



**HAL**  
open science

## Two-isometries and de Branges-Rovnyak spaces

Karim Kellay, Mohamed Zarrabi

► **To cite this version:**

Karim Kellay, Mohamed Zarrabi. Two-isometries and de Branges-Rovnyak spaces. 2014. hal-00962422v1

**HAL Id: hal-00962422**

**<https://hal.science/hal-00962422v1>**

Preprint submitted on 21 Mar 2014 (v1), last revised 10 Jul 2014 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# TWO-ISOMETRIES AND de BRANGES-ROVNYAK SPACES

KARIM KELLAY, MOHAMED ZARRABI

ABSTRACT. We characterize the symbols of the de Branges–Rovnyak spaces for which the shift operator is 2-isometry in de Brange–Rovnyak. As consequence we prove by a different method a result of Chevrot, Guillot and Ransford, about equality between a de Branges–Rovnyak space and a Dirichlet-type space.

## 1. INTRODUCTION

Let  $\mathbb{D}$  be the open unit disc in the complex plane, and let  $\mathbb{T} = \partial\mathbb{D}$  be the unit circle. Denote  $H^\infty$  the space of bounded analytic function on  $\mathbb{D}$  and  $H^2$  the classical *Hardy space* of analytic functions on  $\mathbb{D}$  having square-summable Taylor coefficients at the origin.

If  $\mu$  is a finite positive measure on  $\mathbb{T}$ , the *Dirichlet-type space*  $\mathcal{D}(\mu)$  is the set of analytic functions  $f \in H^2$ , such that

$$\mathcal{D}_\mu(f) := \int_{\mathbb{D}} |f'(z)|^2 P_\mu(z) dA(z) < \infty,$$

where  $dA(z) = dx dy / \pi$  stands for the normalized area measure in  $\mathbb{D}$  and  $P_\mu$  is the Poisson integral of  $\mu$

$$P_\mu(z) := \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} d\mu(\zeta), \quad z \in \mathbb{D}.$$

The space  $\mathcal{D}(\mu)$  is endowed with the norm

$$\|f\|_{\mathcal{D}_\mu}^2 := \|f\|_{H^2}^2 + \mathcal{D}_\mu(f).$$

Note that if  $\mu = 0$ , then  $\mathcal{D}(\mu) = H^2$  and if  $\mu$  is the Lebesgue measure on  $\mathbb{T}$ , then  $P_\mu = 1$  and  $\mathcal{D}(\mu)$  is the classical *Dirichlet space*. These spaces were introduced by Richter [3] by considering the 2-isometries on Hilbert spaces. Let  $T$  be a bounded operator in a Hilbert space  $\mathcal{H}$ .  $T$  is said *2-isometry* if  $T^{*2}T^2 - 2T^*T + I = 0$ ,  $T$  is *analytic* if  $\bigcap_{n \geq 0} T^n \mathcal{H} = \{0\}$  and  $T$  is *cyclic* if  $\text{span}\{T^n x, n \in \mathbb{N}\}$  is dense in  $\mathcal{H}$  for some  $x \in \mathcal{H}$ .

Let  $S$  denote the shift operator on  $H^2$ ,

$$S : f \rightarrow zf.$$

A typical example of 2-isometry is given by the operator  $S|_{\mathcal{D}_\mu}$ , the restriction of  $S$  on  $\mathcal{D}_\mu$ . Richter gave the following representation for 2-isometries:

---

2000 *Mathematics Subject Classification*. Primary 46E22; Secondary 30H10.

*Key words and phrases*. Dirichlet space; de Branges–Rovnyak space; two-isometry.

**Theorem** (Richter [3]). *For every cyclic, analytic and 2–isometry operator  $T$  on a Hilbert space, there exists a unique finite measure  $\mu$  on  $\mathbb{T}$  such that  $T$  is unitarily equivalent to  $S|_{\mathcal{D}_\mu}$ .*

In this paper we investigate when the shift operator is a 2–isometry on the de Brange–Rovnyak spaces. This allows us to find a different proof of N. Chevrot, D. Guillot, T. Ransford result where they give conditions to have equality between a de Branges–Rovnyak space and a Dirichlet–type space, see [1, 2].

Now we introduce the de Brange–Rovnyak spaces. Let  $P_+$  be the orthogonal projection of  $L^2(\mathbb{T})$  onto  $H^2$ . For  $\varphi \in L^\infty(\mathbb{T})$ , the Toeplitz operator  $T_\varphi : H^2 \rightarrow H^2$  is given by

$$T_\varphi f := P_+(\varphi f), \quad f \in H^2.$$

Let  $b \in H^\infty$  such that  $\|b\|_\infty \leq 1$ , where  $\|\cdot\|_\infty$  is the supremum norm on  $\mathbb{D}$ . The de Brange–Rovnyak space  $\mathcal{H}(b)$  is the image of  $H^2$  under the operator  $(I - T_b T_{\bar{b}})^{1/2}$ , that is  $\mathcal{H}(b) = (I - T_b T_{\bar{b}})^{1/2}(H^2)$ , endowed with the inner product

$$\langle (I - T_b T_{\bar{b}})^{1/2}(f), (I - T_b T_{\bar{b}})^{1/2}(g) \rangle_b = \langle f, g \rangle_{H^2}, \quad f, g \in (\ker(I - T_b T_{\bar{b}})^{1/2})^\perp.$$

where  $\langle \cdot, \cdot \rangle_{H^2}$  is the inner product in  $H^2$  given by

$$\langle f, g \rangle_{H^2} = \frac{1}{2\pi} \int_{\mathbb{T}} f(\zeta) \bar{g}(\zeta) |d\zeta|,$$

here we identify  $f$  and  $g$  with their non-tangential limits. Note that when  $\|b\|_\infty < 1$  then  $\mathcal{H}_b = H^2$  and when  $b$  is an inner function, that is  $|b| = 1$  a.e. on  $\mathbb{T}$ , then  $\mathcal{H}_b = H^2 \ominus bH^2$ .

Let  $S$  be the shift operator on  $H^2$ . Note that  $\mathcal{H}(b)$  is invariant by  $S$  if and only if  $b$  is nonextreme in the unit ball of  $H^\infty$  see [5, p. 23]. When  $b$  is nonextreme we set that

$$T = S|_{\mathcal{H}(b)},$$

the restriction of  $S$  to the space  $\mathcal{H}(b)$ . Now we state our main result.

**Theorem 1.** *Suppose that  $b$  is nonextreme in the unit ball of  $H^\infty$ . Then  $T$  is 2–isometry on  $\mathcal{H}(b)$  if and only if  $b = c$ , where  $c$  is a constant such that  $|c| < 1$  or*

$$b(z) = \frac{c + \gamma z}{1 - \beta z}, \quad z \in \mathbb{D},$$

where  $c, \gamma, \beta \in \mathbb{C}$  such that  $|\beta| < 1$  and  $1 + |\beta|^2 - |c|^2 - |\gamma|^2 = 2|\beta + \bar{c}\gamma| > 0$ .

N. Chevrot, D. Guillot, T. Ransford in [1] investigate for which pairs  $(b, \mu)$  the de Branges–Rovnyak space  $\mathcal{H}_b$  is equal to the Dirichlet–type space  $\mathcal{D}_\mu$  with equality of norms. As consequence of our main Theorem 1 we obtain their Theorem [1, Theorem 3.1].

**Corollary 2** (Chevrot, Guillot, Ransford [1]). *Let  $\mu$  be a finite positive measure on  $\mathbb{T}$  and let  $b$  be in the unit ball of  $H^\infty$ . Then  $\mathcal{H}(b) = \mathcal{D}(\mu)$  with equality of norms, if and only if*

$$\mu = c\delta_\lambda \quad \text{and} \quad b(z) = \frac{\gamma z}{1 - \beta z},$$

where  $\beta \neq 0$ ,  $|\beta| + |\gamma| = 1$ ,  $c = |\gamma|^2/|\beta|^2$  and  $\lambda = \bar{\beta}/|\beta|$ .

## 2. PROOFS

To give the proofs, we need some additional notations and properties. The space  $\mathcal{H}(b)$  is a reproducing kernel space :

$$f(w) = \langle f, k_w^b \rangle_b, \quad w \in \mathbb{D}, \quad f \in \mathcal{H}(b),$$

where

$$k_w^b(z) := \frac{1 - \overline{b(w)}b(z)}{1 - \overline{w}z}, \quad w, z \in \mathbb{D}.$$

Let  $S^*$  be the backward shift operator on  $H^2$ ,

$$S^*f(z) = \frac{f(z) - f(0)}{z}, \quad f \in H^2.$$

The space  $\mathcal{H}(b)$  is always invariant by  $S^*$  [5, p. 11]. Note also that the following properties are equivalent [5, p. 24-25-26]

**Proposition 3** (Sarason [5]). *Let  $b$  be in the unit ball of  $H^\infty$ .*

- (1)  $b \in \mathcal{H}(b)$
- (2)  $b$  is nonextreme
- (3)  $\log(1 - |b|) \in L^1(\mathbb{T})$
- (4)  $\mathcal{H}(b)$  is invariant by  $S$
- (5) The polynomials are dense in  $\mathcal{H}(b)$ .

Let  $b$  be nonextreme function in the unit ball of  $H^\infty$ . Then there exists a unique outer function  $a$  such that  $a(0) > 0$  and

$$|a(\zeta)|^2 + |b(\zeta)|^2 = 1 \quad \text{a.e. } \zeta \in \mathbb{T}.$$

We set  $\mathcal{M}(\bar{a}) = T_{\bar{a}}H^2$ . Since  $b$  is nonextreme

$$f \in \mathcal{H}(b) \iff T_{\bar{b}}f \in \mathcal{M}(\bar{a})$$

and  $\mathcal{M}(\bar{a})$  is dense in  $\mathcal{H}(b)$  (see [5, p. 24-25]). Moreover  $T_{\bar{a}}$  is one-to-one, thus for every  $f \in \mathcal{H}(b)$  there is a unique function  $f^+$  such that

$$T_{\bar{b}}f = T_{\bar{a}}f^+.$$

Note that for  $f, g \in \mathcal{H}(b)$ ,

$$\langle f, g \rangle_b = \langle f, g \rangle_{H^2} + \langle f^+, g^+ \rangle_{H^2}. \quad (1)$$

**Lemma 4.** *Suppose that  $b$  is nonextreme, then*

$$\langle b, S^*b \rangle_b = -\frac{\overline{a'(0)}}{a(0)}.$$

*Proof.* We have  $T_{\bar{b}}b = T_{\bar{a}}((1/\overline{a(0)}) - a)$  and  $T_{\bar{b}}S^*b = -T_{\bar{a}}S^*a$ , hence

$$b^+ = \frac{1}{a(0)} - a, \quad (S^*b)^+ = -S^*a.$$

So by (1)

$$\begin{aligned}
\langle b, S^*b \rangle_b &= \langle b, S^*b \rangle_{H^2} + \langle b^+, (S^*b)^+ \rangle_{H^2} \\
&= \langle b, S^*b \rangle_{H^2} + \langle a, S^*a \rangle_{H^2} - \langle 1/\overline{a(0)}, S^*a \rangle_{H^2} \\
&= \frac{1}{2\pi} \int_{\mathbb{T}} \zeta (|b(\zeta)|^2 + |a(\zeta)|^2) |d\zeta| - \frac{\overline{a'(0)}}{a(0)} \\
&= \frac{1}{2\pi} \int_{\mathbb{T}} \zeta d|\zeta| - \frac{\overline{a'(0)}}{a(0)} = -\frac{\overline{a'(0)}}{a(0)}.
\end{aligned}$$

□

**Lemma 5.** Let  $b(z) = (c + \gamma z)/(1 - \beta z)$ ,  $z \in \mathbb{D}$ , where  $c, \gamma, \beta \in \mathbb{C}$ ,  $|\beta| < 1$ . Then

- (a)  $\|b\|_\infty \leq 1 \iff 2|\beta + \bar{c}\gamma| \leq 1 + |\beta|^2 - |c|^2 - |\gamma|^2$ .  
(b) If  $\|b\|_\infty \leq 1$ , then

$$b \text{ is nonextreme} \iff 1 + |\beta|^2 - |c|^2 - |\gamma|^2 > 0.$$

In this case we have

$$a(z) = \frac{\rho - \sigma z}{1 - \beta z}, \quad z \in \mathbb{D},$$

where  $\rho$  and  $\sigma$  are defined by the following

- (i)  $\rho \geq |\sigma|$ ,  
(ii)  $\rho^2 + |\sigma|^2 = 1 + |\beta|^2 - |c|^2 - |\gamma|^2$  and  $\rho^2|\sigma|^2 = |\beta + \bar{c}\gamma|^2$ ,  
(iii)  $\arg(\sigma) = \arg(\beta + \bar{c}\gamma)$ .

*Proof.* (a) We have

$$\begin{aligned}
\|b\|_\infty \leq 1 &\iff |c + \gamma\zeta|^2 \leq |1 - \beta\zeta|^2, \quad \forall \zeta \in \mathbb{T}, \\
&\iff |c|^2 + |\gamma|^2 + 2\operatorname{Re}(\bar{c}\gamma\zeta) \leq 1 + |\beta|^2 - 2\operatorname{Re}(\beta\zeta), \quad \forall \zeta \in \mathbb{T}, \\
&\iff 2\operatorname{Re}((\beta + \bar{c}\gamma)\zeta) \leq 1 + |\beta|^2 - |c|^2 - |\gamma|^2, \quad \forall \zeta \in \mathbb{T}, \\
&\iff 2|\beta + \bar{c}\gamma| \leq 1 + |\beta|^2 - |c|^2 - |\gamma|^2.
\end{aligned}$$

(b) Assume that  $\|b\|_\infty \leq 1$ . So there exist  $\rho$  and  $\sigma$  satisfying (i), (ii) and (iii). In fact  $\rho^2$  and  $|\sigma|^2$  are the solutions of the equation  $x^2 - (1 + |\beta|^2 - |c|^2 - |\gamma|^2)x + |\beta + \bar{c}\gamma|^2 = 0$ . For every  $\zeta \in \mathbb{T}$ ,

$$\begin{aligned}
|(1 - \beta\zeta)|^2(1 - |b(\zeta)|^2) &= |1 - \beta\zeta|^2 - |c + \gamma\zeta|^2 \\
&= 1 + |\beta|^2 - |c|^2 - |\gamma|^2 - 2\operatorname{Re}((\beta + \bar{c}\gamma)\zeta) \\
&= \rho^2 + |\sigma|^2 - 2\operatorname{Re}(\rho\sigma\zeta) \\
&= |\rho - \sigma\zeta|^2.
\end{aligned}$$

Since  $|\beta| < 1$ , we have

$$\begin{aligned} \log(1 - |b|^2) \in L^1(\mathbb{T}) &\iff \int_{\mathbb{T}} \log(|1 - \beta\zeta|^2(1 - |b(\zeta)|^2)) |d\zeta| > -\infty \\ &\iff \int_{\mathbb{T}} \log |\rho - \sigma\zeta|^2 |d\zeta| > -\infty, \end{aligned}$$

Since  $\rho \geq |\sigma|$ ,

$$\begin{aligned} \int_{\mathbb{T}} \log |\rho - \sigma\zeta|^2 |d\zeta| > -\infty &\iff \rho > 0 \\ &\iff 1 + |\beta|^2 - |c|^2 - |\gamma|^2 > 0. \end{aligned}$$

which proves the first part of (b). Assume now that  $b$  is nonextreme and let  $a$  be the outer function associated to  $b$ . Since the two outer functions  $(1 - \beta z)a(z)$  and  $\rho - \sigma z$  take positive values at  $z = 0$  and

$$|(1 - \beta\zeta)a(\zeta)|^2 = |(1 - \beta\zeta)|^2(1 - |b(\zeta)|^2) = |\rho - \sigma\zeta|^2, \quad \text{a.e } \zeta \in \mathbb{T},$$

we get  $(1 - \beta\zeta)a(z) = \rho - \sigma z$  on  $\mathbb{D}$ . □

**Remarks.** Let  $b(z) = (c + \gamma z)/(1 - \beta z)$ ,  $z \in \mathbb{D}$ , with  $|\beta| < 1$ .

1.  $b(\mathbb{T})$  is the circle of radius  $|\gamma + \beta\bar{c}|/(1 - |\beta|^2)$  centered at the point  $(c + \gamma\bar{\beta})/(1 - |\beta|^2)$ . So

$$\|b\|_{\infty} = \frac{|c + \gamma\bar{\beta}|}{1 - |\beta|^2} + \frac{|\gamma + \beta\bar{c}|}{1 - |\beta|^2}.$$

2. Suppose  $\|b\|_{\infty} \leq 1$ . It follows from Lemma 5 that  $b$  is extreme if and only if  $1 + |\beta|^2 - |c|^2 - |\gamma|^2 = |\beta + \bar{c}\gamma| = 0$  which is equivalent to  $b(z) = e^{i\theta}(\bar{\beta} - z)/(1 - \beta z)$  for some  $\theta \in \mathbb{R}$ .

**Proof of Theorem 1.** We set  $X = S^*|\mathcal{H}(b)$ . We have

$$X^*(\cdot) = S(\cdot) - \langle \cdot, S^*b \rangle_b b \quad \text{and} \quad T = X^* + \langle \cdot, S^*b \rangle_b b,$$

see [5, p. 12]. Therefore

$$T^* = X + \langle \cdot, b \rangle_b S^*b = S^*|\mathcal{H}(b) + \langle \cdot, b \rangle_b S^*b.$$

So, for every  $h \in \mathcal{H}_b$

$$\begin{aligned} T^*T(h) &= T^*(Sh) \\ &= S^*Sh + \langle Sh, b \rangle_b S^*b \\ &= h + \langle h, T^*b \rangle_b S^*b, \end{aligned} \tag{2}$$

and

$$\begin{aligned}
T^{*2}T^2(h) &= T^{*2}(S^2h) \\
&= T^*(S^*S^2h + \langle S^2h, b \rangle_b S^*b) \\
&= T^*(Sh + \langle h, T^{*2}b \rangle_b S^*b) \\
&= S^*Sh + \langle h, T^{*2}b \rangle_b S^{*2}b + \left\langle Sh + \langle h, T^{*2}b \rangle_b S^*b, b \right\rangle_b S^*b \\
&= h + \langle h, T^{*2}b \rangle_b S^{*2}b + \left( \langle h, T^*b \rangle_b + \langle S^*b, b \rangle_b \langle h, T^{*2}b \rangle_b \right) S^*b \\
&= h + \langle h, T^{*2}b \rangle_b S^{*2}b + \left\langle h, T^*b + \langle b, S^*b \rangle_b T^{*2}b \right\rangle_b S^*b. \tag{3}
\end{aligned}$$

By (2) and (3), we get

$$(T^{*2}T^2 - 2T^*T + I)(h) = \langle h, T^{*2}b \rangle_b S^{*2}b + \left\langle h, \langle b, S^*b \rangle_b T^{*2}b - T^*b \right\rangle_b S^*b \tag{4}$$

Moreover, we have

$$T^*b = S^*b + \langle b, b \rangle_b S^*b = (1 + \|b\|_b^2)S^*b$$

and

$$T^{*2}b = (1 + \|b\|_b^2) \left( S^{*2}b + \langle S^*b, b \rangle_b S^*b \right). \tag{5}$$

Assume that  $T$  is 2-isometry. Then (4) and (5) imply that  $S^*b = 0$  or be an eigenvector of  $S^*$ . Hence there exist  $\alpha, \beta \in \mathbb{C}$  such that  $|\beta| < 1$  and

$$S^*b(z) = \frac{\alpha}{1 - \beta z}, \quad z \in \mathbb{D}.$$

Then there exist  $c, \gamma \in \mathbb{C}$  such that

$$b(z) = \frac{c + \gamma z}{1 - \beta z}, \quad z \in \mathbb{D}.$$

Note that the complex numbers  $c, \gamma, \beta$  satisfy the conditions found in Lemma 5 corresponding to the fact that  $\|b\|_\infty \leq 1$  and  $b$  is nonextreme. We have  $S^{*2}b = \beta S^*b$  and by Lemma 4, equality (4) become

$$(T^{*2}T^2 - 2T^*T + I)(h) = \left\langle h, \overline{\beta}T^{*2}b - \frac{\overline{a'(0)}}{a(0)}T^{*2}b - T^*b \right\rangle_b S^*b. \tag{6}$$

By (5), we have  $T^{*2}b = (1 + \|b\|_b^2) \left( \beta - \frac{a'(0)}{a(0)} \right) S^*b$ . Thus equality (6) become

$$(T^{*2}T^2 - 2T^*T + I)(h) = (1 + \|b\|_b^2) \left\langle h, \left[ \left| \beta - \frac{a'(0)}{a(0)} \right|^2 - 1 \right] \right\rangle_b S^*b.$$

Since  $T$  is 2-isometry, we have  $S^*b = 0$  (that is  $b = c$  with  $|c| < 1$ ) or

$$\left| \beta - \frac{a'(0)}{a(0)} \right| = 1. \tag{7}$$

Recall that

$$a(z) = \frac{\rho - \sigma z}{1 - \beta z}$$

where  $\rho$  and  $\sigma$  are given by Lemma 5. Note that  $a(0) = \rho > 0$  and  $a'(0) = -\sigma + \rho\beta$ . So (7) holds if and only if

$$\frac{|\sigma|}{\rho} = 1 \iff 1 + |\beta|^2 - |c|^2 - |\gamma|^2 = 2|\beta + \bar{c}\gamma|.$$

Since  $b$  is nonextreme,  $1 + |\beta|^2 - |c|^2 - |\gamma|^2 > 0$ .

The converse is clear and now, the proof is complete.

For the proof of Corollary 2, we need the following lemma.

**Lemma 6.** *Let  $\mu$  be a finite positive measure on  $\mathbb{T}$  and let  $b \in H^\infty$  be a nonextreme function in the unit ball of  $H^\infty$ . Let  $k_w(z) = 1/(1 - \bar{w}z)$ ,  $w, z \in \mathbb{D}$ . Then*

- (i)  $\langle k_z, k_w \rangle_{\mathcal{D}(\mu)} = \left(1 + \bar{z}w \int_{\mathbb{T}} \frac{d\mu(\zeta)}{(1 - \bar{z}\zeta)(1 - w\bar{\zeta})}\right) k_z(w)$ .
- (ii)  $\langle k_z, k_w \rangle_{\mathcal{H}(b)} = \left(1 + \frac{\bar{b}(z)b(w)}{a(z)a(w)}\right) k_z(w)$ .
- (iii)  $\text{span}\{k_w, w \in \mathbb{D}\}$  is dense in  $\mathcal{D}(\mu)$ .
- (iv)  $\text{span}\{k_w, w \in \mathbb{D}\}$  is dense in  $\mathcal{H}(b)$ .

*Proof.* By Douglas formula, for every  $f, g \in \mathcal{D}(\mu)$  we have

$$\langle f, g \rangle_{\mathcal{D}(\mu)} = \langle f, g \rangle_{H^2} + \frac{1}{2\pi} \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{(f(\zeta) - f(\zeta'))(\overline{g(\zeta) - g(\zeta')})}{|\zeta - \zeta'|^2} |d\zeta| d\mu(\zeta'),$$

(see [3]). Using now Cauchy formula we get (i). For the proof of (ii) see [5, p. 32].

Now we proof (iii). Let  $f \in \mathcal{D}(\mu)$  and set  $f_r(z) = f(rz)$ ,  $0 < r < 1$ . Note that

$$\|f_{r^2} - f\|_{\mathcal{D}(\mu)} \rightarrow 0, \quad \text{as } r \rightarrow 1-$$

(see [4, Theorem 5.2] and [5, Corollary]). Since the map  $\zeta \rightarrow f_r(\zeta)k_{r\zeta}$  is continuous from  $\mathbb{T}$  to  $\mathcal{D}(\mu)$  and

$$f_{r^2} = \frac{1}{2\pi} \int_{\mathbb{T}} f(r\zeta)k_{r\zeta} |d\zeta|,$$

we see that  $f_{r^2}$  can be approximated by finite combinations of the functions  $k_w$ . So  $f$  belong to the closure of  $\text{span}\{k_w, w \in \mathbb{D}\}$ .

To prove (iv), let  $h \in \mathcal{H}(b)$ . We have

$$\langle f, k_w \rangle_b = h(w) + (b(w)/a(w))h^+(w) \tag{8}$$

(see [5, p. 32]).

Suppose that  $h \perp \text{span}\{k_w : w \in \mathbb{D}\}$ , then by (8)

$$h = -(b/a)h^+. \tag{9}$$



So  $(h^+/a) - \bar{a}h^+ = \bar{b}h \in L^2(\mathbb{T})$  and then  $h^+/a \in L^2(\mathbb{T})$ . Since  $h^+/a$  belong to Smirnov class,  $h^+/a \in H^2$ . We know that  $T_b h = T_{\bar{a}} h^+$ . Then using (9)

$$P_+\left(\frac{|b|^2}{a}h\right) = -T_{\bar{a}}h^+ \iff P_+\left(\frac{1-|a|^2}{a}h^+\right) = -T_{\bar{a}}h^+ \iff P_+\left(\frac{h^+}{a} - \bar{a}h^+\right) = -T_{\bar{a}}h^+.$$

Therefore  $P_+(h^+/a) = 0$  and then  $h^+ = 0$ . We get from (8) that  $h = 0$ .  $\square$

**Remark.** The polynomials are dense in  $\mathcal{D}(\mu)$  and in  $\mathcal{H}(b)$ , [3, 5]. We can deduce (iii) and (iv) from this fact. Note that for  $f \in \mathcal{H}(b)$  we have not in general  $\|f_r - f\|_{\mathcal{H}(b)} \rightarrow 0$  as  $r \rightarrow 1-$ , see [5, 1].

**Proof of Corollary.** Suppose that  $\mathcal{H}(b) = \mathcal{D}(\mu)$  for some finite positive measure with equality of norms. Then  $b$  is nonextreme since  $\mathcal{H}(b)$  is invariant by  $S$ , see Proposition 3. By Richter Theorem [3],  $T$  is 2-isometry and by Theorem 1

$$b(z) = \frac{c + \gamma z}{1 - \beta z}, \quad z \in \mathbb{D},$$

where  $c, \gamma, \beta \in \mathbb{C}$  such that  $|\beta| < 1$  and  $1 + |\beta|^2 - |c|^2 - |\gamma|^2 = 2|\beta + \bar{c}\gamma| > 0$ .

Let  $k_w(z) = 1/(1 - \bar{w}z)$ . If  $\mathcal{H}(b) = \mathcal{D}(\mu)$  with equality of norms then

$$\|k_w\|_{\mathcal{D}(\mu)}^2 = \|k_w\|_b^2, \quad w \in \mathbb{D}. \quad (10)$$

By Lemma 6,

$$\|k_w\|_{\mathcal{D}(\mu)}^2 = \frac{1}{1 - |w|^2} \left(1 + \frac{|w|^2}{1 - |w|^2} P_\mu(w)\right), \quad w \in \mathbb{D},$$

$$\|k_w\|_b^2 = \frac{1}{1 - |w|^2} \left(1 + \frac{|b(w)|^2}{|a(w)|^2}\right), \quad w \in \mathbb{D}.$$

By (10), we obtain  $1 = \|k_0\|_{\mathcal{D}(\mu)}^2 = \|k_0\|_b^2 = 1 + |b(0)|^2/|a(0)|^2$ , so  $b(0) = 0$ . Therefore  $b(z) = \gamma z/(1 - \beta z)$ , with  $\beta \neq 0$  and  $|\beta| + |\gamma| = 1$ . It follows from Lemma 5 that

$$a(z) = \frac{|\beta| - \beta z}{1 - \beta z}, \quad z \in \mathbb{D}. \quad (11)$$

Again by (10)

$$P_\mu(w) = \frac{|\gamma|^2}{|\beta|^2} \frac{1 - |w|^2}{|1 - (\beta/|\beta|)w|^2}.$$

So  $\mu = (|\gamma|^2/|\beta|^2)\delta_\lambda$  where  $\lambda = \bar{\beta}/|\beta|$ .

Conversely, assume that  $b(z) = \gamma z/(1 - \beta z)$  with  $|\beta| + |\gamma| = 1$  and let  $\mu = (|\gamma|^2/|\beta|^2)\delta_\lambda$  where  $\lambda = \bar{\beta}/|\beta|$ . By (11) and Lemma 6,

$$\langle k_z, k_w \rangle_{\mathcal{D}(\mu)} = \left(1 + \frac{|\gamma|^2 \bar{z}w}{(|\beta| - \bar{\beta}\bar{z})(|\beta| - \beta w)}\right) k_z(w) = \langle k_z, k_w \rangle_{\mathcal{H}(b)}.$$

Thus  $\langle f, g \rangle_{\mathcal{D}(\mu)} = \langle f, g \rangle_{\mathcal{H}(b)}$  for every  $f, g \in \text{span}\{k_w, w \in \mathbb{D}\}$ . By (iii) and (iv) of Lemma 6, we have  $\mathcal{H}(b) = \mathcal{D}(\mu)$  with equality of norms.

## REFERENCES

- [1] N. Chevrot, D. Guillot, T. Ransford, De Branges-Rovnyak spaces and Dirichlet spaces, *J. Funct. Anal.* 259 (2010), 2366-2383.
- [2] C. Costara, T. Ransford, Which de Branges-Rovnyak spaces are Dirichlet spaces (and vice versa)?, *J. Funct. Anal.* 265 (2013), 3204-3218.
- [3] S. Richter, A representation theorem for cyclic analytic two-isometries, *Trans. Am. Math. Soc.*, 328 (1991) 325-349.
- [4] Richter and C. Sundberg, Multipliers and invariant subspaces in the Dirichlet space, *J. Operator Theory* 28 (1992), 167-186.
- [5] D. Sarason, *Sub-Hardy Hilbert Spaces in the Unit Disk*, John Wiley, Sons Inc., New York, 1994.
- [6] D. Sarason, Local Dirichlet spaces as de Branges-Rovnyak spaces, *Proc. Amer. Math. Soc.* 125 (1997), 2133-2139.

IMB, UNIVERSITÉ DE BORDEAUX, 351 COURS DE LA LIBERATION, 33405 TALENCE, FRANCE

*E-mail address:* `kkellay@math.u-bordeaux1.fr`

*E-mail address:* `Mohamed.Zarrabi@math.u-bordeaux1.fr`