Two-isometries and de Branges-Rovnyak spaces

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Abstract. We characterize the symbols of the de Branges–Rovnyak spaces for which the shift operator is 2–isometry in de Brange–Rovnyak. As consequence we prove by a different method a result of Chevrot, Guillot and Ransford, about equality between a de Branges–Rovnyak space and a Dirichlet-type space.

1. Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane, and let $\mathbb{T} = \partial\mathbb{D}$ be the unit circle. Denote H^∞ the space of bounded analytic function on $\mathbb{D}$ and H^2 the classical *Hardy space* of analytic functions on $\mathbb{D}$ having square–summable Taylor coefficients at the origin.

If μ is a finite positive measure on $\mathbb{T}$, *the Dirichlet-type space* $\mathcal{D}(\mu)$ is the set of analytic functions $f \in H^2$, such that

$$\mathcal{D}_\mu(f) := \int_{\mathbb{D}} |f'(z)|^2 P\mu(z) dA(z) < \infty,$$

where $dA(z) = dxdy/\pi$ stands for the normalized area measure in $\mathbb{D}$ and $P\mu$ is the Poisson integral of μ

$$P\mu(z) := \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu(\zeta), \qquad z \in \mathbb{D}.$$

The space $\mathcal{D}(\mu)$ is endowed with the norm

$$\|f\|^2_{\mathcal{D}_\mu} := \|f\|^2_{H^2} + \mathcal{D}_\mu(f).$$

Note that if $\mu = 0$, then $\mathcal{D}(\mu) = H^2$ and if μ is the Lebesgue measure on $\mathbb{T}$, then $P_\mu = 1$ and $\mathcal{D}(\mu)$ is the classical *Dirichlet space*. These spaces were introduced by Richter [3] by considering the 2–isometries on Hilbert spaces. Let T be a bounded operator in a Hilbert space $\mathcal{H}$. T is said 2*–isometry* if $T^{*2}T^2 - 2T^*T + I = 0$, T is *analytic* if $\bigcap_{n\geq 0} T^n\mathcal{H} = \{0\}$ and T is *cyclic* if $\text{span}\{T^n x,\ n \in \mathbb{N}\}$ is dense in $\mathcal{H}$ for some $x \in \mathcal{H}$.

Let S denote the shift operator on H^2,

$$S\ :\ f \to zf.$$

A typical example of 2–isometry is given by the operator $S|\mathcal{D}_\mu$, the restriction of S on $\mathcal{D}_\mu$. Richter gave the following representation for 2–isometries:

2000 *Mathematics Subject Classification.* Primary 46E22; Secondary 30H10.

Key words and phrases. Dirichlet space; de Branges–Rovnyak space; two–isometry.

Theorem (Richter [3]). *For every cyclic, analytic and 2–isometry operator T on a Hilbert space, there exists a unique finite measure μ on $\mathbb{T}$ such that T is unitarily equivalent to $S|\mathcal{D}_\mu$.*

In this paper we investigate when the shift operator is a 2–isometry on the de Brange–Rovnyak spaces. This allows us to find a different proof of N. Chevrot, D. Guillot, T. Ransford result where they give conditions to have equality between a de Branges–Rovnyak space and a Dirichlet–type space, see [1, 2].

Now we introduce the de Brange–Rovnyak spaces. Let P_+ be the orthogonal projection of $L^2(\mathbb{T})$ onto H^2. For $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator $T_\varphi : H^2 \to H^2$ is given by

$$T_\varphi f := P_+(\varphi f), \qquad f \in H^2.$$

Let $b \in H^\infty$ such that $\|b\|_\infty \leq 1$, where $\|\cdot\|_\infty$ is the supremum norm on $\mathbb{D}$. The de Brange–Rovnyak space $\mathcal{H}(b)$ is the image of H^2 under the operator $(I - T_b T_{\overline{b}})^{1/2}$, that is $\mathcal{H}(b) = (I - T_b T_{\overline{b}})^{1/2}(H^2)$, endowed with the inner product

$$\langle (I - T_b T_{\overline{b}})^{1/2}(f), (I - T_b T_{\overline{b}})^{1/2}(g)\rangle_b = \langle f, g\rangle_{H^2}, \qquad f, g \in (\ker(I - T_b T_{\overline{b}})^{1/2})^\perp.$$

where $\langle \cdot, \cdot \rangle_{H^2}$ is the inner product in H^2 given by

$$\langle f, g\rangle_{H^2} = \frac{1}{2\pi}\int_{\mathbb{T}} f(\zeta)\overline{g}(\zeta)|d\zeta|,$$

here we identify f and g with there non-tangential limits. Note that when $\|b\|_\infty < 1$ then $\mathcal{H}_b = H^2$ and when b is an inner function, that is $|b| = 1$ a.e. on $\mathbb{T}$, then $\mathcal{H}_b = H^2 \ominus bH^2$.

Let S be the shift operator on H^2. Note that $\mathcal{H}(b)$ is invariant by S if and only if b is nonextreme in the unit ball of H^∞ see [5, p. 23]. When b is nonextreme we set that

$$T = S|\mathcal{H}(b),$$

the restriction of S to the space $\mathcal{H}(b)$. Now we state our main result.

Theorem 1. *Suppose that b is nonextreme in the unit ball of H^∞. Then T is 2–isometry on $\mathcal{H}(b)$ if and only if $b = c$, where c is a constant such that $|c| < 1$ or*

$$b(z) = \frac{c + \gamma z}{1 - \beta z}, \qquad z \in \mathbb{D},$$

where $c, \gamma, \beta \in \mathbb{C}$ such that $|\beta| < 1$ and $1 + |\beta|^2 - |c|^2 - |\gamma|^2 = 2|\beta + \overline{c}\gamma| > 0$.

N. Chevrot, D. Guillot, T. Ransford in [1] investigate for which pairs (b, μ) the de Branges–Rovnyak space $\mathcal{H}_b$ is equal to the Dirichlet–type space $\mathcal{D}_\mu$ with equality of norms. As consequence of our main Theorem 1 we obtain their Theorem [1, Theorem 3.1].

Corollary 2 (Chevrot, Guillot, Ransford [1]). *Let μ be a finite positive measure on $\mathbb{T}$ and let b be in the unit ball of H^∞. Then $\mathcal{H}(b) = \mathcal{D}(\mu)$ with equality of norms, if and only if*

$$\mu = c\delta_\lambda \quad \text{and} \quad b(z) = \frac{\gamma z}{1 - \beta z},$$

where $\beta \neq 0$, $|\beta| + |\gamma| = 1$, $c = |\gamma|^2/|\beta|^2$ and $\lambda = \bar{\beta}/|\beta|$.

2. Proofs

To give the proofs, we need some additional notations and properties. The space $\mathcal{H}(b)$ is a reproducing kernel space :

$$f(w) = \langle f, k_w^b \rangle_b, \qquad w \in \mathbb{D},\ f \in \mathcal{H}(b),$$

where

$$k_w^b(z) := \frac{1 - \overline{b(w)}b(z)}{1 - \overline{w}z}, \qquad w, z \in \mathbb{D}.$$

Let S^* be the backward shift operator on H^2,

$$S^* f(z) = \frac{f(z) - f(0)}{z}, \qquad f \in H^2.$$

The space $\mathcal{H}(b)$ is always invariant by S^* [5, p. 11]. Note also that the following properties are equivalent [5, p. 24-25-26]

Proposition 3 (Sarason [5])**.** *Let b be in the unit ball of H^∞.*

(1) $b \in \mathcal{H}(b)$
(2) b is nonextreme
(3) $\log(1 - |b|) \in L^1(\mathbb{T})$
(4) $\mathcal{H}(b)$ is invariant by S
(5) The polynomials are dense in $\mathcal{H}(b)$.

Let b be nonextreme function in the unit ball of H^∞. Then there exists a unique outer function a such that $a(0) > 0$ and

$$|a(\zeta)|^2 + |b(\zeta)|^2 = 1 \quad \text{a.e. } \zeta \in \mathbb{T}.$$

We set $\mathcal{M}(\bar{a}) = T_{\bar{a}} H^2$. Since b is nonextreme

$$f \in \mathcal{H}(b) \iff T_{\bar{b}} f \in \mathcal{M}(\bar{a})$$

and $\mathcal{M}(\overline{a})$ is dense in $\mathcal{H}(b)$ (see [5, p. 24–25]). Moreover $T_{\bar{a}}$ is one–to–one, thus for every $f \in \mathcal{H}(b)$ there is a unique function f^+ such that

$$T_{\bar{b}} f = T_{\bar{a}} f^+.$$

Note that for $f, g \in \mathcal{H}(b)$,

$$\langle f, g \rangle_b = \langle f, g \rangle_{H^2} + \langle f^+, g^+ \rangle_{H^2}. \tag{1}$$

Lemma 4. *Suppose that b is nonextreme, then*

$$\langle b, S^* b \rangle_b = -\frac{\overline{a'(0)}}{\overline{a(0)}}.$$

Proof. We have $T_{\bar{b}} b = T_{\bar{a}}((1/\overline{a(0)}) - a)$ and $T_{\bar{b}} S^* b = -T_{\bar{a}} S^* a$, hence

$$b^+ = \frac{1}{\overline{a(0)}} - a, \qquad (S^* b)^+ = -S^* a.$$

So by (1)

$$\begin{aligned}\langle b, S^*b\rangle_b &= \langle b, S^*b\rangle_{H^2} + \langle b^+, (S^*b)^+\rangle_{H^2} \\ &= \langle b, S^*b\rangle_{H^2} + \langle a, S^*a\rangle_{H^2} - \langle 1/\overline{a(0)}, S^*a\rangle_{H^2} \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} \zeta(|b(\zeta)|^2 + |a(\zeta)|^2)|d\zeta| - \frac{\overline{a'(0)}}{\overline{a(0)}} \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} \zeta d|\zeta| - \frac{\overline{a'(0)}}{\overline{a(0)}} = -\frac{\overline{a'(0)}}{\overline{a(0)}}.\end{aligned}$$

□

Lemma 5. *Let $b(z) = (c+\gamma z)/(1-\beta z)$, $z \in \mathbb{D}$, where $c, \gamma, \beta \in \mathbb{C}$, $|\beta| < 1$. Then*

(a) $\|b\|_\infty \le 1 \iff 2|\beta + \bar{c}\gamma| \le 1 + |\beta|^2 - |c|^2 - |\gamma|^2$.
(b) If $\|b\|_\infty \le 1$, *then*

$$b \text{ is nonextreme} \iff 1 + |\beta|^2 - |c|^2 - |\gamma|^2 > 0.$$

In this case we have

$$a(z) = \frac{\rho - \sigma z}{1 - \beta z}, \qquad z \in \mathbb{D},$$

where ρ and σ are defined by the following

(i) $\rho \ge |\sigma|$,
(ii) $\rho^2 + |\sigma|^2 = 1 + |\beta|^2 - |c|^2 - |\gamma|^2$ *and* $\rho^2|\sigma|^2 = |\beta + \bar{c}\gamma|^2$,
(iii) $\arg(\sigma) = \arg(\beta + \bar{c}\gamma)$.

Proof. (a) We have

$$\begin{aligned}\|b\|_\infty \le 1 &\iff |c+\gamma\zeta|^2 \le |1-\beta\zeta|^2, \qquad \forall \zeta \in \mathbb{T}, \\ &\iff |c|^2 + |\gamma|^2 + 2\mathrm{Re}(\bar{c}\gamma\zeta) \le 1 + |\beta|^2 - 2\mathrm{Re}(\beta\zeta), \qquad \forall \zeta \in \mathbb{T}, \\ &\iff 2\mathrm{Re}((\beta + \bar{c}\gamma)\zeta) \le 1 + |\beta|^2 - |c|^2 - |\gamma|^2, \qquad \forall \zeta \in \mathbb{T}, \\ &\iff 2|\beta + \bar{c}\gamma| \le 1 + |\beta|^2 - |c|^2 - |\gamma|^2.\end{aligned}$$

(b) Assume that $\|b\|_\infty \le 1$. So there exist ρ and σ satisfying (i), (ii) and (iii). In fact ρ^2 and $|\sigma|^2$ are the solutions of the equation $x^2 - (1 + |\beta|^2 - |c|^2 - |\gamma|^2)x + |\beta + \bar{c}\gamma|^2 = 0$. For every $\zeta \in \mathbb{T}$,

$$\begin{aligned}|(1-\beta\zeta)|^2(1 - |b(\zeta)|^2) &= |1-\beta\zeta|^2 - |c+\gamma\zeta|^2| \\ &= 1 + |\beta|^2 - |c|^2 - |\gamma|^2 - 2\mathrm{Re}((\beta + \bar{c}\gamma)\zeta) \\ &= \rho^2 + |\sigma|^2 - 2\mathrm{Re}(\rho\sigma\zeta) \\ &= |\rho - \sigma\zeta|^2.\end{aligned}$$

Since $|\beta| < 1$, we have

$$\begin{aligned} \log(1-|b|^2) \in L^1(\mathbb{T}) &\iff \int_{\mathbb{T}} \log(|1-\beta\zeta|^2(1-|b(\zeta)|^2))|d\zeta| > -\infty \\ &\iff \int_{\mathbb{T}} \log|\rho-\sigma\zeta|^2|d\zeta| > -\infty, \end{aligned}$$

Since $\rho \geq |\sigma|$,

$$\begin{aligned} \int_{\mathbb{T}} \log|\rho-\sigma\zeta|^2|d\zeta| > -\infty &\iff \rho > 0 \\ &\iff 1+|\beta|^2-|c|^2-|\gamma|^2 > 0. \end{aligned}$$

which proves the first part of (b). Assume now that b is nonextreme and let a be the outer function associated to b. Since the two outer functions $(1-\beta z)a(z)$ and $\rho-\sigma z$ take positive values at $z=0$ and

$$|(1-\beta\zeta)a(\zeta)|^2 = |(1-\beta\zeta)|^2(1-|b(\zeta)|^2) = |\rho-\sigma\zeta|^2, \qquad \text{a.e } \zeta \in \mathbb{T},$$

we get $(1-\beta\zeta)a(z) = \rho-\sigma z$ on $\mathbb{D}$. □

Remarks. Let $b(z) = (c+\gamma z)/(1-\beta z)$, $z \in \mathbb{D}$, with $|\beta| < 1$.
1. $b(\mathbb{T})$ is the circle of radius $|\gamma+\beta c|/(1-|\beta|^2)$ centered at the point $(c+\gamma\bar{\beta})/(1-|\beta|^2)$. So

$$\|b\|_\infty = \frac{|c+\gamma\bar{\beta}|}{1-|\beta|^2} + \frac{|\gamma+\beta c|}{1-|\beta|^2}.$$

2. Suppose $\|b\|_\infty \leq 1$. It follows from Lemma 5 that b is extreme if and only if $1+|\beta|^2-|c|^2-|\gamma|^2 = |\beta+\bar{c}\gamma| = 0$ wich is equivalent to $b(z) = e^{i\theta}(\bar{\beta}-z)/(1-\beta z)$ for some $\theta \in \mathbb{R}$.

Proof of Theorem 1. We set $X = S^*|\mathcal{H}(b)$. We have

$$X^*(\cdot) = S(\cdot) - \langle \cdot, S^*b\rangle_b b \quad \text{ and } \quad T = X^* + \langle \cdot, S^*b\rangle_b b\,,$$

see [5, p. 12]. Therefore

$$T^* = X + \langle \cdot, b\rangle_b S^*b = S^*|\mathcal{H}(b) + \langle \cdot, b\rangle_b S^*b.$$

So, for every $h \in \mathcal{H}_b$

$$\begin{aligned} T^*T(h) &= T^*(Sh) \\ &= S^*Sh + \langle Sh, b\rangle_b S^*b \\ &= h + \langle h, T^*b\rangle_b S^*b, \end{aligned} \tag{2}$$

and

$$\begin{aligned} T^{*2}T^2(h) &= T^{*2}(S^2h) \\ &= T^*\big(S^*S^2h + \langle S^2h, b\rangle_b S^*b\big) \\ &= T^*\big(Sh + \langle h, T^{*2}b\rangle_b S^*b\big) \\ &= S^*Sh + \langle h, T^{*2}b\rangle_b S^{*2}b + \Big\langle Sh + \langle h, T^{*2}b\rangle_b S^*b, b\Big\rangle_b S^*b \\ &= h + \langle h, T^{*2}b\rangle_b S^{*2}b + \Big(\langle h, T^*b\rangle_b + \langle S^*b, b\rangle_b \langle h, T^{*2}b\rangle_b\Big) S^*b \\ &= h + \langle h, T^{*2}b\rangle_b S^{*2}b + \Big\langle h, T^*b + \langle b, S^*b\rangle_b T^{*2}b\Big\rangle_b S^*b. \end{aligned} \tag{3}$$

By (2) and (3), we get

$$(T^{*2}T^2 - 2T^*T + I)(h) = \langle h, T^{*2}b\rangle S^{*2}b + \Big\langle h, \langle b, S^*b\rangle_b T^{*2}b - T^*b\Big\rangle_b S^*b \tag{4}$$

Moreover, we have

$$T^*b = S^*b + \langle b, b\rangle_b S^*b = (1 + \|b\|_b^2) S^*b$$

and

$$T^{*2}b = (1 + \|b\|_b^2)\Big(S^{*2}b + \langle S^*b, b\rangle_b S^*b\Big). \tag{5}$$

Assume that T is 2–isometry. Then (4) and (5) imply that $S^*b = 0$ or be an eigenvector of S^*. Hence there exist $\alpha, \beta \in \mathbb{C}$ such that $|\beta| < 1$ and

$$S^*b(z) = \frac{\alpha}{1 - \beta z}, \qquad z \in \mathbb{D}.$$

Then there exist $c, \gamma \in \mathbb{C}$ such that

$$b(z) = \frac{c + \gamma z}{1 - \beta z}, \qquad z \in \mathbb{D}.$$

Note that the complex numbers c, γ, β satisfy the conditions found in Lemma 5 corresponding to the fact that $\|b\|_\infty \leq 1$ and b is nonextreme. We have $S^{*2}b = \beta S^*b$ and by Lemma 4, equality (4) become

$$(T^{*2}T^2 - 2T^*T + I)(h) = \Big\langle h, \overline{\beta} T^{*2}b - \frac{\overline{a'(0)}}{\overline{a(0)}} T^{*2}b - T^*b\Big\rangle_b S^*b. \tag{6}$$

By (5), we have $T^{*2}b = (1 + \|b\|_b^2)\big(\beta - \frac{a'(0)}{a(0)}\big) S^*b$. Thus equality (6) become

$$(T^{*2}T^2 - 2T^*T + I)(h) = (1 + \|b\|_b^2)\Big\langle h, \Big[\Big|\beta - \frac{a'(0)}{a(0)}\Big|^2 - 1\Big]\Big\rangle_b S^*b.$$

Since T is 2–isometry, we have $S^*b = 0$ (that is $b = c$ with $|c| < 1$) or

$$\Big|\beta - \frac{a'(0)}{a(0)}\Big| = 1. \tag{7}$$

Recall that

$$a(z) = \frac{\rho - \sigma z}{1 - \beta z}$$

where ρ and σ are given by Lemma 5. Note that $a(0) = \rho > 0$ and $a'(0) = -\sigma + \rho\beta$. So (7) holds if and only if

$$\frac{|\sigma|}{\rho} = 1 \iff 1 + |\beta|^2 - |c|^2 - |\gamma|^2 = 2|\beta + \bar{c}\gamma|.$$

Since b is nonextreme, $1 + |\beta|^2 - |c|^2 - |\gamma|^2 > 0$.

The converse is clear and now, the proof is complete.

For the proof of Corollary 2, we need the following lemma.

Lemma 6. *Let μ be a finite positive measure on $\mathbb{T}$ and let $b \in H^\infty$ be a nonextreme function in the unit ball of H^∞. Let $k_w(z) = 1/(1 - \bar{w}z)$, $w, z \in \mathbb{D}$. Then*

(i) $\langle k_z, k_w\rangle_{\mathcal{D}(\mu)} = \Big(1 + \bar{z}w \displaystyle\int_{\mathbb{T}} \frac{d\mu(\zeta)}{(1 - \bar{z}\zeta)(1 - w\bar{\zeta})}\Big) k_z(w).$

(ii) $\langle k_z, k_w\rangle_{\mathcal{H}(b)} = \Big(1 + \dfrac{\overline{b(z)}b(w)}{\overline{a(z)}a(w)}\Big) k_z(w).$

(iii) $\operatorname{span}\{k_w,\ w \in \mathbb{D}\}$ *is dense in* $\mathcal{D}(\mu)$.

(iv) $\operatorname{span}\{k_w,\ w \in \mathbb{D}\}$ *is dense in* $\mathcal{H}(b)$.

Proof. By Douglas formula, for every $f, g \in \mathcal{D}(\mu)$ we have

$$\langle f, g\rangle_{\mathcal{D}(\mu)} = \langle f, g\rangle_{H^2} + \frac{1}{2\pi} \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{(f(\zeta) - f(\zeta'))\overline{(g(\zeta) - g(\zeta'))}}{|\zeta - \zeta'|^2} |d\zeta| d\mu(\zeta'),$$

(see [3]). Using now Cauchy formula we get (i). For the proof of (ii) see [5, p. 32].

Now we proof (iii). Let $f \in \mathcal{D}(\mu)$ and set $f_r(z) = f(rz)$, $0 < r < 1$. Note that

$$\|f_{r^2} - f\|_{\mathcal{D}(\mu)} \to 0, \quad \text{as} \quad r \to 1-$$

(see [4, Theorem 5.2] and [5, Corollary]). Since the map $\zeta \to f_r(\zeta)k_{r\zeta}$ is continuous from $\mathbb{T}$ to $\mathcal{D}(\mu)$ and

$$f_{r^2} = \frac{1}{2\pi} \int_{\mathbb{T}} f(r\zeta) k_{r\zeta}\ |d\zeta|,$$

we see that f_{r^2} can be approximated by finite combinations of the functions k_w. So f belong to the closure of $\operatorname{span}\{k_w,\ w \in \mathbb{D}\}$.

To prove (iv), let $h \in \mathcal{H}(b)$. We have

$$\langle f, k_w\rangle_b = h(w) + (b(w)/a(w))h^+(w) \tag{8}$$

(see [5, p. 32]).

Suppose that $h \perp \operatorname{span}\{k_w : w \in \mathbb{D}\}$, then by (8)

$$h = -(b/a)h^+. \tag{9}$$

So $(h^+/a) - \bar{a}h^+ = \bar{b}h \in L^2(\mathbb{T})$ and then $h^+/a \in L^2(\mathbb{T})$. Since h^+/a belong to Smirnov class, $h^+/a \in H^2$. We know that $T_{\bar{b}}h = T_{\bar{a}}h^+$. Then using (9)

$$P_+\Big(\frac{|b|^2}{a}h\Big) = -T_{\bar{a}}h^+ \iff P_+\Big(\frac{1-|a|^2}{a}h^+\Big) = -T_{\bar{a}}h^+ \iff P_+\Big(\frac{h^+}{a} - \bar{a}h^+\Big) = -T_{\bar{a}}h^+.$$

Therefore $P_+(h^+/a) = 0$ and then $h^+ = 0$. We get from (8) that $h = 0$. □

Remark. The polynomials are dense in $\mathcal{D}(\mu)$ and in $\mathcal{H}(b)$, [3, 5]. We can deduce (iii) and (iv) from this fact. Note that for $f \in \mathcal{H}(b)$ we have not in general $\|f_r - f\|_{\mathcal{H}(b)} \to 0$ as $r \to 1-$, see [5, 1].

Proof of Corollary. Suppose that $\mathcal{H}(b) = \mathcal{D}(\mu)$ for some finite positive measure with equality of norms. Then b is nonextreme since $\mathcal{H}(b)$ is invariant by S, see Proposition 3. By Richter Theorem [3], T is 2–isometry and by Theorem 1

$$b(z) = \frac{c + \gamma z}{1 - \beta z}, \qquad z \in \mathbb{D},$$

where $c, \gamma, \beta \in \mathbb{C}$ such that $|\beta| < 1$ and $1 + |\beta|^2 - |c|^2 - |\gamma|^2 = 2|\beta + \overline{c}\gamma| > 0$.

Let $k_w(z) = 1/(1 - \bar{w}z)$. If $\mathcal{H}(b) = \mathcal{D}(\mu)$ with equality of norms then

$$\|k_w\|^2_{\mathcal{D}(\mu)} = \|k_w\|^2_b, \qquad w \in \mathbb{D}. \tag{10}$$

By Lemma 6,

$$\|k_w\|^2_{\mathcal{D}_\mu} = \frac{1}{1-|w|^2}\Big(1 + \frac{|w|^2}{1-|w|^2}P_\mu(w)\Big), \qquad w \in \mathbb{D},$$

$$\|k_w\|^2_b = \frac{1}{1-|w|^2}\Big(1 + \frac{|b(w)|^2}{|a(w)|^2}\Big), \qquad w \in \mathbb{D}.$$

By (10), we obtain $1 = \|k_0\|^2_{\mathcal{D}(\mu)} = \|k_0\|^2_b = 1 + |b(0)|^2/|a(0)|^2$, so $b(0) = 0$. Therefore $b(z) = \gamma z/(1 - \beta z)$, with $\beta \neq 0$ and $|\beta| + |\gamma| = 1$. It follows from Lemma 5 that

$$a(z) = \frac{|\beta| - \beta z}{1 - \beta z}, \quad z \in \mathbb{D}. \tag{11}$$

Again by (10)

$$P_\mu(w) = \frac{|\gamma|^2}{|\beta|^2}\frac{1-|w|^2}{|1-(\beta/|\beta|)w|^2}.$$

So $\mu = (|\gamma|^2/|\beta|^2)\delta_\lambda$ where $\lambda = \bar{\beta}/|\beta|$.

Conversely, assume that $b(z) = \gamma z/(1 - \beta z)$ with $|\beta| + |\gamma| = 1$ and let $\mu = (|\gamma|^2/|\beta|^2)\delta_\lambda$ where $\lambda = \bar{\beta}/|\beta|$. By (11) and Lemma 6,

$$\langle k_z, k_w\rangle_{\mathcal{D}(\mu)} = \Big(1 + \frac{|\gamma|^2 \bar{z}w}{(|\beta| - \bar{\beta}\bar{z})(|\beta| - \beta w)}\Big)k_z(w) = \langle k_z, k_w\rangle_{\mathcal{H}(b)}.$$

Thus $\langle f, g\rangle_{\mathcal{D}(\mu)} = \langle f, g\rangle_{\mathcal{H}(b)}$ for every $f, g \in \operatorname{span}\{k_w,\ w \in \mathbb{D}\}$. By (iii) and (iv) of Lemma 6, we have $\mathcal{H}(b) = \mathcal{D}(\mu)$ with equality of norms.

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