{H}(b)$ is invariant by $S|_{\mathcal{H}(b)}$ only in the case where b is nonextreme. Note also that $S|_{\mathcal{H}(b)}$ is an isometry if and only if $b = c$ for some constant with $|c| < 1$ and in this case $\mathcal{H}(b) = H^2$.

The invariant subspaces lattice of the restriction of the shift operator $S|_{\mathcal{H}(b)}$, when b is non extreme, is not known [10, p.35]. Sarason has studied this question ([9]) only for an example, namely $b(z) = (1 + z)/2$. By Theorem 1 and Richter wandering subspace Theorem [6, Theorem 1] we have the following

Corollary 2. *Let b has the form given by (1), the closed invariant subspaces of $S|_{\mathcal{H}(b)}$ are wandering that is*

$$M = \text{Clos}_{\mathcal{H}(b)} \left(\text{Span}\{z^n(M \ominus zM) : n \geq 0\} \right).$$

We do not know for which nonextreme functions b the z -closed invariant subspaces of $\mathcal{H}(b)$ are wandering. Also we do not know for which z -closed invariant subspaces are singly generated.

N. Chevrot, D. Guillot, T. Ransford in [3] investigate for which b the de Branges–Rovnyak space $\mathcal{H}(b)$ coincide with some Dirichlet–type space with equality of norms. As consequence of our main theorem we obtain their Theorem [3, Theorem 3.1], see Corollary 3 below. Let us now introduce the Dirichlet type spaces. If μ is a finite positive measure on \mathbb{T} , the *Dirichlet-type space* $\mathcal{D}(\mu)$ is the set of analytic functions $f \in H^2$, such that

$$\mathcal{D}_\mu(f) := \int_{\mathbb{D}} |f'(z)|^2 P_\mu(z) dA(z) < \infty,$$

where $dA(z) = dx dy / \pi$ stands for the normalized area measure in \mathbb{D} and P_μ is the Poisson integral of μ

$$P_\mu(z) := \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} d\mu(\zeta), \quad z \in \mathbb{D}.$$

The space $\mathcal{D}(\mu)$ is endowed with the norm

$$\|f\|_{\mathcal{D}_\mu}^2 := \|f\|_{H^2}^2 + \mathcal{D}_\mu(f).$$

Note that if $\mu = 0$, $\mathcal{D}(\mu) = H^2$ and if μ is the Lebesgue measure on \mathbb{T} , $P_\mu = 1$ and then $\mathcal{D}(\mu)$ is the classical *Dirichlet space*, see [5]. These spaces were introduced by Richter [7] by considering the 2-isometries on Hilbert spaces. A bounded operator T on a Hilbert space \mathcal{H} is called *analytic* if $\bigcap_{n \geq 0} T^n \mathcal{H} = \{0\}$. Note that Richter proved in [7] that for every *cyclic, analytic* and *2-isometry* operator T on a Hilbert space, there exists a unique finite measure

μ on \mathbb{T} such that T is unitarily equivalent to $S|\mathcal{D}_\mu$. We deduce from Theorem 1 the following Corollary

Corollary 3. [Chevrot, Guillot, Ransford [3]] *Let μ be a finite positive measure on \mathbb{T} and let b be in the unit ball of H^∞ . Then $\mathcal{H}(b) = \mathcal{D}(\mu)$ with equality of norms, if and only if*

$$\mu = c\delta_\lambda \quad \text{and} \quad b(z) = \frac{\gamma z}{1 - \beta z},$$

where $\beta \neq 0$, $|\beta| + |\gamma| = 1$, $c = |\gamma|^2/|\beta|$ and $\lambda = \bar{\beta}/|\beta|$.

2. PROOFS

To give the proofs, we need some additional notations and properties. The space $\mathcal{H}(b)$ is a reproducing kernel space :

$$f(w) = \langle f, k_w^b \rangle_b, \quad w \in \mathbb{D}, \quad f \in \mathcal{H}(b),$$

where

$$k_w^b(z) := \frac{1 - \overline{b(w)}b(z)}{1 - \overline{w}z}, \quad w, z \in \mathbb{D}.$$

Let S^* be the backward shift operator on H^2 ,

$$S^*f(z) = \frac{f(z) - f(0)}{z}, \quad f \in H^2.$$

The space $\mathcal{H}(b)$ is always invariant by S^* [10, p. 11]. We briefly recall some basic notions about the de Brange–Rovnyak space due to Sarason.

Proposition 4. *Let b be in the unit ball of H^∞ . The following properties are equivalent :*

- (1) $b \in \mathcal{H}(b)$
- (2) b is nonextreme
- (3) $\log(1 - |b|) \in L^1(\mathbb{T})$
- (4) $\mathcal{H}(b)$ is invariant by S
- (5) The polynomials are dense in $\mathcal{H}(b)$.

Proof. see [10, p. 24-25-26]. □

Let b be a nonextreme function in the unit ball of H^∞ . Then there exists a unique outer function a such that $a(0) > 0$ and

$$|a(\zeta)|^2 + |b(\zeta)|^2 = 1 \quad \text{a.e. } \zeta \in \mathbb{T}.$$

We set $\mathcal{M}(\bar{a}) = T_{\bar{a}}H^2$. Since b is nonextreme

$$f \in \mathcal{H}(b) \iff T_{\bar{b}}f \in \mathcal{M}(\bar{a})$$

and $\mathcal{M}(\bar{a})$ is dense in $\mathcal{H}(b)$ (see [10, p. 24–25]). Moreover $T_{\bar{a}}$ is one-to-one, thus for every $f \in \mathcal{H}(b)$ there is a unique function f^+ such that

$$T_{\bar{b}}f = T_{\bar{a}}f^+.$$

Note that for $f, g \in \mathcal{H}(b)$,

$$\langle f, g \rangle_b = \langle f, g \rangle_{H^2} + \langle f^+, g^+ \rangle_{H^2}. \quad (2)$$

Lemma 5. *Suppose that b is nonextreme, then*

$$\langle b, S^*b \rangle_b = -\frac{\overline{a'(0)}}{a(0)}.$$

Proof. We have $T_{\bar{b}}b = T_{\bar{a}}((1/\overline{a(0)}) - a)$ and $T_{\bar{b}}S^*b = -T_{\bar{a}}S^*a$, hence

$$b^+ = \frac{1}{a(0)} - a, \quad (S^*b)^+ = -S^*a.$$

So by (2)

$$\begin{aligned} \langle b, S^*b \rangle_b &= \langle b, S^*b \rangle_{H^2} + \langle b^+, (S^*b)^+ \rangle_{H^2} \\ &= \langle b, S^*b \rangle_{H^2} + \langle a, S^*a \rangle_{H^2} - \langle 1/\overline{a(0)}, S^*a \rangle_{H^2} \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} \zeta (|b(\zeta)|^2 + |a(\zeta)|^2) |d\zeta| - \frac{\overline{a'(0)}}{a(0)} \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} \zeta d|\zeta| - \frac{\overline{a'(0)}}{a(0)} = -\frac{\overline{a'(0)}}{a(0)}. \end{aligned}$$

□

Lemma 6. *Let $b(z) = (c + \gamma z)/(1 - \beta z)$, $z \in \mathbb{D}$, where $c, \gamma, \beta \in \mathbb{C}$, $|\beta| < 1$. Then*

$$(a) \quad \|b\|_{\infty} \leq 1 \iff 2|\beta + \bar{c}\gamma| \leq 1 + |\beta|^2 - |c|^2 - |\gamma|^2.$$

(b) *If $\|b\|_{\infty} \leq 1$, then*

$$b \text{ is nonextreme} \iff 1 + |\beta|^2 - |c|^2 - |\gamma|^2 > 0.$$

In this case we have

$$a(z) = \frac{\rho - \sigma z}{1 - \beta z}, \quad z \in \mathbb{D},$$

where ρ and σ are defined by the following

- (i) $\rho \geq |\sigma|$,
- (ii) $\rho^2 + |\sigma|^2 = 1 + |\beta|^2 - |c|^2 - |\gamma|^2$ and $\rho^2|\sigma|^2 = |\beta + \bar{c}\gamma|^2$,
- (iii) $\arg(\sigma) = \arg(\beta + \bar{c}\gamma)$.

Proof. (a) We have

$$\begin{aligned} \|b\|_{\infty} \leq 1 &\iff |c + \gamma\zeta|^2 \leq |1 - \beta\zeta|^2, \quad \forall \zeta \in \mathbb{T}, \\ &\iff |c|^2 + |\gamma|^2 + 2\operatorname{Re}(\bar{c}\gamma\zeta) \leq 1 + |\beta|^2 - 2\operatorname{Re}(\beta\zeta), \quad \forall \zeta \in \mathbb{T}, \\ &\iff 2\operatorname{Re}((\beta + \bar{c}\gamma)\zeta) \leq 1 + |\beta|^2 - |c|^2 - |\gamma|^2, \quad \forall \zeta \in \mathbb{T}, \\ &\iff 2|\beta + \bar{c}\gamma| \leq 1 + |\beta|^2 - |c|^2 - |\gamma|^2. \end{aligned}$$

(b) Assume that $\|b\|_{\infty} \leq 1$. So there exist ρ and σ satisfying (i), (ii) and (iii). In fact ρ^2 and $|\sigma|^2$ are the solutions of the equation $x^2 - (1 + |\beta|^2 - |c|^2 - |\gamma|^2)x + |\beta + \bar{c}\gamma|^2 = 0$. For every $\zeta \in \mathbb{T}$,

$$\begin{aligned} |(1 - \beta\zeta)|^2(1 - |b(\zeta)|^2) &= |1 - \beta\zeta|^2 - |c + \gamma\zeta|^2 \\ &= 1 + |\beta|^2 - |c|^2 - |\gamma|^2 - 2\operatorname{Re}((\beta + \bar{c}\gamma)\zeta) \\ &= \rho^2 + |\sigma|^2 - 2\operatorname{Re}(\rho\sigma\zeta) \\ &= |\rho - \sigma\zeta|^2. \end{aligned}$$

Recall that $|\beta| < 1$. Then we have

$$\begin{aligned} \log(1 - |b|^2) \in L^1(\mathbb{T}) &\iff \int_{\mathbb{T}} \log(|1 - \beta\zeta|^2(1 - |b(\zeta)|^2)) |d\zeta| > -\infty \\ &\iff \int_{\mathbb{T}} \log |\rho - \sigma\zeta|^2 |d\zeta| > -\infty. \end{aligned}$$

Since $\rho \geq |\sigma|$,

$$\begin{aligned} \int_{\mathbb{T}} \log |\rho - \sigma\zeta|^2 |d\zeta| > -\infty &\iff \rho > 0 \\ &\iff 1 + |\beta|^2 - |c|^2 - |\gamma|^2 > 0, \end{aligned}$$

which proves the first part of (b). Assume now that b is nonextreme and let a be the outer function associated to b . Since the two outer functions $(1 - \beta z)a(z)$ and $\rho - \sigma z$ take positive values at $z = 0$ and

$$|(1 - \beta\zeta)a(\zeta)|^2 = |(1 - \beta\zeta)|^2(1 - |b(\zeta)|^2) = |\rho - \sigma\zeta|^2, \quad \text{a.e } \zeta \in \mathbb{T},$$

we get $(1 - \beta z)a(z) = \rho - \sigma z$ on \mathbb{D} . □

Remarks. Let $b(z) = (c + \gamma z)/(1 - \beta z)$, $z \in \mathbb{D}$, with $|\beta| < 1$.

1. $b(\mathbb{T})$ is the circle of radius $|\gamma + \beta c|/(1 - |\beta|^2)$ centered at the point $(c + \gamma\bar{\beta})/(1 - |\beta|^2)$. So

$$\|b\|_{\infty} = \frac{|c + \gamma\bar{\beta}|}{1 - |\beta|^2} + \frac{|\gamma + \beta c|}{1 - |\beta|^2}.$$

2. Suppose $\|b\|_{\infty} \leq 1$. It follows from Lemma 6 that b is extreme if and only if $1 + |\beta|^2 - |c|^2 - |\gamma|^2 = |\beta + \bar{c}\gamma| = 0$ which is equivalent to $b(z) = e^{i\theta}(\bar{\beta} - z)/(1 - \beta z)$ for some $\theta \in \mathbb{R}$.

For the proof of Corollary 3, we need the following lemma.

Lemma 7. Let μ be a finite positive measure on \mathbb{T} and let $b \in H^{\infty}$ be a nonextreme function in the unit ball of H^{∞} . Let $k_w(z) = 1/(1 - \bar{w}z)$, $w, z \in \mathbb{D}$. Then

- (i) $\langle k_z, k_w \rangle_{\mathcal{D}(\mu)} = \left(1 + \bar{z}w \int_{\mathbb{T}} \frac{d\mu(\zeta)}{(1 - \bar{z}\zeta)(1 - w\bar{\zeta})}\right) k_z(w)$.
- (ii) $\langle k_z, k_w \rangle_{\mathcal{H}(b)} = \left(1 + \frac{\overline{b(z)b(w)}}{a(z)a(w)}\right) k_z(w)$.
- (iii) $\text{Span}\{k_w, w \in \mathbb{D}\}$ is dense in $\mathcal{D}(\mu)$.
- (iv) $\text{Span}\{k_w, w \in \mathbb{D}\}$ is dense in $\mathcal{H}(b)$.

Proof. By Douglas formula, for every $f, g \in \mathcal{D}(\mu)$ we have

$$\langle f, g \rangle_{\mathcal{D}(\mu)} = \langle f, g \rangle_{H^2} + \frac{1}{2\pi} \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{(f(\zeta) - f(\zeta'))(\overline{g(\zeta) - g(\zeta')})}{|\zeta - \zeta'|^2} |d\zeta| d\mu(\zeta'),$$

(see [7]). Using now Cauchy formula we get (i). For the proof of (ii) see [10, p. 32].

Now we proof (iii). Let $f \in \mathcal{D}(\mu)$ and set $f_r(z) = f(rz)$, $0 < r < 1$. Note that

$$\|f_{r^2} - f\|_{\mathcal{D}(\mu)} \rightarrow 0, \quad \text{as } r \rightarrow 1-,$$

(see [8, Theorem 5.2] and [10, Corollary]). Since the map $\zeta \rightarrow f_r(\zeta)k_{r\zeta}$ is continuous from \mathbb{T} to $\mathcal{D}(\mu)$ and

$$f_{r^2} = \frac{1}{2\pi} \int_{\mathbb{T}} f(r\zeta)k_{r\zeta} |d\zeta|,$$

we see that $f_{r,2}$ can be approximated by finite combinations of the functions k_w . So f belong to the closure of $\text{Span}\{k_w, w \in \mathbb{D}\}$.

To prove (iv), let $h \in \mathcal{H}(b)$. We have

$$\langle f, k_w \rangle_b = h(w) + (b(w)/a(w))h^+(w) \quad (3)$$

(see [10, p. 32]).

Suppose that $h \perp \text{Span}\{k_w : w \in \mathbb{D}\}$, then by (3) $h = -(b/a)h^+$. So

$$(h^+/a) - \bar{a}h^+ = -\bar{b}h. \quad (4)$$

It follows that $(h^+/a) - \bar{a}h^+ \in L^2(\mathbb{T})$ and then $h^+/a \in L^2(\mathbb{T})$. Since h^+/a belong to Smirnov class, $h^+/a \in H^2$. By (4) we get $P_+\left(\frac{h^+}{a} - \bar{a}h^+\right) = -P_+(\bar{b}h) = -T_{\bar{b}}h$. Since $T_{\bar{b}}h = T_{\bar{a}}h^+$, we obtain $P_+\left(\frac{h^+}{a} - \bar{a}h^+\right) = -P_+(\bar{a}h^+)$. Therefore $P_+(h^+/a) = 0$ and then $h^+ = 0$. \square

Remark. The polynomials are dense in $\mathcal{D}(\mu)$ and in $\mathcal{H}(b)$, [7, 10]. We can deduce (iii) and (iv) from this fact. Note that for $f \in \mathcal{H}(b)$ we have not in general $\|f_r - f\|_{\mathcal{H}(b)} \rightarrow 0$ as $r \rightarrow 1-$, see [10, 3].

Proof of Theorem 1. We set $T = S|\mathcal{H}(b)$ and $X = S^*|\mathcal{H}(b)$. We have

$$X^*(\cdot) = S(\cdot) - \langle \cdot, S^*b \rangle_b b \quad \text{and} \quad T = X^* + \langle \cdot, S^*b \rangle_b b,$$

see [10, p. 12]. Therefore

$$T^* = X + \langle \cdot, b \rangle_b S^*b = S^*|\mathcal{H}(b) + \langle \cdot, b \rangle_b S^*b.$$

So, for every $h \in \mathcal{H}(b)$

$$\begin{aligned} T^*T(h) &= T^*(Sh) \\ &= S^*Sh + \langle Sh, b \rangle_b S^*b \\ &= h + \langle h, T^*b \rangle_b S^*b, \end{aligned} \quad (5)$$

and

$$\begin{aligned} T^{*2}T^2(h) &= T^{*2}(S^2h) \\ &= T^*(S^*S^2h + \langle S^2h, b \rangle_b S^*b) \\ &= T^*(Sh + \langle h, T^{*2}b \rangle_b S^*b) \\ &= S^*Sh + \langle h, T^{*2}b \rangle_b S^{*2}b + \left\langle Sh + \langle h, T^{*2}b \rangle_b S^*b, b \right\rangle_b S^*b \\ &= h + \langle h, T^{*2}b \rangle_b S^{*2}b + \left\langle h, T^*b + \langle b, S^*b \rangle_b T^{*2}b \right\rangle_b S^*b. \end{aligned} \quad (6)$$

By (5) and (6), we get

$$(T^{*2}T^2 - 2T^*T + I)(h) = \langle h, T^{*2}b \rangle_b S^{*2}b + \left\langle h, \langle b, S^*b \rangle_b T^{*2}b - T^*b \right\rangle_b S^*b. \quad (7)$$

Moreover, we have

$$T^*b = S^*b + \langle b, b \rangle_b S^*b = (1 + \|b\|_b^2)S^*b \quad (8)$$

and

$$T^{*2}b = (1 + \|b\|_b^2)(S^{*2}b + \langle S^*b, b \rangle_b S^*b). \quad (9)$$

So by (7), (8) and (9), T is concave if and only if

$$\begin{aligned} \langle (T^{*2}T^2 - 2T^*T + I)h, h \rangle_b &= (1 + \|b\|_b^2) \times \\ &\left[|\langle h, S^{*2}b \rangle_b|^2 + 2\operatorname{Re} \left(\langle h, S^*b \rangle_b \overline{\langle h, S^{*2}b \rangle_b} \langle b, S^*b \rangle_b \right) - \left(1 - |\langle b, S^*b \rangle_b|^2 \right) |\langle h, S^*b \rangle_b|^2 \right] \leq 0. \end{aligned} \quad (10)$$

If $h \in \{S^*b\}^\perp$, it follows from (10), $|\langle h, S^{*2}b \rangle_b|^2 \leq 0$ and then $h \in \{S^{*2}b\}^\perp$, where $\{f\}^\perp$ stands for the set of all functions that are orthogonal to f . So $\{S^*b\}^\perp \subset \{S^{*2}b\}^\perp$ and then $S^{*2}b = \beta S^*b$ for some $\beta \in \mathbb{C}$. So there exist $\alpha, \beta \in \mathbb{C}$ such that

$$S^*b(z) = \frac{\alpha}{1 - \beta z}, \quad z \in \mathbb{D}.$$

Note that necessarily $|\beta| < 1$ if $S^*b \neq 0$ and we take $\beta = 0$ if $S^*b = 0$. Then there exist $c, \gamma, \beta \in \mathbb{C}$ such that $|\beta| < 1$ and

$$b(z) = \frac{c + \gamma z}{1 - \beta z}, \quad z \in \mathbb{D}.$$

We recall that the complex numbers c, γ and β satisfy the conditions found in Lemma 6 corresponding to the facts that $\|b\|_\infty \leq 1$ and b nonextreme.

By (9), we have $T^{*2}b = (1 + \|b\|_b^2) \left(\beta - \frac{a'(0)}{a(0)} \right) S^*b$. Thus by (10) we get

$$(T^{*2}T^2 - 2T^*T + I)(h) = (1 + \|b\|_b^2) \left\langle h, \left[\left| \beta - \frac{a'(0)}{a(0)} \right|^2 - 1 \right] S^*b \right\rangle_b S^*b.$$

So T is concave if and only if $S^*b = 0$, that is b is constant function, or

$$\left| \beta - \frac{a'(0)}{a(0)} \right|^2 - 1 \leq 0. \quad (11)$$

Recall that

$$a(z) = \frac{\rho - \sigma z}{1 - \beta z}$$

where ρ and σ are given by Lemma 6. Note that $a(0) = \rho > 0$ and $a'(0) = -\sigma + \rho\beta$. So (11) holds if and only if

$$\frac{|\sigma|}{\rho} \leq 1,$$

which is satisfied by Lemma 6.

On the other hand, T is 2-isometry if and only if b is constant, or

$$\left| \beta - \frac{a'(0)}{a(0)} \right|^2 - 1 = 0$$

It follows that T is 2-isometry if and only if $\frac{|\sigma|}{\rho} = 1$, which is equivalent to

$$1 + |\beta|^2 - |c|^2 - |\gamma|^2 = 2|\beta + \bar{c}\gamma|.$$

This completes the proof.

Proof of Corollary 2. We set $T = S|_{\mathcal{H}(b)}$. Note that T is concave if and only if, for every $f \in \mathcal{H}(b)$,

$$\|T^2 f\|^2 - 2\|Tf\| + \|f\|^2 \leq 0. \quad (12)$$

Let M be a z -closed invariant subspace of $\mathcal{H}(b)$. It follows from (12) that $T|_M$ is concave. Now the corollary follows from [6, Theorem 1].

Proof of Corollary 3. Suppose that $\mathcal{H}(b) = \mathcal{D}(\mu)$ for some finite positive measure with equality of norms. Then b is nonextreme since $\mathcal{H}(b)$ is invariant by S , see Proposition 4. Since $S|_{\mathcal{D}(\mu)}$ is 2-isometry, T is also 2-isometry and by Theorem 1,

$$b(z) = \frac{c + \gamma z}{1 - \beta z}, \quad z \in \mathbb{D},$$

where $c, \gamma, \beta \in \mathbb{C}$ such that $|\beta| < 1$ and $1 + |\beta|^2 - |c|^2 - |\gamma|^2 = 2|\beta + \bar{c}\gamma| > 0$.

Let $k_w(z) = 1/(1 - \bar{w}z)$. Assume now that $\mathcal{H}(b) = \mathcal{D}(\mu)$ with equality of norms. Then

$$\|k_w\|_{\mathcal{D}(\mu)}^2 = \|k_w\|_b^2, \quad w \in \mathbb{D}. \quad (13)$$

By Lemma 7,

$$\|k_w\|_{\mathcal{D}(\mu)}^2 = \frac{1}{1 - |w|^2} \left(1 + \frac{|w|^2}{1 - |w|^2} P_\mu(w) \right), \quad w \in \mathbb{D},$$

$$\|k_w\|_b^2 = \frac{1}{1 - |w|^2} \left(1 + \frac{|b(w)|^2}{|a(w)|^2} \right), \quad w \in \mathbb{D}.$$

Using (13), we obtain $1 = \|k_0\|_{\mathcal{D}(\mu)}^2 = \|k_0\|_b^2 = 1 + |b(0)|^2/|a(0)|^2$, so $b(0) = 0$. Therefore $b(z) = \gamma z/(1 - \beta z)$, with $\beta \neq 0$ and $|\beta| + |\gamma| = 1$. It follows from Lemma 6 that

$$a(z) = \frac{1}{|\beta|^{1/2}} \frac{|\beta| - \beta z}{1 - \beta z}, \quad z \in \mathbb{D}. \quad (14)$$

Again by (13)

$$P_\mu(w) = \frac{|\gamma|^2}{|\beta|} \frac{1 - |w|^2}{|1 - (\beta/|\beta|)w|^2}.$$

So $\mu = (|\gamma|^2/|\beta|)\delta_\lambda$ where $\lambda = \bar{\beta}/|\beta|$.

Conversely, assume that $b(z) = \gamma z/(1 - \beta z)$ with $|\beta| + |\gamma| = 1$ and let $\mu = (|\gamma|^2/|\beta|)\delta_\lambda$ where $\lambda = \bar{\beta}/|\beta|$. By (14) and Lemma 7,

$$\langle k_z, k_w \rangle_{\mathcal{D}(\mu)} = \left(1 + \frac{|\gamma|^2 \bar{z}w}{(|\beta| - \bar{\beta}\bar{z})(|\beta| - \beta w)} \right) k_z(w) = \langle k_z, k_w \rangle_{\mathcal{H}(b)}.$$

Thus $\langle f, g \rangle_{\mathcal{D}(\mu)} = \langle f, g \rangle_{\mathcal{H}(b)}$ for every $f, g \in \text{Span}\{k_w, w \in \mathbb{D}\}$. By (iii) and (iv) of Lemma 7, we have $\mathcal{H}(b) = \mathcal{D}(\mu)$ with equality of norms.

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