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# THE ABEL TRANSFORMATION ON SYMMETRIC POLYGONAL GRAPHS

ALAA JAMAL EDDINE

ABSTRACT. Let  $\Gamma$  be a polygonal symmetric graph of type  $k$  and order  $r$ , where  $k, r \geq 2$ . In this paper we give explicit expressions of the horocyclic Abel transformation and its dual as well as their inverses on  $\Gamma$ .

## 1. INTRODUCTION

Let  $\Gamma$  be a symmetric polygonal graph, and let  $G$  a group acting isometrically and simply transitively on  $\Gamma$ . Iozzi and Picardello [16],[17] have generalized to the case of groups acting isometrically and simply transitively on symmetric polygonal graph  $\Gamma$  the theory of representation, spherical functions and convolution operators studied in [13],[12] in the case of free groups and homogenous trees. They considered the convolution algebra of radial functions  $C^*(\mu_1)$ , where  $\mu_1$  is the uniformly distributed measure over words of length one, and they computed the spectrum  $S$  of  $\mu_1$  in the regular  $C^*$ -algebra. It turns out that the Plancherel measure is the only positive measure supported on  $S$ , and that verifies  $f(o) = \int_S \hat{f}(\lambda) d\lambda$ , for all  $f \in C^*(\mu_1)$ , where  $\hat{f}(\lambda)$  is the Helgason Fourier transform of  $f$ , which in this case coincides with the spherical Fourier transform of  $f$  as  $f$  is radial, and  $o$  denotes a reference vertex in  $\Gamma$ . The Plancherel measure has been computed on symmetric graphs in [11] and [19]. In this paper we extend to the case of symmetric graphs, the definition of the horocyclic Abel transformation, a variant of the horocyclic Radon transformation. The Radon transformation on trees has been first studied by Cartier [6], and was further investigated by Betori and Pagliacci [5], Betori, Faraut and Pagliacci [3], by E. Casadio Tarabusi, M. Cohen and F. Colona [9], and by M. Cowling, S. Meda and A. Setti [10].

We will deal with radial functions. We will first define the horocycles on the symmetric graph  $\Gamma$ , then we will give an explicit expression of the Abel transform. Our argument

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is similar to that used on semihomogeneous trees in [5]. In this case we will derive an inversion formula for the Abel transformation. We then study the mapping properties of the Abel transformation, and characterize the Abel transform of Schwartz spaces.

Using the definition of the Abel transformation, we derive an explicit expression of the dual Abel transformation and its inverse. The dual Abel transformation has been used extensively, in order to derive an explicit expression of the solution to the wave equation, (see [15] for the symmetric space case, [2] for the Damek–Ricci spaces and for the homogeneous trees cases, and [18] for the symmetric graph case).

In the last section, we show that the Helgason–Fourier transformation factors into the Abel transformation and the Euclidean Fourier transformation as in the symmetric space case, and the homogeneous trees case. As an application of this relation, we derive an explicit expression of the heat kernel on  $\Gamma$ . We then derive the Plancherel formula and the inversion formula of the Helgason–Fourier transformation using the Plancherel and the inversion formulae of the spherical Fourier transformation already derived in [19] and [11]. We then deduce a version of the Kunze–Stein phenomenon on  $G$ .

## 2. SYMMETRIC GRAPHS

A graph  $\Gamma$  is symmetric of type  $k \geq 2$  and order  $r \geq 2$  if every vertex  $v$  belongs exactly to  $r$  polygons, with  $k$  sides each, contained in the graph, with no sides and no vertex in common except  $v$ , and if every nontrivial loop in  $\Gamma$  runs through all the edges of at least one polygon. In other words, a symmetric graph of type  $k$  and order  $r$  can be thought of as a homogeneous tree of order  $r$  built up with polygons with  $k$  sides. Notice that, if  $k = 2$ ,  $\Gamma$  is a homogeneous tree of degree  $2r$ .

Different notions of distance on  $\Gamma$  have been introduced in [16]. We define the distance between two vertices  $v_1$  and  $v_2$  as the minimal number of polygons crossed by a path connecting  $v_1$  with  $v_2$ . Here we define the length  $|v|$  of a vertex  $v$ , with respect to a reference vertex  $o$ , as the distance between  $v$  and  $o$ . We can prove easily that the group  $G$  that acts isometrically and simply transitively on  $\Gamma$  is isomorphic to the free product of  $r$  copies of  $\mathbb{Z}/k\mathbb{Z}$  if  $k > 2$ , while, for  $k = 2$ , i.e. when  $\Gamma$  is a homogeneous tree,  $G$  is isomorphic to the free product of  $t$  copies of  $\mathbb{Z}$  and  $s$  copies of  $\mathbb{Z}/2\mathbb{Z}$ , where  $2t + s = r$ .

Let us denote  $\mathcal{V}$  the set of vertices of  $\Gamma$ . There is a natural probability measure on  $\mathcal{V}$  which we will denote by  $\mu$ . Every element of  $\mathcal{V}$  can be identified with an element of  $G$ , and every polygon of the graph corresponds, under this identification to an orbit under

right translations by one of the factors  $\mathbb{Z}/k\mathbb{Z}$ . Under this identification, we can define the length  $|g|$  of an element  $g \in G$  as follows. Let  $a_1, \dots, a_r$  denote the generators of  $G$ . Then, an element  $g \in G$  can be written as  $g = a_{i_1}^{m_1} \dots a_{i_n}^{m_n}$  and  $|g| = n$ . Denote  $S(o, n)$  the sphere of centre  $o$  and radius  $n$  in  $G$ . we have

$$|S(o, n)| = \delta(n) = \begin{cases} 1 & \text{if } n = 0 \\ r(k-1)Q^{n-1} & \text{if } n \geq 1 \end{cases}$$

where  $Q = (r-1)(k-1)$ .

Between any two points  $v_1$  and  $v_2$  in  $\Gamma$  such that  $d(v_1, v_2) = n$ , there is a unique geodesic path of the form  $(P_1, \dots, P_n)$  where  $(P_i)_{1 \leq i \leq n}$  are polygons in  $\Gamma$  and such that  $v_1 \in P_1 \setminus P_2$  and  $v_2 \in P_n \setminus P_{n-1}$ .

A geodesic ray  $\mathcal{P}$  in  $\Gamma$  is a one sided sequence  $\{P_n, n \in \mathbb{N}\}$  of polygons. We say that  $v \in \mathcal{P}$  if  $v \in P_n$  for some  $n \in \mathbb{N}$ . We denote by  $\Omega$  the set of all geodesic rays starting with  $o$ .  $\Omega$  is called the boundary of  $\Gamma$ .  $G$  acts by left translation on words in  $\Omega$ . For  $x \in G$ , let  $E(x)$  denotes the subset of  $\Omega$  of words that begin with the reduced word  $x$ . Then,  $\{E(x), x \in G\}$  is a base of the topology of  $\Omega$  making  $\Omega$  a compact topological space. Let  $\nu$  denotes a probability measure on  $\Omega$  defined by

$$\nu(E(x)) = \frac{1}{\delta(|x|)} \text{ if } |x| = n.$$

We then denote  $(\Omega, \nu)$  the Poisson boundary of  $G$  with respect to  $\mu_1$ . The group  $G$  acts on measures on  $\Omega$ , particularly on  $\nu$  by

$$\nu_x(A) = \nu(x^{-1}A)$$

for all  $x \in G$ , and for all Borelian set  $A \in \Omega$ .  $\nu_x$  is absolutely continuous with respect to  $\nu$ , and we have, thus for  $\omega \in \Omega$ , such that  $\omega = a_{i_1}^{n_1} a_{i_2}^{n_2} \dots$  we denote  $\omega_m = a_{i_1}^{n_1} \dots a_{i_m}^{n_m}$ . The Radon Nikodym derivative

$$(1) \quad \nu(x^{-1}E(\omega_m))/\nu(E(\omega_m)) = Q^{\zeta(x, \omega)}$$

where  $\zeta$  denotes the Busemann function on  $\Gamma$

$$\zeta(x, \omega) = d(o, \omega_m) - d(x, \omega_m).$$

We remark that  $\zeta(x, \omega)$  doesn't depend on  $\omega$  if  $m > |x|$ . The quantity obtained in (1) denotes the Poisson kernel  $P(x, \omega)$ , it verifies the following cocycle identities :

$$(2) \quad P(o, \omega) = 1 \quad \text{and} \quad P(xy, \omega) = P(y, x^{-1}\omega) P(x, \omega).$$

We shall need some function spaces and related notations. Given a discrete space  $X$ , we denote by  $\mathcal{D}(X)$  the space of all finitely supported functions on  $X$ . A function  $f$  on  $G$  is radial if it is constant on  $S(o, n)$  for all  $n \in \mathbb{N}$ . If  $E(G)$  is a space of functions on  $G$ , then  $E(G)^\#$  will denote the subset of  $E(G)$  of radial functions. If  $E(\mathbb{Z})$  is a space of functions on  $\mathbb{Z}$ , then  $E(\mathbb{Z})_{\text{even}}$  will denote the subspace of  $E(\mathbb{Z})$  of even functions therein.

We define the convolution product on  $G$  by

$$f * g(x) = \sum_{y \in G} f(y)g(y^{-1}x).$$

If  $g$  is radial, we have

$$f * g(x) = \sum_{n \in \mathbb{N}} g(n) \sum_{d(x,y)=n} f(y)$$

We use the variable constants convention, and denote by  $C$  a constant who will depend only on unvariable data.

### 3. THE ABEL TRANSFORMATION

In this section, we give an explicite expression of the Abel transformation, which is the horocyclic Radon transformation on radial functions. We first begin by describing the horocycles on  $\Gamma$ . Recall that  $\mathcal{V}$  denotes the set of vertices of  $\Gamma$  and  $\Omega$  its boundary. For  $\omega \in \Omega$  and  $x \in \mathcal{V}$ , there exists a unique geodesic ray issued from  $x$  and joining  $\omega$  that we will denote by  $[x, \omega]$ . Let  $x, y \in \Gamma$  and  $\omega \in \Omega$ . We denote by  $z$  the confluence point of the geodesic rays  $[x, \omega]$ ,  $[y, \omega]$ , that is the last point on  $[x, \omega]$  laying on the geodesic ray  $[y, \omega]$ . We then define  $\zeta_\omega(x, y) = d(x, z) - d(y, z)$ . Thus,  $\forall \omega \in \Omega$ , the relation  $\zeta_\omega(x, y) = 0$  defines an equivalence relation on  $\mathcal{V}$ , and the equivalence classes denote the horocycles of  $\Gamma$ . The choice of  $o$  as an origin will permit us to enumerate the horocycles and to prove that the set of horocycles  $H_h(\omega)$  is one-to-one with  $\Omega \times \mathbb{Z}$ . More precisely,  $\forall \omega \in \Omega$ ,  $\forall h \in \mathbb{Z}$ ,

$$H_h(\omega) = \{x \in \mathcal{V}, \zeta_\omega(o, x) = h\}.$$

The function  $\zeta_\omega(o, x)$  is nothing else than the Busemann function  $\zeta(x, \omega)$  defined above. Thus, horocycles are the level sets of the Poisson kernel. We may prove easily that  $\mathcal{V}$  decomposes disjointly as :

$$\mathcal{V} = \bigcup_{h \in \mathbb{Z}} H_h(\omega),$$

in particular,  $o \in H_0(\omega)$ . The horocyclic Radon transformation is defined on  $\mathcal{D}(\mathcal{V})$  by

$$\mathcal{R}f(\omega, h) = \sum_{x \in H_h(\omega)} f(x)$$

and the Abel transformation on  $\mathcal{D}(\mathcal{V})^\sharp$  by

$$\mathcal{A}f(\omega, h) = Q^{\frac{h}{2}} \mathcal{R}f(\omega, h).$$

The following proposition gives an explicit formula for the Abel transform of a radial function  $f$ . Such a function may be identified with a function on  $\mathbb{N}$ , and we denote by  $f(n)$  the common value  $f(x)$  when  $|x| = n$ .

**Proposition 3.1.** *Denote  $\sigma = k - 2$ . If  $f \in \mathcal{D}(\mathcal{V})^\sharp$ , then*

$$\mathcal{A}f(\omega, h) = Q^{\frac{|h|}{2}} f(|h|) + \sigma \sum_{j \geq 1} Q^{\frac{|h|}{2} + j - 1} f(|h| + 2j - 1) + \frac{r - 2}{r - 1} \sum_{j \geq 1} Q^{\frac{|h|}{2} + j} f(|h| + 2j)$$

$$\forall (\omega, h) \in \Omega \times \mathbb{Z}.$$

Consequently,  $\mathcal{A}f$  is constant in the first variable, and is even in the second. The following lemma is crucial in the proof of this proposition.

**Lemma 3.2.** *Given  $\omega \in \Omega$ . For all  $n \in \mathbb{N}$  and  $h \in \mathbb{Z}$ , let  $b(n, h) = \text{Card} \{H_h(\omega) \cap S(o, n)\}$ . Then,*

$$b(n, h) = \begin{cases} 0 & \text{if } n < |h| \\ Q^{-h_-} & \text{if } n = |h| \\ \sigma Q^{-h_- + j - 1} & \text{if } n = |h| + 2j - 1 \text{ where } j \geq 1 \\ (r - 2)(k - 1) Q^{-h_- + j - 1} & \text{if } n = |h| + 2j \text{ where } j \geq 1, \end{cases}$$

where  $h_- = \min(0, h)$ .

*Proof of lemma.* The proof is similar to that on semihomogeneous trees (see [5]). Given  $\omega \in \Omega$ , and let  $\omega = a_{i_1}^{n_1} a_{i_2}^{n_2} \dots$  be the reduced word representation of  $\omega$ . We denote by  $\omega_0 = a_{i_1}^{n_1}$ ,  $\omega_1 = \omega_0 a_{i_2}^{n_2}, \dots$  the nodes of  $[o, \omega]$ , that is the points on  $[o, \omega]$  where we change of horocycles (see fig. ). when  $h \geq 0$ ,  $\omega_h$  is the only element of  $H_h(\omega) \cap S(o, h)$ , thus  $b(h, h) = 1$ . Now, if  $h < 0$ , a point  $x$  belongs to  $H_h(\omega) \cap S(o, |h|)$  iff  $o$  is the confluence point of the geodesics  $[x, \omega]$  and  $[o, \omega]$ , or

$$\text{Card}\{x \in S(o, -h) \mid o \notin [x, \omega] \cap [o, \omega]\} = (k - 1)^{-h} (r - 1)^{-h - 1},$$

then  $b(-h, h) = \text{Card}\{S(o, -h)\} - (k-1)^{-h} (r-1)^{-h-1} = Q^{-h}$ . Clearly  $b(n, h) = 0$  when  $|h| > n$ . In the other cases, we proceed by recurrence on  $n$ , using the following equality

$$(3) \quad b(n+2, h) = Qb(n, h)$$

in both cases. We consider first the case  $h = n - 1$ . In this case  $h$  and  $n$  are of different parity. The intersection points of  $S(o, n)$  and  $H_h(\omega)$  are situated on the polygon that has  $\omega_h$  and  $\omega_{h+1}$  as nodes, thus  $\omega_h \in H_h(\omega) \setminus S(o, n)$ , and  $\omega_{h+1} \in H_{h+1}(\omega)$ . Clearly the other points of this polygon belongs to  $S(o, n) \cap H_h(\omega)$  (see fig...), so  $b(n, n-1) = \sigma$ , thus we have proven the formula for  $h = n - 1$ . If  $h = n - 2$ ,  $h$  and  $n$  have the same parity. In this case, we consider the points  $v \in \mathcal{V}$  such that  $|v| = n = |\omega_h + 2|$ , and  $v \in H_h(\omega)$ . These points are vertices of polygons issued from  $\omega_{h+1}$ . Or, all vertice of polygons situated between  $\omega_{h+1}$  and  $\omega_{h+2}$  belong to  $H_{h+1}(\omega)$ . It's clear that the  $(r-2)$  other polygons have vertices that belong to  $S(o, n) \cap H_h(\omega)$  for  $h = n - 2$  (see fig 1). Then, if  $h = n - 2$ ,  $b(n, h) = (r-2)(k-1)$ . We conclude using (3).  $\square$

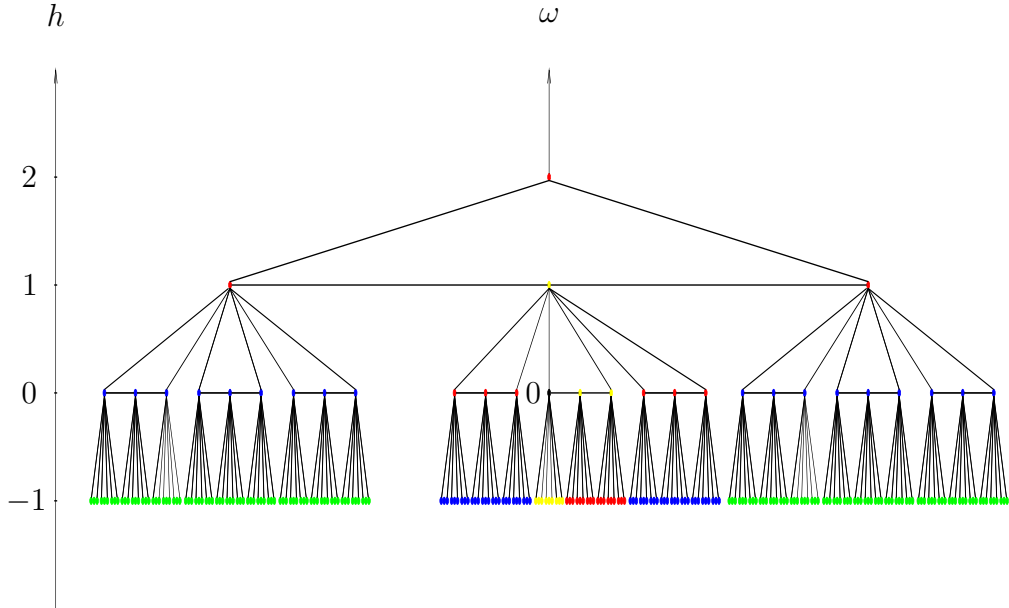


FIGURE 1. Spheres in the upper-half space of  $(\mathbb{Z}/4\mathbb{Z})^{*4}$

By definition of the Abel transformation,

$$\mathcal{A}f(\omega, h) = Q^{\frac{h}{2}} \sum_{n \in \mathbb{N}} b(n, h) f(n),$$

thus we prove the proposition 3.1 by substituting the value of  $b(n, h)$  in this formula.

**Proposition 3.3.** *For all  $k \leq r$ ,  $\mathcal{A} : \mathcal{D}(G)^\# \rightarrow \mathcal{D}(\mathbb{Z})_{\text{even}}$  is an isomorphism, and its inverse is given by*

$$(4) \quad \mathcal{A}^{-1}g(n) = \frac{1}{k} Q^{-\frac{n-1}{2}} \left\{ \sum_{m=1}^{\infty} Q^{-\frac{m}{2}} [g(n+m-1) - g(n+m+1)] \right. \\ \left. + \sum_{m=1}^{\infty} (-1)^{m-1} (k-1)^m Q^{-\frac{m}{2}} [g(n+m-1) - g(n+m+1)] \right\}.$$

which is equivalent to the following expression

$$(5) \quad \mathcal{A}^{-1}g(n) = Q^{-\frac{n}{2}} \left\{ g(n) - (k-2) Q^{-\frac{1}{2}} g(n+1) \right. \\ \left. - \frac{Q-1}{k} \sum_{m=1}^{\infty} Q^{-\frac{m}{2}} g(n+m) \right. \\ \left. - \frac{r-k}{r} \sum_{m=1}^{\infty} (-1)^m (k-1)^m Q^{-\frac{m}{2}} g(n+m) \right\}.$$

Note that these sums are finite as long as we consider functions  $g$  of finite supports.

*Proof.* By definition of  $\mathcal{A}$ , we have

$$Q^{-\frac{n}{2}} \left\{ \mathcal{A}f(n) - \mathcal{A}f(n+2) + Q^{-\frac{1}{2}} [\mathcal{A}f(n+1) - \mathcal{A}f(n+3)] \right\} = \\ f(n) - f(n+2) + (k-1) [f(n+1) - f(n+3)],$$

or  $f$  is of finite support, so

$$f(n) + (k-1) f(n+1) = \sum_{j \geq 0} \{f(n+2j) - f(n+2j+2)\} + \\ (k-1) \sum_{j \geq 1} \{f(n+2j-1) - f(n+2j+1)\}$$

thus

$$(6) \quad f(n) + (k-1) f(n+1) = \sum_{j \geq 0} Q^{-\frac{n}{2}-j} \left\{ \mathcal{A}f(n+2j) - \mathcal{A}f(n+2j+2) \right\} \\ + \sum_{j \geq 1} Q^{-\frac{n-1}{2}-j} \left\{ \mathcal{A}f(n+2j-1) - \mathcal{A}f(n+2j+1) \right\}.$$

Let  $F(n) = (k-1)^n f(n)$ , so multiplying both members of (6) by  $(k-1)^n$ , we obtain the following recurrence formula

$$(7) \quad F(n) + F(n+1) = (k-1)^n G(n),$$



where

$$(8) \quad G(n) = \sum_{j \geq 0} Q^{-\frac{n}{2}-j} \{g(n+2j) - g(n+2j+2)\} \\ + \sum_{j \geq 0} Q^{-\frac{n+1}{2}-j} \{g(n+2j+1) - g(n+2j+3)\}.$$

We rewrite (7) as

$$F(n) = (k-1)^n G(n) - F(n+1).$$

Then

$$(9) \quad F(n) = \sum_{\ell \geq 0} (-1)^\ell (k-1)^{n+\ell} G(n+\ell).$$

Thus

$$(10) \quad f(n) = \sum_{\ell \geq 0} (-1)^\ell (k-1)^\ell G(n+\ell).$$

Note that this sum is finite for functions  $g = \mathcal{A}f$ , consequently  $G$  is of finite support. Substituing (8) in (10), then letting  $m = \ell + 2j + 1$ , respectively  $m = \ell + 2j$ , and distinguishing the case  $m$  even and  $m$  odd, we have

$$(11) \quad \mathcal{A}^{-1}g(n) = Q^{-\frac{n}{2}} \{g(n) - g(n+2)\} \\ - \sum_{m \text{ even} \geq 2} \frac{(k-1)^m - 1}{k} Q^{-\frac{n+m-1}{2}} \{g(n+m-1) - g(n+m+1)\} \\ + \sum_{m \text{ odd} \geq 3} \frac{(k-1)^m + 1}{k} Q^{-\frac{n+m-1}{2}} \{g(n+m-1) - g(n+m+1)\} \\ = \frac{1}{k} \left\{ \sum_{m \text{ odd} > 0} [(k-1)^m + 1] Q^{-\frac{n+m-1}{2}} \{g(n+m-1) - g(n+m+1)\} \right. \\ \left. - \sum_{m \text{ even} > 0} [(k-1)^m - 1] Q^{-\frac{n+m-1}{2}} \{g(n+m-1) - g(n+m+1)\} \right\}.$$

Rearranging the terms, we obtain (4), then (5). □

**Corollary 3.4.**  $\text{supp } f \subset B'(0, n)$  iff  $\text{supp } \mathcal{A}f \subset [-n, +n]$ .

Our next goal is to extend the definition of the Abel transformation to Schwartz spaces. For  $0 < p < \infty$ , let  $\mathcal{S}_p(G)$  the set of functions  $f : G \rightarrow \mathbb{C}$  such that

$$(12) \quad \|f\|_{(p,m)} = \sup_{x \in G} (1+|x|)^m Q^{|x|/p} |f(x)| < \infty \quad \forall m \in \mathbb{N}.$$

It's a Frechet space for the increasing norms family (12). We will also consider the space  $\mathcal{S}(\mathbb{Z})_{\text{even}}$  of even functions  $g : \mathbb{Z} \rightarrow \mathbb{C}$  such that

$$\|g\|_{(m)} = \sup_{n \in \mathbb{Z}} (1+|n|)^m |g(n)| < \infty \quad \forall m \in \mathbb{N}.$$

- Proposition 3.5.** (1) For all  $1 \leq p \leq 2$ , the Abel transformation extends to a continuous homomorphism from  $\mathcal{S}_p(G)^\sharp$  to  $Q^{-(\frac{1}{p}-\frac{1}{2})|\cdot|} \mathcal{S}(\mathbb{Z})_{\text{even}}$   
(2) If  $k \leq r$ , then  $\mathcal{A}^{-1}$  is a continuous homomorphism from  $Q^{-(\frac{1}{p}-\frac{1}{2})|\cdot|} \mathcal{S}(\mathbb{Z})_{\text{even}}$  to  $\mathcal{S}_p(G)^\sharp$ , for all  $1 \leq p \leq 2$   
(3) If  $k > r$ , then  $\mathcal{A}^{-1}$  is a continuous homomorphism from  $Q^{-\frac{1}{2}|\cdot|} \mathcal{S}(\mathbb{Z})_{\text{even}}$  to  $\mathcal{S}_1(G)^\sharp$ .

*Proof.* Let  $1 \leq p \leq 2$ . We will show that, for all  $m \in \mathbb{N}$  there exist  $C > 0$  such that, for  $f \in \mathcal{S}_p(G)^\sharp$ ,

$$(13) \quad \left\| Q^{(\frac{1}{p}-\frac{1}{2})|\cdot|} \mathcal{A}f \right\|_{(m)} \leq C \|f\|_{(p,m+2)}.$$

Or by hypothesis we have

$$|f(n)| \leq C'(1+n)^{-m-2} Q^{-n/p} \|f\|_{(p,m+2)}$$

for all  $f \in \mathcal{S}_p(G)^\sharp$  and  $n \in \mathbb{N}$ . Then

$$\begin{aligned} (1+h)^m Q^{(\frac{1}{p}-\frac{1}{2})h} |\mathcal{A}f(h)| &\leq (1+h)^m \sum_{j \in \mathbb{N}} Q^{\frac{h}{p}+\frac{j}{2}} |f(h+j)| \\ &\leq C \sum_{j \in \mathbb{N}} (1+j)^{-2} \|f\|_{(p,m+2)} \end{aligned}$$

for all  $h \in \mathbb{N}$ , and this end the proof of the first part. For the second part, we will show that, for all  $m \in \mathbb{N}$ , there exist  $C > 0$  such that

$$\|\mathcal{A}^{-1}g\|_{(p,m)} \leq C \|g\|_{(m+2)} \quad \forall g \in Q^{-(\frac{1}{p}-\frac{1}{2})|\cdot|} \mathcal{S}(\mathbb{Z})_{\text{even}}.$$

By hypothesis, for all  $g \in Q^{-(\frac{1}{p}-\frac{1}{2})|\cdot|} \mathcal{S}(\mathbb{Z})_{\text{even}}$  and for all  $m \in \mathbb{N}$ , we have

$$|g(n)| \leq (1+n)^{-m-2} Q^{-(\frac{1}{p}-\frac{1}{2})n} \left\| Q^{-(\frac{1}{p}-\frac{1}{2})|\cdot|} g \right\|_{(m+2)} \quad \forall n \in \mathbb{N}.$$

Or  $|\mathcal{A}^{-1}g(n)| \leq \sum_{i \geq 0} (k-1)^i |G(n+i)| \quad \forall n \in \mathbb{N}$ , where

$$(k-1)^i \leq Q^{\frac{i}{2}} \leq Q^{\frac{i}{p}} \text{ and}$$

$$|G(n+i)| \leq C \sum_{j \geq 0} Q^{-\frac{n+i+j}{2}} |g(n+i+j)|,$$

so we deduce the following estimations

$$\begin{aligned} (1+n+i)^{m+2} Q^{\frac{n+i}{p}} |G(n+i)| &\leq C \sum_{j \geq 0} Q^{-\frac{j}{p}} (1+n+i+j)^{m+2} Q^{(\frac{1}{p}-\frac{1}{2})(n+i+j)} |g(n+i+j)| \\ &\leq C \left\| Q^{(\frac{1}{p}-\frac{1}{2})|\cdot|} g \right\|_{(m+2)} \quad \forall n, i, \in \mathbb{N}, \end{aligned}$$

then

$$\begin{aligned} (1+n)^m Q^{\frac{n}{p}} |\mathcal{A}^{-1}g(n)| &\leq \sum_{i \geq 0} (1+i)^{-2} (1+n+i)^{m+2} Q^{\frac{n+i}{p}} |G(n+i)| \\ &\leq C \left\| Q^{(\frac{1}{p}-\frac{1}{2})|\cdot|} g \right\|_{(m+2)} \quad \forall n \in \mathbb{N}. \end{aligned}$$

Thus we have proved the second part. The last part can be proven in the same way.  $\square$

**Corollary 3.6.** (1) *If  $k \leq r$ , then the Abel transformation is a topologic isomorphism between  $\mathcal{S}_p(G)^\sharp$  and  $Q^{-(\frac{1}{p}-\frac{1}{2})|\cdot|} \mathcal{S}(\mathbb{Z})_{\text{even}}$ , for all  $1 \leq p \leq 2$ .*  
(2) *If  $k > r$ , then the Abel transformation is a topologic isomorphism between  $\mathcal{S}_1(G)^\sharp$  and  $Q^{-\frac{1}{2}|\cdot|} \mathcal{S}(\mathbb{Z})_{\text{even}}$ .*

Next, we will derive an explicite expression of the dual Abel transformation and its inverse. The dual Abel transform  $\mathcal{A}^*g$  of even functions  $g : \mathbb{Z} \rightarrow \mathbb{C}$  is defined as follows :

$$(14) \quad \sum_{n \in \mathbb{N}} \mathcal{A}^*g(n) f(n) \delta(n) = \sum_{h \in \mathbb{Z}} g(|h|) \mathcal{A}f(|h|)$$

for all  $f \in \mathcal{D}(\mathcal{V})^\sharp$ . Recall that  $\delta(n)$  denotes the cardinal of sphere of radius  $n$ , and  $\sigma = k - 2$ .

**Theorem 3.7.** *We have*

$$\mathcal{A}^*g(0) = g(0)$$

and

$$(15) \quad \mathcal{A}^*g(n) = 2 \frac{r-1}{r} Q^{-\frac{n}{2}} g(n) + \sigma \frac{r-1}{r} Q^{-\frac{n-1}{2}} \sum_{\substack{-n < j < n \\ j \text{ and } n \text{ of different parity}}} g(\pm j) + \frac{r-2}{r} Q^{-\frac{n}{2}} \sum_{\substack{-n < j < n \\ j \text{ and } n \text{ of different parity}}} g(\pm j)$$

for all  $n > 0$ .

*Proof.*

$$(16) \quad \sum_{h \in \mathbb{Z}} \mathcal{A}f(h) g(h) = \mathcal{A}f(0)g(0) + 2 \sum_{h \geq 1} \mathcal{A}f(h) g(h) = \sum_{n \geq 0} Q^{\frac{n}{2}} f(n)G(n).$$

where  $G(0) = g(0)$  and

$$G(n) = 2g(n) + \frac{r-2}{r-1} \sum_{\substack{-n < j < n \\ j \text{ et } n \text{ of different parity}}} g(\pm j) + \sigma Q^{-\frac{1}{2}} \sum_{\substack{-n < j < n \\ j \text{ et } n \text{ of different parity}}} g(\pm j),$$

we then conclude easily.  $\square$

We will next establish the expression of the inverse dual Abel transformation.

**Theorem 3.8.** *The dual Abel transformation is an isomorphism from the space of even functions  $g : \mathbb{Z} \rightarrow \mathbb{C}$  on the space of radial functions  $f : \mathcal{V} \rightarrow \mathbb{C}$ . Its inverse is given by*

$$g(0) = f(0), \quad g(1) = -\frac{\sigma}{2} Q^{-\frac{1}{2}} f(0) + \frac{r(k-1)}{2} Q^{-\frac{1}{2}} f(1)$$

et

$$\begin{aligned} g(n) = & -\frac{1}{2k} \{q-1+(r-k)(1-k)^n\} Q^{-\frac{n}{2}} f(0) \\ & -\frac{r(k-1)}{2k} \sum_{0 < j < n-1} \{q-1+(r-k)(1-k)^{n-j}\} Q^{j-\frac{n}{2}-1} f(j) \\ & -\frac{1}{2} r(k-1) \sigma Q^{\frac{n}{2}-2} f(n-1) + \frac{1}{2} r(k-1) Q^{\frac{n}{2}-1} f(n) \end{aligned}$$

for all  $n \geq 2$ , with the usual convention that sum on empty sets is equal to zero.

*Proof.* Given  $g \in \mathcal{D}(\mathbb{Z})_{\text{even}}$ , let

$$\begin{aligned} G(n) &= Q^{\frac{n}{2}} g(n) \quad \text{and} \\ F(n) &= \frac{1}{2} r(k-1) Q^{\frac{n}{2}} \left\{ Q^{\frac{n+2}{2}} \mathcal{A}^* f(n+2) - Q^{\frac{n}{2}} \mathcal{A}^* g(n) \right\}. \end{aligned}$$

From theorem 3.7 we have,

$$(17) \quad G(n+2) + \sigma G(n+1) - (k-1)G(n) = F(n)$$

for all  $n \in \mathbb{N}$ . The equation (17) is a inhomogeneous linear recurrence relation of second order. Its homogeneous solution is given by

$$G_{\text{hom}}(n) = c_1 + c_2(1-k)^n.$$

We then use the constant variation method to resolve the inhomogeneous equation :

$$(18) \quad G(n) = c_1(n) + c_2(n)(1-k)^n.$$

with the additionnal condition :

$$(19) \quad c_1(n+1) - c_1(n) = \{c_2(n+1) - c_2(n)\} (1-k)^{n+1} = 0$$

so we obtain

$$(20) \quad G(n+1) = c_1(n) + c_2(n) (1 - k)^{n+1},$$

hence

$$(21) \quad G(n+2) = c_1(n+1) + c_2(n+1) (1 - k)^{n+2}.$$

Substituing (18), (20), (21) in (17), we obtain

$$(22) \quad c_1(n+1) - c_1(n) + \{c_2(n+1) - c_2(n)\} (1 - k)^{n+2} = F(n).$$

The system of equations formed by (19) and (22) have the following solution

$$\begin{cases} c_1(n+1) - c_1(n) = k^{-1} F(n) \\ c_2(n+1) - c_2(n) = -k^{-1} (1 - k)^{-n-1} F(n) \end{cases}$$

hence

$$\begin{cases} c_1(n) = c_1 + k^{-1} \sum_{0 \leq j < n} F(j) \\ c_2(n) = c_2 - k^{-1} \sum_{0 \leq j < n} (1 - k)^{-j-1} F(j) \end{cases}$$

and

$$(23) \quad G(n) = c_1 + c_2(1 - k)^n + \sum_{0 \leq j \leq n-2} \frac{1 - (1 - k)^{n-j-1}}{k} F(j).$$

We compute the constants

$$\begin{cases} c_1 = \frac{1}{2} \mathcal{A}^* g(0) + \frac{r(k-1)}{2k} \mathcal{A}^* g(1), \\ c_2 = \frac{1}{2} \mathcal{A}^* g(0) - \frac{r(k-1)}{2k} \mathcal{A}^* g(1), \end{cases}$$

using the initial values

$$\begin{cases} G(0) = g(0) = \mathcal{A}^* g(0), \\ G(1) = Q^{\frac{1}{2}} g(1) = \frac{r(k-1)}{2} \mathcal{A}^* g(1) - \frac{\sigma}{2} \mathcal{A}^* g(0). \end{cases}$$

We conclude by substituting in (23) the expressions of  $c_1$ ,  $c_2$  and  $F$ . □

We will now describe some consequences of the results obtained above. Let us recall first the definition of the spherical Fourier transformation on  $\Gamma$ . There is a natural Laplace operator  $\mathcal{L}$  on  $\Gamma$ , defined by

$$\mathcal{L}f(x) = f(x) - \frac{1}{r(k-1)} \sum_{y:d(x,y)=1} f(y),$$

and the spherical functions are the radial eigenfunctions  $\varphi$  of  $\mathcal{L}$ , normalized with the condition  $\varphi(o) = 1$ . Each of these functions may be represented (see e.g., [17]) in terms of the Poisson kernel by the formula

$$(24) \quad \varphi_\lambda(x) = \int_{\Omega} P(x, \omega)^{\frac{1}{2}+i\lambda} d\omega,$$

so that  $\varphi_\lambda(x)$  is an eigenfunction with eigenvalue

$$\gamma(\lambda) = \frac{Q^{\frac{1}{2}+i\lambda} + Q^{\frac{1}{2}-i\lambda} + \sigma}{r(k-1)}.$$

The spherical Fourier transform of a function  $f \in \mathcal{D}(\Gamma)^\sharp$  is defined by

$$\mathcal{H}f(\lambda) = \sum_{x \in \Gamma} f(x) \varphi_\lambda(x) \quad \forall \lambda \in \mathbb{C}.$$

This definition of the spherical Fourier transformation may be extended to some classes of functions of infinite support. For example, as

$$|\varphi_\lambda(x)| \leq 1 \quad \forall x \in G, \lambda \in \mathbb{C} \text{ with } |\operatorname{Im} \lambda| \leq \frac{1}{2},$$

$\mathcal{H}f(\lambda)$  still can be defined for functions  $f \in L^1(G)^\sharp$  and for  $\lambda \in \mathbb{C}$  with  $|\operatorname{Im} \lambda| \leq \frac{1}{2}$ .

Let  $\tau = \frac{2\pi}{\ln Q}$ .  $\mathcal{H}f$  is even and  $\tau$ -periodic. More generally, we can define the Helgason-Fourier transform of a function  $f \in \mathcal{D}(G)$  (not necessarily radial) by

$$(25) \quad \hat{f}(\lambda, \omega) = \sum_{x \in G} f(x) P(x, \omega)^{\frac{1}{2}+i\lambda}, \quad \forall (\omega, \lambda) \in \Omega \times \mathbb{C}.$$

Using the expression of  $P(x, \omega)$ , we obtain

$$(26) \quad \hat{f}(\lambda, \omega) = \sum_{h \in \mathbb{Z}} Q^{(\frac{1}{2}+i\lambda)h} \sum_{x \in H_h(\omega)} f(x) = \mathcal{F}_h \left\{ Q^{\frac{h}{2}} \mathcal{R}f(\omega, h) \right\}(\lambda),$$

where  $\mathcal{F}$  is the Fourier transform on  $\mathbb{Z}$ , given by :

$$\mathcal{F}g(\lambda) = \sum_{n \in \mathbb{Z}} Q^{in\lambda} g(n),$$

and its inverse is given by

$$g(n) = \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \mathcal{F}g(\lambda) Q^{-in\lambda} d\lambda.$$

Recall that for radial functions  $f$ ,  $\mathcal{A}f$  doesn't depend on  $\omega$ , whence, using (26), we observe that, for radial functions  $f$ ,  $\hat{f}$  doesn't depend on  $\omega$ . In this case, using the

definition (24) of spherical functions, we have

$$\hat{f}(\lambda, \omega) = \int_{\Omega} \hat{f}(\lambda, \omega) d\nu(\omega) = \sum_{x \in G} f(x) \int_{\Omega} P^{\frac{1}{2}+i\lambda}(x, \omega) d\nu(\omega) = \mathcal{H}f(\lambda).$$

This shows that, the spherical Fourier transformation and the Helgason-Fourier transformation coincide on radial functions, and we have

$$(27) \quad \mathcal{H} = \mathcal{F} \circ \mathcal{A}.$$

The next section is devoted to the Kunze-Stein phenomenon on  $\Gamma$ . To this end, we will establish the Plancherel formula and the inversion formula of the Helgason-Fourier transformation. Recall that the Plancherel formula and the inversion formula for the spherical Fourier transformation were obtained by different methods in [11], [19]. More precisely, Iozzi and Picardello [17] have computed the Plancherel measure  $\mu$  on  $\Gamma$  and they have shown that  $\mu$  is supported on the set  $D \cup E$ , where  $D$  is the segment  $[\frac{\sigma-2Q^{\frac{1}{2}}}{r(k-1)}, \frac{\sigma+2Q^{\frac{1}{2}}}{r(k-1)}]$  and  $E$  is empty when  $k \leq r$  and is equal to  $\frac{1}{1-k}$  if  $k > r$ . More precisely, from [11], [19] we have the following expression of the Plancherel measure

$$(28) \quad f(o) = \frac{1}{2\pi} \frac{q \ln(q)}{r(k-1)} \int_0^{\frac{\pi}{2}} \mathcal{H}f(\lambda) |c(\lambda)|^{-2} d\lambda + \left[ \frac{(k-r)_+}{k} \mathcal{H}f(\lambda_0) \right]$$

where the usual notation  $(k-r)_+$  stands to 0 if  $k \leq r$  and to  $(k-r)$  if  $k > r$ , and  $\gamma(\lambda_0) = \frac{1}{1-k}$ . We then can deduce the following.

- **Plancherel formula** : For  $f \in \mathcal{D}(\mathcal{V})^\sharp$ , we have

$$(29) \quad \|f\|_{L^2}^2 = \frac{1}{2\pi} \frac{q \ln(q)}{r(k-1)} \int_0^{\frac{\pi}{2}} |\mathcal{H}f(\lambda)|^2 |c(\lambda)|^{-2} d\lambda + \left[ \frac{(k-r)_+}{k} |\mathcal{H}f(\lambda_0)|^2 \right]$$

- **Inversion formula** : For all  $f \in \mathcal{D}(\mathcal{V})^\sharp$  et  $x \in \mathcal{V}$ , we have

$$(30) \quad f(x) = \frac{1}{2\pi} \frac{q \ln(q)}{r(k-1)} \int_0^{\frac{\pi}{2}} \mathcal{H}f(\lambda) \varphi_\lambda(x) |c(\lambda)|^{-2} d\lambda + \left[ \frac{(k-r)_+}{k} \mathcal{H}f(\lambda_0) \varphi_{\lambda_0}(x) \right].$$

We will now generalize in the usual way these results to the nonradial case. The proof is like the proof of the inversion formula for the Helgason Fourier transformation on a symmetric space. More precisely, we first obtain the expression for  $f(o)$  using the inversion formula established above for the spherical Fourier transformation. Then we derive the expression of  $f(x)$  by noting that  $f(x)$  is the value at  $o$  of a suitable translate

of  $f$ . To this end, we define the spherical means of a function  $f : G \rightarrow \mathbb{C}$  by

$$(31) \quad f^\sharp(z) = \frac{1}{\delta(z)} \sum_{y: |y|=|z|} f(y).$$

The mean operator  $f \rightarrow f^\sharp$  is the projection over radial functions. It verifies

$$(32) \quad \begin{aligned} \langle f^\sharp, g^\sharp \rangle &= \langle f^\sharp, g \rangle = \langle f, g^\sharp \rangle, \\ \langle f, g \rangle &= \sum_{x \in \mathcal{V}} f(x) g(x) \end{aligned}$$

Let  $f \in \mathcal{D}(\mathcal{V})$ . Since  $\varphi_\lambda$  is radial, we have

$$(33) \quad \langle f^\sharp, \varphi_\lambda \rangle = \langle f, (\varphi_\lambda)^\sharp \rangle = \langle f, \varphi_\lambda \rangle.$$

Applying (33) and (31) on  $f^\sharp$ , we have

$$(34) \quad f(o) = f^\sharp(o) = \frac{1}{2\pi} \frac{Q \ln(Q)}{r(k-1)} \int_0^{\frac{\pi}{2}} \langle f, \varphi_\lambda \rangle |c(\lambda)|^{-2} d\lambda + \left[ \frac{(k-r)_+}{r} \langle f, \varphi_{\lambda_0} \rangle \right].$$

Recall the principal series of representations  $\pi_\lambda$  of  $G$  given in [17]. For all  $\eta \in L^2(\Omega, d\nu)$ ,

$$(35) \quad \pi_\lambda(x)\eta(\omega) = P(x, \omega)^{\frac{1}{2}+i\lambda} \eta(x^{-1}\omega) \quad \forall x \in G, \forall \omega \in \Omega,$$

where  $P(x, \omega)$  design the Poisson kernel, and  $\lambda \in \mathbb{C}$ . These representations were defined and studied on homogeneous trees in [13], and then on symmetric graphs in [17]. Note also that  $\pi_\lambda$  stands also to the representation on the convolution algebra  $\mathcal{D}(\mathcal{V})$

$$\pi_\lambda(f) = \sum_{x \in \mathcal{V}} f(x) \pi_\lambda(x).$$

Spherical functions are matrix coefficients of  $\pi_\lambda$ . They verify

$$\varphi_\lambda(x) = (\pi_\lambda(x) \mathbb{1}, \mathbb{1})$$

where  $(\cdot | \cdot)$  is the scalar product in  $L^2(\Omega, d\nu)$ , and  $\mathbb{1}$  is the constant function that equals to 1 on  $\Omega$ . We now can define the Helgason–Fourier transformation in terms of  $\pi_\lambda$  by

$$(36) \quad \hat{f}(\lambda, \omega) = \sum_{x \in \mathcal{V}} f(x) P(x, \omega)^{\frac{1}{2}+i\lambda} = [\pi_\lambda(f) \mathbb{1}](\omega).$$

**Lemma 3.9.** *for all  $x, y \in G$  we have*

$$\varphi_\lambda(xy^{-1}) = \varphi_\lambda(y^{-1}x) = \int_{\Omega} P(x, \omega)^{\frac{1}{2}+i\lambda} P(y, \omega)^{\frac{1}{2}-i\lambda} d\nu(\omega).$$

*Proof.* From the cocycle identity (2), we have, for all  $x, y \in G$  and  $\omega \in \Omega$ ,

$$P(y^{-1}x, \omega) = P(x, y\omega) P(y^{-1}, \omega) \quad \text{et} \quad 1 = P(o, \omega) = P(y, y\omega) P(y^{-1}\omega),$$



whence

$$P(y^{-1}x, \omega) = P(x, y\omega) P(y, y\omega)^{-1}.$$

We may then deduce that

$$\begin{aligned} \varphi_\lambda(y^{-1}x) &= \int_{\Omega} P(x, y\omega)^{\frac{1}{2}+i\lambda} P(y, y\omega)^{-\frac{1}{2}-i\lambda} d\nu(\omega) \\ &= \int_{\Omega} P(x, y\omega)^{\frac{1}{2}+i\lambda} P(y, y\omega)^{-\frac{1}{2}-i\lambda} \frac{d\nu(\omega)}{d\nu(y\omega)} d\nu(y\omega) \\ &= \int_{\Omega} P(x, \omega)^{\frac{1}{2}+i\lambda} P(y, \omega)^{-\frac{1}{2}-i\lambda} P(y, \omega) d\nu(\omega) \\ &= \int_{\Omega} P(x, \omega)^{\frac{1}{2}+i\lambda} P(y, \omega)^{\frac{1}{2}-i\lambda} d\nu(\omega), \end{aligned}$$

where we have used the fact that  $P(y, \omega) = \frac{d\nu(y^{-1}\omega)}{d\nu(\omega)}$  is a Radon-Nikodym derivative.  $\square$

**Lemma 3.10.** *Let  $k \leq r$ . In this case we have the following :*

(a) **Plancherel formula** : *For all  $f \in \mathcal{D}(\mathcal{V})$ , we have*

$$(37) \quad \sum_{x \in \mathcal{V}} |f(x)|^2 = \frac{1}{2\pi} \frac{q \ln q}{r(k-1)} \int_0^{\frac{\pi}{2}} \int_{\Omega} |\hat{f}(\lambda, \omega)|^2 |c(\lambda)|^{-2} d\nu(\omega) d\lambda.$$

(b) **Inversion formula** : *For all  $f \in \mathcal{D}(\mathcal{V})$  and  $x \in \mathcal{V}$ , we have*

$$(38) \quad f(x) = \frac{1}{2\pi} \frac{q \ln q}{r(k-1)} \int_0^{\frac{\pi}{2}} \int_{\Omega} \hat{f}(\lambda, \omega) P(x, \omega)^{\frac{1}{2}-i\lambda} |c(\lambda)|^{-2} d\nu(\omega) d\lambda.$$

*Note that, when  $k > r$ , there is an additive term that appears on the right hand side of the equality, which corresponds to the parameter  $\lambda_0$ .*

*Proof.* The proof is similar to the symmetric space case (see [15]) and to the homogeneous space case (see [13]). More precisely, we apply (28) on  $(f * f^*)^\sharp$ , where  $f^*(x) = \overline{f(x^{-1})}$ . On one hand we have

$$(f * f^*)(o) = \sum_{x \in G} |f(x)|^2 = \|f\|_{L^2(\mathcal{V})}^2.$$

On the other hand, from lemma 3.9,

$$\begin{aligned} \mathcal{H} [(f * f^*)^\sharp] (\lambda) &= \sum_{x,y \in G} f(x) \overline{f(y)} \varphi_\lambda(xy^{-1}) \\ &= \int_{\Omega} \left[ \sum_{x \in G} f(x) P(x, \omega)^{\frac{1}{2} + i\lambda} \right] \times \left[ \sum_{y \in G} f(y) P(y, \omega)^{\frac{1}{2} + i\lambda} \right] d\nu(\omega) \\ &= \int_{\Omega} |\hat{f}(\lambda, \omega)|^2 d\nu(\omega) \end{aligned}$$

for all  $\lambda \in \mathbb{R}$ . We conclude by using (3.9), proven (a). In order to prove (b), we apply (28) to  $f_x^\sharp$ . On one hand,  $(f_x^\sharp)(o) = f(x)$ . On the other hand, using (31) and lemma 3.9, one can show that

$$\begin{aligned} \mathcal{H} [(f_x)^\sharp] (\lambda) &= \sum_{\substack{x,y \in G \\ |y|=|x|}} f(xy) \frac{\varphi_\lambda(x,y)}{\delta(|z|)} \\ &= \sum_{y \in G} f(y) \varphi_\lambda(x^{-1}y) = \int_{\Omega} \left[ \sum_{y \in G} f(y) P(y, \omega)^{\frac{1}{2} + i\lambda} \right] P(x, \omega)^{\frac{1}{2} - i\lambda} \\ &= \int_{\Omega} \hat{f}(\lambda, \omega) P(x, \omega)^{\frac{1}{2} - i\lambda} d\nu(\omega) \end{aligned}$$

for all  $\lambda \in \mathbb{R}$ . We conclude using (28). □

Now let  $f \in \mathcal{D}(\mathcal{V})$  and  $\chi \in \mathcal{D}(\mathcal{V})^\sharp$ . One can easily show that

$$(39) \quad \widehat{(f * \chi)}(\lambda, \omega) = \hat{f}(\lambda, \omega) \mathcal{H}\chi(\lambda).$$

Using the Plancherel formula (37), and the fact that  $|\varphi_\lambda(x)| \leq |\varphi_0(x)| \forall \lambda \in \mathbb{R}, \forall x \in \mathcal{V}$ , one can show that

$$\|f * \chi\|_{L^2} \leq \|f\|_{L^2} \sum_{x \in \mathcal{V}} \chi(x) \varphi_0(x).$$

Now using the fact that  $\varphi_0(x) \leq C(1+|x|)Q^{-\frac{|x|}{2}} \forall x \in \mathcal{V}$ , we obtain the following version of the Kunze–Stein phenomenon :

$$(40) \quad \|f * \chi\|_{L^2(\mathcal{V})} \leq C \|f\|_{L^2(\mathcal{V})} \sum_{n \geq 0} \chi(n) (1+n) q^{\frac{n}{2}}.$$

In the next proposition we generalize this result to general  $L^p$  spaces.

**Proposition 3.11.** *For all  $2 \leq p, \tilde{p} < \infty$ , there exist  $C > 0$  such that*

$$\|f * \chi\|_{L^p} \leq C \|f\|_{L^{\tilde{p}'}} \left\{ \sum_{n \geq 0} |\chi(n)|^r (1+n)^{2s} q^{(1+s)n} \right\}^{\frac{1}{r}},$$

where  $r = \frac{p\tilde{p}}{p+\tilde{p}}$  and  $s = \frac{\min\{p, \tilde{p}\}}{p+\tilde{p}}$ .

The same result has been obtained on symmetric spaces and more generally on Damek-Ricci spaces in [1]. The proof is based on an interpolation argument between the following inequalities,

$$\|f * \chi\|_{L^p} \leq \|f\|_{L^1} \|\chi\|_{L^p}, \quad \|f * \chi\|_{L^\infty} \leq \|f\|_{L^{\tilde{p}'}} \|\chi\|_{L^p}.$$

The interpolation argument is well detailed in ([1], lemma 5.1).

We show now how to derive an explicite expression of the heat kernel on symmetric graphs in terms of the heat kernel on  $\mathbb{Z}$ , using the relation (26). Recall that the heat kernel  $h(x, t)$  is the radial,  $t$ -analytic solution of the equation

$$\partial_t h(x, t) + \mathcal{L}h(x, t) = 0 \quad \text{and} \quad h(x, 0) = \delta_0.$$

We resolve this equation by the standard way, applying first the spherical Fourier transform, and then its inverse. So, using (26), one obtain , for  $k \leq r$ ,

$$\begin{aligned} h^\Gamma(x, t) = h^\Gamma(n, t) &= \frac{r(k-1)}{k} \frac{e^{-(\alpha-\beta)t}}{t} q^{-\frac{n}{2}} \left\{ \sum_{m=1}^{\infty} q^{-\frac{m}{2}} (n+m) h^{\mathbb{Z}}(n+m, \beta t) \right. \\ &\quad \left. + \sum_{m=1}^{\infty} (-1)^{m-1} (k-1)^m q^{-\frac{m}{2}} (n+m) h^{\mathbb{Z}}(n+m, \beta t) \right\} \end{aligned}$$

for all  $t > 0$  and  $n \in \mathbb{N}$  such that  $|x| = n$ , where

$$h^{\mathbb{Z}}(n, t) = \frac{e^{-t}}{\pi} \int_0^\pi e^{t \cos \lambda} \cos(n\lambda) d\lambda = e^{-t} I_n(t)$$

where  $I_n(t)$  is the modified Bessel function of the first kind. The heat kernel will be studied in a forthcoming paper, we will be able to establish sharp estimates for the real time heat kernel, and to the Schrödinger kernel.

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