NUMERICAL SIMULATIONS OF THE PERIODIC INVISCID BURGERS EQUATION WITH STOCHASTIC FORCING

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NUMERICAL SIMULATIONS OF THE PERIODIC INVISCID BURGERS EQUATION WITH STOCHASTIC FORCING

Emmanuel Audusse[1], Sébastien Boyaval[2], Yueyuan Gao[3] and Danielle Hilhorst[4]

Abstract. We perform numerical simulations of the first order Burgers equation with a stochastic source term in the one-dimensional torus. To that purpose we apply a Finite-Volume scheme combining Godunov numerical flux with Euler-Maruyama integrator in time. We perform the numerical tests with different regularities of the source term in space, while it has the regularity of a white noise in time. Our computations exhibit features of the solution, in particular their large time behavior, for various regularities in space. The expectation always converges to the space-average of the initial function as the time tends to infinity in all cases (even when the regularity in space is rougher than what is covered by the existing large-time theories). Moreover, the variance stabilizes, at a value depending on the space regularity and on the intensity of the noise. We perform Monte Carlo simulations for which we visualize both statistical averages and single realizations.

Résumé. Nous proposons dans cet article une étude détaillée de la résolution numérique de l'équation de Burgers stochastique non-visqueuse en dimension un avec des conditions aux limites priodiques. Nous utilisons un schéma de Volumes Finis combinant une intégration en temps de type Euler-Maruyama avec un flux numérique de Godunov. Dans tous les tests numériques, le terme stochastique possède la régularité d'un bruit blanc en temps, tandis que nous considérons différentes régularités en espace. Nous effectuons des simulations de Monte-Carlo et présentons et analysons les solutions obtenues avec ces différentes régularités, en accordant une attention particulière au comportement en temps long. Il apparaît que la moyenne des réalisations converge toujours vers la moyenne en espace de la solution initiale, même si la régularité du terme stochastique est plus faible que ce qui est couvert par les théories existantes. De plus, la variance se stabilise elle aussi vers une valeur, qui dépend de la régularité et de l'amplitude du terme stochastique. Enfin, nous présentons également les résultats obtenus au niveau des réalisations individuelles, ce qui nous permet là encore de mettre en évidence l'influence de la régularité du terme stochastique.

1. Introduction

We would like to numerically approximate solutions to the stochastically-forced inviscid Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = g \tag{1}$$

in a bounded domain of unit length with periodic boundary conditions, equivalently on the torus $x \in \mathbb{S}^1$.

The stochastic forcing has zero space average

$$\int_{\mathbb{S}^1} g = 0 \tag{2}$$

to preserve the conservative character of the inviscid Burgers equation on the torus. In particular, in a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider the Cauchy problem for (1) together with the initial condition $u(t=0) = u_0$

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of deterministic space-average $\int_{\mathbb{S}^1} u_0$. Then, (2) implies

$$\int_{\mathbb{S}^1} u = \int_{\mathbb{S}^1} u_0 \quad \forall t > 0.$$

Moreover, the stochastic source term g is assumed to behave as a white noise with respect to time $t \in [0,T)$. A notion of global solution is possible only if one admits discontinuous solutions, as in the deterministic case. We briefly discuss below in this introduction some existing results concerning the (non obvious) interpretation of (1), depending on the regularity in space of the stochastic source term, for general cases. As in the deterministic case, a notion of entropic solution is necessary.

In this note, we only consider space-time discrete versions of (1), with obvious interpretation. Of course, we take care of the difficulties inherent to the nature of (1) through our numerical scheme. In particular, it is well known that one cannot expect global smooth solutions to conservation laws as (1) in the deterministic case $g = 0$, and that a notion of stochastic differential evolution should take into account the adaptation to a filtration in time. Our numerical scheme thus combines standard discretization techniques for scalar first order conservation laws like the inviscid Burgers equation with periodic boundary conditions [7, 28] with standard discretization techniques for stochastic differential equations similar to what one obtains after discretizing the inviscid Burgers equation in space.

More precisely, we consider a Finite-Volume (FV) discretization of the flux difference. Given $I \in \mathbb{N}^\star$, we split the one-dimensional torus $\mathbb{S}^1$ regularly into cells of uniform volume $\Delta x := 1/I$. Then, a discretization of (1) in between two times $t^{n+1} > t^n > 0$ is typically obtained from the integral formula

$$\int_{(i-1/2)\Delta x}^{(i+1/2)\Delta x} u(x,t^{n+1})dx = \int_{(i-1/2)\Delta x}^{(i+1/2)\Delta x} u(x,t^n)dx + \int_{t^n}^{t^{n+1}} \big((u^2/2)((i-1/2)\Delta x,t) - (u^2/2)((i+1/2)\Delta x,t)\big)\,dt + \int_{t^n}^{t^{n+1}} \int_{(i-1/2)\Delta x}^{(i+1/2)\Delta x} g dx dt \quad (3)$$

by defining a numerical solution u^n for $n > 0$ through the relation

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(F^n_{i+1/2} - F^n_{i-1/2}\right) + G_i^n \quad \forall n \in \mathbb{N} \quad \forall i \in \{1,...,I\}, \quad (4)$$

where the space derivative in approximated in a conservative way through the definition of the fluxes $F^n_{i+1/2}$ [28] and the time process is discretized by an explicit Euler-Maruyama time integrator [22, 24], recalling that g is a white noise in time. We have thereby defined a Markov process $(u^n)_{n\in\mathbb{N}}$ with values in $\mathbb{R}^I$ to discretize the cell-averages of the solution of (1). In the following, we fix a deterministic initial condition u_0 approximated as

$$u_i^0 = \frac{1}{\Delta x}\int_{(i-1/2)\Delta x}^{(i+1/2)\Delta x} u_0(x).$$

We make precise the definition of the flux $F^n_{i+1/2}$ and of the stochastic source term G_i^n in the next Section. We recall that the source term G^n is the image by a linear mapping of a Gaussian vector with zero mean (i.e. expectation) $\mathbb{E}(G_i^n) = 0$ uniformly in time and space, and with correlations

$$\mathbb{E}(G_i^n G_j^m) = \delta_{n,m} B^T B \quad (5)$$

where the matrix B can vary as a function of two parameters $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{N}$. We refer to Section 2 for a more precise formula defining G^n, which covers the space-time white noise $\mathbb{E}(G_i^n G_j^m) = \delta_{n,m}\delta_{i,j}\frac{\Delta t}{\Delta x}$ when $\beta = 0$.

A question is of course whether, and how, the Markov process defined above in (4) allows one to approximate solutions of (1). Note that in the deterministic case with $G^n = 0$ the discrete sequences $(u^n)_{n\in\mathbb{N}}$ are known to converge, as the space and time steps tend to zero under a CFL condition [14, 25], to entropic solutions of the inviscid Burgers equation with periodic boundary conditions

$$\frac{\partial u}{\partial t}(x,t) + \frac{\partial}{\partial x}\left(\frac{u^2}{2}(x,t)\right) = 0 \quad \text{for all } (x,t) \in \mathbb{S}^1 \times [0,T), \quad (6)$$

for all $T > 0$. We recall that in order to consider stochastic cases where $G^n \neq 0$, one should first define a notion of entropic solution of (1) that holds in a probabilistic setting. For the "space-time white noise" case in the sense of zero correlation-length in space and time which is among the cases considered here, the existence of some kind of entropic solutions has been shown in e.g. [15], and more recently in [10]. Though it is not clear whether our (very natural) constructive numerical scheme does indeed approximate any kind of entropic solutions.

Note that the *viscous* Burgers equation with stochastic forcing, with an additional diffusion term $-\nu\partial_{xx}^2 u$ on the left-hand-side of (1) ($\nu > 0$), has been much more studied than the inviscid one, from the pure theoretical as well as numerical [1] viewpoints. In one space dimension, all viscous solutions are known to converge to the entropic solutions when $\nu \to 0$ in the deterministic case. So one might expect a similar behaviour in the stochastic case. This is used for instance in [11]. Though, it has been shown in [19] that the discretization of the nonlinear term has a lot of impact on the limit of the viscous case. It is thus not fully clear yet how to obtain univoque, physically-meaningful, entropic solutions of the stochastic Burgers equation. Note that as opposed to [19], we use here a more usual discretization of the nonlinear (flux) term, which actually allows one to define univoque, physically-meaningful, entropic solutions of the deterministic Burgers equation. However, we are not able to pass to the continuous limit $I \to \infty$ in the stochastic case.

Having fixed the discretization of (1), that is with a positive constant I being fixed, another important question with respect to the stochastic dynamics is the behaviour of u^n as $n \to \infty$. In this work, we numerically study the large time behaviour when the noise G^n is defined with various regularities in space and various intensities (parameterized by $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{N}$, see Section 2).

In order to obtain detailed information about the large time properties, existing theories for Markov processes [23] would for instance consider the Kolmogorov forward equation satisfied by the probability density functional $\psi^n(u)$ on $\mathbb{R}^I$ of the Markov process $u^n \in \mathbb{R}^I$ at step $n \in \mathbb{N}$

$$\psi^{n+1}(u) = \psi^n(u) + I\Delta t \nabla_u \cdot \left(\left(F_{+1/2}(u, \tau_+ u) - F_{-1/2}(\tau_- u, u) \right) \psi^n(u) + \nabla_u \psi^n(u) \right) . \tag{7}$$

where $\tau_\pm$ denote translation operators of length $\pm\Delta x$. Unfortunately, the nonlinear term $F_{+1/2} - F_{-1/2}$ cannot be written as a gradient of u, and it is not even useful to compute stationary measures (invariant by the Markov dynamics) solutions of e.g. $\left(F_{+1/2} - F_{-1/2}\right) \psi^\infty + \nabla_u \psi^\infty \equiv C \in \mathbb{R}$. We thus limit ourselves to numerical simulations.

Note however that, in deterministic cases without noise [8] or with a time-independent forcing with zero space average [2], the solutions of (6) are known to converge for large times to the constant $\int_{\mathbb{S}^1} u_0(x)dx$. Moreover, the existence of an invariant measure has been proved for sufficiently regular noise in space (the case $\beta \geq 2$ in the formula (10) below), using the equivalence with a Hamilton-Jacobi equation in the celebrated article [11], and with kinetic formalism in [9].

The stochastic Burgers equation with periodic boundary conditions has already been studied numerically by numerous authors in the past. Let us simply mention [12,13] here as an example, where some physical motivations are explained. The numerical scheme which we propose here does not seem to have been used yet, in particular for a long-time study.

2. Numerical method

We define the two-point numerical approximation of the flux $(u^2/2)((i-1/2)\Delta x, t)$ by using the Godunov scheme [28]

$$F_{i-1/2} = \begin{cases} \dfrac{u_i^2}{2} & \text{if} \quad u_{i-1} \leq u_i \leq 0 \\ \dfrac{u_{i-1}^2}{2} & \text{if} \quad 0 \leq u_{i-1} \leq u_i \\ 0 & \text{if} \quad u_{i-1} \leq 0 \leq u_i \\ \dfrac{u_{i-1}^2}{2} & \text{if} \quad u_i \leq u_{i-1} \text{ and } \dfrac{u_i + u_{i-1}}{2} \geq 0 \\ \dfrac{u_i^2}{2} & \text{if} \quad u_i \leq u_{i-1} \text{ and } \dfrac{u_i + u_{i-1}}{2} \leq 0. \end{cases} \tag{8}$$

In the deterministic case, this flux is known to be consistent, stable and entropy-satisfying under the classical CFL condition

$$\Delta t^n \leq \frac{\Delta x}{\max\limits_{i \in \{1,2,..I\}} \{u_i^n\}}, \tag{9}$$

We now turn to the stochastic term. We generate a noise with the formula

$$G_i^n = \alpha \sqrt{\frac{2}{I}} \sum_{k=1}^{\frac{I}{2}-1} \frac{1}{|k|^\beta} \left\{ C_k^n cos(2\pi k x_i) - S_k^n sin(2\pi k x_i) \right\} + \alpha \frac{(-1)^i}{\sqrt{I}} \frac{1}{|\frac{I}{2}|^\beta} C_{\frac{I}{2}}^n \tag{10}$$

where C_k^n and S_k^n are i.i.d. random variables that follow the normal distribution $\mathcal{N}(0, \Delta t)$ for all k and n ($\mathcal{N}(\mu, \sigma^2)$ denotes the normal distribution with expected value μ and variance σ^2). When $\alpha = 1$ and $\beta = 0$, a straightforward computation yields

$$\mathbb{E}(G_i^n G_i^n) = \frac{I-1}{I}\Delta t \tag{11}$$

Note that (10) has the same scaling as in [15, 19], but is restricted to a combination of basis functions with zero average in space as in [12] in order to preserve conservativity.

We compute empirical mean and variance estimators of the Markov chain entries (4) by the Monte-Carlo method

$$\mathrm{E}(u_i^n) := \frac{1}{M}\sum_{m=1}^{M} u_i^n(\omega_m), \qquad \mathrm{Var}(u_i^n) := \frac{1}{M}\sum_{m=1}^{M} |u_i^n(\omega_m) - \mathrm{E}(u_i^n)|^2,$$

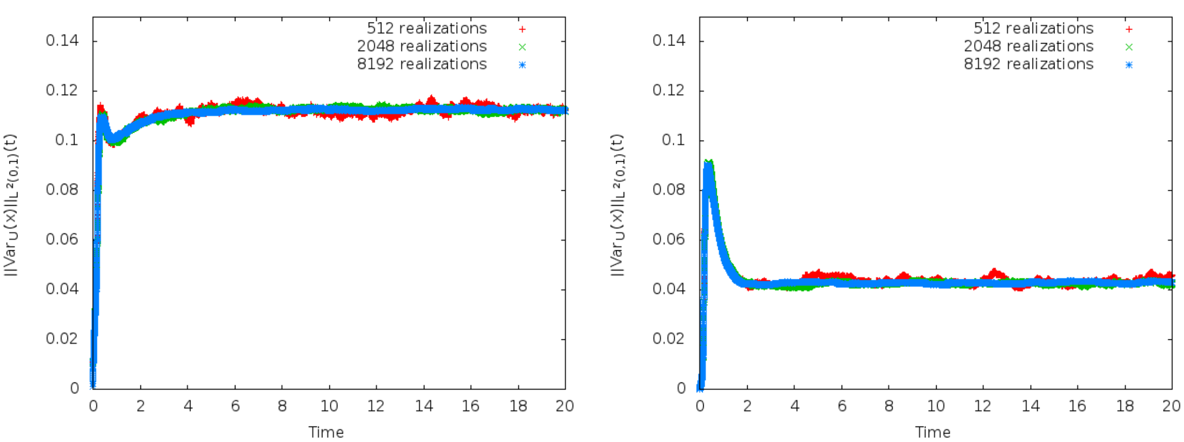
invoking M i.i.d. realizations $u_i^n(\omega_m)$, $m = 1 \dots M$ computed with INM Gaussian numbers $\mathcal{N}(0, \Delta t)$ (cf. (10)). Here N denotes the number of time iterations that one needs to reach the stationary distribution, see next Section.

The number of volume elements is first fixed equal to $I = 80$, so the number of Fourier modes in (10) is equal to 79 (recall that the constant mode is eliminated to ensure a zero space average.)

The CFL condition (9) is naturally stochastic, i.e. dependent on the realization. To maintain a fixed number of time steps for each realization, we assume that $|u_i^n| \le \bar{u}$ with $\bar{u} := 10$ set arbitrarily, which allows to fix the CFL condition as $\Delta t = 0.1\Delta x$. Of course, we are aware that this may introduce a bias in the probability law which is simulated.

We have performed numerical tests with a variety of values of α and β. In each case, we chose the number of realizations M large enough so that confidence intervals for $\mathrm{E}(u_i^n)$ and $\mathrm{Var}(u_i^n)$ are small enough, and our assumption $|u_i^n| \le \bar{u}$ was hardly violated (at a frequency less than 1/2000).

The influence of the number of realizations on the evaluations of $\|\mathrm{Var}\, u(\cdot, t)\|_{L^2(0,1)}$, in the cases that $\alpha = 1$, $\beta = 0$ and $\alpha = 1$, $\beta = 1$ is presented in Figure 1.



(A) $\alpha = 1$ and $\beta = 0$ (B) $\alpha = 1$ and $\beta = 1$

FIGURE 1. Time evolution of the L^2 norm of the variance

It is remarkable that long-time convergence seems to occur while $\beta < 2$. However, to conclude about the continuous limit, one should still discuss the space and time discretization in those cases. And a complete understanding of ergodic properties would require the study of higher-order moments as well.

3. Results and discussion

Numerical tests are all performed with the deterministic initial condition $u_0(x) = sin(2\pi x), x \in \mathbb{S}^1$. We performed 2048 realizations for each computation. In the figures 2, 3 and 4, we present on the left-hand side comparisons between $\mathrm{E}(u_i^n)$, and u_i^n in the case without noise, for different values of the discrete time $t^n = n\Delta t$. On the right-hand side, we show comparisons between u_i^n for some realization and u_i^n in the case without noise, at the same values of the discrete time $t^n = n\Delta t$.

We first fix two cases $\alpha = 0.1$, $\beta = 0$ and $\alpha = 0.1$, $\beta = 1$.

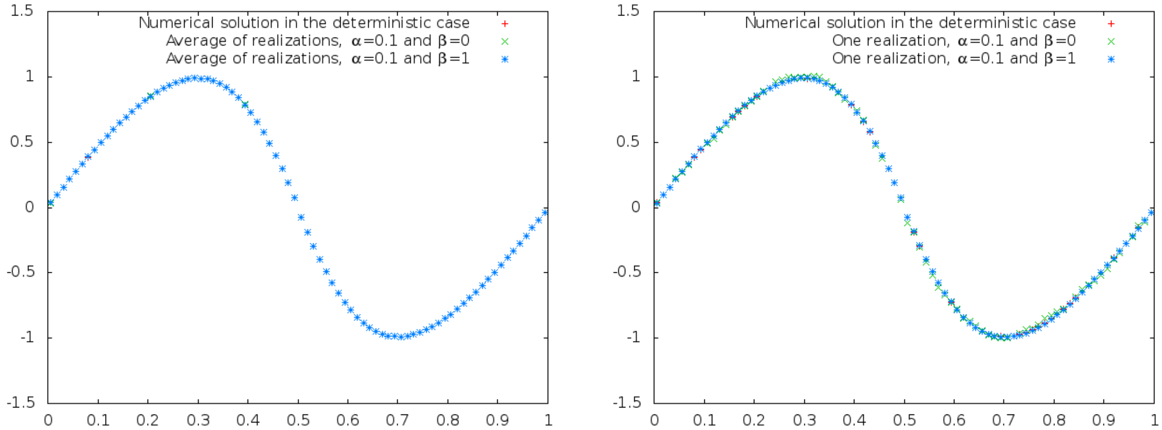


FIGURE 2. Empirical average (left) and one realization (right) at $t = 0.05$

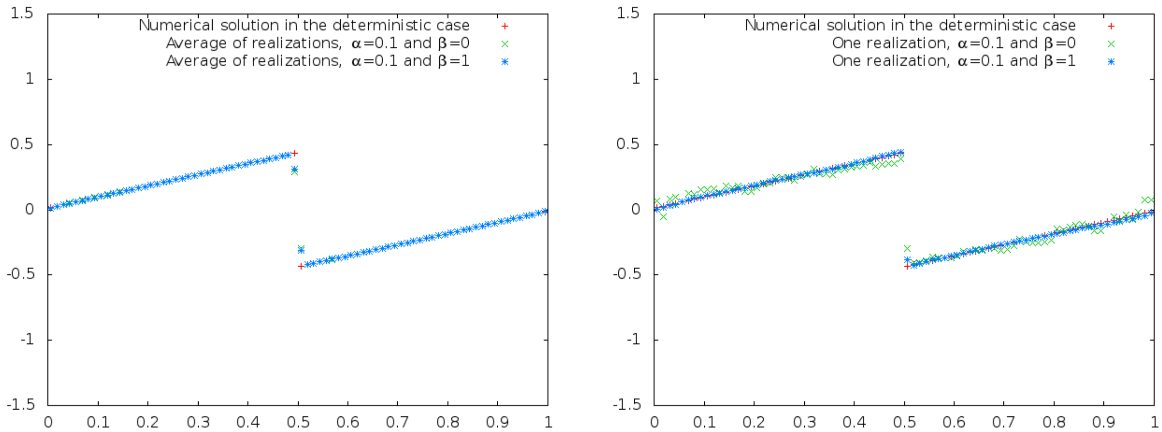


FIGURE 3. Empirical average (left) and one realization (right) at $t = 1$

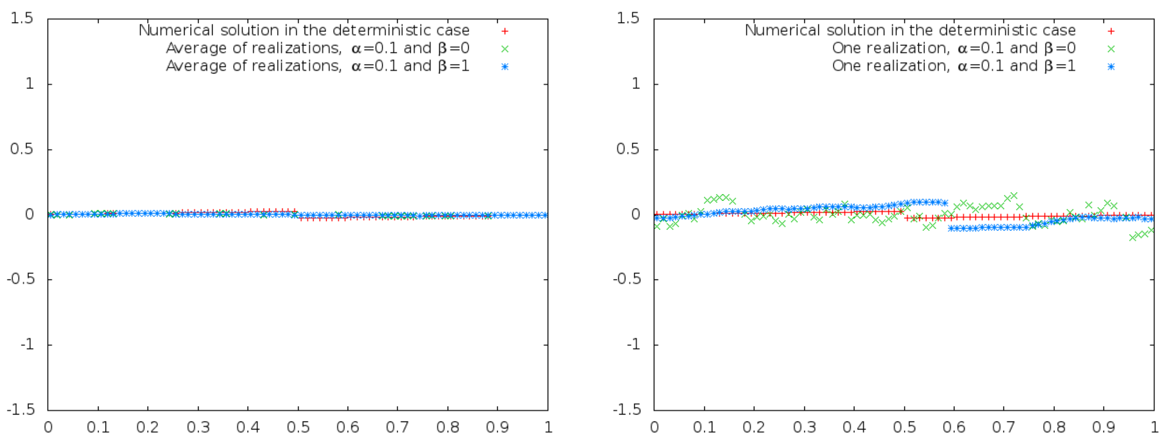
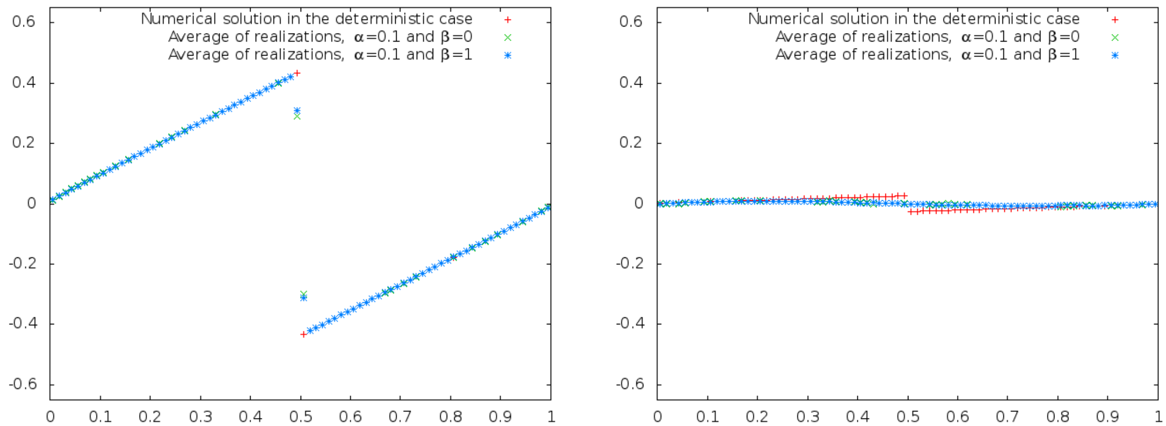


FIGURE 4. Empirical average (left) and one realization (right) at $t = 20$

In Figure 5 we rescale the comparisons for the empirical average at the times $t = 1$ and $t = 20$.



(A) $t = 1$ (B) $t = 20$

FIGURE 5. Rescaled graphs for $t = 1$ and $t = 20$

We then fix $\alpha = 1$, $\beta = 0$ and $\alpha = 1$, $\beta = 1$. We present the corresponding results in the figures 6, 7 and 8.

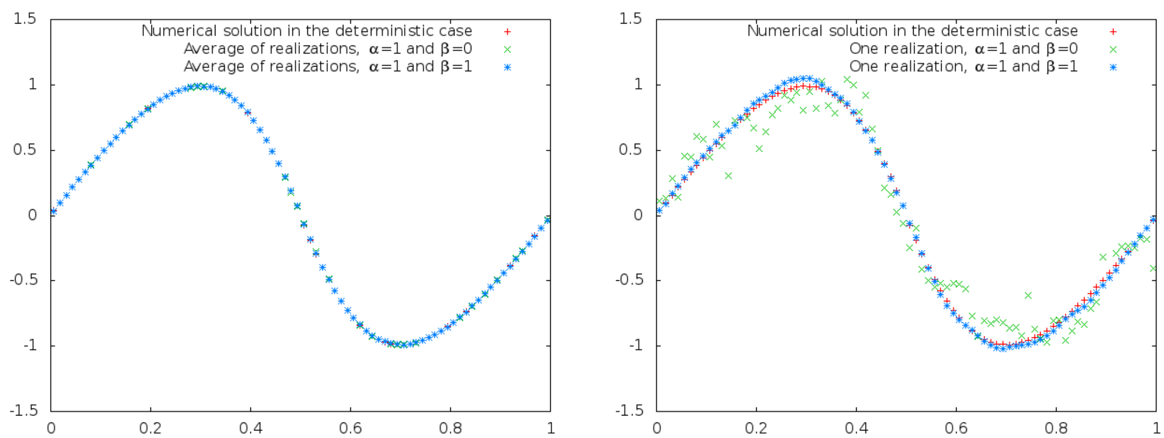


FIGURE 6. Empirical average (left) and one realization (right) at $t = 0.05$

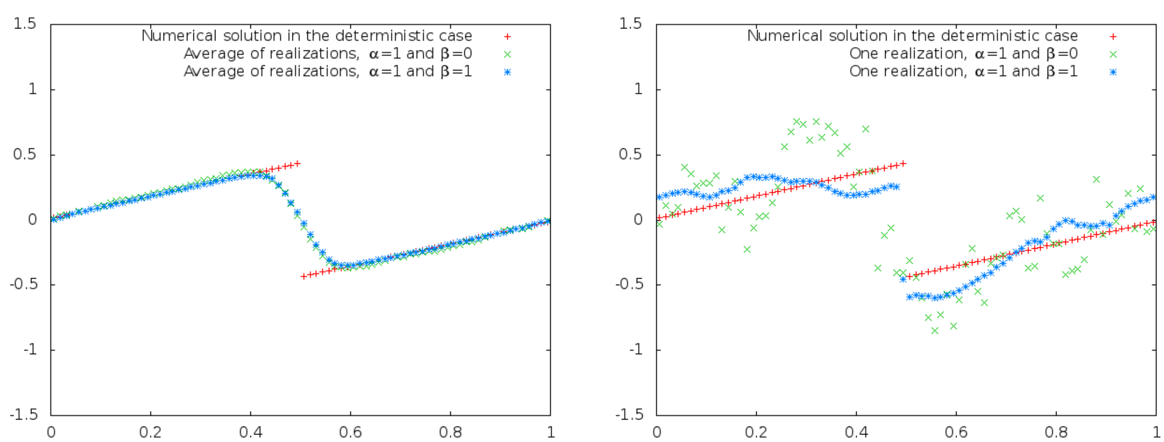


FIGURE 7. Empirical average (left) and one realization (right) at $t = 1$

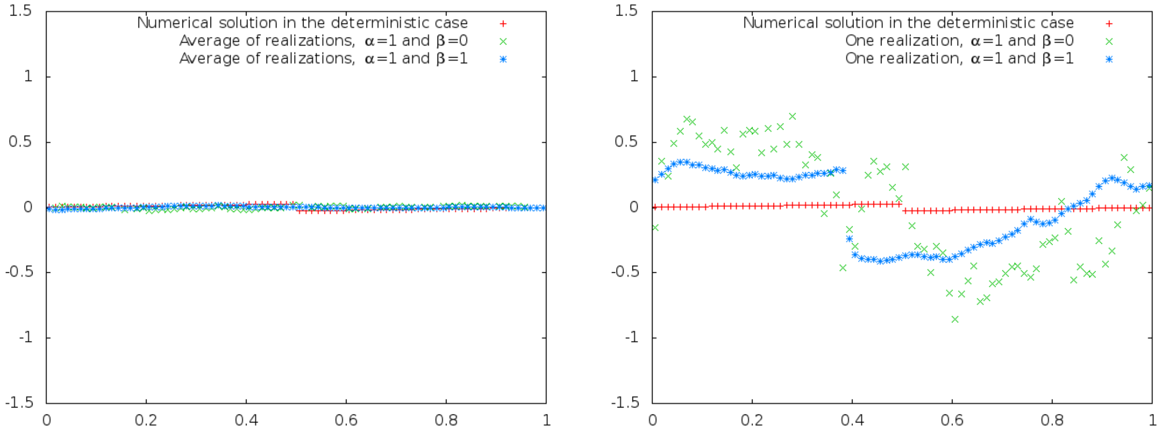
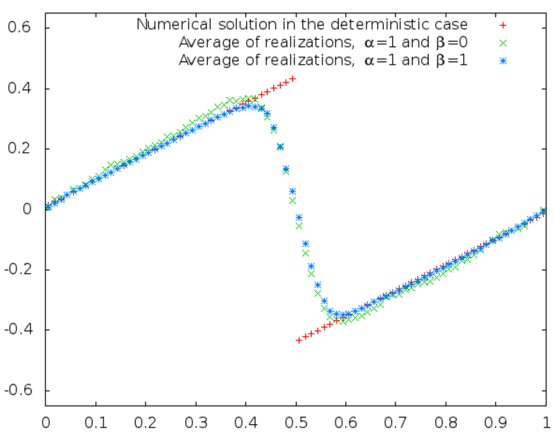
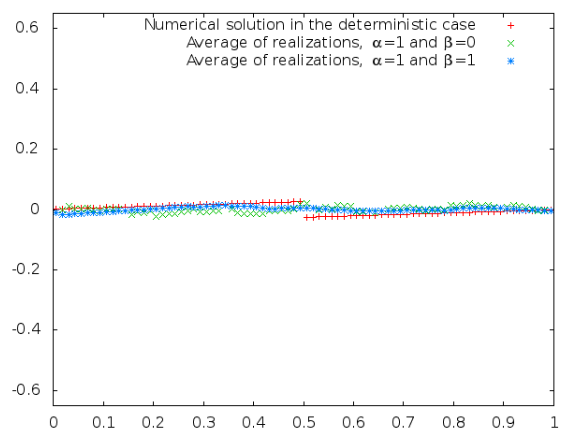


FIGURE 8. Empirical average (left) and one realization (right) at $t = 20$

In Figure 9 we rescale again the comparisons at the times $t = 1$ and $t = 20$.



(A) $t = 1$



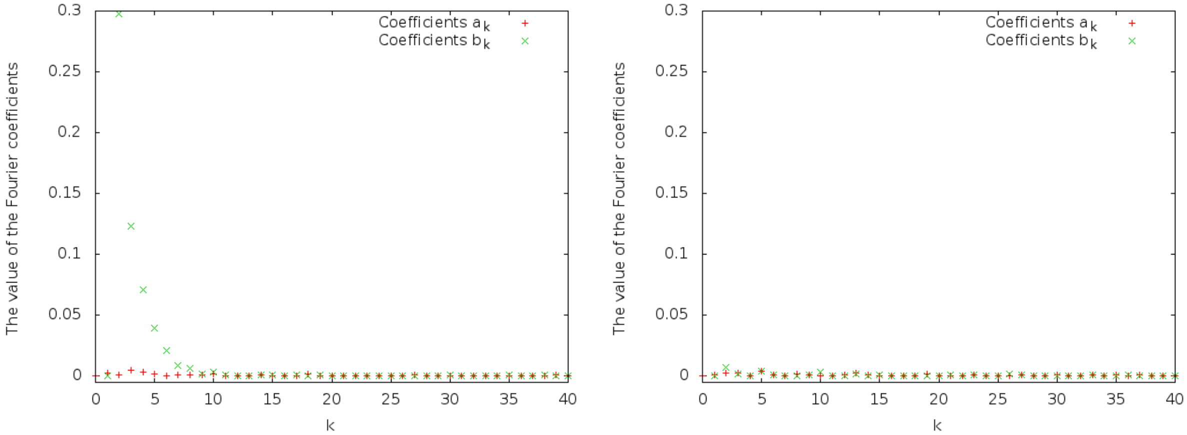
(B) $t = 20$

FIGURE 9. Rescaled graphs for $t = 1$ and $t = 20$

It seems that solutions of the periodic inviscid Burgers equation (1) have definitely different innterpretation depending on the regularity in space of the stochastic forcing.

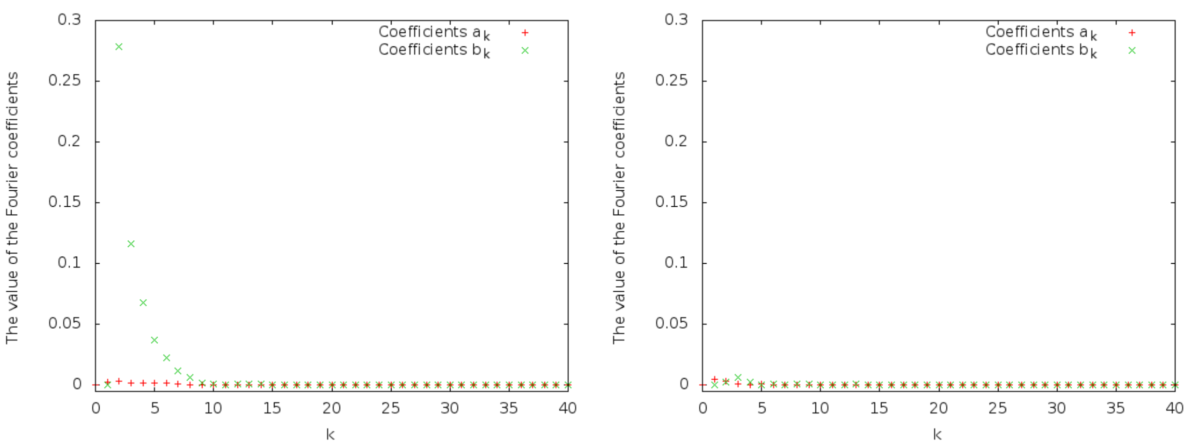
Whereas realizations look like discontinuous functions in the case that $\beta = 1$, they do not even look like functions in the case that $\beta = 0$. However, the expectation seems to behave in long times as the solution of the deterministic case without noise, or equivalently as the statistical average of deterministic solutions forced by time-independent periodic forces with zero statistical average.

We have tried to characterize the regularity of the solution by a Fourier analysis of the empirical mean when $\alpha = 1$, $\beta = 0$ and $\alpha = 1$, $\beta = 1$ in the figures (10) and (11).



(A) $E_U(\cdot, 1)$ (B) $E_U(\cdot, 20)$

Figure 10. Fourier coefficients for $E(u(\cdot,1))$ and $E(u(\cdot,20))$ when $\alpha = 1$, $\beta = 0$



(A) $E_U(\cdot, 1)$ (B) $E_U(\cdot, 20)$

Figure 11. Fourier coefficients for $E(u(\cdot,1))$ and $E(u(\cdot,20))$ when $\alpha = 1$, $\beta = 1$

We now compare the ways in which the expectation and the variance converge towards their stationary values when the parameters α and β are varying. In particular we observe in the figures (12) and (13) faster convergence in time as α and β increase, and larger variance in the long time limit as α increases and β decreases.

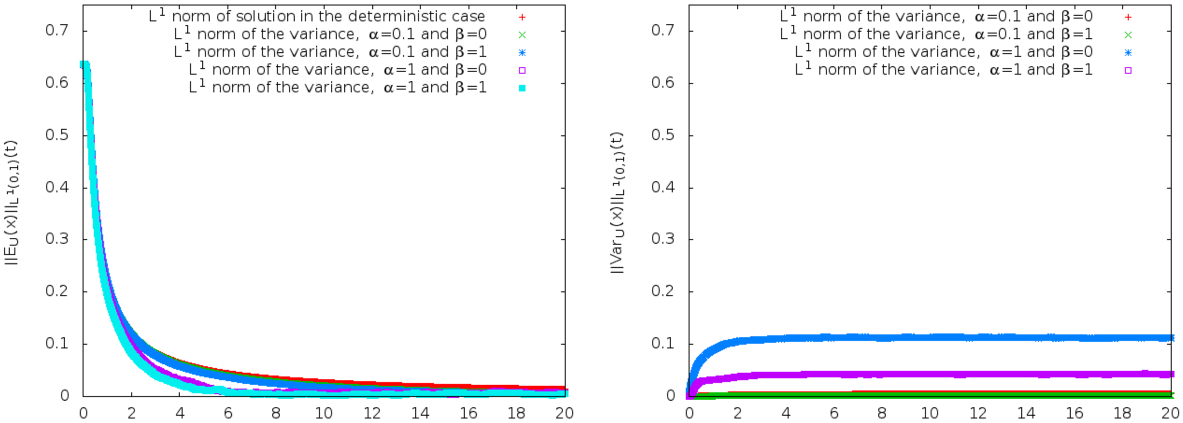


FIGURE 12. Norm L^1 of the expectation (left) and variance (right) of the solution

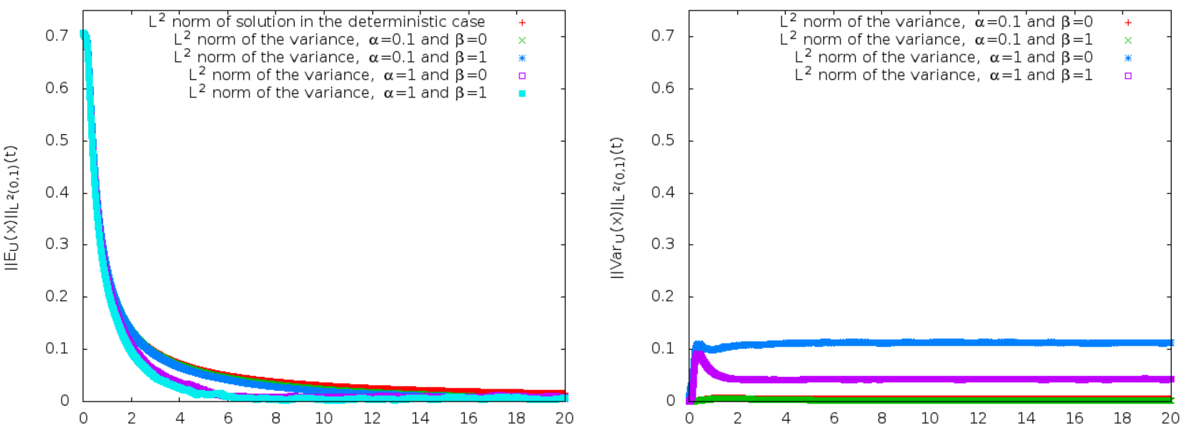


FIGURE 13. Norm L^2 of the expectation (left) and variance (right) of the solution

4. Some conclusions

We have considered cases of a white noise in space and time ($\alpha = 0.1$, $\beta = 0$) and that of a smoother in space version ($\alpha = 0.1$, $\beta = 1$). In both cases, the average of the realizations is a good approximation of the deterministic solution; as time tends to infinity it converges to the space-average of the initial function [8]. However, while the deterministic solution is discontinuous, the average of the realizations smoothens it out.

We have also considered corresponding cases with a larger amplitude, namely ($\alpha = 1$, $\beta = 0$) and ($\alpha = 1$, $\beta = 1$); then our numerical results for one realization are very dispersed. The average smoothens the deterministic solution and goes faster to equilibrium than in the cases where $\alpha = 0.1$. However, when $\beta = 0$, the expectation at $t = 20$ does not seem to be a function anymore, which is consistent with the fact that the existence proofs of an invariant measure for the PDE solution only holds with smooth enough in space noise [6, 9, 11].

Our results are still far from complete. A forthcoming work will also involve a numerical study of the limit $I \to \infty$, as well as a more detailed study of single realizations. We also propose to compute various distribution functions.

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