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THE MAXIMAL POTENTIAL ENERGY OF BIASED RANDOM WALKS ON TREES

YUEYUN HU AND ZHAN SHI

ABSTRACT. The biased random walk on supercritical Galton–Watson trees is known to exhibit a multiscale phenomenon in the slow regime: the maximal displacement of the walk in the first n steps is of order $(\log n)^3$, whereas the typical displacement of the walk at the n -th step is of order $(\log n)^2$. Our main result reveals another multiscale property of biased walks: the maximal potential energy of the biased walks is of order $(\log n)^2$ in contrast with its typical size, which is of order $\log n$. The proof relies on analyzing the intricate multiscale structure of the potential energy.

1. INTRODUCTION

Let \mathbb{T} be a supercritical Galton–Watson tree rooted at \emptyset . Let $\omega := (\omega(x), x \in \mathbb{T})$ be a sequence of vectors: for each $x \in \mathbb{T}$, $\omega(x) := (\omega(x, y), y \in \mathbb{T})$ is such that $\omega(x, y) \geq 0$ ($\forall y \in \mathbb{T}$) and that $\sum_{y \in \mathbb{T}} \omega(x, y) = 1$.

Given ω , we define a random walk $(X_n, n \geq 0)$ on \mathbb{T} , started at $X_0 = \emptyset$, with transition probabilities given by

$$P_\omega\{X_{n+1} = y \mid X_n = x\} = \omega(x, y).$$

We assume that for each pair of vertices x and y , $\omega(x, y) > 0$ if and only if $y \sim x$, i.e., y is either a child, or the parent, of x ; in particular, the walk is nearest-neighbour.

We are going to study a *slow regime* of the random walk $(X_n, n \geq 0)$. In order to observe such a slow regime, the transition probabilities $\omega(x, y)$ are *random*; i.e., given a realisation of ω , we run a (conditional) Markov chain (X_n) . So (X_n) is a randomly biased walk on the Galton–Watson tree \mathbb{T} , and can also be considered as a random walk in random environment.

We use \mathbf{P} to denote the law of the environment ω , and $\mathbb{P} := \mathbf{P} \otimes P_\omega$ the annealed probability measure. It is convenient to consider (ω, \mathbb{T}) as a marked tree. For brevity, we say “for almost all environment ω ” to mean “for almost all (ω, \mathbb{T}) ”.

Randomly biased walks on trees have a large literature. The model is introduced by Lyons and Pemantle [44], extending the previous model of deterministically biased walks studied in Lyons [41]–[42]. In [44], a general recurrence vs. transience criterion is obtained; for walks on Galton–Watson trees, the question is later also studied by Menshikov and Petritis [50] and Faraud [28]. Ben Arous and Hammond [14] prove that in some sense, randomly biased walks on \mathbb{T} are more regular than deterministically biased walks on \mathbb{T} , preventing some “cyclic phenomena” from happening. Often motivated by results and questions in Lyons, Pemantle and Peres [46] and [47], the transient case has received much research attention ([1], [2], [4], [12], [13], [15]), while the recurrent case has been studied in [5], [8], [9], [10], [23], [24], [26], [28], [29], [31], [32], [33], [36], [37] and [38]. For a more general account of study on biased walks on trees and random walks in random environments, we refer to the books by Lyons and Peres [48], Révész [54], as well as the Saint-Flour lecture notes of [53] and [55]. Biased random walks in other types of random

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environments are also studied in the literature, such as [11] for biased random walks on dynamical percolation clusters, or [19] for random walks among dynamical random conductances.

Although it is not necessary, we add a special vertex, $\overleftarrow{\emptyset}$, which is the parent of \emptyset ; this simplifies our representation. The values of the transition probabilities at a finite number of vertices bringing no change to results of the paper, we can modify the value of $\omega(\emptyset, \bullet)$, the transition probability at \emptyset , in such a way that $(\omega(x, y), y \sim x)$, for $x \in \mathbb{T}$, are an i.i.d. family of random variables.

A crucial notion in the study of the behaviour of the random walk (X_n) is the **potential** on \mathbb{T} , which we define by $V(\overleftarrow{\emptyset}) := 0$, $V(\emptyset) := 0$ and

$$(1.1) \quad V(x) := - \sum_{y \in \llbracket \emptyset, x \rrbracket} \log \frac{\omega(\overleftarrow{y}, y)}{\omega(\overleftarrow{y}, \overleftarrow{y})}, \quad x \in \mathbb{T} \setminus \{\emptyset\},$$

where \overleftarrow{y} is the parent of y , and $\llbracket \emptyset, x \rrbracket := \llbracket \emptyset, x \rrbracket \setminus \{\emptyset\}$, with $\llbracket \emptyset, x \rrbracket$ denoting the set of vertices on the unique shortest path connecting \emptyset to x .

Since $(\omega(x, y), y \sim x)$, for $x \in \mathbb{T}$, are i.i.d., the potential process $(V(x), x \in \mathbb{T})$ is a branching random walk in the usual sense of Biggins [16], and is also well studied in the physics literature (see for example Derrida and Spohn [25], represented in terms of directed polymers on trees).

Throughout the paper, we assume

$$(1.2) \quad \mathbf{E} \left(\sum_{x: |x|=1} e^{-V(x)} \right) = 1, \quad \mathbf{E} \left(\sum_{x: |x|=1} V(x) e^{-V(x)} \right) = 0.$$

We also assume the existence of $\delta > 0$ such that

$$(1.3) \quad \mathbf{E} \left(\sum_{x: |x|=1} e^{-(1+\delta)V(x)} \right) + \mathbf{E} \left(\sum_{x: |x|=1} e^{\delta V(x)} \right) + \mathbf{E} \left[\left(\sum_{x: |x|=1} 1 \right)^{1+\delta} \right] < \infty.$$

A general result of Lyons and Pemantle [44], applied to our special setting of the Galton–Watson tree, implies that under (1.2), the random walk (X_n) is almost surely recurrent. This is proved in [44] under an additional condition on the exchangeability of $(V(x), |x| = 1)$; the condition is removed in Faraud [28]. See also Menshikov and Petritis [50] for another proof, using Mandelbrot’s multiplicative cascades, modulo some additional assumptions. In the language of branching random walks, (1.2) refers to the “boundary case” in the sense of Biggins and Kyprianou [17]. In the boundary case, the biased walk (X_n) exhibits slow movement and reveals a somehow surprising multiscale behaviour: under (1.2) and (1.3) and upon the system’s survival, $\max_{0 \leq i \leq n} |X_i|$ is of order $(\log n)^3$ whereas the typical size of $|X_n|$ is of order $(\log n)^2$:

$$(1.4) \quad \lim_{n \rightarrow \infty} \frac{1}{(\log n)^3} \max_{0 \leq i \leq n} |X_i| = \frac{8}{3\pi^2 \sigma^2}, \quad \mathbb{P}^* \text{-a.s.},$$

where $\mathbb{P}^* := \mathbb{P}(\cdot | \text{non-extinction})$ and

$$(1.5) \quad \sigma^2 := \mathbf{E} \left(\sum_{|x|=1} V(x)^2 e^{-V(x)} \right) \in (0, \infty).$$

On the other hand, it is shown in [34] that for any $t > 0$,

$$(1.6) \quad \lim_{n \rightarrow \infty} \mathbb{P}^* \left(\frac{\sigma^2 |X_n|}{(\log n)^2} \leq t \right) = \int_0^t \frac{1}{(2\pi s)^{1/2}} \mathbf{P} \left(\mathbf{m}_1^\# \leq \frac{1}{s^{1/2}} \right) ds,$$

where $\mathbf{m}_1^\# := \sup_{0 \leq r \leq s \leq 1} (\mathbf{m}_r - \mathbf{m}_s)$, and $(\mathbf{m}_s)_{0 \leq s \leq 1}$ is a Brownian meander (for the definition and basic properties of the Brownian meander, see Yen and Yor [57]).

In dimension 1 (which corresponds heuristically to the case that every vertex has one child), a well known result of Sinai [56] tells that $\frac{X_n}{(\log n)^2}$ converges weakly to a non-degenerate limit; so (1.6) can be considered as a kind of companion of Sinai's theorem for the Galton–Watson tree.

Indeed, the proof of (1.6) relies on a localization of X_n by some barriers constructed from the potential V . Using such a localization ([34, Theorem 2.1]), we may deduce that assuming (1.2) and (1.3), under \mathbb{P}^* , $\frac{V(X_n)}{\log n}$ converges in law to a finite and positive random variable. For any $t > 0$, under $\mathbf{P}(\cdot \mid \text{non-extinction})$,

$$(1.7) \quad \lim_{n \rightarrow \infty} P_\omega \left(\frac{V(X_n)}{\log n} \leq t \right) = \left(\frac{2}{\pi} \right)^{1/2} \mathbf{E} \left(\min \left(\frac{1}{\mathfrak{m}_1^\#}, \frac{t}{\mathfrak{m}_1} \right) \right), \quad \text{in probability.}$$

We refer to Appendix B for a proof of (1.7). Note that $\mathbf{E}(\frac{1}{\mathfrak{m}_1^\#}) = (\frac{\pi}{2})^{1/2}$, see [35], so both right-hand sides in (1.6) and (1.7) correspond to the partition functions of certain probability distributions on $(0, \infty)$.

In this paper, we are interested in the **maximal potential energy**,

$$\max_{0 \leq k \leq n} V(X_k),$$

of the random walk (X_i) in the first n steps. In the literature, results on the maximal energy of random walks in random environment or related models are obtained in the one-dimensional case by Monthus and Le Doussal [52], and for the Metropolis algorithm by Aldous [7], and also by Maillard and Zeitouni [49].

In the one-dimensional recurrent case, it is proved by Monthus and Le Doussal [52] that $\log n$ is the common order of magnitude for both $V(X_n)$ and $\max_{0 \leq k \leq n} V(X_k)$. In fact, for one-dimensional random walks in random environment, it is known (Sinai [56]) that in n steps, the maximal potential energy is bounded by $(1 + o(1)) \log n$ (for $n \rightarrow \infty$); more precisely, the ratio between the maximal potential energy and $\log n$ converges to a non-degenerate random variable taking values in $[0, 1]$. For the tree-valued random walk (X_i) , its restriction to each branch of \mathbb{T} being a one-dimensional walk in random environment, the maximal potential energy along a given branch is thus bounded by $(1 + o(1)) \log n$, for $n \rightarrow \infty$.

Let us present the main result of the paper.

Theorem 1.1. *Assume (1.2) and (1.3). We have, on the set of non-extinction,*

$$\lim_{n \rightarrow \infty} \frac{1}{(\log n)^2} \max_{0 \leq k \leq n} V(X_k) = \frac{1}{2}, \quad \mathbb{P}\text{-a.s.}$$

Theorem 1.1 can be viewed as another multiscale phenomenon of the randomly biased walk (X_n) from the perspective of potential energy.

The rest of the paper is as follows. Section 3 recalls some known techniques of branching random walks which are going to be used in the proof of Theorem 1.1. The section is preceded by a brief Section 2, where we outline the main ideas in the proof of Theorem 1.1. It turns out that the proof relies essentially on a quenched tail estimate of excursion heights of biased walks. This tail estimate, stated in (2.8), is proved in Section 4 by means of a second moment argument. The second moment argument being rather involving, we present it by means of two lemmas (Lemmas 4.1 and 4.2), serving as the key step in the proof of the upper and lower bounds, respectively, in (2.8). Lemma 4.2 is quite technical; its proof is the heart of the paper.

Throughout the paper, we write $f(r) \sim g(r)$, $r \rightarrow \infty$, to denote $\lim_{r \rightarrow \infty} \frac{f(r)}{g(r)} = 1$, and $f(r) = o(1)$, $r \rightarrow \infty$, to denote $\lim_{r \rightarrow \infty} f(r) = 0$. For any pair of vertices x and y in \mathbb{T} ,

we write $x < y$ (or $y > x$) to say that y is a descendant of x , and $x \leq y$ (or $y \geq x$) to say that y is either a descendant of x or is x itself.

2. PROOF OF THEOREM 1.1: AN OUTLINE

We assume (1.2) and (1.3), and briefly describe the proof of Theorem 1.1. Let $\varrho_0 := 0$ and let

$$(2.1) \quad \varrho_n := \inf\{i > \varrho_{n-1} : X_i = \overleftarrow{\emptyset}\}, \quad n \geq 1.$$

In words, ϱ_n denotes the n -th hits to $\overleftarrow{\emptyset}$ by the walk (X_i) . It turns out that $\varrho_n = n^{1+o(1)}$ \mathbb{P} -a.s. for $n \rightarrow \infty$:

Lemma 2.1. *Assume (1.2) and (1.3). On the set of non-extinction,*

$$\lim_{n \rightarrow \infty} \frac{\log \varrho_n}{\log n} = 1, \quad \mathbb{P}\text{-a.s.}$$

The lemma is (implicitly) in [33] or [9]. We present the proof at the end of this section, for the sake of completeness, and also to justify the passage from hitting times at \emptyset to hitting times at $\overleftarrow{\emptyset}$.

In view of Lemma 2.1, Theorem 1.1 is equivalent to the following estimate: for \mathbf{P} -almost all ω in the set of non-extinction,

$$(2.2) \quad \frac{1}{(\log n)^2} \max_{0 \leq k \leq \varrho_n} V(X_k) \rightarrow \frac{1}{2}, \quad P_\omega\text{-a.s.}$$

At this stage, we recall an elementary result:

Fact 2.2. *Let $\alpha > 0$. Let $(\xi_n)_{n \geq 1}$ be a sequence of i.i.d. real-valued random variables such that $\mathbf{P}(\xi_1 \geq u) = \exp[-(\alpha + o(1))u]$, $u \rightarrow \infty$. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{\log n} \max_{1 \leq k \leq n} \xi_k = \frac{1}{\alpha}, \quad \mathbf{P}\text{-a.s.}$$

Let us go back to (2.2). For given ω , $\max_{0 \leq k \leq \varrho_n} V(X_k)$ is the maximum of n independent copies of $\max_{0 \leq k \leq \varrho_1} V(X_k)$; so applying Fact 2.2 to $\xi := [\max_{0 \leq k \leq \varrho_1} V(X_k)]^{1/2}$ (on the set of non-extinction) and $\alpha := 2^{1/2}$, we see that the proof of (2.2) is reduced to verifying the following: for \mathbf{P} -almost all ω in the set of non-extinction,

$$(2.3) \quad P_\omega \left(\max_{0 \leq k \leq \varrho_1} V(X_k) \geq r \right) = \exp \left(- (1 + o(1)) (2r)^{1/2} \right), \quad r \rightarrow \infty.$$

For any $r > 0$, let us consider the following subset of the genealogical tree:

$$(2.4) \quad \mathcal{H}_r := \{x \in \mathbb{T} : V(x) \geq r, \overline{V}(\overleftarrow{x}) < r\},$$

where \overleftarrow{x} denotes as before the parent of x , and for any vertex $y \in \mathbb{T}$,

$$(2.5) \quad \overline{V}(y) := \max_{z \in [\emptyset, y]} V(z),$$

which is the maximal value of the potential $V(\cdot)$ along the path $[\emptyset, y]$.

By definition, $\{\max_{0 \leq k \leq \varrho_1} V(X_k) \geq r\} = \{T_{\mathcal{H}_r} < T_{\overleftarrow{\emptyset}}\}$, where

$$(2.6) \quad T_{\mathcal{H}_r} := \inf\{i \geq 0 : X_i \in \mathcal{H}_r\},$$

$$(2.7) \quad T_{\overleftarrow{\emptyset}} := \inf\{i \geq 0 : X_i = \overleftarrow{\emptyset}\} = \varrho_1.$$

In words, $T_{\mathcal{H}_r}$ is the first hitting time of the set \mathcal{H}_r by the biased walk (X_i) . We mention that \mathcal{H}_r depends only on the environment, whereas $T_{\mathcal{H}_r}$ involves also the behaviour of the biased walk.

We observe that (2.3) is equivalent to saying that \mathbf{P} -almost surely on the set of non-extinction,

$$(2.8) \quad P_\omega(T_{\mathcal{H}_r} < T_{\varnothing}^-) = \exp\left(- (1 + o(1)) (2r)^{1/2}\right), \quad r \rightarrow \infty.$$

It is (2.8) we are going to prove, in Section 4.

Let us close this section with the proof of Lemma 2.1.

Proof of Lemma 2.1. For any $j \geq 1$, we have

$$P_\omega\left\{\max_{0 \leq i \leq \varrho_1} |X_i| \geq j\right\} = \sum_{k=1}^{\infty} P_\omega\left\{\max_{0 \leq i \leq \varrho_1} |X_i| \geq j, \sum_{i=1}^{\varrho_1} \mathbf{1}_{\{X_i = \varnothing\}} = k\right\}.$$

Observe that

$$P_\omega\left\{\sum_{i=1}^{\varrho_1} \mathbf{1}_{\{X_i = \varnothing\}} = k\right\} = [1 - \omega(\varnothing, \overleftarrow{\varnothing})]^k \omega(\varnothing, \overleftarrow{\varnothing}),$$

and that

$$\begin{aligned} P_\omega\left\{\max_{0 \leq i \leq \varrho_1} |X_i| \geq j \mid \sum_{i=1}^{\varrho_1} \mathbf{1}_{\{X_i = \varnothing\}} = k\right\} &= 1 - \left(1 - P_\omega\left\{\max_{0 \leq i \leq \varrho^\varnothing} |X_i| \geq j \mid |X_1| = 1\right\}\right)^k \\ &= 1 - \left(1 - \frac{P_\omega\{\max_{0 \leq i \leq \varrho^\varnothing} |X_i| \geq j\}}{1 - \omega(\varnothing, \overleftarrow{\varnothing})}\right)^k, \end{aligned}$$

where $\varrho^\varnothing := \inf\{i \geq 1 : X_i = \varnothing\}$. Thus

$$\begin{aligned} P_\omega\left\{\max_{0 \leq i \leq \varrho_1} |X_i| \geq j\right\} &= \sum_{k=1}^{\infty} \left(P_\omega\{\max_{0 \leq i \leq \varrho^\varnothing} |X_i| \geq j\}\right)^k \omega(\varnothing, \overleftarrow{\varnothing}) \\ &= \frac{P_\omega\{\max_{0 \leq i \leq \varrho^\varnothing} |X_i| \geq j\}}{\omega(\varnothing, \overleftarrow{\varnothing}) + P_\omega\{\max_{0 \leq i \leq \varrho^\varnothing} |X_i| \geq j\}} \\ &\leq \frac{P_\omega\{\max_{0 \leq i \leq \varrho^\varnothing} |X_i| \geq j\}}{\omega(\varnothing, \overleftarrow{\varnothing})}. \end{aligned}$$

So for any $n \geq 1$,

$$\begin{aligned} P_\omega\left\{\max_{0 \leq i \leq \varrho_n} |X_i| \geq j\right\} &= 1 - \left[1 - P_\omega\left\{\max_{0 \leq i \leq \varrho_1} |X_i| \geq j\right\}\right]^n \\ &\leq 1 - \left[1 - \frac{P_\omega\{\max_{0 \leq i \leq \varrho^\varnothing} |X_i| \geq j\}}{\omega(\varnothing, \overleftarrow{\varnothing})}\right]^n. \end{aligned}$$

By [29], $\frac{1}{j^{1/3}} \log P_\omega\{\max_{0 \leq i \leq \varrho^\varnothing} |X_i| \geq j\} \rightarrow -(\frac{3\pi^2\sigma^2}{8})^{1/3}$ (for $j \rightarrow \infty$) \mathbf{P} -almost surely on the set of non-extinction. Taking $j := \lceil (1 + \varepsilon)^3 \frac{8}{3\pi^2\sigma^2} (\log n)^3 \rceil$ with $\varepsilon > 0$, we immediately see that \mathbf{P} -a.s. on the set of non-extinction, $\sum_{\ell} P_\omega\{\max_{0 \leq i \leq \varrho_{n_\ell}} |X_i| \geq (1 + \varepsilon)^3 \frac{8}{3\pi^2\sigma^2} (\log n_\ell)^3\} < \infty$ if we take the subsequence $n_\ell := \lfloor \ell^{2/\varepsilon} \rfloor$, $\ell \geq 1$. By the Borel–Cantelli lemma, this yields that \mathbb{P} -almost surely, on the set of non-extinction and for all sufficiently large ℓ ,

$$\max_{0 \leq i \leq \varrho_{n_\ell}} |X_i| < (1 + \varepsilon)^3 \frac{8}{3\pi^2\sigma^2} (\log n_\ell)^3,$$

which, in turn, implies that for $n \in [n_{\ell-1}, n_\ell]$,

$$\max_{0 \leq i \leq \varrho_n} |X_i| < (1 + \varepsilon)^3 \frac{8}{3\pi^2\sigma^2} (\log n_\ell)^3 \leq (1 + 2\varepsilon)^3 \frac{8}{3\pi^2\sigma^2} (\log n)^3.$$

Therefore, on the set of non-extinction,

$$\limsup_{n \rightarrow \infty} \frac{1}{(\log n)^3} \max_{0 \leq i \leq \varrho_n} |X_i| \leq \frac{8}{3\pi^2\sigma^2}, \quad \mathbb{P}\text{-a.s.}$$

On the other hand, since $\varrho_n \rightarrow \infty$ \mathbb{P} -a.s., it follows from (1.4) that on the set of non-extinction,

$$\liminf_{n \rightarrow \infty} \frac{1}{(\log \varrho_n)^3} \max_{0 \leq i \leq \varrho_n} |X_i| \geq \frac{8}{3\pi^2\sigma^2}, \quad \mathbb{P}\text{-a.s.}$$

Combining the last two displayed formulas yields $\limsup_{n \rightarrow \infty} \frac{\log \varrho_n}{\log n} \leq 1$ \mathbb{P} -a.s. on the set of non-extinction. This is the desired upper bound in Lemma 2.1. The lower bound is trivial since $\varrho_n \geq 2n - 1$, $\forall n \geq 1$. \square

3. PRELIMINARIES: SPINAL DECOMPOSITIONS

We recall a useful consequence of the spinal decomposition for branching random walks. The idea of the spinal decomposition, of which we find roots in [39] and [18], has been developed in the literature independently by various groups of researchers in different contexts and forms. We use here the formulation of Lyons, Pemantle and Peres [45] and Lyons [43], based on a change-of-probabilities technique on the space of trees. We only give a brief description, referring to [45] and [43] for more details.

Throughout this section, we assume $\mathbf{E}(\sum_{|x|=1} e^{-V(x)}) = 1$, which is guaranteed by (1.2). Let

$$W_n := \sum_{x: |x|=n} e^{-V(x)}, \quad n \geq 0,$$

which is an (\mathcal{F}_n) -martingale, where \mathcal{F}_n denotes the σ -field generated by the branching random walk $(V(x))$ in the first n generations. Kolmogorov's extension theorem ensures the existence of a probability measure \mathbf{Q} on \mathcal{F}_∞ , the σ -field generated by the entire branching random walk, such that for any n and any $A \in \mathcal{F}_n$,

$$(3.1) \quad \mathbf{Q}(A) = \mathbf{E}(W_n \mathbf{1}_A).$$

The distribution of $(V(x))$ under the new probability \mathbf{Q} is called the distribution of a *size-biased* branching random walk. It is immediately observed that the size-biased branching random walk survives with probability one. For future use, we record here a consequence of Hölder's inequality: the assumption (1.3) implies the existence of a constant $c_1 > 0$ such that

$$(3.2) \quad \mathbf{E}_{\mathbf{Q}} \left[\left(\sum_{x: |x|=1} e^{-V(x)} \right)^{c_1} \right] = \mathbf{E} \left[\left(\sum_{x: |x|=1} e^{-V(x)} \right)^{1+c_1} \right] < \infty.$$

We identify a branching random walk $(V(x))$ with a marked tree. On the enlarged probability space formed by marked trees with distinguished rays,¹ it is possible to construct a probability \mathbf{Q} satisfying (3.1), and an infinite ray $\{w_0 = \emptyset, w_1, \dots, w_n, \dots\}$ (i.e., w_n is the parent of w_{n+1} , and $|w_n| = n$, $\forall n \geq 0$) such that for any $n \geq 0$ and any vertex x with $|x| = n$,

$$(3.3) \quad \mathbf{Q}\{w_n = x \mid \mathcal{F}_n\} = \frac{e^{-V(x)}}{W_n}.$$

¹Strictly speaking, the enlarged probability space is a product space: the first coordinate concerns the branching random walk, and the second concerns the distinguished ray (= spine). In order to keep the notation as simple as possible, we choose to work formally on the same space, while bearing in mind that the spine (w_n) is not measurable with respect to the σ -field generated by the branching random walk.

Let us write from now on

$$S_n := V(w_n), \quad n \geq 0.$$

For any vertex $x \in \mathbb{T} \setminus \{\emptyset\}$, we define

$$(3.4) \quad \Delta V(x) := V(x) - V(\overleftarrow{x}).$$

Let $f : \mathbb{R} \rightarrow [0, \infty)$ be a Borel function, and write

$$\eta_i^{(f)} := \sum_{y: \overleftarrow{y}=w_{i-1}} f(\Delta V(y)).$$

[In particular, $\eta_1^{(f)} := \sum_{y: |y|=1} f(V(y))$.] According to the spinal decomposition (see Lyons [43]), $(S_i - S_{i-1}, \eta_i^{(f)})$, $i \geq 1$, are i.i.d. under \mathbf{Q} .

For any vertex $x \in \mathbb{T}$, let x_i be the ancestor of x in the i -th generation for $0 \leq i \leq |x|$ (so $x_0 = \emptyset$, and $x_{|x|} = x$). Let $n \geq 1$, and let $g : \mathbb{R}^{2n} \rightarrow [0, \infty)$ be a Borel function. By definition of \mathbf{Q} , we have

$$\begin{aligned} & \mathbf{E} \left[\sum_{x: |x|=n} g \left(V(x_i), \sum_{y: \overleftarrow{y}=x_{i-1}} f(\Delta V(y)), 1 \leq i \leq n \right) \right] \\ &= \mathbf{E}_{\mathbf{Q}} \left[\frac{1}{W_n} \sum_{x: |x|=n} g \left(V(x_i), \sum_{y: \overleftarrow{y}=x_{i-1}} f(\Delta V(y)), 1 \leq i \leq n \right) \right], \end{aligned}$$

which, according to (3.3), is

$$\begin{aligned} &= \mathbf{E}_{\mathbf{Q}} \left[\sum_{x: |x|=n} e^{V(x)} \mathbf{1}_{\{w_n=x\}} g \left(V(x_i), \sum_{y: \overleftarrow{y}=x_{i-1}} f(\Delta V(y)), 1 \leq i \leq n \right) \right] \\ &= \mathbf{E}_{\mathbf{Q}} \left[e^{V(w_n)} g \left(V(w_i), \sum_{y: \overleftarrow{y}=w_{i-1}} f(\Delta V(y)), 1 \leq i \leq n \right) \right]. \end{aligned}$$

In our notation, this means that

$$(3.5) \quad \begin{aligned} & \mathbf{E} \left[\sum_{x: |x|=n} g \left(V(x_i), \sum_{y: \overleftarrow{y}=x_{i-1}} f(\Delta V(y)), 1 \leq i \leq n \right) \right] \\ &= \mathbf{E}_{\mathbf{Q}} \left[e^{S_n} g \left(S_i, \eta_i^{(f)}, 1 \leq i \leq n \right) \right]. \end{aligned}$$

A special case of (3.5) is of particular interest: for any $n \geq 1$ and any Borel function $g : \mathbb{R}^n \rightarrow \mathbb{R}_+$,

$$(3.6) \quad \mathbf{E} \left[\sum_{x: |x|=n} g(V(x_1), \dots, V(x_n)) \right] = \mathbf{E}_{\mathbf{Q}} \left[e^{S_n} g(S_1, \dots, S_n) \right].$$

This is the so-called many-to-one formula, and can also be directly checked by induction on n without using (3.3). An immediate consequence of (3.6) is that the assumption (1.2) yields $\mathbf{E}_{\mathbf{Q}}(S_1) = 0$, whereas the assumption (1.3) implies that

$$\mathbf{E}_{\mathbf{Q}}(e^{aS_1}) < \infty, \quad \forall 0 \leq a < \delta.$$

The existence of some finite exponential moments allows us to use the last displayed formula² on page 1229 of Chang [22] to see that there exists a constant $c_2 > 0$ satisfying

$$(3.7) \quad \sup_{b>0} \mathbf{E}_{\mathbf{Q}} \left[\exp(c_2 \Delta S_{H_b^{(S)}}) \right] < \infty,$$

²More precisely, we apply the formula of Chang [22] to the ladder height of our mean-zero random walk via the Theorem on page 250 of Doney [27].

where

$$(3.8) \quad \Delta S_i := S_i - S_{i-1}, \quad i \geq 1,$$

$$(3.9) \quad H_r^{(S)} := \inf\{i \geq 0 : S_i \geq r\}, \quad r \geq 0.$$

The formula (3.5), stated for any given generation n , remains valid if n is replaced by \mathcal{H}_r , with $\mathcal{H}_r := \{x \in \mathbb{T} : V(x) \geq r, \bar{V}(\overleftarrow{x}) < r\}$ as in (2.4). Indeed, according to Proposition 3 of [6], for any $r > 0$ and any measurable functions f and g ,

$$(3.10) \quad \begin{aligned} & \mathbf{E} \left[\sum_{x \in \mathcal{H}_r} g(V(x_i), \sum_{y: \overleftarrow{y}=x_{i-1}} f(\Delta V(y)), 1 \leq i \leq |x|) \right] \\ &= \mathbf{E}_{\mathbf{Q}} \left[\exp(S_{H_r^{(S)}}) g(S_i, \eta_i^{(f)}, 1 \leq i \leq H_r^{(S)}) \right], \end{aligned}$$

where $\eta_i^{(f)} := \sum_{y: \overleftarrow{y}=w_{i-1}} f(\Delta V(y))$ as before. We recall that $(S_i - S_{i-1}, \eta_i^{(f)})$, $i \geq 1$, are i.i.d. under \mathbf{Q} .

In particular, we have the following analogue of the many-to-one formula for \mathcal{H}_r :

$$(3.11) \quad \mathbf{E} \left[\sum_{x \in \mathcal{H}_r} g(V(x_1), \dots, V(x_{|x|})) \right] = \mathbf{E}_{\mathbf{Q}} \left[\exp(S_{H_r^{(S)}}) g(S_1, \dots, S_{H_r^{(S)}}) \right].$$

4. MULTISCALE POTENTIAL ENERGY OPTIMIZATION AND PROOF OF THEOREM 1.1

The proof of Theorem 1.1 relies on a multiscale optimization analysis of the potential theory. The main ingredients are summarized in a couple of lemmas, stated as Lemmas 4.1 and 4.2 below. Lemma 4.2, rather technical, consists of three estimates, namely, (4.10), (4.11) and (4.12). Here is how the proofs are organized:

- Subsection 4.1: proof of Theorem 1.1, by admitting Lemmas 4.1 and 4.2.
- Subsection 4.2: proof of Lemma 4.1.
- Subsection 4.3: proof of Lemma 4.2, the inequality (4.10).
- Subsection 4.4: proof of Lemma 4.2, the inequality (4.11).
- Subsection 4.5: proof of Lemma 4.2, the inequality (4.12).

Throughout the section, we assume (1.2) and (1.3).

For any $x \in \mathbb{T} \cup \{\overleftarrow{\emptyset}\}$, let

$$(4.1) \quad T_x := \inf\{n \geq 0 : X_n = x\}, \quad (\inf \emptyset := \infty)$$

which stands for the first hitting time of the vertex x by the biased walk. [In the special case $x := \overleftarrow{\emptyset}$, (4.1) is in agreement with (2.7).] For $r > 0$, we recall from (2.6) that

$$T_{\mathcal{H}_r} := \inf\{i \geq 0 : X_i \in \mathcal{H}_r\},$$

where $\mathcal{H}_r := \{x \in \mathbb{T} : V(x) \geq r, \bar{V}(\overleftarrow{x}) < r\}$ as in (2.4).

Our first preliminary result is as follows. Its innocent-looking statement masks the multiscale nature.

Lemma 4.1. *Assume (1.2) and (1.3). We have³*

$$\limsup_{r \rightarrow \infty} \frac{1}{(2r)^{1/2}} \log \mathbf{E}[P_\omega(T_{\mathcal{H}_r} < T_{\overleftarrow{\emptyset}})] \leq -1.$$

³Of course, $\mathbf{E}[P_\omega(\dots)]$ is nothing else but $\mathbf{E}(\dots)$.

The statement of the second lemma requires the introduction of the basic multiscale setting in our analysis of the potential energy. Let $r > 0$. Let $\chi \in (\frac{1}{2}, 1)$. Let

$$(4.2) \quad k := \lfloor r^{1-\chi} \rfloor,$$

$$(4.3) \quad h_m := \frac{r}{k} m, \quad 0 \leq m \leq k,$$

$$(4.4) \quad \lambda_m := (2r)^{1/2} \left(\frac{k-m+1}{k} \right)^{1/2}, \quad 1 \leq m \leq k.$$

[The value of λ_m defined in (4.4) represents, in fact, the solution to an optimization problem.] For any $x \in \mathbb{T}$ and any $0 \leq s \leq \bar{V}(x)$ (for definition of $\bar{V}(x)$, see (2.5)), let

$$(4.5) \quad H_s^{(x)} = \inf \left\{ i \geq 0 : V(x_i) \geq s, V(x_j) < s, \forall j \in [0, i) \right\}.$$

In words, $H_s^{(x)}$ is the generation of the oldest vertex in the path $[\emptyset, x]$ such that the value of the branching random walk $V(\cdot)$ is at least s .

For $x \in \mathcal{H}_r := \{x \in \mathbb{T} : V(x) \geq r, \bar{V}(x) < r\}$, we set⁴

$$(4.6) \quad a_i^{(x)} := \lambda_m, \quad \text{if } H_{h_{m-1}}^{(x)} \leq i < H_{h_m}^{(x)} \text{ for } m \in [1, k].$$

In order to control the increments of the potential along the children of vertices in the spine, we introduce, for any vertex $x \in \mathbb{T}$, the following quantity

$$(4.7) \quad \Lambda(x) := \sum_{y: \overleftarrow{y}=x} e^{-\Delta V(y)} = \sum_{y: \overleftarrow{y}=x} e^{-[V(y)-V(x)]}.$$

Let $c_1 > 0$ be the constant in (3.2). Fix $\varepsilon > 0$, $\beta \geq 0$, $0 < \varepsilon_1 < c_1 \varepsilon$ and $\theta \in (\frac{1}{2}, \chi)$.⁵ We consider the following subset of \mathcal{H}_r :

$$(4.8) \quad \mathcal{H}_r^* := \left\{ x \in \mathcal{H}_r : \max_{1 \leq m < k} \Delta V(x_{H_{h_m}^{(x)}}) \leq r^\theta, \underline{V}(x) \geq -\beta, |x| < \lfloor e^{\varepsilon_1 r^{1/2}} \rfloor, \right. \\ \left. \bar{V}(x_j) - V(x_j) \leq a_j^{(x)}, \forall 0 \leq j < |x|, \max_{0 \leq j < |x|} \Lambda(x_j) \leq e^{\varepsilon r^{1/2}} \right\},$$

where $\Delta V(y) := V(y) - V(\overleftarrow{y})$ as in (3.4), $\Lambda(x) := \sum_{y: \overleftarrow{y}=x} e^{-\Delta V(y)}$ as in (4.7), and

$$\underline{V}(y) := \min_{z \in [\emptyset, y]} V(z),$$

for all $y \in \mathbb{T}$. See Figure 1.

Define $Z_r = Z_r(\varepsilon, \varepsilon_1, \beta, \theta, \chi)$ by

$$(4.9) \quad Z_r := \sum_{x \in \mathcal{H}_r^*} \mathbf{1}_{\{T_x < T_{\overleftarrow{\emptyset}}\}}.$$

The reason for which we are interested in Z_r is the obvious relation $\{T_{\mathcal{H}_r} < T_{\overleftarrow{\emptyset}}\} \supset \{Z_r > 0\}$.

In the definition of Z_r , everything depends only on the random potential $V(\cdot)$, except for T_x and $T_{\overleftarrow{\emptyset}}$, both of which depend also on the movement of the biased random walk (X_i) .

The following lemma, which is the main technical result of the paper, summarizes the moment properties of Z_r we need in the multiscale analysis of the potential energy.

⁴As such, $a_i^{(x)}$ is well defined for all $0 \leq i < H_r^{(x)} = |x|$ (for $x \in \mathcal{H}_r$). The value of $a_i^{(x)}$ for $i = H_r^{(x)}$ plays no role. [One can, for example, set $a_i^{(x)} := a_{i-1}^{(x)}$ for $i = H_r^{(x)}$.]

⁵For Lemma 4.2, we can take any $\theta \in (\frac{\chi}{2}, \chi)$, but condition $\max_{1 \leq m \leq k} \Delta V(x_{H_{h_{m-1}}^{(x)}}) \leq r^\theta$ is also exploited in Section 4.2 in the proof of Lemma 4.1, where θ needs to be greater than $\frac{1}{2}$. In order to avoid any possibility of confusion, we take $\theta \in (\frac{1}{2}, \chi)$ once for all.

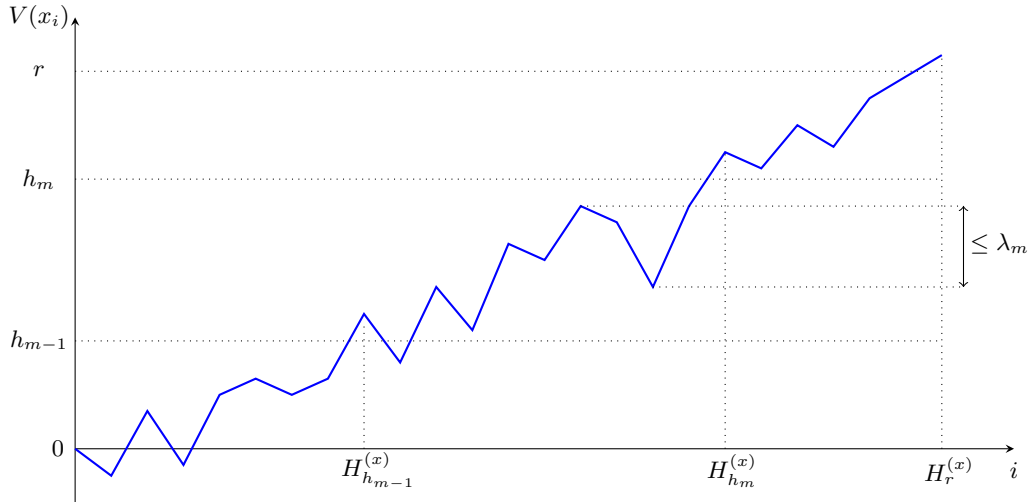


FIGURE 1. An illustration of $(V(x_i), 0 \leq i \leq H_r^{(x)})$ when $x \in \mathcal{H}_r^*$. We have $h_k = r$ and $1 \leq m \leq k$. For $H_{h_{m-1}}^{(x)} \leq i < H_{h_m}^{(x)}$, $a_i^{(x)} = \lambda_m$, and $\bar{V}(x_i) - V(x_i) \leq \lambda_m$.

Lemma 4.2. *Assume (1.2) and (1.3). For any $0 < \varepsilon_1 < c_1 \varepsilon$, $\beta \geq 0$ and $\frac{1}{2} < \theta < \chi < 1$, we have*

$$(4.10) \quad \liminf_{r \rightarrow \infty} \frac{1}{(2r)^{1/2}} \log \mathbf{E}[E_\omega(Z_r)] \geq -1 - \frac{\varepsilon_1}{2^{1/2}},$$

$$(4.11) \quad \limsup_{r \rightarrow \infty} \frac{1}{(2r)^{1/2}} \log \mathbf{E}[E_\omega(Z_r^2)] \leq -1 + 2^{1/2}(\varepsilon + \varepsilon_1),$$

$$(4.12) \quad \limsup_{r \rightarrow \infty} \frac{1}{(2r)^{1/2}} \log \mathbf{E}[(E_\omega Z_r)^2] \leq -2 + 2^{1/2} \varepsilon.$$

By admitting Lemmas 4.1 and 4.2 for the time being, we are ready to prove Theorem 1.1.

4.1. Proof of Theorem 1.1. We have seen in Section 2 that the proof of Theorem 1.1 consists of verifying (2.8), of which we recall the statement: under the assumptions (1.2) and (1.3), \mathbf{P} -almost surely on the set of non-extinction,

$$(2.8) \quad \lim_{r \rightarrow \infty} \frac{1}{(2r)^{1/2}} \log P_\omega(T_{\mathcal{H}_r} < T_{\emptyset}^-) = -1.$$

Lemma 4.1 is useful in the proof of the upper bound in (2.8), and Lemma 4.2 the lower bound.

We start with the proof of the upper bound, by means of Lemma 4.1. Let

$$\mathbf{P}^*(\cdot) := \mathbf{P}(\cdot \mid \text{non-extinction}).$$

By Lemma 4.1 and the Markov inequality,

$$\mathbf{P}^*\{P_\omega(T_{\mathcal{H}_r} < T_{\emptyset}^-) > e^{-(1-\varepsilon)(2r)^{1/2}}\} \leq e^{-c_3(2r)^{1/2}},$$

for some $c_3 = c_3(\varepsilon) > 0$ and all sufficiently large r . An application of the Borel–Cantelli lemma yields that with \mathbf{P}^* -probability 1, for all sufficiently large *integer* numbers $r > 0$, $P_\omega(T_{\mathcal{H}_r} < T_{\emptyset}^-) \leq e^{-(1-\varepsilon)(2r)^{1/2}}$. Since $r \rightarrow T_{\mathcal{H}_r}$ is non-decreasing, we can remove the condition that r be integer. As a consequence,

$$\limsup_{r \rightarrow \infty} \frac{1}{(2r)^{1/2}} \log P_\omega(T_{\mathcal{H}_r} < T_{\emptyset}^-) \leq -1, \quad \mathbf{P}^*\text{-a.s.},$$

which is the desired upper bound in (2.8).

We now turn to the proof of the lower bound. Since $\mathbf{E}[P_\omega\{Z_r > 0\}] = (\mathbf{P} \otimes P_\omega)\{Z_r > 0\}$, it follows from the Cauchy–Schwarz inequality that

$$\mathbf{E}[P_\omega\{Z_r > 0\}] \geq \frac{\{\mathbf{E}[E_\omega(Z_r)]\}^2}{\mathbf{E}[E_\omega(Z_r^2)]}.$$

Applying (4.10) and (4.11) of Lemma 4.2 yields that

$$(4.13) \quad \liminf_{r \rightarrow \infty} \frac{1}{(2r)^{1/2}} \log \mathbf{E}[P_\omega\{Z_r > 0\}] \geq -1 - 2^{1/2}(\varepsilon + \varepsilon_1) - 2^{1/2} \varepsilon_1.$$

On the other hand, by the Markov inequality, $P_\omega\{Z_r > 0\} \leq E_\omega(Z_r)$, so it follows from (4.12) of Lemma 4.2 that

$$(4.14) \quad \limsup_{r \rightarrow \infty} \frac{1}{(2r)^{1/2}} \log \mathbf{E}[(P_\omega\{Z_r > 0\})^2] \leq -2 + 2^{1/2} \varepsilon.$$

Recall (a special case of) the Paley–Zygmund inequality: for any non-negative random variable ξ , we have $\mathbf{P}\{\xi > \frac{1}{2}\mathbf{E}(\xi)\} \geq \frac{1}{4} \frac{\{\mathbf{E}(\xi)\}^2}{\mathbf{E}(\xi^2)}$. We apply it to $\xi := P_\omega\{Z_r > 0\}$. In view of (4.13) and (4.14), we obtain: for any $\varepsilon_2 > 6\varepsilon + 8\varepsilon_1$ and all sufficiently large r ,

$$\mathbf{P}\{P_\omega\{Z_r > 0\} > e^{-(1+\varepsilon_2)(2r)^{1/2}}\} \geq e^{-\varepsilon_2 r^{1/2}}.$$

Let

$$(4.15) \quad \gamma_r := P_\omega(T_{\mathcal{H}_r} < T_{\emptyset}^{\leftarrow}).$$

Since $\{T_{\mathcal{H}_r} < T_{\emptyset}^{\leftarrow}\} \supset \{Z_r > 0\}$, we have $\gamma_r \geq P_\omega\{Z_r > 0\}$. Consequently, for all sufficiently large $r > 0$,

$$(4.16) \quad \mathbf{P}\{\gamma_r > e^{-(1+\varepsilon_2)(2r)^{1/2}}\} \geq e^{-\varepsilon_2 r^{1/2}}.$$

As this stage, it is convenient to have the following preliminary estimate. Recall from (2.5) that $\bar{V}(x) := \max_{z \in [\emptyset, x]} V(z)$.

Claim 4.3. *Let $c_4 > 0$ be a constant satisfying (4.21) below. Let $0 < \alpha < \frac{1}{2}$. Let*

$$\mu_L := \mathbf{E} \left(\sum_{x: |x|=L} \mathbf{1}_{\{V(x) \geq L^\alpha\}} \mathbf{1}_{\{\bar{V}(x) < 2L^\alpha\}} \mathbf{1}_{\{\prod_{j=0}^{L-1} [1 + \Lambda(x_j)] \leq e^{c_4 L}\}} \right),$$

where $\Lambda(x) := \sum_{y: \overleftarrow{y}=x} e^{-\Delta V(y)}$ as in (4.7). Then $\lim_{L \rightarrow \infty} \mu_L = \infty$.

Proof of Claim 4.3. By (3.5), we have

$$\mu_L = \mathbf{E}_{\mathbf{Q}} \left(e^{S_L} \mathbf{1}_{\{S_L \geq L^\alpha\}} \mathbf{1}_{\{\bar{S}_L < 2L^\alpha\}} \mathbf{1}_{\{\prod_{j=1}^L (1 + \eta_j) \leq e^{c_4 L}\}} \right),$$

where $(S_j - S_{j-1}, \eta_j)$, $j \geq 1$, are i.i.d. random vectors under \mathbf{Q} , with $\eta_1 := \sum_{y: |y|=1} e^{-V(y)}$, and

$$(4.17) \quad \bar{S}_j := \max_{0 \leq i \leq j} S_i, \quad j \geq 0.$$

Hence

$$(4.18) \quad \begin{aligned} \mu_L &\geq e^{L^\alpha} \mathbf{Q} \left\{ S_L \geq L^\alpha, \bar{S}_L < 2L^\alpha, \prod_{j=1}^L (1 + \eta_j) \leq e^{c_4 L} \right\} \\ &\geq e^{L^\alpha} \left[\mathbf{Q} \{ S_L \geq L^\alpha, \bar{S}_L < 2L^\alpha \} - \mathbf{Q} \left\{ \prod_{j=1}^L (1 + \eta_j) > e^{c_4 L} \right\} \right]. \end{aligned}$$

We claim that for some constants $c_5 > 0$ and $c_6 > 0$,

$$(4.19) \quad \liminf_{L \rightarrow \infty} L^{\frac{3}{2}-2\alpha} \mathbf{Q}\{S_L \geq L^\alpha, \bar{S}_L < 2L^\alpha\} \geq c_5,$$

$$(4.20) \quad \limsup_{L \rightarrow \infty} \frac{1}{L} \log \mathbf{Q}\left\{\prod_{j=1}^L (1 + \eta_j) > e^{c_4 L}\right\} \leq -c_6.$$

It is clear that Claim 4.3 will follow from (4.19) and (4.20).

To check (4.19), we use $\mathbf{Q}\{S_L \geq L^\alpha, \bar{S}_L < 2L^\alpha\} \geq \mathbf{Q}\{L^\alpha \leq S_L < 2L^\alpha, \bar{S}_{L-1} \leq S_L\}$. Since $(S_L - S_{L-i}, 0 \leq i \leq L)$ is distributed as $(S_i, 0 \leq i \leq L)$, the latter probability equals $\mathbf{Q}\{L^\alpha \leq S_L < 2L^\alpha, S_i \geq 0, \forall 1 \leq i \leq L\}$, which can be written as $\mathbf{Q}\{S_i \geq 0, \forall 1 \leq i \leq L\} \times \mathbf{Q}\{L^\alpha \leq S_L < 2L^\alpha \mid S_i \geq 0, \forall 1 \leq i \leq L\}$. It is known (Kozlov [40]) that $L^{1/2} \mathbf{Q}\{S_i \geq 0, \forall 1 \leq i \leq L\}$ converges (when $L \rightarrow \infty$) to a positive limit, whereas according to Caravenna [21], $\liminf_{L \rightarrow \infty} L^{1-2\alpha} \mathbf{Q}\{L^\alpha \leq S_L < 2L^\alpha \mid S_i \geq 0, \forall 1 \leq i \leq L\} > 0$. This yields (4.19).

The proof of (4.20) is also elementary. Let $\delta_1 \in (0, 1]$. By the Markov inequality,

$$\mathbf{Q}\left\{\prod_{j=1}^L (1 + \eta_j) > e^{c_4 L}\right\} \leq \left\{e^{-\delta_1 c_4} \mathbf{E}_{\mathbf{Q}}[(1 + \eta_1)^{\delta_1}]\right\}^L \leq \left\{e^{-\delta_1 c_4} [1 + \mathbf{E}_{\mathbf{Q}}(\eta_1^{\delta_1})]\right\}^L.$$

Note that $\mathbf{E}_{\mathbf{Q}}(\eta_1^{\delta_1}) = \mathbf{E}_{\mathbf{Q}}[(\sum_{|y|=1} e^{-V(y)})^{\delta_1}] < \infty$ if we choose $\delta_1 := \min\{c_1, 1\}$ (see (3.2)). So, as long as

$$(4.21) \quad c_4 > \frac{\log[1 + \mathbf{E}_{\mathbf{Q}}(\eta_1^{\delta_1})]}{\delta_1},$$

we have $e^{-\delta_1 c_4} [1 + \mathbf{E}_{\mathbf{Q}}(\eta_1^{\delta_1})] < 1$, which yields (4.20). Claim 4.3 is proved. \square

We continue with our proof of Theorem 1.1, or more precisely, of the lower bound in (2.8). By Claim 4.3, we are entitled to choose and **fix** an integer L such that $\mu_L > 1$.

Let us construct a super-critical Galton–Watson $\mathbb{G}^{(L)}$ which is a sub-tree of \mathbb{T} . The vertices in $\mathbb{G}_1^{(L)}$, the first generation of $\mathbb{G}^{(L)}$, are those $x \in \mathbb{T}$ with $|x| = L$ such that

$$V(x) \geq L^\alpha, \quad \bar{V}(x) < 2L^\alpha, \quad \prod_{j=0}^{L-1} [1 + \Lambda(x_j)] \leq e^{c_4 L},$$

where $\Lambda(x) := \sum_{y: \overleftarrow{y}=x} e^{-\Delta V(y)}$ as in (4.7). More generally, for any $n \geq 2$, the vertices in $\mathbb{G}_n^{(L)}$, the n -th generation of $\mathbb{G}^{(L)}$, are those $x \in \mathbb{T}$ with $|x| = nL$ such that $V(x) - V(x^*) \geq L^\alpha$, that $\max_{(n-1)L \leq i \leq nL} [V(x_i) - V(x^*)] < 2L^\alpha$ and that $\prod_{j=(n-1)L}^{nL-1} [1 + \Lambda(x_j)] \leq e^{c_4 L}$, where x^* is the parent in $\mathbb{G}_{n-1}^{(L)}$ of x (so $x^* = x_{(n-1)L}$ as a matter of fact).

Let $c_4 > 0$ be a constant satisfying (4.21). Let $\mathcal{H}_s := \{x \in \mathbb{T} : V(x) \geq s, \bar{V}(\overleftarrow{x}) < s\}$ as defined in (2.4). Let

$$\mathcal{H}_s := \left\{x \in \mathcal{H}_s : \prod_{j=0}^{|x|-1} [1 + \Lambda(x_j)] \leq e^{2c_4 L^{1-\alpha} s}, |x| \leq 2L^{1-\alpha} s, V(x) \leq 4s\right\}.$$

We need an elementary result.

Claim 4.4. *For $n \geq 1$ and $s \in [2nL^\alpha, 2(n+1)L^\alpha]$,*

$$(4.22) \quad \#\mathcal{H}_s \geq \sum_{y \in \mathbb{G}_n^{(L)}} \mathbf{1}_{\{\exists z \in \mathbb{G}_{2n+2}^{(L)} : y < z\}}.$$

Proof of Claim 4.4. Let $y \in \mathbb{G}_n^{(L)}$ be such that there exists $z \in \mathbb{G}_{2n+2}^{(L)}$ with $y < z$. By definition of $\mathbb{G}^{(L)}$, we have $V(y) < 2nL^\alpha \leq s$ and $V(z) \geq (2n+2)L^\alpha \geq s$. So there exists $x \in \llbracket y, z \rrbracket$ such that $x \in \mathcal{H}_s$. Since x is a descendant of y , all we need is to check that $x \in \mathcal{H}_s$.

Since $z \in \mathbb{G}_{2n+2}^{(L)}$, we have, by definition of $\mathbb{G}^{(L)}$, $\prod_{j=0}^{|z|-1} [1 + \Lambda(z_j)] \leq e^{c_4(2n+2)L}$, and a fortiori (using $x \leq z$), $\prod_{j=0}^{|x|-1} [1 + \Lambda(x_j)] \leq e^{c_4(2n+2)L} \leq e^{4c_4nL} \leq e^{2c_4L^{1-\alpha}s}$.

On the other hand, $|x| \leq |z| = (2n+2)L \leq 4nL \leq 2L^{1-\alpha}s$.

Finally, $V(x) \leq (2n+2)2L^\alpha \leq 8nL^\alpha \leq 4s$. As a conclusion, $x \in \mathcal{H}_s$. \square

We come back to the proof of the lower bound in (2.8). We use the trivial inequality

$$\sum_{y \in \mathbb{G}_n^{(L)}} \mathbf{1}_{\{\exists z \in \mathbb{G}_{2n+2}^{(L)}: y < z\}} \geq \sum_{y \in \mathbb{G}_n^{(L)}} \mathbf{1}_{\{\text{the sub-tree in } \mathbb{G}^{(L)} \text{ rooted at } y \text{ survives}\}}.$$

Since $\mathbb{G}^{(L)}$ is supercritical, there exist constants $c_7 > 0$ and $c_8 > 0$ such that for all sufficiently large n ,

$$\mathbf{P}\left\{ \sum_{y \in \mathbb{G}_n^{(L)}} \mathbf{1}_{\{\exists z \in \mathbb{G}_{2n+2}^{(L)}: y < z\}} \geq e^{c_7 n} \right\} \geq c_8.$$

Applying Claim 4.4, we see that there exists a constant $c_9 > 0$ such that for all sufficiently large s ,

$$(4.23) \quad \mathbf{P}\{\#\mathcal{H}_s \geq e^{c_9 s}\} \geq c_8.$$

Let $r > 4s$. We have

$$\gamma_r := P_\omega(T_{\mathcal{H}_r} < T_{\emptyset}^{\leftarrow}) \geq \sum_{x \in \mathcal{H}_s} P_\omega\{T_{\mathcal{H}_s} < T_{\emptyset}^{\leftarrow}, X_{T_{\mathcal{H}_s}} = x\} \gamma_{r-V(x)}^{(x)},$$

where, conditionally on $\mathcal{F}_{\mathcal{H}_s}$, $(\gamma_t^{(x)}, t \geq 0)$, for $x \in \mathcal{H}_s$, are independent copies of $(\gamma_t, t \geq 0)$, and are independent of $\mathcal{F}_{\mathcal{H}_s}$. [For $x \in \mathcal{H}_s$, we have $V(x) \leq 4s < r$, so $\gamma_{r-V(x)}^{(x)}$ is well defined.] For $x \in \mathcal{H}_s$, and with the notation $\Lambda(x) := \sum_{y: \overleftarrow{y}=x} e^{-\Delta V(y)}$ from (4.7),

$$P_\omega\{T_{\mathcal{H}_s} < T_{\emptyset}^{\leftarrow}, X_{T_{\mathcal{H}_s}} = x\} \geq \prod_{j=1}^{|x|} \omega(x_{j-1}, x_j) = \frac{e^{-V(x)}}{\prod_{j=0}^{|x|-1} [1 + \Lambda(x_j)]};$$

on the other hand, by definition of \mathcal{H}_s , we have $\prod_{j=0}^{|x|-1} [1 + \Lambda(x_j)] \leq e^{2c_4L^{1-\alpha}s}$ and $V(x) \leq 4s$ for $x \in \mathcal{H}_s$. Consequently, for $x \in \mathcal{H}_s$,

$$P_\omega\{T_{\mathcal{H}_s} < T_{\emptyset}^{\leftarrow}, X_{T_{\mathcal{H}_s}} = x\} \geq e^{-(4+2c_4L^{1-\alpha})s}.$$

Hence, writing $c_{10} := 4 + 2c_4L^{1-\alpha}$, we have

$$\gamma_r \geq e^{-c_{10}s} \sum_{x \in \mathcal{H}_s} \gamma_{r-s}^{(x)} \geq e^{-c_{10}s} \max_{x \in \mathcal{H}_s} \gamma_{r-s}^{(x)}.$$

Applying (4.16) to γ_{r-s} implies that if $r-s$ is sufficiently large,

$$\begin{aligned} \mathbf{P}\{\gamma_r \geq e^{-c_{10}s} e^{-(1+\varepsilon_2)(2(r-s))^{1/2}}\} &\geq 1 - \mathbf{E}\{(1 - e^{-\varepsilon_2(r-s)^{1/2}})^{\#\mathcal{H}_s}\} \\ &\geq 1 - \mathbf{E}\{e^{-e^{-\varepsilon_2(r-s)^{1/2}} \#\mathcal{H}_s}\} \\ &\geq (1 - e^{-e^{-\varepsilon_2(r-s)^{1/2}} e^{c_9 s}}) \mathbf{P}\{\#\mathcal{H}_s \geq e^{c_9 s}\}. \end{aligned}$$

By (4.23), $\mathbf{P}\{\#\mathcal{H}_s \geq e^{c_9 s}\} \geq c_8$ if s is sufficiently large. As a consequence, for all sufficiently large s and $r-s$,

$$\mathbf{P}\{\gamma_r \geq e^{-c_{10}s} e^{-(1+\varepsilon_2)(2(r-s))^{1/2}}\} \geq c_8 [1 - e^{-e^{-\varepsilon_2(r-s)^{1/2}} e^{c_9 s}}].$$

We take $s := \frac{2}{c_9} \varepsilon_2 r^{1/2}$, and see that for $\varepsilon_3 := (1 + \frac{2^{1/2} c_{10}}{c_9}) \varepsilon_2$, there exists $c_{11} \in (0, 1)$ such that for all sufficiently large r , say $r \geq r_0$,

$$(4.24) \quad \mathbf{P}\{\gamma_r \geq e^{-(1+\varepsilon_3)(2r)^{1/2}}\} \geq c_{11}.$$

Let J_1 be an integer such that $(1 - c_{11})^{J_1} < \varepsilon_3$. Let $\mathbf{P}^*(\cdot) := \mathbf{P}(\cdot \mid \text{non-extinction})$ as before. Under \mathbf{P}^* , the system survives almost surely, so there exists an integer J_2 such that $\mathbf{P}^*\{\sum_{|x|=J_2} \mathbf{1} > J_1\} > 1 - \varepsilon_3$. Let r_1 be sufficiently large such that $\mathbf{P}^*\{\sum_{|x|=J_2} \mathbf{1}_{\{V(x) < r_1\}} \geq J_1\} \geq 1 - \varepsilon_3$. We observe that for $r \geq r_1$,

$$\begin{aligned} \gamma_r &\geq \max_{y: |y|=J_2, V(y) < r_1} P_\omega\{T_y < T_\emptyset^{\leftarrow}\} P_\omega^y\{T_{\mathcal{H}_r} < T_\emptyset^{\leftarrow}\} \\ &\geq c_{12}(\omega) \max_{y: |y|=J_2, V(y) < r_1} P_\omega^y\{T_{\mathcal{H}_r} < T_\emptyset^{\leftarrow}\}, \end{aligned}$$

where $c_{12}(\omega) := \min_{y: |y|=J_2, V(y) < r_1} P_\omega\{T_y < T_\emptyset^{\leftarrow}\} > 0$ \mathbf{P} -a.s. (notation: $\min_\emptyset := 1$, $\max_\emptyset := 0$).

For $|y| = J_2$ with $V(y) < r_1$, conditionally on $V(y)$, $P_\omega^y\{T_{\mathcal{H}_r} < T_\emptyset^{\leftarrow}\}$ is distributed as $\gamma_{r-V(y)}$, which is greater than or equal to γ_r . It follows from (4.24) that for $r \geq \max\{r_1, r_0\}$,

$$\begin{aligned} \mathbf{P}\{\gamma_r \geq c_{12}(\omega) e^{-(1+\varepsilon_3)(2r)^{1/2}}\} &\geq \mathbf{P}\left\{\max_{y: |y|=J_2, V(y) < r_1} P_\omega^y\{T_{\mathcal{H}_r} < T_\emptyset^{\leftarrow}\} \geq e^{-(1+\varepsilon_3)(2r)^{1/2}}\right\} \\ &\geq (1 - (1 - c_{11})^{J_1}) \mathbf{P}\left\{\sum_{|x|=J_2} \mathbf{1}_{\{V(x) < r_1\}} \geq J_1\right\}. \end{aligned}$$

By definition of r_1 , we have $\mathbf{P}\{\sum_{|x|=J_2} \mathbf{1}_{\{V(x) < r_1\}} \geq J_1\} \geq (1 - \varepsilon_3)(1 - q)$, where $q := \mathbf{P}\{\text{extinction}\} < 1$. Therefore, for $r \geq \max\{r_1, r_0\}$,

$$\mathbf{P}\{\gamma_r \geq c_{12}(\omega) e^{-(1+\varepsilon_3)(2r)^{1/2}}\} \geq (1 - (1 - c_{11})^{J_1})(1 - \varepsilon_3)(1 - q) \geq (1 - \varepsilon_3)^2(1 - q),$$

the last inequality following from the definition of J_1 . Since $c_{12}(\omega) > 0$ \mathbf{P} -a.s., we have proved that

$$\mathbf{P}^*\left\{\liminf_{r \rightarrow \infty} \frac{\log \gamma_r}{(2r)^{1/2}} \geq -1 - \varepsilon_3\right\} \geq (1 - \varepsilon_3)^2.$$

Recall the definition $\varepsilon_3 := (1 + \frac{2^{1/2} c_{10}}{c_9}) \varepsilon_2$, with $\varepsilon_2 > 6\varepsilon + 8\varepsilon_1$, $\varepsilon > 0$ and $\varepsilon_1 \in (0, c_1 \varepsilon)$; so $\varepsilon_3 > 0$ can be taken arbitrarily small. This yields the lower bound in (2.8), and thus completes the proof of Theorem 1.1 by admitting Lemmas 4.1 and 4.2. \square

The rest of the section is devoted to the proof of Lemmas 4.1 and 4.2.

4.2. Proof of Lemma 4.1. In the study of one-dimensional random walks, a frequent type of technical difficulties is to handle the overshoots. Such difficulties are, unfortunately, present throughout the proof of both Lemmas 4.1 and 4.2.

Let $r > 0$. Let $\chi \in (0, 1)$. Recall from (4.2)–(4.3) that

$$k := \lfloor r^{1-\chi} \rfloor, \quad h_m := \frac{r}{k} m, \quad 0 \leq m \leq k.$$

Recall from (2.4) that $\mathcal{H}_r := \{x \in \mathbb{T} : V(x) \geq r, \bar{V}(x) < r\}$. We distinguish the vertices x of \mathcal{H}_r according to whether there are some ‘‘large overshoots’’ of the random potential $V(\cdot)$ along the path $[\emptyset, x]$: let $\theta \in (\frac{1}{2}, \chi)$, and let

$$\begin{aligned} \mathcal{H}_{r,+} &:= \left\{x \in \mathcal{H}_r : \max_{1 \leq m < k} \Delta V(x_{H_{hm}^{(x)}}) > r^\theta\right\}, \\ \mathcal{H}_{r,-} &:= \left\{x \in \mathcal{H}_r : \max_{1 \leq m < k} \Delta V(x_{H_{hm}^{(x)}}) \leq r^\theta\right\}, \end{aligned}$$

where, as before, $\Delta V(y) := V(y) - V(\overleftarrow{y})$ for any vertex $y \in \mathbb{T} \setminus \{\emptyset\}$.

Recall from (2.6) that

$$T_{\mathcal{H}_r} = \inf_{x \in \mathcal{H}_r} T_x = \min \left\{ \inf_{x \in \mathcal{H}_{r,+}} T_x, \inf_{x \in \mathcal{H}_{r,-}} T_x \right\},$$

where $T_x := \inf\{i \geq 0 : X_i = x\}$ as in (4.1). So

$$(4.25) \quad P_\omega(T_{\mathcal{H}_r} < T_{\overleftarrow{\emptyset}}) \leq \sum_{x \in \mathcal{H}_{r,+}} P_\omega(T_x < T_{\overleftarrow{\emptyset}}) + P_\omega\left(\inf_{x \in \mathcal{H}_{r,-}} T_x < T_{\overleftarrow{\emptyset}}\right).$$

We first bound $\sum_{x \in \mathcal{H}_{r,+}} P_\omega(T_x < T_{\overleftarrow{\emptyset}})$. By a one-dimensional argument (Goloso [30]), for any $x, y \in \mathbb{T}$ with $y < x$,

$$(4.26) \quad P_\omega\{T_x < T_{\overleftarrow{\emptyset}} \mid X_0 = y\} = \frac{\sum_{u \in [\emptyset, y]} e^{V(u)}}{\sum_{u \in [\emptyset, x]} e^{V(u)}}.$$

In particular, for any $x \in \mathbb{T} \setminus \{\emptyset\}$,

$$(4.27) \quad P_\omega\{T_x < T_{\overleftarrow{\emptyset}}\} = \frac{1}{\sum_{u \in [\emptyset, x]} e^{V(u)}} \leq e^{-\overline{V}(x)}.$$

Hence

$$\sum_{x \in \mathcal{H}_{r,+}} P_\omega(T_x < T_{\overleftarrow{\emptyset}}) \leq \sum_{x \in \mathcal{H}_{r,+}} e^{-\overline{V}(x)} = \sum_{x \in \mathcal{H}_{r,+}} e^{-V(x)},$$

the last identity following from the fact that $\overline{V}(x) = V(x)$ for all $x \in \mathcal{H}_{r,+}$. Taking expectation with respect to \mathbf{E} on both sides, we obtain, by means of (3.11),

$$\mathbf{E}\left[\sum_{x \in \mathcal{H}_{r,+}} P_\omega(T_x < T_{\overleftarrow{\emptyset}})\right] \leq \mathbf{Q}\left[\max_{1 \leq m < k} \Delta S_{H_{h_m}^{(s)}} > r^\theta\right] \leq \sum_{m=1}^{k-1} \mathbf{Q}\left[\Delta S_{H_{h_m}^{(s)}} > r^\theta\right].$$

We use (3.7) to see that for some constant $c_{13} > 0$,

$$\mathbf{E}\left[\sum_{x \in \mathcal{H}_{r,+}} P_\omega(T_x < T_{\overleftarrow{\emptyset}})\right] \leq c_{13} (k-1) e^{-c_2 r^\theta} = c_{13} ([r^{1-\chi}] - 1) e^{-c_2 r^\theta}.$$

Recall that $\theta > \frac{1}{2}$. In view of (4.25), the proof of Lemma 4.1 is reduced to showing the following:

$$(4.28) \quad \limsup_{r \rightarrow \infty} \frac{1}{(2r)^{1/2}} \log \mathbf{E}\left[P_\omega\left(\inf_{x \in \mathcal{H}_{r,-}} T_x < T_{\overleftarrow{\emptyset}}\right)\right] \leq -1.$$

For any vertex $x \in \mathcal{H}_r$, let us recall $a_j^{(x)}$ from (4.6), and define

$$\tau_x := \inf\{j : 1 \leq j \leq |x|, \overline{V}(x_j) - V(x_j) \geq a_j^{(x)}\}. \quad (\inf \emptyset := \infty)$$

For $x \in \mathcal{H}_r$, we have either $\tau_x < |x|$ (with strict inequality), or $\tau_x = \infty$. We observe that

$$\begin{aligned} \inf_{x \in \mathcal{H}_{r,-}} T_x &= \min \left\{ \inf_{x \in \mathcal{H}_{r,-} : \tau_x < |x|} T_x, \inf_{x \in \mathcal{H}_{r,-} : \tau_x = \infty} T_x \right\} \\ &\geq \min \left\{ \inf_{x \in \mathcal{H}_{r,-} : \tau_x < |x|} T_{y(x)}, \inf_{x \in \mathcal{H}_{r,-} : \tau_x = \infty} T_x \right\}, \end{aligned}$$

where $y(x) := x_{\tau_x}$. Hence

$$(4.29) \quad \begin{aligned} P_\omega\left(\inf_{x \in \mathcal{H}_{r,-}} T_x < T_{\overleftarrow{\emptyset}}\right) &\leq P_\omega\left(\inf_{x \in \mathcal{H}_{r,-} : \tau_x < |x|} T_{y(x)} < T_{\overleftarrow{\emptyset}}\right) + \sum_{x \in \mathcal{H}_{r,-} : \tau_x = \infty} P_\omega\{T_x < T_{\overleftarrow{\emptyset}}\} \\ &=: \Sigma_1 + \Sigma_2, \end{aligned}$$

with obvious notation. It is easy to get an upper bound for Σ_2 : by (4.27), $P_\omega\{T_x < T_{\frac{x}{\vartheta}}\} \leq e^{-\bar{V}(x)}$ (which is $e^{-V(x)}$ for $x \in \mathcal{H}_{r,-}$), whereas $\tau_x = \infty$ implies $\bar{V}(x_i) - V(x_i) < a_i^{(x)}$, $\forall i < |x|$, so

$$(4.30) \quad \Sigma_2 \leq \sum_{x \in \mathcal{H}_r} e^{-V(x)} \mathbf{1}_{\{\bar{V}(x_i) - V(x_i) < a_i^{(x)}, \forall i < |x|\}} \prod_{m=1}^{k-1} \mathbf{1}_{\{\Delta V(x_{H_{h_m}^{(x)}})} \leq r^\theta\}}.$$

To bound Σ_1 , we note that

$$\inf_{x \in \mathcal{H}_{r,-} : \tau_x < |x|} T_{y(x)} = \inf\{T_y : \exists x \in \mathcal{H}_{r,-}, y = x_{\tau_x}, \tau_x < |x|\}.$$

Let $y \in \mathbb{T}$ with $j := |y| \geq 1$ such that $h_{m-1} \leq \bar{V}(y) < h_m$ for some $m \in [1, k]$. We define

$$a_i^{(y)} := \begin{cases} \lambda_\ell, & \text{if } H_{h_{\ell-1}}^{(y)} \leq i < H_{h_\ell}^{(y)} \text{ for } \ell \in [1, m), \\ \lambda_m, & \text{if } H_{h_{m-1}}^{(y)} \leq i \leq j. \end{cases}$$

Clearly, if $y = x_{\tau_x}$ for some $x \in \mathcal{H}_{r,-}$ satisfying $\tau_x < |x|$, then $\bar{V}(y_i) - V(y_i) < a_i^{(y)}$, $\forall i < j$, and $\bar{V}(y_j) - V(y_j) \geq \lambda_m$, and moreover $\Delta V(y_{H_{h_\ell}^{(y)}}) \leq r^\theta$, $\forall 1 \leq \ell < m$. Accordingly,

$$\begin{aligned} \Sigma_1 &\leq \sum_{m=1}^k \sum_{j=1}^{\infty} \sum_{|y|=j} \mathbf{1}_{\{h_{m-1} \leq \bar{V}(y) < h_m\}} \mathbf{1}_{\{\bar{V}(y_i) - V(y_i) < a_i^{(y)}, \forall i < j; \bar{V}(y_j) - V(y_j) \geq \lambda_m\}} \times \\ &\quad \times \left(\prod_{\ell=1}^{m-1} \mathbf{1}_{\{\Delta V(y_{H_{h_\ell}^{(y)}})} \leq r^\theta\}} \right) P_\omega\{T_y < T_{\frac{y}{\vartheta}}\}. \end{aligned}$$

Again, by (4.27), we have $P_\omega\{T_y < T_{\frac{y}{\vartheta}}\} \leq e^{-\bar{V}(y)}$. This gives the analogue of (4.30) for Σ_1 .

We apply the many-to-one formula in (3.6). Recall from (3.9) that $H_u^{(S)} := \inf\{i \geq 0 : S_i \geq u\}$ (for $u \geq 0$), and from (3.8) that $\Delta S_i := S_i - S_{i-1}$. Define

$$(4.31) \quad a_i^{(S)} := \lambda_m, \quad \text{if } H_{h_{m-1}}^{(S)} \leq i < H_{h_m}^{(S)} \text{ and } 1 \leq m \leq k.$$

By (3.6),

$$\begin{aligned} \mathbf{E}(\Sigma_1) &\leq \sum_{m=1}^k \sum_{j=1}^{\infty} \mathbf{E}_{\mathbf{Q}} \left[e^{-(\bar{S}_j - S_j)} \mathbf{1}_{\{h_{m-1} \leq \bar{S}_j < h_m\}} \mathbf{1}_{\{\bar{S}_i - S_i < a_i^{(S)}, \forall i < j; \bar{S}_j - S_j \geq \lambda_m\}} \times \right. \\ &\quad \left. \times \prod_{\ell=1}^{m-1} \mathbf{1}_{\{\Delta S_{H_{h_\ell}^{(S)}}} \leq r^\theta\}} \right] \\ (4.32) \quad &\leq \sum_{m=1}^k \sum_{j=1}^{\infty} e^{-\lambda_m} \mathbf{E}_{\mathbf{Q}} \left[\mathbf{1}_{\{h_{m-1} \leq \bar{S}_j < h_m\}} \mathbf{1}_{\{\bar{S}_i - S_i < a_i^{(S)}, \forall i < j\}} \prod_{\ell=1}^{m-1} \mathbf{1}_{\{\Delta S_{H_{h_\ell}^{(S)}}} \leq r^\theta\}} \right]. \end{aligned}$$

Similarly, applying (3.11) in place of (3.6) to $\mathbf{E}(\Sigma_2)$, we obtain:

$$(4.33) \quad \mathbf{E}(\Sigma_2) \leq \mathbf{Q} \left\{ \bar{S}_i - S_i < a_i^{(S)}, \forall 1 \leq i < H_r^{(S)}; \max_{1 \leq \ell < k} \Delta S_{H_{h_\ell}^{(S)}} \leq r^\theta \right\}.$$

At this stage, we have two preliminary results.

Claim 4.5. *For any integers $1 \leq m_0 \leq m < k$ and any $s \in (-\infty, h_{m_0})$, we define*

$$(4.34) \quad f_{m_0, m}(s) := \mathbf{Q} \left(\bigcap_{\ell=m_0+1}^{m+1} \left\{ \max_{i \in [H_{h_{\ell-1}-s}^{(S)}, H_{h_\ell}^{(S)}]} (\bar{S}_i - S_i) < \lambda_\ell \right\} \cap \bigcap_{\ell=m_0}^m \left\{ \Delta S_{H_{h_\ell}^{(S)}} \leq r^\theta \right\} \right).$$

Then, as $r \rightarrow \infty$,

$$(4.35) \quad \sup_{s < h_{m_0}} f_{m_0, m}(s) \leq e^{-(1+o(1)) \sum_{\ell=m_0+1}^{m+1} \frac{r^\chi}{\lambda_\ell}},$$

uniformly in $1 \leq m_0 \leq m < k$. Furthermore,

$$(4.36) \quad \mathbf{Q} \left(\bigcap_{\ell=1}^{m+1} \left\{ \max_{i \in [H_{h_{\ell-1}}^{(S)}, H_{h_\ell}^{(S)}]} (\bar{S}_i - S_i) < \lambda_\ell \right\} \cap \bigcap_{\ell=1}^m \left\{ \Delta S_{H_{h_\ell}^{(S)}} \leq r^\theta \right\} \right) \leq e^{-(1+o(1)) \sum_{\ell=1}^{m+1} \frac{r^\chi}{\lambda_\ell}},$$

uniformly in $1 \leq m < k$.

Claim 4.6. *There exists a constant $c_{14} > 0$ such that for $r \rightarrow \infty$,*

$$(4.37) \quad \sum_{j=1}^{\infty} \mathbf{E}_{\mathbf{Q}} \left[\mathbf{1}_{\{h_{m-1} \leq \bar{S}_j < h_m\}} \mathbf{1}_{\{\bar{S}_i - S_i < a_i^{(S)}, \forall i < j\}} \prod_{\ell=1}^{m-1} \mathbf{1}_{\{\Delta S_{H_{h_\ell}^{(S)}} \leq r^\theta\}} \right] \\ \leq c_{14} r \exp \left(- (1 + o(1)) \sum_{\ell=1}^{m-1} \frac{r^\chi}{\lambda_\ell} \right),$$

uniformly in $m \in [1, k]$.

Proof of Claim 4.5. Applying the strong Markov property successively at $H_{h_{m-s}}^{(S)}$, $H_{h_{m-1-s}}^{(S)}$, \dots , $H_{h_{m_0-s}}^{(S)}$, we obtain that

$$f_{m_0, m}(s) \leq \prod_{\ell=m_0+1}^{m+1} \sup_{u \in [0, r^\theta]} \mathbf{Q} \left(\max_{0 \leq i < H_{h_\ell - h_{\ell-1} - u}^{(S)}} (\bar{S}_i - S_i) < \lambda_\ell \right).$$

By Lemma A.3, we arrive at the following estimate: when $r \rightarrow \infty$,

$$f_{m_0, m}(s) \leq \exp \left(- (1 + o(1)) \sum_{\ell=m_0+1}^{m+1} \frac{h_\ell - h_{\ell-1} - r^\theta}{\lambda_\ell} \right) \leq \exp \left(- (1 + o(1)) \sum_{\ell=m_0+1}^{m+1} \frac{r^\chi}{\lambda_\ell} \right),$$

uniformly in $s < h_{m_0}$ and in $1 \leq m_0 \leq m < k$;⁶ this yields (4.35). The proof of (4.36) is along the same lines. \square

Proof of Claim 4.6. Let $\text{LHS}_{(4.37)}$ denote the sum on the left-hand side of (4.37). Then

$$\text{LHS}_{(4.37)} = \mathbf{E}_{\mathbf{Q}} \left[\left(\prod_{\ell=1}^{m-1} \mathbf{1}_{\{\Delta S_{H_{h_\ell}^{(S)}} \leq r^\theta\}} \right) \sum_{j=H_{h_{m-1}}^{(S)}}^{H_{h_m}^{(S)}-1} \mathbf{1}_{\{\bar{S}_i - S_i < a_i^{(S)}, \forall i < j\}} \right].$$

By definition of $a_i^{(S)}$ in (4.31), this yields that

$$\text{LHS}_{(4.37)} = \mathbf{E}_{\mathbf{Q}} \left[\sum_{j=H_{h_{m-1}}^{(S)}}^{H_{h_m}^{(S)}-1} \mathbf{1}_{\{\bar{S}_i - S_i < \lambda_m, \forall i \in [H_{h_{m-1}}^{(S)}, j]\}} \times \right. \\ \left. \times \left(\prod_{\ell=1}^{m-1} \mathbf{1}_{\{\max_{i \in [H_{h_{\ell-1}}^{(S)}, H_{h_\ell}^{(S)}]} (\bar{S}_i - S_i) < \lambda_\ell\}} \cap \{\Delta S_{H_{h_\ell}^{(S)}} \leq r^\theta\} \right) \right].$$

⁶Since $h_m - h_{m-1} = \frac{r}{k}$ (by (4.3)), it is here we use the condition $\theta < \chi$ to ensure $h_m - h_{m-1} - r^\theta > 0$.

We proceed to get rid of the sum over j on the right-hand side. Applying the strong Markov property at time $H_{h_{m-1}}^{(S)}$, we have

$$(4.38) \quad \text{LHS}_{(4.37)} \leq \mathbf{E}_{\mathbf{Q}} \left[\left(\prod_{\ell=1}^{m-1} \mathbf{1}_{\{\max_{i \in [H_{h_{\ell-1}}^{(S)}, H_{h_{\ell}}^{(S)}]} (\bar{S}_i - S_i) < \lambda_{\ell}\} \cap \{\Delta S_{H_{h_{\ell}}^{(S)}} \leq r^{\theta}\}} \right) \Xi_m \right],$$

where

$$\begin{aligned} \Xi_m &:= \sup_{x \in [h_m - h_{m-1} - r^{\theta}, h_m - h_{m-1}]} \mathbf{E}_{\mathbf{Q}} \left(\sum_{j=0}^{H_x^{(S)} - 1} \mathbf{1}_{\{\bar{S}_i - S_i < \lambda_m, \forall i \in [0, j]\}} \right) \\ &\leq \mathbf{E}_{\mathbf{Q}} \left(\sum_{j=0}^{\infty} \mathbf{1}_{\{\bar{S}_i - S_i < \lambda_m, \forall i \in [0, j]\}} \right). \end{aligned}$$

To estimate the expectation on the right-hand side, we write $\sum_{j=0}^{\infty} = \sum_{n=1}^{\infty} \sum_{j=(n-1)\lambda_m^2}^{n\lambda_m^2-1}$ (by implicitly treating λ_m^2 as an integer; otherwise we replace λ_m by $\lceil \lambda_m \rceil$, and the next three paragraphs will still go through with obvious modifications), so that

$$\begin{aligned} \Xi_m &\leq \sum_{n=1}^{\infty} \mathbf{E}_{\mathbf{Q}} \left(\sum_{j=(n-1)\lambda_m^2}^{n\lambda_m^2-1} \mathbf{1}_{\{\bar{S}_i - S_i < \lambda_m, \forall i \in [0, j]\}} \right) \\ &\leq \sum_{n=1}^{\infty} \lambda_m^2 \mathbf{Q} \left\{ \max_{0 \leq i < (n-1)\lambda_m^2} (\bar{S}_i - S_i) < \lambda_m \right\}. \end{aligned}$$

By the Markov property, $\mathbf{Q} \{ \max_{0 \leq i < (n-1)\lambda_m^2} (\bar{S}_i - S_i) < \lambda_m \} \leq [\mathbf{Q} \{ \max_{0 \leq i < \lambda_m^2} (\bar{S}_i - S_i) < \lambda_m \}]^{n-1}$. So

$$\Xi_m \leq \sum_{n=1}^{\infty} \lambda_m^2 \left[\mathbf{Q} \left\{ \max_{0 \leq i < \lambda_m^2} (\bar{S}_i - S_i) < \lambda_m \right\} \right]^{n-1}.$$

We let $r \rightarrow \infty$ (so that $\lambda_m \rightarrow \infty$ uniformly in $m \in [1, k]$). By Donsker's theorem, $\mathbf{Q} \{ \max_{0 \leq i < \lambda_m^2} (\bar{S}_i - S_i) < \lambda_m \} \rightarrow \mathbb{P} \{ \sup_{s \in [0, 1]} (\bar{W}_s - W_s) < \frac{1}{\sigma} \} < 1$, where $(W_s, s \geq 0)$ under \mathbb{P} is a standard Brownian motion, and $\bar{W}_s := \sup_{u \in [0, s]} W_u$. So there exists a constant $0 < c_{15} < 1$ such that for all sufficiently large r and all $m \in [1, k]$, $\mathbf{Q} \{ \max_{0 \leq i < \lambda_m^2} (\bar{S}_i - S_i) < \lambda_m \} \leq 1 - c_{15}$, which, in turn, yields that

$$\Xi_m \leq \sum_{n=1}^{\infty} \lambda_m^2 (1 - c_{15})^{n-1} = \frac{\lambda_m^2}{c_{15}} \leq \frac{2r}{c_{15}}.$$

Going back to (4.38), this yields that for all sufficiently large r (writing $c_{16} := \frac{2}{c_{15}}$),

$$\text{LHS}_{(4.37)} \leq c_{16} r \mathbf{Q} \left(\bigcap_{\ell=1}^{m-1} \left\{ \max_{i \in [H_{h_{\ell-1}}^{(S)}, H_{h_{\ell}}^{(S)}]} (\bar{S}_i - S_i) < \lambda_{\ell} \right\} \cap \left\{ \Delta S_{H_{h_{\ell}}^{(S)}} \leq r^{\theta} \right\} \right).$$

This implies Claim 4.6 in case $2 \leq m < k$ by means of (4.36), and trivially in case $m = 1$. \square

We continue with the proof of Lemma 4.1. By (4.32) and Claim 4.6, we have

$$\mathbf{E}(\Sigma_1) \leq c_{14} r \sum_{m=1}^k \exp \left(-\lambda_m - (1 + o(1)) \sum_{\ell=1}^{m-1} \frac{r^{\lambda}}{\lambda_{\ell}} \right).$$

By definition, $k := \lfloor r^{1-\chi} \rfloor$ and $\lambda_m := (2r)^{1/2} \left(\frac{k-m+1}{k}\right)^{1/2}$; hence for $r \rightarrow \infty$,

$$(4.39) \quad \sum_{\ell=1}^{m+1} \frac{r^\chi}{\lambda_\ell} = (2r^\chi)^{1/2} [k^{1/2} - (k-m)^{1/2}] + o((2r)^{1/2}),$$

uniformly in $1 \leq m_0 \leq m < k$. In particular,

$$(4.40) \quad \sum_{\ell=1}^k \frac{r^\chi}{\lambda_\ell} \sim (2r)^{1/2}.$$

So uniformly in $m \in [1, k]$,

$$\begin{aligned} & \lambda_m + (2r^\chi)^{1/2} [k^{1/2} - (k-m+1)^{1/2}] \\ & \geq (1+o(1))(2r)^{1/2} \inf_{s \in [0,1]} \left((1-s)^{1/2} + [1 - (1-s)^{1/2}] \right), \end{aligned}$$

and the infimum equals 1 because the function $s \mapsto (1-s)^{1/2} + [1 - (1-s)^{1/2}]$ is identically 1 on $[0, 1]$. Therefore,

$$\mathbf{E}(\Sigma_1) \leq c_{14} r k e^{-(1+o(1))(2r)^{1/2}} \leq e^{-(1+o(1))(2r)^{1/2}},$$

the second inequality being a consequence of definition $k := \lfloor r^{1-\chi} \rfloor$.

On the other hand, by (4.33) and (4.36) (applied to $m := k-1$), we have

$$\mathbf{E}(\Sigma_2) \leq e^{-(1+o(1)) \sum_{\ell=1}^k \frac{r^\chi}{\lambda_\ell}} \leq e^{-(1+o(1))(2r)^{1/2}},$$

the second inequality being a consequence of (4.39) (applied to $m := k-1$). Since $P_\omega(\inf_{x \in \mathcal{H}_r} T_x < T_\emptyset) \leq \Sigma_1 + \Sigma_2$ (see (4.29)), this yields (4.28), and completes the proof of Lemma 4.1. \square

The rest of the section is devoted to the proof of Lemma 4.2, which is more technical. For the sake of clarity, we prove the three parts (4.10), (4.11) and (4.12) separately.

4.3. Proof of Lemma 4.2: the inequality (4.10). Recall from (4.9) the definition $Z_r := \sum_{x \in \mathcal{H}_r^*} \mathbf{1}_{\{T_x < T_\emptyset\}}$, where

$$\begin{aligned} \mathcal{H}_r^* & := \left\{ x \in \mathcal{H}_r : \max_{1 \leq m < k} \Delta V(x_{H_{hm}^{(x)}}) \leq r^\theta, \underline{V}(x) \geq -\beta, |x| < \lfloor e^{\varepsilon_1 r^{1/2}} \rfloor, \right. \\ & \left. \overline{V}(x_j) - V(x_j) \leq a_j^{(x)}, \forall 0 \leq j < |x|, \max_{0 \leq \ell < |x|} \Lambda(x_\ell) \leq e^{\varepsilon r^{1/2}} \right\}, \end{aligned}$$

with $\Lambda(x) := \sum_{y: \overleftarrow{y}=x} e^{-\Delta V(y)}$ as in (4.7). For brevity, we write, in this subsection,

$$n = n(\varepsilon_1, r) := \lfloor e^{\varepsilon_1 r^{1/2}} \rfloor;$$

so $|x| + 1 \leq n$ for all $x \in \mathcal{H}_r^*$. Since only T_x and T_\emptyset depend on the biased walk (X_i) , we have

$$(4.41) \quad E_\omega(Z_r) = \sum_{x \in \mathcal{H}_r^*} P_\omega\{T_x < T_\emptyset\}.$$

By the identity in (4.27), we have $P_\omega\{T_x < T_{\varnothing}\} \geq \frac{1}{|x|+1}e^{-\bar{V}(x)}$, which is $\geq \frac{1}{n}e^{-\bar{V}(x)} = \frac{1}{n}e^{-V(x)}$ for all $x \in \mathcal{H}_r^*$. Taking expectation with respect to \mathbf{E} on both sides leads to:

$$\begin{aligned} \mathbf{E}[E_\omega(Z_r)] &\geq \frac{1}{n} \mathbf{E} \left[\sum_{x \in \mathcal{H}_r^*} e^{-V(x)} \right] \\ &= \frac{1}{n} \mathbf{E} \left[\sum_{x \in \mathcal{H}_r} e^{-V(x)} \mathbf{1}_{\{\bar{V}(x_j) - V(x_j) < a_j^{(x)}, \forall 0 \leq j < |x|\}} \mathbf{1}_{\{V(x) \geq -\beta\}} \times \right. \\ &\quad \left. \times \mathbf{1}_{\{|x| < n\}} \mathbf{1}_{\{\Lambda(x_\ell) \leq e^{\varepsilon r^{1/2}}, \forall 0 \leq \ell < |x|\}} \prod_{m=1}^{k-1} \mathbf{1}_{\{\Delta V(x_{H_{h_m}^{(x)}})} \leq r^\theta\}} \right]. \end{aligned}$$

The expression on the right-hand side is, according to formula (3.10),

$$\begin{aligned} &= \frac{1}{n} \mathbf{Q} \left[\bigcap_{j=0}^{H_r^{(S)}-1} \{\bar{S}_j - S_j < a_j^{(S)}, S_j \geq -\beta\} \cap \right. \\ &\quad \left. \cap \{H_r^{(S)} < n\} \cap \bigcap_{\ell=1}^{H_r^{(S)}} \{\eta_\ell \leq e^{\varepsilon r^{1/2}}\} \cap \bigcap_{m=1}^{k-1} \{\Delta S_{H_{h_m}^{(S)}} \leq r^\theta\} \right], \end{aligned}$$

where $H_r^{(S)} := \inf\{i \geq 0 : S_i \geq r\}$ as in (3.9), $\bar{S}_j := \max_{0 \leq i \leq j} S_i$ as in (4.17), $\Delta S_j := S_j - S_{j-1}$ as before (with $S_0 := 0$), and $\eta_\ell := \sum_{y: \leftarrow y = w_{\ell-1}} e^{-\Delta V(y)}$. [In particular, $\eta_1 := \sum_{y: |y|=1} e^{-V(y)}$.] Recall from Section 3 that $(\Delta S_i, \eta_i)$, $i \geq 1$, are i.i.d. random vectors under \mathbf{Q} . Hence

$$(4.42) \quad \mathbf{E}[E_\omega(Z_r)] \geq \frac{1}{n} [q_1(r) - q_2(r)],$$

where

$$\begin{aligned} q_1(r) &:= \mathbf{Q} \left[\bigcap_{j=0}^{H_r^{(S)}-1} \{\bar{S}_j - S_j < a_j^{(S)}, S_j \geq -\beta\} \cap \{H_r^{(S)} < n\} \cap \bigcap_{m=1}^{k-1} \{\Delta S_{H_{h_m}^{(S)}} \leq r^\theta\} \right], \\ q_2(r) &:= \mathbf{Q} \left[\bigcap_{j=0}^{H_r^{(S)}-1} \{\bar{S}_j - S_j < a_j^{(S)}\} \cap \bigcap_{m=1}^{k-1} \{\Delta S_{H_{h_m}^{(S)}} \leq r^\theta\} \cap \bigcup_{\ell=1}^{H_r^{(S)} \wedge n} \{\eta_\ell > e^{\varepsilon r^{1/2}}\} \right]. \end{aligned}$$

By definition of $(a_j^{(S)})$ in (4.31) (with notation $\Delta S_0 := 0$ for the term $m = 1$ below),

$$\begin{aligned} q_1(r) &= \mathbf{Q} \left(\{H_r^{(S)} < n\} \cap \right. \\ &\quad \left. \cap \bigcap_{m=1}^k \bigcap_{j=H_{h_{m-1}}^{(S)}}^{H_{h_m}^{(S)}-1} \{\bar{S}_j - S_j < \lambda_m, S_j \geq -\beta\} \cap \{\Delta S_{H_{h_{m-1}}^{(S)}} \leq r^\theta\} \right). \end{aligned}$$

Since $\{H_r^{(S)} < n\} \supset \cap_{m=1}^k \{H_{h_m}^{(S)} - H_{h_{m-1}}^{(S)} < \lfloor \frac{n}{k} \rfloor\}$, we have

$$\begin{aligned} q_1(r) &\geq \mathbf{Q} \left\{ \bigcap_{m=1}^k \bigcap_{j=H_{h_{m-1}}^{(S)}}^{H_{h_m}^{(S)}-1} \{\bar{S}_j - S_j < \lambda_m, S_j \geq -\beta\} \cap \right. \\ &\quad \left. \cap \{\Delta S_{H_{h_{m-1}}^{(S)}} \leq r^\theta, H_{h_m}^{(S)} - H_{h_{m-1}}^{(S)} < \lfloor \frac{n}{k} \rfloor\} \right\} \\ &\geq \mathbf{Q} \left\{ \bigcap_{m=1}^k \bigcap_{j=H_{h_{m-1}}^{(S)}}^{H_{h_m}^{(S)}-1} \{\bar{S}_j - S_j < \lambda_m, S_j - S_{H_{h_{m-1}}^{(S)}} \geq -\beta\} \cap \right. \\ &\quad \left. \cap \{\Delta S_{H_{h_{m-1}}^{(S)}} \leq r^\theta, H_{h_m}^{(S)} - H_{h_{m-1}}^{(S)} < \lfloor \frac{n}{k} \rfloor\} \right\}. \end{aligned}$$

Recall that $h_m - h_{m-1} = h_1$. Applying the strong Markov property successively at times $H_{h_{k-1}}^{(S)}, H_{h_{k-2}}^{(S)}, \dots, H_{h_1}^{(S)}$, this gives that⁷

$$(4.43) \quad q_1(r) \geq \prod_{m=1}^k \inf_{x \in (r^\theta, h_1]} \mathbf{Q} \left\{ \bigcap_{j=0}^{H_x^{(S)}-1} \{\bar{S}_j - S_j < \lambda_m, S_j \geq -\beta\} \cap \right. \\ \left. \cap \{\Delta S_{H_x^{(S)}} \leq r^\theta, H_x^{(S)} < \lfloor \frac{n}{k} \rfloor\} \right\}.$$

We let $r \rightarrow \infty$. By Lemma A.2, uniformly in $m \in [1, k]$ and $x \in (r^\theta, h_1]$,

$$\begin{aligned} \mathbf{Q} \left\{ \bigcap_{j=0}^{H_x^{(S)}-1} \{\bar{S}_j - S_j < \lambda_m, S_j \geq -\beta\} \right\} &\geq \exp \left[- (1 + o(1)) \frac{x}{\lambda_m} \right] \\ &\geq \exp \left[- (1 + o(1)) \frac{r^\theta}{\lambda_m} \right]. \end{aligned}$$

On the other hand, (3.7) tells us that $c_{17} := \sup_{b>0} \mathbf{E}_{\mathbf{Q}}[\exp(c_2 \Delta S_{H_b^{(S)}})] < \infty$. By the Markov inequality, for $r \rightarrow \infty$, uniformly in $m \in [1, k]$ and $x \in (r^\theta, h_1]$,

$$\mathbf{Q}\{\Delta S_{H_x^{(S)}} > r^\theta\} \leq c_{17} e^{-c_2 r^\theta} \leq \frac{1}{3} \exp \left[- (1 + o(1)) \frac{r^\theta}{\lambda_m} \right].$$

[The last inequality, valid for all sufficiently large r , relies on the facts that $\theta > \frac{\chi}{2}$ and that $\lambda_m \geq (2r^\chi)^{1/2}$.] Also, for some constant $c_{18} > 0$ and all sufficiently large r and all $m \in [1, k]$, $\sup_{x \in (r^\theta, h_1]} \mathbf{Q}\{H_x^{(S)} \geq \lfloor \frac{n}{k} \rfloor\} \leq c_{18} \frac{h_1}{(\lfloor \frac{n}{k} \rfloor)^{1/2}}$ (see Theorem A of Kozlov [40]), which is bounded by $\frac{1}{3} \exp[-(1 + o(1)) \frac{r^\chi}{\lambda_m}]$ as well for some constant $\varepsilon_1 > 0$ (for $r \rightarrow \infty$; recalling that $n := \lfloor e^{\varepsilon_1} r^{1/2} \rfloor$). [We use the fact that $\frac{1}{2} > \frac{\chi}{2}$.] As a consequence, for $r \rightarrow \infty$,

⁷For the term $m = k$ on the right-hand side, there is no need to consider $\{\Delta S_{H_x^{(S)}} \leq r^\theta\}$, whereas the $m = 1$ term has only the value $x = h_1$. The current form of the inequality is used to give a compact expression for the lower bound.

uniformly in $m \in [1, k]$ and $x \in (r^\theta, h_1]$,

$$\begin{aligned}
& \mathbf{Q} \left\{ \bigcap_{j=0}^{H_x^{(S)}-1} \{\bar{S}_j - S_j < \lambda_m, S_j \geq -\beta\} \cap \{\Delta S_{H_x^{(S)}} \leq r^\theta, H_x^{(S)} < \lfloor \frac{n}{k} \rfloor\} \right\} \\
& \geq \mathbf{Q} \left\{ \bigcap_{j=0}^{H_x^{(S)}-1} \{\bar{S}_j - S_j < \lambda_m, S_j \geq -\beta\} \right\} - \mathbf{Q} \{\Delta S_{H_x^{(S)}} > r^\theta\} - \mathbf{Q} \{H_x^{(S)} \geq \lfloor \frac{n}{k} \rfloor\} \\
& \geq \frac{1}{3} \exp \left[- (1 + o(1)) \frac{r^\chi}{\lambda_m} \right],
\end{aligned}$$

which is still $\exp[-(1 + o(1))\frac{r^\chi}{\lambda_m}]$ by changing the value of $o(1)$. Going back to (4.43), we see that for $r \rightarrow \infty$,

$$(4.44) \quad q_1(r) \geq \exp \left[- (1 + o(1)) \sum_{m=1}^k \frac{r^\chi}{\lambda_m} \right] = e^{-(1+o(1))(2r)^{1/2}},$$

the last identity following from the observation in (4.40) that $\sum_{m=1}^k \frac{r^\chi}{\lambda_m} \sim (2r)^{1/2}$, $r \rightarrow \infty$.

We now estimate $q_2(r)$. By definition,

$$\begin{aligned}
(4.45) \quad q_2(r) & \leq \sum_{\ell=1}^n \mathbf{Q} \left[\bigcap_{j=0}^{H_r^{(S)}-1} \{\bar{S}_j - S_j < a_j^{(S)}\}; \max_{1 \leq i < k} \Delta S_{H_{h_i}^{(S)}} \leq r^\theta; \eta_\ell > e^{\varepsilon r^{1/2}}; \ell \leq H_r^{(S)} \right] \\
& = \sum_{\ell=1}^n \sum_{m=1}^k q_2^{(\ell, m)}(r),
\end{aligned}$$

where

$$\begin{aligned}
& q_2^{(\ell, m)}(r) \\
& := \mathbf{Q} \left[\bigcap_{j=0}^{H_r^{(S)}-1} \{\bar{S}_j - S_j < a_j^{(S)}\}; \max_{1 \leq i < k} \Delta S_{H_{h_i}^{(S)}} \leq r^\theta; \eta_\ell > e^{\varepsilon r^{1/2}}; H_{h_{m-1}}^{(S)} < \ell \leq H_{h_m}^{(S)} \right] \\
& = \mathbf{Q} \left[\bigcap_{i=1}^k \bigcap_{j=H_{h_{i-1}}^{(S)}}^{H_{h_i}^{(S)}-1} \{\bar{S}_j - S_j < \lambda_i\}; \max_{1 \leq i < k} \Delta S_{H_{h_i}^{(S)}} \leq r^\theta; \eta_\ell > e^{\varepsilon r^{1/2}}; H_{h_{m-1}}^{(S)} < \ell \leq H_{h_m}^{(S)} \right].
\end{aligned}$$

We apply the strong Markov property at $H_{h_{k-1}}^{(S)}$, to see that for $1 \leq m < k$,

$$\begin{aligned}
q_2^{(\ell, m)}(r) & \leq \mathbf{Q} \left[\bigcap_{i=1}^{k-1} \bigcap_{j=H_{h_{i-1}}^{(S)}}^{H_{h_i}^{(S)}-1} \{\bar{S}_j - S_j < \lambda_i\}; \max_{1 \leq i < k} \Delta S_{H_{h_i}^{(S)}} \leq r^\theta; \eta_\ell > e^{\varepsilon r^{1/2}}; \right. \\
& \quad \left. H_{h_{m-1}}^{(S)} < \ell \leq H_{h_m}^{(S)} \right] \times \sup_{x \in [h_{k-1}, h_{k-1} + r^\theta]} \mathbf{Q} \left[\bigcap_{j=0}^{H_{h_k-x}^{(S)}-1} \{\bar{S}_j - S_j < \lambda_k\} \right].
\end{aligned}$$

Let $r \rightarrow \infty$. By Lemma A.3, we have, uniformly in $x \in [h_{k-1}, h_{k-1} + r^\theta]$,

$$\begin{aligned} \mathbf{Q} \left[\bigcap_{j=0}^{H_{h_{k-x}}^{(S)}-1} \{\bar{S}_j - S_j < \lambda_k\} \right] &\leq \exp \left[- (1 + o(1)) \frac{h_k - h_{k-1} - r^\theta}{\lambda_k} \right] \\ &\leq \exp \left[- (1 + o(1)) \frac{r^\chi}{\lambda_k} \right]. \end{aligned}$$

We iterate the argument and apply the strong Markov property successively at $H_{h_{k-2}}^{(S)}$, $H_{h_{k-3}}^{(S)}$, \dots , $H_{h_m}^{(S)}$, to see that

$$\begin{aligned} q_2^{(\ell, m)}(r) &\leq \mathbf{Q} \left[\bigcap_{i=1}^m \bigcap_{j=H_{h_{i-1}}^{(S)}}^{H_{h_i}^{(S)}-1} \{\bar{S}_j - S_j < \lambda_i\}; \max_{1 \leq i \leq m} \Delta S_{H_{h_i}^{(S)}} \leq r^\theta; \eta_\ell > e^{\varepsilon r^{1/2}}; \right. \\ &\quad \left. H_{h_{m-1}}^{(S)} < \ell \leq H_{h_m}^{(S)} \right] \times \exp \left[- (1 + o(1)) \sum_{i=m+1}^k \frac{r^\chi}{\lambda_i} \right] \\ &\leq \mathbf{Q} \left[\bigcap_{i=1}^{m-1} \bigcap_{j=H_{h_{i-1}}^{(S)}}^{H_{h_i}^{(S)}-1} \{\bar{S}_j - S_j < \lambda_i\}; \max_{1 \leq i \leq m-2} \Delta S_{H_{h_i}^{(S)}} \leq r^\theta; \eta_\ell > e^{\varepsilon r^{1/2}}; \right. \\ &\quad \left. H_{h_{m-1}}^{(S)} < \ell \right] \times \exp \left[- (1 + o(1)) \sum_{i=m+1}^k \frac{r^\chi}{\lambda_i} \right]. \end{aligned}$$

To bound the probability expression $\mathbf{Q}[\dots]$ on the right-hand side, we note that under \mathbf{Q} , given $H_{h_{m-1}}^{(S)} < \ell$, η_ℓ is independent of everything concerning the potential $V(\cdot)$ until $H_{h_{m-1}}^{(S)}$, and has the law of η_1 . Consequently,

$$\begin{aligned} q_2^{(\ell, m)}(r) &\leq \mathbf{Q} \left[\bigcap_{i=1}^{m-1} \bigcap_{j=H_{h_{i-1}}^{(S)}}^{H_{h_i}^{(S)}-1} \{\bar{S}_j - S_j < \lambda_i\}; \max_{1 \leq i \leq m-2} \Delta S_{H_{h_i}^{(S)}} \leq r^\theta; H_{h_{m-1}}^{(S)} < \ell \right] \times \\ &\quad \times \mathbf{Q}(\eta_1 > e^{\varepsilon r^{1/2}}) \times \exp \left[- (1 + o(1)) \sum_{i=m+1}^k \frac{r^\chi}{\lambda_i} \right] \\ &\leq \mathbf{Q} \left[\bigcap_{i=1}^{m-1} \bigcap_{j=H_{h_{i-1}}^{(S)}}^{H_{h_i}^{(S)}-1} \{\bar{S}_j - S_j < \lambda_i\}; \max_{1 \leq i \leq m-2} \Delta S_{H_{h_i}^{(S)}} \leq r^\theta \right] \times \\ &\quad \times \mathbf{Q}(\eta_1 > e^{\varepsilon r^{1/2}}) \times \exp \left[- (1 + o(1)) \sum_{i=m+1}^k \frac{r^\chi}{\lambda_i} \right]. \end{aligned}$$

Looking at the two probability expressions $\mathbf{Q}[\bigcap_{i=1}^{m-1} \dots]$ and $\mathbf{Q}(\eta_1 > e^{\varepsilon r^{1/2}})$ on the right-hand side. The first probability expression is, according to (4.36), bounded by $\exp[-(1 + o(1)) \sum_{\ell=1}^{m-1} \frac{r^\chi}{\lambda_\ell}]$. For the second probability expression, let us recall that $\eta_1 = \sum_{y: |y|=1} e^{-V(y)}$ by definition; so by (3.2), there exists a constant $c_{19} > 0$ such that $\mathbf{Q}(\eta_1 >$

$e^{\varepsilon r^{1/2}}) \leq c_{19} e^{-c_1 \varepsilon r^{1/2}}$. We have thus proved that, for $1 \leq m \leq k$,

$$\begin{aligned} q_2^{(\ell, m)}(r) &\leq c_{19} e^{-c_1 \varepsilon r^{1/2}} \exp \left[- (1 + o(1)) \sum_{i: 1 \leq i \leq k, i \neq m} \frac{r^\chi}{\lambda_i} \right] \\ &\leq c_{19} e^{-c_1 \varepsilon r^{1/2} - (1+o(1))(2r)^{1/2}}. \end{aligned}$$

Since $q_2(r) \leq \sum_{\ell=1}^n \sum_{m=1}^k q_2^{(\ell, m)}(r)$ (see (4.45)), and $n := \lfloor e^{\varepsilon_1 r^{1/2}} \rfloor \leq e^{\varepsilon_1 r^{1/2}}$, this yields that

$$q_2(r) \leq c_{19} k e^{-(c_1 \varepsilon - \varepsilon_1) r^{1/2} - (1+o(1))(2r)^{1/2}}.$$

Recall that $\mathbf{E}[E_\omega(Z_r)] \geq \frac{q_1(r) - q_2(r)}{n}$ (see (4.42)) and that $q_1(r) \geq e^{-(1+o(1))(2r)^{1/2}}$ (see (4.44)), we obtain that for $r \rightarrow \infty$,

$$\mathbf{E}[E_\omega(Z_r)] \geq \frac{1}{n} \left[e^{-(1+o(1))(2r)^{1/2}} - c_{19} k e^{-(c_1 \varepsilon - \varepsilon_1) r^{1/2} - (1+o(1))(2r)^{1/2}} \right].$$

Since $\varepsilon_1 \in (0, c_1 \varepsilon)$, the term $c_{19} k e^{-(c_1 \varepsilon - \varepsilon_1) r^{1/2} - (1+o(1))(2r)^{1/2}}$ does not play any role when taking the limit $r \rightarrow \infty$ (recalling that $k := \lfloor r^{1-\chi} \rfloor$). By definition, $n := \lfloor e^{\varepsilon_1 r^{1/2}} \rfloor$, this readily yields (4.10). \square

4.4. Proof of Lemma 4.2: the inequality (4.11). Recall definition again from (4.9):

$Z_r := \sum_{x \in \mathcal{H}_r^*} \mathbf{1}_{\{T_x < T_{\varnothing}^{\leftarrow}\}}$, where

$$\begin{aligned} \mathcal{H}_r^* &:= \left\{ x \in \mathcal{H}_r : \max_{1 \leq m < k} \Delta V(x_{H_{hm}^{(x)}}) \leq r^\theta, \underline{V}(x) \geq -\beta, |x| < \lfloor e^{\varepsilon_1 r^{1/2}} \rfloor, \right. \\ &\quad \left. \overline{V}(x_j) - V(x_j) \leq a_j^{(x)}, \forall 0 \leq j < |x|, \max_{0 \leq j < |x|} \Lambda(x_j) \leq e^{\varepsilon r^{1/2}} \right\}, \end{aligned}$$

with $\Lambda(x) := \sum_{y: \overleftarrow{y}=x} e^{-\Delta V(y)}$ as in (4.7). By definition,

$$\begin{aligned} E_\omega(Z_r^2) &= \sum_{x, y \in \mathcal{H}_r^*} P_\omega \{T_x < T_{\varnothing}^{\leftarrow}, T_y < T_{\varnothing}^{\leftarrow}\} \\ (4.46) \quad &= E_\omega(Z_r) + \sum_{x \neq y \in \mathcal{H}_r^*} P_\omega \{T_x < T_{\varnothing}^{\leftarrow}, T_y < T_{\varnothing}^{\leftarrow}\}. \end{aligned}$$

By (4.27), $P_\omega \{T_x < T_{\varnothing}^{\leftarrow}\} \leq e^{-\overline{V}(x)}$. On the other hand, by the definition of \mathcal{H}_r , we have $\overline{V}(x) = V(x)$ for $x \in \mathcal{H}_r^* \subset \mathcal{H}_r$. So

$$\begin{aligned} E_\omega(Z_r) &\leq \sum_{x \in \mathcal{H}_r^*} e^{-V(x)} \\ &\leq \sum_{x \in \mathcal{H}_r} e^{-V(x)} \mathbf{1}_{\{\max_{1 \leq m < k} \Delta V(x_{H_{hm}^{(x)}}) \leq r^\theta\}} \mathbf{1}_{\{\overline{V}(x_j) - V(x_j) \leq a_j^{(x)}, \forall 0 \leq j < |x|\}}. \end{aligned}$$

Taking expectation on both sides, we obtain that

$$\mathbf{E}[E_\omega(Z_r)] \leq \mathbf{E} \left(\sum_{x \in \mathcal{H}_r} e^{-V(x)} \mathbf{1}_{\{\max_{1 \leq m < k} \Delta V(x_{H_{hm}^{(x)}}) \leq r^\theta\}} \mathbf{1}_{\{\overline{V}(x_j) - V(x_j) \leq a_j^{(x)}, \forall 0 \leq j < |x|\}} \right),$$

which, by formula (3.11), is

$$= \mathbf{Q} \left(\max_{1 \leq m < k} \Delta S_{H_{hm}^{(S)}}, \overline{S}_j - S_j \leq a_j^{(S)}, \forall 0 \leq j < H_r^{(S)} \right).$$

Applying (4.36), we get that $\mathbf{E}[E_\omega(Z_r)] \leq e^{-(1+o(1)) \sum_{\ell=1}^k \frac{r^\chi}{\lambda_\ell}}$. Since $\sum_{\ell=1}^k \frac{r^\chi}{\lambda_\ell} \sim (2r)^{1/2}$ (see (4.40)), we arrive at:

$$(4.47) \quad \mathbf{E}[E_\omega(Z_r)] \leq e^{-(1+o(1))(2r)^{1/2}}.$$

Also, since $V(x) \geq r$ for $x \in \mathcal{H}_r^*$, we have $\sum_{x \in \mathcal{H}_r^*} e^{-2V(x)} \leq e^{-r} \sum_{x \in \mathcal{H}_r^*} e^{-V(x)}$, so for all sufficiently large r ,

$$(4.48) \quad \mathbf{E} \left(\sum_{x \in \mathcal{H}_r^*} e^{-2V(x)} \right) \leq e^{-r}.$$

By (4.47) and (4.46), we have

$$(4.49) \quad \mathbf{E}[E_\omega(Z_r^2)] \leq e^{-(1+o(1))(2r)^{1/2}} + \mathbf{E} \left[\sum_{x \neq y \in \mathcal{H}_r^*} P_\omega \{T_x < T_{\overleftarrow{\emptyset}}, T_y < T_{\overleftarrow{\emptyset}}\} \right].$$

For any pair of distinct vertices $x \neq y$, let $x \wedge y$ denote their youngest common ancestor; equivalently, $x \wedge y$ is the unique vertex satisfying $[\emptyset, x \wedge y] = [\emptyset, x] \cap [\emptyset, y]$. Consider the quenched probability expression

$$P_\omega \{T_x < T_y < T_{\overleftarrow{\emptyset}}\}.$$

To realize $T_x < T_y < T_{\overleftarrow{\emptyset}}$, the biased walk first needs to hit $x \wedge y$ before hitting $\overleftarrow{\emptyset}$, then, starting from $x \wedge y$, it should hit x before hitting $\overleftarrow{\emptyset}$, (and then, starting from x , it hits automatically $x \wedge y$ before hitting $\overleftarrow{\emptyset}$), and then, starting from $x \wedge y$, it should hit y before hitting $\overleftarrow{\emptyset}$. Applying the strong Markov property, we obtain that

$$P_\omega \{T_x < T_y < T_{\overleftarrow{\emptyset}}\} \leq P_\omega \{T_{x \wedge y} < T_{\overleftarrow{\emptyset}}\} P_\omega^{x \wedge y} \{T_x < T_{\overleftarrow{\emptyset}}\} P_\omega^{x \wedge y} \{T_y < T_{\overleftarrow{\emptyset}}\},$$

where, for any vertex z , P_ω^z denotes the (quenched) probability under which the biased walk starts at z . By exchanging x and y , we also have

$$P_\omega \{T_y < T_x < T_{\overleftarrow{\emptyset}}\} \leq P_\omega \{T_{x \wedge y} < T_{\overleftarrow{\emptyset}}\} P_\omega^{x \wedge y} \{T_y < T_{\overleftarrow{\emptyset}}\} P_\omega^{x \wedge y} \{T_x < T_{\overleftarrow{\emptyset}}\}.$$

Hence

$$\begin{aligned} P_\omega \{T_x < T_{\overleftarrow{\emptyset}}, T_y < T_{\overleftarrow{\emptyset}}\} &= P_\omega \{T_x < T_y < T_{\overleftarrow{\emptyset}}\} + P_\omega \{T_y < T_x < T_{\overleftarrow{\emptyset}}\} \\ &\leq 2P_\omega \{T_{x \wedge y} < T_{\overleftarrow{\emptyset}}\} P_\omega^{x \wedge y} \{T_x < T_{\overleftarrow{\emptyset}}\} P_\omega^{x \wedge y} \{T_y < T_{\overleftarrow{\emptyset}}\}. \end{aligned}$$

[Although we have implicitly assumed $x \wedge y$ is different from the root \emptyset , the last inequality remains trivially valid even if $x \wedge y$ is the root.] By (4.27), $P_\omega \{T_{x \wedge y} < T_{\overleftarrow{\emptyset}}\} \leq e^{-\overline{V}(x \wedge y)}$.

More generally, we use (4.26) to see that

$$P_\omega^{x \wedge y} \{T_x < T_{\overleftarrow{\emptyset}}\} \leq (|x \wedge y| + 1) e^{-[\overline{V}(x) - \overline{V}(x \wedge y)]}.$$

We also have $P_\omega^{x \wedge y} \{T_y < T_{\overleftarrow{\emptyset}}\} \leq (|x \wedge y| + 1) e^{-[\overline{V}(y) - \overline{V}(x \wedge y)]}$ by interchanging the roles of x and y . As a consequence,

$$P_\omega \{T_x < T_{\overleftarrow{\emptyset}}, T_y < T_{\overleftarrow{\emptyset}}\} \leq 2(|x \wedge y| + 1)^2 e^{\overline{V}(x \wedge y) - \overline{V}(x) - \overline{V}(y)},$$

which is bounded by $2(|x \wedge y| + 1)^2 e^{\overline{V}(x \wedge y) - V(x) - V(y)}$. Moreover, for $x \in \mathcal{H}_r^*$, we have $|x \wedge y| + 1 \leq |x| + 1 \leq \lfloor e^{\varepsilon_1 r^{1/2}} \rfloor$. Going back to (4.49), we obtain that

$$(4.50) \quad \begin{aligned} &\mathbf{E}[E_\omega(Z_r^2)] \\ &\leq e^{-(1+o(1))(2r)^{1/2}} + 2e^{2\varepsilon_1 r^{1/2}} \mathbf{E} \left(\sum_{z: \overline{V}(z) < r} \sum_{x, y \in \mathcal{H}_r^*: x \wedge y = z} e^{\overline{V}(z) - V(x) - V(y)} \right) \end{aligned}$$

$$(4.51) \quad = e^{-(1+o(1))(2r)^{1/2}} + 2e^{2\varepsilon_1 r^{1/2}} \mathbf{E} \left(\sum_{n=0}^{\infty} \sum_{m=1}^k \Sigma_3^{(n,m)} \right),$$

where

$$\Sigma_3^{(n,m)} := \sum_{z: |z|=n} e^{\bar{V}(z)} \mathbf{1}_{\{h_{m-1} \leq \bar{V}(z) < h_m\}} \sum_{x, y \in \mathcal{H}_r^*: x \wedge y = z} e^{-V(x) - V(y)}.$$

For further use, we also see from the inequality $E_\omega(Z_r) \leq \sum_{x \in \mathcal{H}_r^*} e^{-V(x)}$ that, for all sufficiently large r ,

$$(4.52) \quad \mathbf{E}[(E_\omega Z_r)^2] \leq e^{-r} + \mathbf{E} \left(\sum_{z: \bar{V}(z) < r} \mathbf{1}_{\{\underline{V}(z) \geq -\beta\}} \sum_{x, y \in \mathcal{H}_r^*: x \wedge y = z} e^{-V(x) - V(y)} \right).$$

The term e^{-r} comes from $\mathbf{E}(\sum_{x \in \mathcal{H}_r^*} e^{-2V(x)})$ and (4.48). The indicator function $\mathbf{1}_{\{\underline{V}(z) \geq -\beta\}}$ was implicitly present in $x \in \mathcal{H}_r^*$; it is written explicitly here because it is going to play a crucial role later. We note that the expectation expressions on the right-hand side of (4.50) and (4.52) are very similar to each other, except that there is no $\bar{V}(z)$ term on the right-hand side of (4.52).

For each pair (n, m) , we estimate $\mathbf{E}(\Sigma_3^{(n,m)})$. By definition (recalling that x_i is the ancestor of x in generation i for $i \leq |x|$),

$$\begin{aligned} \Sigma_3^{(n,m)} &= \sum_{z: |z|=n} e^{\bar{V}(z)} \mathbf{1}_{\{h_{m-1} \leq \bar{V}(z) < h_m\}} \sum_{u \neq v, \bar{u} = z = \bar{v}} e^{-V(u) - V(v)} \times \\ &\quad \times \sum_{x \in \mathcal{H}_r^*: x_{n+1} = u} e^{-[V(x) - V(u)]} \sum_{y \in \mathcal{H}_r^*: y_{n+1} = v} e^{-[V(y) - V(v)]}. \end{aligned}$$

We first take expectation conditioning on $\mathcal{F}_{n+1} := \sigma\{V(w) : |w| \leq n+1\}$, the σ -field generated by the random potential in the first $n+1$ generations:

$$(4.53) \quad \begin{aligned} &\mathbf{E}(\Sigma_3^{(n,m)} \mid \mathcal{F}_{n+1}) \\ &\leq \sum_{z: |z|=n} e^{\bar{V}(z)} \mathbf{1}_{\{h_{m-1} \leq \bar{V}(z) < h_m\}} \mathbf{1}_{\{\bar{V}(z_i) - V(z_i) < a_i^{(z)}, \forall 0 \leq i \leq n\}} \mathbf{1}_{\{\max_{1 \leq \ell < m} \Delta V(z_{H_{h_\ell}^{(z)}}) \leq r^\theta\}} \times \\ &\quad \times \mathbf{1}_{\{\Lambda(z) \leq e^{\varepsilon r^{1/2}}\}} \sum_{(u,v): u \neq v, \bar{u} = z = \bar{v}} e^{-V(u) - V(v)} f_m(V(u)) f_m(V(v)), \end{aligned}$$

where $\Lambda(x) := \sum_{y: \bar{y} = x} e^{-\Delta V(y)}$ as in (4.7), and for $s < h_{m+1}$,

$$f_m(s) := \mathbf{E} \left\{ \sum_{x \in \mathcal{H}_{r-s}^*} e^{-V(x)} \left(\prod_{\ell=m+1}^{k-1} \mathbf{1}_{\{\Delta V(x_{H_{h_\ell}^{(x)}})} \leq r^\theta\}} \right) \left(\prod_{\ell=m+2}^k \prod_{i=H_{h_{\ell-1}-s}^{(x)}}^{H_{h_\ell}^{(x)}-1} \mathbf{1}_{\{\bar{V}(x_i) - V(x_i) < \lambda_\ell\}} \right) \right\}.$$

Some care needs to be taken in order to make (4.53) valid in all situations. On the right-hand side of (4.53), $V(u) < h_m$ for **most** u with $\bar{u} = z$ (and $V(u) < r$ for most v with $\bar{v} = z$); however, there is a possible situation when $V(u) \geq h_m$: this is when $u \in \mathcal{H}_{h_m}$ (for some $1 \leq m \leq k$), in which case we only have $V(u) \leq h_m + r^\theta$ (which is strictly smaller than h_{m+1}). In order to take care of this situation, only overshoots $\Delta V(x_{H_{h_\ell}^{(x)}})$ for $\ell > m$ are involved in the definition of $f_m(s)$. In particular, $f_{k-1}(s) = 1$ for $s < r$, and $f_k(s)$ should be defined as 1 for all $s \in \mathbb{R}$.

By the formula (3.11), this gives that for $s < h_{m+1}$,

$$f_m(s) = \mathbf{Q} \left(\bigcap_{\ell=m+1}^{k-1} \mathbf{1}_{\{\Delta S_{H_{h_\ell}^{(s)}}} \leq r^\theta\}} \cap \bigcap_{\ell=m+2}^k \bigcap_{i=H_{h_{\ell-1}-s}^{(s)}}^{H_{h_\ell}^{(s)}-1} \{\bar{S}_i - S_i < \lambda_\ell\} \right),$$

where $H_t^{(S)} := \inf\{i \geq 0 : S_i \geq t\}$ (for any $t \geq 0$) as in (3.9). By Claim 4.5, we arrive at the following estimate: when $r \rightarrow \infty$,

$$f_m(s) \leq \exp\left(- (1 + o(1)) \sum_{\ell=m+2}^k \frac{r^\chi}{\lambda_\ell}\right),$$

uniformly in $s < h_{m+1}$ and $m \in [1, k]$ (and in $n \geq 1$).

Let us go back to (4.53), and first look at the double sum $\sum_{(u,v): u \neq v, \overleftarrow{u}=z=\overleftarrow{v}}$ on the right-hand side. Thanks to the upper bound for $f_m(s)$ we have just obtained that is valid uniformly in $s \geq 0$, we get that, on the right-hand side of (4.53),

$$\begin{aligned} & \mathbf{1}_{\{\Lambda(z) \leq e^{\varepsilon r^{1/2}}\}} \sum_{(u,v): u \neq v, \overleftarrow{u}=z=\overleftarrow{v}} e^{-V(u)-V(v)} f_m(V(u)) f_m(V(v)) \\ & \leq \mathbf{1}_{\{\Lambda(z) \leq e^{\varepsilon r^{1/2}}\}} e^{-(2+o(1)) \sum_{\ell=m+2}^k \frac{r^\chi}{\lambda_\ell}} \left[\sum_{u: \overleftarrow{u}=z} e^{-V(u)} \right]^2 \\ & \leq e^{-(2+o(1)) \sum_{\ell=m+2}^k \frac{r^\chi}{\lambda_\ell}} \left[e^{-V(z)} e^{\varepsilon r^{1/2}} \right]^2, \end{aligned}$$

where, in the last inequality, we used the definition of $\Lambda(z) := \sum_{u: \overleftarrow{u}=z} e^{-[V(u)-V(z)]}$ as in (4.7) to see that on the event $\{\Lambda(z) \leq e^{\varepsilon r^{1/2}}\}$, we have $\sum_{u: \overleftarrow{u}=z} e^{-V(u)} = e^{-V(z)} \Lambda(z) \leq e^{-V(z)} e^{\varepsilon r^{1/2}}$. Therefore, (4.53) yields that

$$\begin{aligned} \mathbf{E}(\Sigma_3^{(n,m)} | \mathcal{F}_{n+1}) & \leq e^{2\varepsilon r^{1/2} - (2+o(1)) \sum_{\ell=m+2}^k \frac{r^\chi}{\lambda_\ell}} \sum_{z: |z|=n} e^{\overline{V}(z) - 2V(z)} \mathbf{1}_{\{h_{m-1} \leq \overline{V}(z) < h_m\}} \times \\ & \quad \times \mathbf{1}_{\{\overline{V}(z_i) - V(z_i) < a_i^{(z)}, \forall 0 \leq i \leq n\}} \mathbf{1}_{\{\max_{1 \leq \ell < m} \Delta V(z_{H_{h_\ell}^{(z)}}) \leq r^\theta\}}. \end{aligned}$$

Taking expectation to get rid of the conditioning, and using the many-to-one formula (3.6), we obtain that

$$\begin{aligned} \mathbf{E}(\Sigma_3^{(n,m)}) & \leq e^{2\varepsilon r^{1/2} - (2+o(1)) \sum_{\ell=m+2}^k \frac{r^\chi}{\lambda_\ell}} \mathbf{E}_{\mathbf{Q}} \left[e^{\overline{S}_n - S_n} \mathbf{1}_{\{h_{m-1} \leq \overline{S}_n < h_m\}} \times \right. \\ & \quad \left. \times \mathbf{1}_{\{\overline{S}_i - S_i < a_i^{(S)}, \forall 0 \leq i \leq n\}} \mathbf{1}_{\{\max_{1 \leq \ell < m} \Delta S_{H_{h_\ell}^{(S)}} \leq r^\theta\}} \right]. \end{aligned}$$

Going back to (4.51), this yields that

$$(4.54) \quad \begin{aligned} \mathbf{E}[E_\omega(Z_r^2)] & \leq e^{-(1+o(1))(2r)^{1/2}} + 2e^{2\varepsilon_1 r^{1/2}} \sum_{n=0}^{\infty} \sum_{m=1}^k e^{2\varepsilon r^{1/2} - (2+o(1)) \sum_{\ell=m+2}^k \frac{r^\chi}{\lambda_\ell}} \times \\ & \quad \times \mathbf{E}_{\mathbf{Q}} \left[e^{\overline{S}_n - S_n} \mathbf{1}_{\{h_{m-1} \leq \overline{S}_n < h_m\}} \mathbf{1}_{\{\overline{S}_i - S_i < a_i^{(S)}, \forall 0 \leq i \leq n\}} \mathbf{1}_{\{\max_{1 \leq \ell < m} \Delta S_{H_{h_\ell}^{(S)}} \leq r^\theta\}} \right]. \end{aligned}$$

Similarly, (4.52) leads to: for $r \rightarrow \infty$,

$$(4.55) \quad \begin{aligned} \mathbf{E}[(E_\omega Z_r)^2] & \leq e^{-r} + \sum_{n=0}^{\infty} \sum_{m=1}^k e^{2\varepsilon r^{1/2} - (2+o(1)) \sum_{\ell=m+2}^k \frac{r^\chi}{\lambda_\ell}} \mathbf{E}_{\mathbf{Q}} \left[e^{-S_n} \mathbf{1}_{\{\min_{0 \leq i \leq n} S_i \geq -\beta\}} \times \right. \\ & \quad \left. \times \mathbf{1}_{\{h_{m-1} \leq \overline{S}_n < h_m\}} \mathbf{1}_{\{\overline{S}_i - S_i < a_i^{(S)}, \forall 0 \leq i \leq n\}} \mathbf{1}_{\{\max_{1 \leq \ell < m} \Delta S_{H_{h_\ell}^{(S)}} \leq r^\theta\}} \right]. \end{aligned}$$

We proceed with (4.54). Recall from (4.31) that $a_i^{(S)} := \lambda_\ell$ if $H_{h_{\ell-1}}^{(S)} \leq i < H_{h_\ell}^{(S)}$. In particular, $a_n^{(S)} = \lambda_m$ on the event $\{h_{m-1} \leq \overline{S}_n < h_m\}$, so $e^{\overline{S}_n - S_n} \leq e^{\lambda_m}$ on $\{h_{m-1} \leq \overline{S}_n <$

$h_m\} \cap \{\bar{S}_n - S_n < a_n^{(S)}\}$. Consequently,

$$\begin{aligned} \mathbf{E}[E_\omega(Z_r^2)] &\leq e^{-(1+o(1))(2r)^{1/2}} + 2e^{2\varepsilon_1 r^{1/2}} \sum_{n=0}^{\infty} \sum_{m=1}^k e^{\lambda_m + 2\varepsilon r^{1/2} - (2+o(1)) \sum_{\ell=m+2}^k \frac{r^\chi}{\lambda_\ell}} \times \\ &\times \mathbf{Q}\left(\{h_{m-1} \leq \bar{S}_n < h_m\} \cap \{\bar{S}_i - S_i < a_i^{(S)}, \forall 0 \leq i \leq n\} \cap \left\{\max_{1 \leq \ell < m} \Delta S_{H_{h_\ell}^{(S)}} \leq r^\theta\right\}\right). \end{aligned}$$

According to Claim 4.6, this yields that

$$\begin{aligned} \mathbf{E}[E_\omega(Z_r^2)] &\leq e^{-(1+o(1))(2r)^{1/2}} + \\ &+ 2e^{2\varepsilon_1 r^{1/2}} \sum_{m=1}^k e^{\lambda_m + 2\varepsilon r^{1/2} - (2+o(1)) \sum_{\ell=m+2}^k \frac{r^\chi}{\lambda_\ell}} \times c_{14} r e^{-(1+o(1)) \sum_{\ell=1}^{m-1} \frac{r^\chi}{\lambda_\ell}}. \end{aligned}$$

By definition, $k := \lfloor r^{1-\chi} \rfloor$ and $\lambda_m := (2r)^{1/2} \left(\frac{k-m+1}{k}\right)^{1/2}$. Hence

$$\lambda_m - 2 \sum_{\ell=m+2}^k \frac{r^\chi}{\lambda_\ell} - \sum_{\ell=1}^{m-1} \frac{r^\chi}{\lambda_\ell} \sim -(2r)^{1/2}.$$

This completes the proof of the inequality (4.11) in Lemma 4.2. \square

4.5. Proof of Lemma 4.2: the inequality (4.12). We recall from (4.55) that

$$\begin{aligned} \mathbf{E}[(E_\omega Z_r)^2] &\leq e^{-r} + \sum_{n=0}^{\infty} \sum_{m=1}^k e^{2\varepsilon r^{1/2} - (2+o(1)) \sum_{\ell=m+2}^k \frac{r^\chi}{\lambda_\ell}} \mathbf{E}_{\mathbf{Q}} \left[e^{-S_n} \mathbf{1}_{\{\min_{0 \leq i \leq n} S_i \geq -\beta\}} \times \right. \\ &\left. \times \mathbf{1}_{\{h_{m-1} \leq \bar{S}_n < h_m\}} \mathbf{1}_{\{\bar{S}_i - S_i < a_i^{(S)}, \forall 0 \leq i \leq n\}} \mathbf{1}_{\{\max_{1 \leq \ell < m} \Delta S_{H_{h_\ell}^{(S)}} \leq r^\theta\}} \right]. \end{aligned}$$

On the right-hand side, we throw away $\mathbf{1}_{\{\max_{1 \leq \ell < m} \Delta S_{H_{h_\ell}^{(S)}} \leq r^\theta\}}$ by saying that it is bounded by 1. On the event $\{h_{m-1} \leq \bar{S}_n < h_m\}$, we have $a_n^{(S)} = \lambda_m$, so $\mathbf{1}_{\{\bar{S}_i - S_i < a_i^{(S)}, \forall 0 \leq i \leq n\}} \leq \mathbf{1}_{\{\bar{S}_n - S_n < \lambda_m\}}$. This leads to:

$$\begin{aligned} \mathbf{E}[(E_\omega Z_r)^2] &\leq e^{-r} + \sum_{m=1}^k e^{2\varepsilon r^{1/2} - (2+o(1)) \sum_{\ell=m+2}^k \frac{r^\chi}{\lambda_\ell}} \mathbf{E}_{\mathbf{Q}} \left[\sum_{n=0}^{\infty} e^{-S_n} \mathbf{1}_{\{\min_{0 \leq i \leq n} S_i \geq -\beta\}} \times \right. \\ &\left. \times \mathbf{1}_{\{h_{m-1} \leq \bar{S}_n < h_m\}} \mathbf{1}_{\{\bar{S}_n - S_n < \lambda_m\}} \right] \\ (4.56) \quad &=: e^{-r} + \sum_{m=1}^k \Sigma_4^{(m)}, \end{aligned}$$

with obvious notation.

Fix $0 < \varepsilon_5 < 1$. We use different estimates for $\Sigma_4^{(m)}$ on the right-hand side, depending on whether $m \leq \lceil \varepsilon_5 k \rceil$ or not.

First case: $1 \leq m \leq \lceil \varepsilon_5 k \rceil$. In this case, we simply use $\mathbf{1}_{\{h_{m-1} \leq \bar{S}_n < h_m\}} \leq 1$ and $\mathbf{1}_{\{\bar{S}_n - S_n < \lambda_m\}} \leq 1$, to see that for large r ,

$$\Sigma_4^{(m)} \leq e^{2\varepsilon r^{1/2} - (2+o(1)) \sum_{\ell=m+2}^k \frac{r^\chi}{\lambda_\ell}} \mathbf{E}_{\mathbf{Q}} \left[\sum_{n=0}^{\infty} e^{-S_n} \mathbf{1}_{\{\min_{0 \leq i \leq n} S_i \geq -\beta\}} \right].$$

According to Lemma B.2 of Aidékon [3], for any $b > 0$, there exists a constant $c_{20}(b) > 0$, whose value depends also on β , such that

$$(4.57) \quad \mathbf{E}_{\mathbf{Q}} \left[\sum_{j=1}^{\infty} e^{-b S_j} \mathbf{1}_{\{S_i \geq -\beta, \forall i \leq j\}} \right] \leq c_{20}(b).$$

Consequently, for all sufficiently large r ,

$$\Sigma_4^{(m)} \leq c_{20}(1) e^{2\varepsilon r^{1/2} - (2+o(1)) \sum_{\ell=m+2}^k \frac{r^\chi}{\lambda_\ell}}.$$

By (4.39) and (4.40), for $1 \leq m \leq \lceil \varepsilon_5 k \rceil$, we have

$$\sum_{\ell=m+2}^k \frac{r^\chi}{\lambda_\ell} = \sum_{\ell=1}^k \frac{r^\chi}{\lambda_\ell} - \sum_{\ell=1}^{m+1} \frac{r^\chi}{\lambda_\ell} = (1+o(1))(2r)^{1/2} - (2r^\chi)^{1/2} [k^{1/2} - (k - \lceil \varepsilon_5 k \rceil)^{1/2}],$$

which is $(1+o(1))(1-\varepsilon_5)^{1/2}(2r)^{1/2}$, $r \rightarrow \infty$. Therefore,

$$(4.58) \quad \sum_{m=1}^{\lceil \varepsilon_5 k \rceil} \Sigma_4^{(m)} \leq c_{20}(1) \lceil \varepsilon_5 k \rceil e^{2\varepsilon r^{1/2} - (2+o(1))(1-\varepsilon_5)^{1/2}(2r)^{1/2}}.$$

Second (and last) case: $\lceil \varepsilon_5 k \rceil < m \leq k$. Since $m > \lceil \varepsilon_5 k \rceil$, we have $h_{m-1} = (m-1) \frac{r}{k} \geq \varepsilon_5 r$. So on the event $\{h_{m-1} \leq \bar{S}_n < h_m\} \cap \{\bar{S}_n - S_n < \lambda_m\}$, we have $S_n > \bar{S}_n - \lambda_m \geq h_{m-1} - \lambda_m \geq \varepsilon_5 r - \lambda_m$, which is greater than or equal to $\varepsilon_5 r - \lambda_1 = \varepsilon_5 r - (2r)^{1/2}$. Accordingly,

$$\begin{aligned} \Sigma_4^{(m)} &\leq e^{2\varepsilon r^{1/2} - (2+o(1)) \sum_{\ell=m+2}^k \frac{r^\chi}{\lambda_\ell}} \mathbf{E}_{\mathbf{Q}} \left[\sum_{n=0}^{\infty} e^{-\frac{1}{2} S_n} e^{-\frac{1}{2} [\varepsilon_5 r - (2r)^{1/2}]} \mathbf{1}_{\{\min_{0 \leq i \leq n} S_i \geq -\beta\}} \right] \\ &\leq e^{2\varepsilon r^{1/2}} \mathbf{E}_{\mathbf{Q}} \left[\sum_{n=0}^{\infty} e^{-\frac{1}{2} S_n} e^{-\frac{1}{2} [\varepsilon_5 r - (2r)^{1/2}]} \mathbf{1}_{\{\min_{0 \leq i \leq n} S_i \geq -\beta\}} \right] \\ &= e^{2\varepsilon r^{1/2} - \frac{1}{2} [\varepsilon_5 r - (2r)^{1/2}]} \mathbf{E}_{\mathbf{Q}} \left[\sum_{n=0}^{\infty} e^{-\frac{1}{2} S_n} \mathbf{1}_{\{\min_{0 \leq i \leq n} S_i \geq -\beta\}} \right]. \end{aligned}$$

So by (4.57), we have $\Sigma_4^{(m)} \leq c_{20}(\frac{1}{2}) e^{2\varepsilon r^{1/2} - \frac{1}{2} [\varepsilon_5 r - (2r)^{1/2}]}$ for $\lceil \varepsilon_5 k \rceil < m \leq k$. As a consequence,

$$(4.59) \quad \sum_{m=\lceil \varepsilon_5 k \rceil + 1}^k \Sigma_4^{(m)} \leq c_{20}(1/2) k e^{2\varepsilon r^{1/2} - \frac{1}{2} [\varepsilon_5 r - (2r)^{1/2}]}.$$

Since $\mathbf{E}[(E_\omega Z_r)^2] \leq e^{-r} + \sum_{m=1}^k \Sigma_4^{(m)}$ (see (4.56)), it follows from (4.58) and (4.59) that

$$\begin{aligned} \mathbf{E}[(E_\omega Z_r)^2] &\leq e^{-r} + c_{20}(1) \lceil \varepsilon_5 k \rceil e^{2\varepsilon r^{1/2} - (2+o(1))(1-\varepsilon_5)^{1/2}(2r)^{1/2}} + \\ &\quad + c_{20}(1/2) k e^{2\varepsilon r^{1/2} - \frac{1}{2} [\varepsilon_5 r - (2r)^{1/2}]}. \end{aligned}$$

Recall that $k := \lfloor r^{1-\chi} \rfloor$. Since $\varepsilon_5 > 0$ can be as close to 0 as possible, this yields (4.12), and completes the proof of Lemma 4.2. \square

APPENDIX A. PROBABILITY ESTIMATES FOR ONE-DIMENSIONAL RANDOM WALKS

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $S_0 := 0$ and let $(S_i - S_{i-1}, i \geq 1)$ be a sequence of i.i.d. real-valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}(S_1) = 0$ and $\sigma^2 := \mathbb{E}(S_1^2) \in (0, \infty)$. We write

$$\bar{S}_j := \max_{0 \leq i \leq j} S_i, \quad j \geq 0.$$

For any $b \in \mathbb{R}$, let⁸

$$\mathbb{H}_b := \inf\{i \geq 1 : S_i \geq b\}, \quad \mathbb{H}_b^- := \inf\{i \geq 1 : S_i \leq b\}.$$

⁸For $b > 0$, \mathbb{H}_b is nothing else but $H_b^{(S)}$ defined in (3.9).

Applying (2.6) of Borovkov and Foss [20] to the ladder heights, we immediately see that the assumption $\mathbb{E}(S_1^2) < \infty$ ensures that $\mathbb{E}(S_{\mathbb{H}_b}) < \infty$ for all $b \geq 0$, and that there exists a constant $c_{21} > 0$ satisfying $\mathbb{E}(S_{\mathbb{H}_b} - b) \leq c_{21}(b + 1)$ for all $b \geq 0$.

Lemma A.1. (i) *Assume $\mathbb{E}(|S_1|^3) < \infty$. There exists a constant $c_{22} > 0$ such that for any $a \geq 0$ and $b \geq 0$ with $a + b > 0$,*

$$(A.1) \quad \frac{b - c_{22}}{a + b} \leq \mathbb{P}\{\mathbb{H}_a < \mathbb{H}_{-b}^-\} \leq \frac{b + c_{22}}{a + b}.$$

(ii) *Assume $\mathbb{E}(|S_1|^{3+\delta}) < \infty$ for some $\delta > 0$. Then for any $a \geq 0$,*

$$(A.2) \quad \mathbb{P}\{\mathbb{H}_{-b}^- < \mathbb{H}_a\} \sim \frac{\mathbb{E}(S_{\mathbb{H}_a})}{b}, \quad b \rightarrow \infty.$$

Proof. We follow the same argument as in [6].

(i) Since $\mathbb{E}(|S_1|^3) < \infty$, it is known (Mogulskii [51]) that $\sup_{b>0} \mathbb{E}(-S_{\mathbb{H}_{-b}^-} - b) < \infty$.

By the optional stopping theorem, $0 = \mathbb{E}(S_{\mathbb{H}_a \wedge \mathbb{H}_{-b}^-}) = \mathbb{E}[(S_{\mathbb{H}_a} - S_{\mathbb{H}_{-b}^-}) \mathbf{1}_{\{\mathbb{H}_a < \mathbb{H}_{-b}^-\}}] + \mathbb{E}(S_{\mathbb{H}_{-b}^-}) \geq (a + b) \mathbb{P}\{\mathbb{H}_a < \mathbb{H}_{-b}^-\} - b - \mathbb{E}(-S_{\mathbb{H}_{-b}^-} - b) \geq (a + b) \mathbb{P}\{\mathbb{H}_a < \mathbb{H}_{-b}^-\} - b - c_{23}$ where $c_{23} := \sup_{b>0} \mathbb{E}(-S_{\mathbb{H}_{-b}^-} - b) < \infty$. This yields the second inequality in (A.1). Considering $(-S_n)$ in place of (S_n) (and exchanging the roles of a and b) yields the first inequality.

(ii) Again, by the optional stopping theorem, $0 = \mathbb{E}(S_{\mathbb{H}_a \wedge \mathbb{H}_{-b}^-}) = -b \mathbb{P}\{\mathbb{H}_{-b}^- < \mathbb{H}_a\} + \mathbb{E}(S_{\mathbb{H}_a}) + \mathbb{E}\{[(S_{\mathbb{H}_{-b}^-} + b) - S_{\mathbb{H}_a}] \mathbf{1}_{\{\mathbb{H}_{-b}^- < \mathbb{H}_a\}}\}$, which leads to

$$(A.3) \quad b \mathbb{P}\{\mathbb{H}_{-b}^- < \mathbb{H}_a\} = \mathbb{E}(S_{\mathbb{H}_a}) + \mathbb{E}\{[|S_{\mathbb{H}_{-b}^-} + b| + S_{\mathbb{H}_a}] \mathbf{1}_{\{\mathbb{H}_{-b}^- < \mathbb{H}_a\}}\}.$$

We let $b \rightarrow \infty$. We have $\mathbb{P}\{\mathbb{H}_{-b}^- < \mathbb{H}_a\} \rightarrow 0$ (by (A.1)), whereas $\sup_{b>0} \mathbb{E}(|S_{\mathbb{H}_{-b}^-} + b|^{1+\delta}) < \infty$ and $\mathbb{E}[(S_{\mathbb{H}_a})^{1+\delta}] < \infty$ (which is a consequence of the assumption $\mathbb{E}(|S_1|^{3+\delta}) < \infty$; see Mogulskii [51]). By Hölder's inequality, $\mathbb{E}\{[|S_{\mathbb{H}_{-b}^-} + b| + S_{\mathbb{H}_a}] \mathbf{1}_{\{\mathbb{H}_{-b}^- < \mathbb{H}_a\}}\} \rightarrow 0$. So (A.3) implies (A.2). \square

Lemma A.2. *Assume $\mathbb{E}(|S_1|^3) < \infty$. There exist constants $c_{24} > 0$, $c_{25} > 0$ and $c_{26} > 0$ such that for all $r \geq 1$ and $\lambda \geq c_{24}$, we have*

$$(A.4) \quad \mathbb{P}\left\{\overline{S}_j - S_j < \lambda, S_j \geq 0, \forall 0 \leq j \leq \mathbb{H}_r\right\} \geq c_{25} \exp\left(-\frac{r}{\lambda} - \frac{c_{26} r}{\lambda^{3/2}}\right).$$

Proof. Let $c_{22} > 0$ be the constant in Lemma A.1. Since $\mathbb{E}(S_1) = 0$ and $\mathbb{E}(S_1^2) > 0$, there exist $c_{27} > 0$ and $c_{28} \in (0, 1)$ such that $\mathbb{P}\{S_1 \geq c_{27}\} \geq c_{28}$, so that

$$\mathbb{P}\{\mathbb{H}_{c_{22}+1} < \mathbb{H}_0^-\} \geq \mathbb{P}\left\{S_i - S_{i-1} \geq c_{27}, \forall 1 \leq i \leq \lceil \frac{c_{22}+1}{c_{27}} \rceil\right\} \geq c_{28}^{\lceil \frac{c_{22}+1}{c_{27}} \rceil} =: c_{29} > 0.$$

Let $y > 0$ and let $r_k := (c_{22} + 1) + yk$, for $0 \leq k \leq N := \lceil \frac{r}{y} \rceil$.

Let $E_{(A.4)} := \{\overline{S}_j - S_j < \lambda, S_j \geq 0, \forall 0 \leq j \leq \mathbb{H}_r\}$. Since $r_N \geq r$, $E_{(A.4)}$ will be realized if $\mathbb{H}_{r_0} < \mathbb{H}_0^-$ and if for all $0 \leq k \leq N - 1$, the following is true: after hitting $[r_k, \infty)$ for the first time, the walk (S_n) hits $[r_{k+1}, \infty)$ before hitting $(-\infty, r_k - \lambda]$. Applying the strong Markov property gives that (\mathbb{P}_x being the probability under which the random walk starts at x ; so $\mathbb{P}_0 = \mathbb{P}$)

$$\mathbb{P}(E_{(A.4)}) \geq \mathbb{P}\{\mathbb{H}_{r_0} < \mathbb{H}_0^-\} \times \prod_{k=0}^{N-1} \mathbb{P}_{r_k}\{\mathbb{H}_{r_{k+1}} < \mathbb{H}_{r_k-\lambda}^-\} \geq c_{29} \prod_{k=0}^{N-1} \mathbb{P}_{r_k}\{\mathbb{H}_{r_{k+1}} < \mathbb{H}_{r_k-\lambda}^-\}.$$

[We do not need to worry about overshoots, because $x \mapsto \mathbb{P}_x\{\mathbb{H}_{r_{k+1}} < \mathbb{H}_{r_k-\lambda}^-\}$ is non-decreasing for $x \in [r_k, \infty)$.]

Since $\mathbb{P}_{r_k}\{\mathbb{H}_{r_{k+1}} < \mathbb{H}_{r_k-\lambda}^-\} = \mathbb{P}\{\mathbb{H}_{r_{k+1}-r_k} < \mathbb{H}_{-\lambda}^-\} = \mathbb{P}\{\mathbb{H}_y < \mathbb{H}_{-\lambda}^-\}$, it follows from Lemma A.1 that (with λ sufficiently large such that $\lambda > y + c_{22}$)

$$\mathbb{P}_{r_k}\{\mathbb{H}_{r_{k+1}} < \mathbb{H}_{r_k-\lambda}^-\} \geq \frac{\lambda - c_{22}}{y + \lambda} = 1 - \frac{y + c_{22}}{y + \lambda} \geq 1 - \frac{y + c_{22}}{\lambda},$$

which is greater than or equal to $\exp[-\frac{y+c_{22}}{\lambda} - (\frac{y+c_{22}}{\lambda})^2]$ if $\frac{y+c_{22}}{\lambda} \leq \frac{1}{2}$ (by the elementary inequality that $1 - x \geq e^{-x-x^2}$ for $0 \leq x \leq \frac{1}{2}$). Since $N \leq \frac{r}{y} + 1 = \frac{r+y}{y}$, we obtain:

$$\mathbb{P}(E_{(A.4)}) \geq c_{29} \exp \left[-\frac{y + c_{22}}{\lambda} \frac{r + 1}{y} - \frac{(y + c_{22})^2}{\lambda^2} \frac{r + 1}{y} \right].$$

We choose $\lambda \geq 1$ and $r \geq 1$. We note that $\frac{y+c_{22}}{\lambda} \frac{r+1}{y} = \frac{r}{\lambda} + \frac{1}{\lambda} + \frac{c_{22}}{\lambda} \frac{r+1}{y} \leq \frac{r}{\lambda} + 1 + \frac{2c_{22}r}{\lambda y}$, and that if $y \geq c_{22}$, $\frac{(y+c_{22})^2}{\lambda^2} \frac{r+1}{y} \leq \frac{4y^2}{\lambda^2} \frac{2r}{y} = \frac{8ry}{\lambda^2}$. So, taking $y := \lambda^{1/2}$ yields that

$$\mathbb{P}(E_{(A.4)}) \geq c_{29} \exp \left[-\frac{r}{\lambda} - 1 - \frac{2c_{22}r}{\lambda^{3/2}} - \frac{8r}{\lambda^{3/2}} \right],$$

proving the lemma. \square

The next lemma says that, under sufficient integrability conditions, the main term $\frac{r}{\lambda}$ within the exponential function in Lemma A.2 is, in some sense, optimal:

Lemma A.3. *Assume that $\mathbb{E}(e^{\delta S_1}) < \infty$ for some $\delta > 0$. For any $\varepsilon > 0$, there exist constants $c_{30} > 0$ and $c_{31} > 0$ such that for all $r \geq 1$ and $\lambda \geq c_{30}$, we have*

$$(A.5) \quad \mathbb{P}\left\{\bar{S}_j - S_j < \lambda, \forall 0 \leq j \leq \mathbb{H}_r\right\} \leq c_{31} \exp\left(- (1 - \varepsilon) \frac{r}{\lambda}\right).$$

Proof. Let $\tau_0 := 0$ and for any $k \geq 1$, let $\tau_k := \inf\{i > \tau_{k-1} : S_i \geq S_{\tau_{k-1}}\}$ be the k -th ascending ladder epoch. Let $\mathbb{P}_{(A.5)}$ denote the probability expression on the left-hand side of (A.5). For any $k \geq 1$, we have

$$\mathbb{P}_{(A.5)} \leq \mathbb{P}\{S_{\tau_k} \geq r\} + \mathbb{P}\left\{S_{\tau_{i-1}} - \min_{\tau_{i-1} \leq j \leq \tau_i} S_j < \lambda, \forall 1 \leq i \leq k\right\}.$$

We now estimate the two probability expressions on the right-hand side.

For the first probability expression, we write $S_{\tau_k} = \sum_{i=1}^k (S_{\tau_i} - S_{\tau_{i-1}})$, and observe that $(S_{\tau_i} - S_{\tau_{i-1}}, i \geq 1)$ is a sequence of i.i.d. random variables, with $\mathbb{E}(e^{aS_{\tau_1}}) < \infty$ for all $a < \delta$. So we take

$$k = k(r, \varepsilon) := \left\lceil \frac{1 - \varepsilon}{\mathbb{E}(S_{\tau_1})} r \right\rceil;$$

there exist constants $c_{32} > 0$ and $c_{33} > 0$, depending on ε , such that $\mathbb{P}\{S_{\tau_{k(r, \varepsilon)}} \geq r\} \leq c_{32} e^{-c_{33} r}$ for all $r \geq 1$.

For the second probability expression (now with $k := k(r, \varepsilon)$), we use the fact that $(S_{\tau_{i-1}} - \min_{\tau_{i-1} \leq j \leq \tau_i} S_j, i \geq 1)$ is also a sequence of i.i.d. random variables, having the same distribution as $-\min_{0 \leq j \leq \tau_1} S_j$; accordingly,

$$\mathbb{P}\left\{S_{\tau_{i-1}} - \min_{\tau_{i-1} \leq j \leq \tau_i} S_j < \lambda, \forall 1 \leq i \leq k(r, \varepsilon)\right\} = \left[\mathbb{P}\left\{-\min_{0 \leq j \leq \tau_1} S_j < \lambda\right\} \right]^{k(r, \varepsilon)}.$$

Since $\tau_1 = \mathbb{H}_0$ and $\{-\min_{0 \leq j \leq \tau_1} S_j < \lambda\} = \{\mathbb{H}_0 < \mathbb{H}_{-\lambda}^-\}$, we are entitled to apply (A.2) to see that for all sufficiently large λ (say $\lambda \geq \lambda_0$), $\mathbb{P}\{-\min_{0 \leq j \leq \tau_1} S_j < \lambda\} \leq 1 - (1 - \varepsilon) \frac{\mathbb{E}(S_{\tau_1})}{\lambda}$. Hence for $\lambda \geq \lambda_0$,

$$\mathbb{P}\left\{S_{\tau_{i-1}} - \min_{\tau_{i-1} \leq j \leq \tau_i} S_j < \lambda, \forall 1 \leq i \leq k(r, \varepsilon)\right\} \leq \left(1 - (1 - \varepsilon) \frac{\mathbb{E}(S_{\tau_1})}{\lambda}\right)^{k(r, \varepsilon)},$$

which is bounded by $\exp[-(1-\varepsilon)\frac{\mathbb{E}(S_{\tau_1})}{\lambda}k(r, \varepsilon)]$. Assembling these pieces yields that for $r \geq 1$ and $\lambda \geq \lambda_0$,

$$\mathbb{P}_{(A.5)} \leq c_{32} e^{-c_{33} r} + \exp\left[-(1-\varepsilon)\frac{\mathbb{E}(S_{\tau_1})}{\lambda}k(r, \varepsilon)\right],$$

which yields (A.5) as $\varepsilon > 0$ is arbitrary. \square

APPENDIX B. PROOF OF (1.7)

The proof of (1.7) relies on several results from [34], and we use the notation therein. Let $t > 0$. By [34, Theorems 2.1 and 2.7],

$$P_\omega\left(\frac{V(X_n)}{\log n} \leq t\right) = \frac{\sigma^2}{2D_\infty \log n} \sum_{x \in \mathbb{T}} e^{-V(x)} \mathbf{1}_{\{V(x) \leq t \log n, x < \mathcal{L}_n\}} + o_{\mathbf{P}^*}(1),$$

where $x < \mathcal{L}_n$ means that for all $y \in \llbracket \emptyset, x \rrbracket$, $\sum_{z \in \llbracket \emptyset, y \rrbracket} e^{V(z)-V(y)} \leq n$, and $o_{\mathbf{P}^*}(1)$ denotes a quantity which converges to 0 in \mathbf{P}^* -probability as $n \rightarrow \infty$. Let for any $s, \lambda > 0$,

$$W_n^{(s, \lambda)} := \sum_{|x|=n} e^{-V(x)} \mathbf{1}_{\{V(x) \leq s, \max_{y \in \llbracket \emptyset, x \rrbracket} (\bar{V}(y) - V(y)) \leq \lambda\}}.$$

A line-by-line analogue of the proof of [34, Lemma 4.1] yields that

$$\lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} \sum_{k=1}^{\infty} W_k^{(t\lambda, \lambda)} = \left(\frac{8}{\pi}\right)^{1/2} \frac{D_\infty}{\sigma^2} \mathbf{E}\left(\min\left(\frac{1}{\mathbf{m}_1^\#}, \frac{t}{\mathbf{m}_1}\right)\right), \quad \text{in } \mathbf{P}^*\text{-probability.}$$

From this point, we can closely follow the step-by-step arguments in the proof of Corollary 2.3 of [34]. Let us give an outline. First, by using [34, Equations (4.6) and (4.7)], we have that for $B > b > 0$,

$$\begin{aligned} & \frac{\sigma^2}{2D_\infty \log n} \sum_{x \in \mathbb{T}} e^{-V(x)} \mathbf{1}_{\{V(x) \leq t \log n, x < \mathcal{L}_n\}} \\ &= \frac{\sigma^2}{2D_\infty \log n} \sum_{b(\log n)^2 \leq |x| \leq B(\log n)^2} e^{-V(x)} \mathbf{1}_{\{V(x) \leq t \log n, x < \mathcal{L}_n\}} + o_{\mathbf{P}^*, b, B}(1), \end{aligned}$$

where the term $o_{\mathbf{P}^*, b, B}(1)$ denotes a quantity which converges to 0 in \mathbf{P}^* -probability when first $n \rightarrow \infty$, then $b \rightarrow 0$ and $B \rightarrow \infty$. Exactly as in the proof of Corollary 2.3 of [34], we can use the following inequalities

$$\begin{aligned} \sum_{b(\log n)^2 \leq |x| \leq B(\log n)^2} e^{-V(x)} \mathbf{1}_{\{V(x) \leq t \log n, x < \mathcal{L}_n\}} &\leq \sum_{k=b(\log n)^2}^{B(\log n)^2} W_k^{(t \log n, \log n)}, \\ \sum_{b(\log n)^2 \leq |x| \leq B(\log n)^2} e^{-V(x)} \mathbf{1}_{\{V(x) \leq t \log n, x < \mathcal{L}_n\}} &\geq \sum_{k=b(\log n)^2}^{B(\log n)^2} W_k^{(t \log n, \log(n/B(\log n)^2))}, \end{aligned}$$

to deduce that

$$\frac{\sigma^2}{2D_\infty \log n} \sum_{x \in \mathbb{T}} e^{-V(x)} \mathbf{1}_{\{V(x) \leq t \log n, x < \mathcal{L}_n\}} = \left(\frac{2}{\pi}\right)^{1/2} \mathbf{E}\left(\min\left(\frac{1}{\mathbf{m}_1^\#}, \frac{t}{\mathbf{m}_1}\right)\right) + o_{\mathbf{P}^*}(1).$$

Then (1.7) follows. \square

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