



# Consistency analysis of a 1D Finite Volume scheme for barotropic Euler models

Florent Berthelin, Thierry Goudon, Sébastien Minjeaud

## ► To cite this version:

Florent Berthelin, Thierry Goudon, Sébastien Minjeaud. Consistency analysis of a 1D Finite Volume scheme for barotropic Euler models. J. Fuhrmann, M. Ohlberger, C. Rohde Eds. Finite Volumes for Complex Applications VII-Methods and Theoretical Aspects, 77, Springer Proc. in Math.

Stat., pp. 97-106, 2015. hal-00961615

HAL Id: hal-00961615

<https://hal.science/hal-00961615>

Submitted on 20 Mar 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Consistency analysis of a 1D Finite Volume scheme for barotropic Euler models

F. Berthelin <sup>\*1,3</sup>, T. Goudon <sup>†1,3</sup>, and S. Minjeaud <sup>‡2,3</sup>

<sup>1</sup>*Inria, Sophia Antipolis Méditerranée Research Centre, Project COFFEE*

<sup>2</sup>*Inria, Sophia Antipolis Méditerranée Research Centre, Project CASTOR*

<sup>3</sup>*Univ. Nice Sophia Antipolis, CNRS, LJAD, UMR 7351, France*

## Abstract

This work is concerned with the consistency study of a 1D (staggered kinetic) finite volume scheme for barotropic Euler models. We prove a Lax-Wendroff-like statement: the limit of a converging (and uniformly bounded) sequence of stepwise constant functions defined from the scheme is a weak entropic-solution of the system of conservation laws.

## 1 Introduction

**The model.** This work is concerned with the consistency study of a (staggered kinetic) Finite Volume (FV) scheme for barotropic Euler models

$$\partial_t \rho + \partial_x(\rho V) = 0, \quad (1)$$

$$\partial_t(\rho V) + \partial_x(\rho V^2 + p(\rho)) = 0. \quad (2)$$

The unknowns are the density  $\rho$  and the velocity  $V$ . The pressure ( $\rho \mapsto p(\rho)$ ) is assumed to be  $C^2([0, \infty))$  with  $p(\rho) > 0$ ,  $p'(\rho) > 0$ ,  $p''(\rho) \geq 0$ ,  $\forall \rho > 0$ . Thus, the sound speed  $c : \rho \mapsto \sqrt{p'(\rho)}$  is well defined and is an increasing function.

We consider the problem (1)-(2) on the bounded domain  $(0, L) \times [0, T]$  with the boundary conditions  $V(0, t) = 0 = V(L, t)$ ,  $\forall t > 0$  and the initial conditions  $\rho(x, 0) = \rho_0(x)$ ,  $V(x, 0) = V_0(x)$ ,  $\forall x \in (0, L)$  with  $\rho_0$ ,  $V_0 \in L^\infty(0, L)$ .

Let  $\Phi : \rho > 0 \mapsto \Phi(\rho)$  such that  $\rho\Phi'(\rho) - \Phi(\rho) = p(\rho)$ ,  $\forall \rho > 0$ . The quantity  $\mathcal{S} = \frac{1}{2}\rho|V|^2 + \Phi(\rho)$  is an entropy of the system: entropy solutions to (1)-(2) are required to satisfy: for any  $\varphi \in C_c^\infty((0, L) \times [0, T))$  such that  $\varphi \geq 0$ ,

$$-\int_0^T \int_0^L [\mathcal{S} \partial_t \varphi + (\mathcal{S} + p(\rho)) V \partial_x \varphi](x, t) dx dt - \int_0^L \mathcal{S}_0(x) \varphi(x, 0) dx \leq 0. \quad (3)$$

---

<sup>\*</sup>Florent.Berthelin@unice.fr

<sup>†</sup>thierry.goudon@inria.fr

<sup>‡</sup>minjeaud@unice.fr

**The meshes.** We consider a set of  $J + 1$  points  $0 = x_1 < x_2 < \dots < x_J < x_{J+1} = L$ . The  $x_j$  are the edges of the so-called primal mesh  $\mathcal{T}$ . We set  $\delta x_{j+1/2} = x_{j+1} - x_j$ . The centers of the primal cells,  $x_{j+1/2} = (x_j + x_{j+1})/2$  for  $j \in \{1, \dots, J\}$ , realize the dual mesh  $\mathcal{T}^*$ . We set  $\delta x_j = (\delta x_{j-1/2} + \delta x_{j+1/2})/2$  for  $j \in \{2, \dots, J-1\}$  and  $\delta x = \text{size}(\mathcal{T}) = \max_j \delta x_{j+1/2}$ . The adaptive time step is  $\delta t^k$  and we set  $\delta t = \max_k \delta t^k$ .

**The scheme.** We analyze the scheme introduced in [1]. It works on staggered grids: the densities,  $\rho_{j+1/2}^k$ ,  $j \in \{1, \dots, J\}$ , are evaluated at centers whereas the velocities,  $V_j^k$ ,  $j \in \{1, \dots, J+1\}$ , are evaluated at edges. We set, for  $j \in \{1, \dots, J\}$  and  $i \in \{2, \dots, J\}$

$$\rho_{j+1/2}^0 = \frac{1}{\delta x_{j+1/2}} \int_{x_j}^{x_{j+1}} \rho_0(x) dx, \quad V_i^0 = \frac{1}{\delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} V_0(x) dx. \quad (4)$$

The density is first updated with a FV approximation on the primal mesh

$$\delta x_{j+1/2} (\rho_{j+1/2}^{k+1} - \rho_{j+1/2}^k) + \delta t^k (\mathcal{F}_{j+1}^k - \mathcal{F}_j^k) = 0, \quad \forall j \in \{1, \dots, J\}. \quad (5)$$

Then, the velocity is updated with a FV approximation on the dual mesh:

$$\delta x_j (\rho_j^{k+1} V_j^{k+1} - \rho_j^k V_j^k) + \delta t^k (\mathcal{G}_{j+1/2}^k - \mathcal{G}_{j-1/2}^k + \pi_{j+1/2}^{k+1/2} - \pi_{j-1/2}^{k+1/2}) = 0, \quad (6)$$

for  $j \in \{2, \dots, J\}$ , while  $V_1^{k+1} = V_{J+1}^{k+1} = 0$ . The density on the edges  $\rho_j^k$  is defined by

$$2\delta x_j \rho_j^k = \delta x_{j+1/2} \rho_{j+1/2}^k + \delta x_{j-1/2} \rho_{j-1/2}^k, \quad \forall j \in \{2, \dots, J\}.$$

The definition of the fluxes relies on the kinetic framework. We refer the reader to [1] for details. Let us introduce the two following functions  $\mathcal{F}^+$  and  $\mathcal{F}^-$

$$\mathcal{F}^\pm(\rho, V) = \frac{\rho}{2c(\rho)} \int_{\xi \gtrless 0} \xi \mathbf{1}_{|\xi - V| \leq c(\rho)} d\xi.$$

We adopt the following formulas for mass fluxes:  $\mathcal{F}_1^k = \mathcal{F}_{J+1}^k = 0$ ,

$$\mathcal{F}_j^k = \mathcal{F}^+(\rho_{j-1/2}^k, V_j^k) + \mathcal{F}^-(\rho_{j+1/2}^k, V_j^k), \quad \forall j \in \{2, \dots, J\}, \quad (7)$$

and, for momentum fluxes:  $\mathcal{G}_{3/2}^k = \frac{V_2^k}{2} \mathcal{F}^-(\rho_{5/2}^k, V_2^k)$ ,  $\mathcal{G}_{J+1/2}^k = \frac{V_J^k}{2} \mathcal{F}^+(\rho_{J-1/2}^k, V_J^k)$ ,

$$\begin{aligned} \mathcal{G}_{j+1/2}^k &= \frac{V_j^k}{2} (\mathcal{F}^+(\rho_{j-1/2}^k, V_j^k) + \mathcal{F}^+(\rho_{j+1/2}^k, V_{j+1}^k)) \\ &\quad + \frac{V_{j+1}^k}{2} (\mathcal{F}^-(\rho_{j+1/2}^k, V_j^k) + \mathcal{F}^-(\rho_{j+3/2}^k, V_{j+1}^k)), \quad \forall j \in \{2, \dots, J-1\}. \end{aligned} \quad (8)$$

The discrete pressure gradient combines a space centered scheme and time semi implicit discretization, namely it uses

$$\pi_{j+1/2}^{k+1/2} = \rho_{j+1/2}^k \Phi'(\rho_{j+1/2}^{k+1}) - \Phi(\rho_{j+1/2}^k).$$

**Properties of the scheme.** The analysis is driven by the shapes of the functions  $\mathcal{F}^\pm$ , see [1, Lemma 3.2]. Here, we shall use the following properties

- (i) Smoothness:  $(\rho, V) \in (0, \infty) \times \mathbb{R} \mapsto \mathcal{F}^\pm(\rho, V)$  are of class  $C^1$ ,
- (ii) Consistency:  $\mathcal{F}^+(\rho, V) + \mathcal{F}^-(\rho, V) = \rho V, \quad \forall V \in \mathbb{R}, \forall \rho \geq 0.$

Under CFL conditions, see [1], the scheme preserves the positivity of the discrete density and discrete kinetic and internal energies evolution equations hold.

**Lemma 1.1** *Let  $N \in \mathbb{N}$ . Assume  $\min_i (\rho_{i+1/2}^0) > 0$ . For all  $k \in \{0, \dots, N-1\}$ , there exists  $\mathcal{V}^k > 0$ , which depends only on the state  $(\rho^k, V^k)$ , such that if*

$$\frac{\delta t^k}{\min_j (\delta x_{j+1/2})} \mathcal{V}^k \leq 1, \quad (10)$$

then,  $\min_i (\rho_{i+1/2}^k) > 0, \forall k \in \{0, \dots, N\}$  and

$$0 \leq \sum_{k=0}^{N-1} \sum_{j=2}^J D_j^k \leq C, \quad \text{with} \quad D_j^k = \frac{1}{4} \delta x_j \rho_j^{k+1} (V_j^{k+1} - V_j^k)^2, \quad (11)$$

$$\frac{\delta x_{j+1/2}}{\delta t^k} [e_{j+1/2}^{k+1} - e_{j+1/2}^k] + \bar{G}_{j+1}^k - \bar{G}_j^k + \pi_{j+1/2}^{k+1/2} [V_{j+1}^{k+1} - V_j^{k+1}] \leq \frac{D_j^k}{\delta t^k}, \quad (12)$$

$$\frac{\delta x_j}{\delta t^k} [E_{K,j}^{k+1} - E_{K,j}^k] + \Gamma_{j+1/2}^k - \Gamma_{j-1/2}^k + [\pi_{j+1/2}^{k+1/2} - \pi_{j-1/2}^{k+1/2}] V_j^{k+1} + \frac{D_j^k}{\delta t^k} \leq 0, \quad (13)$$

where  $E_{K,j}^k = \frac{1}{2} \rho_j^k (V_j^k)^2$  and  $e_{j+1/2}^k = \Phi(\rho_{j+1/2}^k)$  are the kinetic and internal energies. The fluxes are defined by  $\bar{G}_1^k = \bar{G}_{J+1}^k = 0$  and

$$\begin{aligned} \bar{G}_j^k &= \Phi(\rho_{j-1/2}^k) V_j^{k+1} - \frac{\delta x_{j-1/2}}{2 \delta t^k} [\bar{\Phi}(\overline{\rho_{j-1/2}^{k+1}}) - \bar{\Phi}(\rho_{j-1/2}^k)], \quad \forall j \in \{2, \dots, J\}, \\ \Gamma_{j+1/2}^k &= \frac{1}{2} V_j^k V_{j+1}^k \frac{\mathcal{F}_j^k + \mathcal{F}_{j+1}^k}{2} + \frac{1}{2} (V_j^k - V_{j+1}^k)^2 \frac{\mathcal{F}_j^{k,|\cdot|} + \mathcal{F}_{j+1}^{k,|\cdot|}}{2}, \quad \forall j \in \{1, \dots, J\}, \\ \overline{\rho_{j-1/2}^{k+1}} &= \rho_{j-1/2}^k - \frac{2 \delta t^k}{\delta x_{j-1/2}} (\mathcal{F}^-(\rho_{j+1/2}^k, V_j^k) - \mathcal{F}^-(\rho_{j-1/2}^k, V_j^k) - \rho_{j-1/2}^k (V_j^{k+1} - V_j^k)), \end{aligned}$$

and  $\mathcal{F}_1^{k,|\cdot|} = \mathcal{F}_{J+1}^{k,|\cdot|} = 0$ ,  $\mathcal{F}_j^{k,|\cdot|} = \mathcal{F}^+(\rho_{j-1/2}^k, V_j^k) - \mathcal{F}^-(\rho_{j+1/2}^k, V_j^k)$ ,  $\forall j \in \{2, \dots, J\}$ . The function  $\bar{\Phi}$  is a  $\mathcal{C}^2$  extension of the function  $\Phi$  (see [1, Section 4.3]).

**Results.** As in [2], we prove a Lax-Wendroff-like statement: the limit of a converging (and uniformly bounded) sequence of stepwise constant functions defined from the scheme is a weak entropic-solution of the system of conservation laws.

## 2 Consistency analysis

**Notation.** Assuming that  $\sum_{k=0}^{N-1} \delta t^k = T$ , we define the reconstructions ( $i = 0, 1$ )

$$\rho_\delta^{(i)} = \sum_{k=0}^{N-1} \sum_{j=1}^J \rho_{j+1/2}^{k+i} \chi_{j+1/2}^{k+1/2}, \quad \pi_\delta = \sum_{k=0}^{N-1} \sum_{j=1}^J \pi_{j+1/2}^{k+1/2} \chi_{j+1/2}^{k+1/2}, \quad V_\delta = \sum_{k=0}^{N-1} \sum_{j=2}^J V_j^k \chi_j^{k+1/2},$$

where  $\chi_j^{k+1/2} = \chi_{[x_{j-1/2}, x_{j+1/2}[ \times [t^k, t^{k+1}[}$ ,  $\chi_{j+1/2}^{k+1/2} = \chi_{[x_j, x_{j+1}[ \times [t^k, t^{k+1}[}$ .

We also introduce the following discrete norms

$$\begin{aligned} \|\rho_\delta\|_{\infty, \tau} &= \max_{0 \leq k \leq N} \max_{1 \leq j \leq J} |\rho_{j+1/2}^k|, \quad \|V_\delta\|_{\infty, \tau^*} = \max_{0 \leq k \leq N} \max_{2 \leq j \leq J} |V_j^k|, \\ \|\rho_\delta\|_{1; \text{BV}, \tau} &= \sum_{k=0}^N \delta t^k \sum_{j=2}^J |\rho_{j+1/2}^k - \rho_{j-1/2}^k|, \quad \|V_\delta\|_{1; \text{BV}, \tau^*} = \sum_{k=0}^N \delta t^k \sum_{j=1}^J |V_{j+1}^k - V_j^k|, \\ \|\rho_\delta\|_{\text{BV}; 1, \tau} &= \sum_{j=1}^J \delta x_{j+1/2} \sum_{k=0}^{N-1} |\rho_{j+1/2}^{k+1} - \rho_{j+1/2}^k|. \end{aligned}$$

For  $\varphi \in \mathcal{C}_c^\infty((0, L) \times [0, T))$ , we set  $\varphi_{j+1/2}^k = \varphi(x_{j+1/2}, t^k)$  and  $\varphi_j^k = \varphi(x_j, t^k)$ . The interpolate  $\varphi_\tau$  of  $\varphi$  on the primal mesh and its discrete derivatives are defined by

$$\begin{aligned} \varphi_\tau(\cdot, 0) &= \sum_{j=1}^J \varphi_{j+1/2}^0 \chi_{j+1/2}^{1/2}(\cdot, 0), \quad \varphi_\tau(\cdot, t) = \sum_{k=0}^{N-1} \sum_{j=1}^J \varphi_{j+1/2}^{k+1} \chi_{j+1/2}^{k+1/2}(\cdot, t), \quad \forall t > 0, \\ \eth_t \varphi_\tau &= \sum_{k=0}^{N-1} \sum_{j=1}^J \frac{\varphi_{j+1/2}^{k+1} - \varphi_{j+1/2}^k}{\delta t^k} \chi_{j+1/2}^{k+1/2}, \quad \eth_x \varphi_\tau = \sum_{k=0}^{N-1} \sum_{j=2}^J \frac{\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}}{\delta x_j} \chi_j^{k+1/2}. \end{aligned}$$

Similarly, the interpolate  $\varphi_{\tau^*}$  of  $\varphi$  on  $\tau^*$  and its discrete derivatives are given by

$$\begin{aligned} \varphi_{\tau^*}(\cdot, 0) &= \sum_{j=2}^J \varphi_j^0 \chi_j^{1/2}(\cdot, 0), \quad \varphi_{\tau^*}(\cdot, t) = \sum_{k=0}^{N-1} \sum_{j=2}^J \varphi_j^{k+1} \chi_j^{k+1/2}(\cdot, t), \quad \forall t > 0, \\ \eth_t^* \varphi_{\tau^*} &= \sum_{k=0}^{N-1} \sum_{j=2}^J \frac{\varphi_j^{k+1} - \varphi_j^k}{\delta t^k} \chi_j^{k+1/2}, \quad \eth_x^* \varphi_{\tau^*} = \sum_{k=0}^{N-1} \sum_{j=1}^J \frac{\varphi_{j+1}^{k+1} - \varphi_j^{k+1}}{\delta x_{j+1/2}} \chi_{j+1/2}^{k+1/2}. \end{aligned}$$

**Assumptions.** Let  $(\mathcal{T}_m)_{m \geq 1}$  be a sequence of meshes s. t.  $\text{size}(\mathcal{T}_m) \rightarrow 0$  and a family of time steps  $(\delta t_m^k)_{k \geq 0, m \geq 1}$  verifying  $\delta t_m \rightarrow 0$  and (10). Assume that there exists  $N_m \in \mathbb{N}$  s. t.  $\sum_{k=0}^{N_m-1} \delta t_m^k = T$ . The scheme defines  $(\rho_{\delta_m}^{(0)}, V_{\delta_m})_{m \geq 1}$ . Suppose that

$$\|\rho_{\delta_m}^{(0)}\|_{\infty, \tau} + \|V_{\delta_m}\|_{\infty, \tau^*} \leq C_\infty, \quad \|\rho_{\delta_m}^{(0)}\|_{1; \text{BV}, \tau} + \|V_{\delta_m}\|_{1; \text{BV}, \tau^*} \leq C_{\text{BV}} \quad (14)$$

holds and, in the case  $(\rho \mapsto \frac{p'(\rho)}{\rho}) \notin L^1_{\text{loc}}(0, \infty)$ ,  $\|(\rho_{\delta_m}^{(0)})^{-1}\|_{\infty, \tau} \leq C$ . We assume that there exists  $(\bar{\rho}, \bar{V}) \in L^\infty((0, T) \times (0, L))^2$  such that

$$(\rho_{\delta_m}^{(0)}, V_{\delta_m}) \longrightarrow (\bar{\rho}, \bar{V}) \text{ in } L^r((0, T) \times (0, L))^2, \quad 1 \leq r < \infty. \quad (15)$$

**Main results.** The uniform bounds imply that there exists constants such that

$$\begin{aligned} \sup_{0 \leq \rho, |V| \leq C_\infty} |\mathcal{A}(\rho, V)| &\leq C_{\mathcal{A}}, \quad \text{with } \mathcal{A} = \mathcal{F}^\pm, \partial_\rho \mathcal{F}^\pm \text{ and } \partial_V \mathcal{F}^\pm, \\ \sup_{0 \leq \rho \leq C_\infty + 4(C_\infty^2 + C_{\mathcal{F}^\pm})} |\mathcal{B}(\rho)| &\leq C_{\mathcal{B}}, \quad \text{with } \mathcal{B} = \Phi, \Phi', \text{ and } \bar{\Phi}'. \end{aligned}$$

Note also that we have  $|\Phi(\rho_{j+1/2}^k)| \leq C_{\Phi,\rho} \rho_{j+1/2}^k$ ,  $\forall j,k$ . Furthermore, we show that  $\|\rho_{\delta_m}^{(0)}\|_{BV;1,\tau} \leq C$  by using (5) which allows to dominate  $\delta x_{j+1/2} |\rho_{j+1/2}^{k+1} - \rho_{j+1/2}^k|$  by

$$\delta t^k [C_{\partial_\rho \mathcal{F}^\pm} (|\rho_{j+1/2}^k - \rho_{j-1/2}^k| + |\rho_{j+3/2}^k - \rho_{j+1/2}^k|) + 2C_{\partial_V \mathcal{F}^\pm} |V_{j+1}^k - V_j^k|].$$

Consequently,  $\rho_{\delta_m}^{(1)} \rightarrow \bar{\rho}$  and  $\pi_{\delta_m} \rightarrow p(\bar{\rho})$  in  $L^r((0,T) \times (0,L))$ ; with (4) and since  $\rho_0, V_0 \in L^\infty(0,L)$ , we get  $\rho_{\delta_m}^{(0)}(\cdot, 0) \rightarrow \rho_0$  and  $V_{\delta_m}(\cdot, 0) \rightarrow V_0$  in  $L^r((0,L))$ ,  $1 \leq r < \infty$ .

Finally, in the sequel, when a function  $\varphi \in \mathcal{C}_c^\infty((0,L) \times [0,T))$  is given, we assume that  $\delta t_m$  and  $\delta x_m$  are sufficiently small so that  $\varphi(x, \cdot) \equiv 0$ ,  $\forall x \in [0, x_{3/2}] \cup [x_{J+1/2}, L]$  and  $\varphi(\cdot, t) \equiv 0$ ,  $\forall t \in [t^{N-1}, t^N]$ . Moreover, since  $\varphi$  is smooth,  $\varphi_{\mathcal{T}_m} \rightarrow \varphi$ ,  $\partial_t \varphi_{\mathcal{T}_m} \rightarrow \partial_t \varphi$ ,  $\partial_x^* \varphi_{\mathcal{T}_m} \rightarrow \partial_x \varphi$ , and  $\partial_x \varphi_{\mathcal{T}_m} \rightarrow \partial_x \varphi$ , in  $L^r((0,T) \times (0,L))$ ,  $1 \leq r \leq \infty$ .

**Theorem 2.1** Assume (14) and (15). Then,  $(\bar{\rho}, \bar{V})$  satisfies (1)-(2) in the distribution sense in  $(\mathcal{C}_c^\infty((0,L) \times [0,T)))'$ , that is

$$-\int_0^T \int_0^L [\bar{\rho} \partial_t \varphi + \bar{\rho} \bar{V} \partial_x \varphi](x, t) dx dt - \int_0^L \rho_0(x) \varphi(x, 0) dx = 0, \quad (16)$$

$$-\int_0^T \int_0^L [\bar{\rho} \bar{V} \partial_t \varphi + (\bar{\rho} \bar{V}^2 + p(\bar{\rho})) \partial_x \varphi](x, t) dx dt - \int_0^L \rho_0(x) V_0(x) \varphi(x, 0) dx = 0. \quad (17)$$

Moreover,  $(\bar{\rho}, \bar{V})$  satisfies the entropy inequality (3).

**Proof.** Let  $\varphi \in \mathcal{C}_c^\infty((0,L) \times [0,T))$ . For the sake of simplicity, the index  $m$  is dropped.

*Mass balance.* We multiply (5) by  $\varphi_{j+1/2}^{k+1}$  and sum the results for  $0 \leq k \leq N-1$  and  $1 \leq j \leq J$  to obtain

$$\underbrace{\sum_{k=0}^{N-1} \sum_{j=1}^J \delta x_{j+1/2} (\rho_{j+1/2}^{k+1} - \rho_{j+1/2}^k) \varphi_{j+1/2}^{k+1}}_{:=T_1} + \underbrace{\sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J (\mathcal{F}_{j+1}^k - \mathcal{F}_j^k) \varphi_{j+1/2}^{k+1}}_{:=T_2} = 0.$$

For  $T_1$ , since  $\varphi_{j+1/2}^N = 0$ , a discrete integration by part w.r.t. time yields

$$T_1 = - \sum_{k=0}^{N-1} \sum_{j=1}^J \delta x_{j+1/2} \rho_{j+1/2}^k (\varphi_{j+1/2}^{k+1} - \varphi_{j+1/2}^k) - \sum_{j=1}^J \delta x_{j+1/2} \rho_{j+1/2}^0 \varphi_{j+1/2}^0.$$

Noting that

$$\int_{t^k}^{t^{k+1}} \int_{x_j}^{x_{j+1}} \rho_\delta^{(0)} \partial_t \varphi_\tau dx dt = \delta x_{j+1/2} \rho_{j+1/2}^k (\varphi_{j+1/2}^{k+1} - \varphi_{j+1/2}^k)$$

for  $k \in \{0, \dots, N-1\}$ ,  $j \in \{1, \dots, J\}$ , we get

$$T_1 = - \int_0^T \int_0^L \rho_\delta^{(0)} \partial_t \varphi_\tau dx dt - \int_0^L \rho_\delta^{(0)}(x, 0) \varphi_\tau(x, 0) dx.$$

For  $T_2$ , by integrating by part w.r.t. space, we readily obtain

$$T_2 = - \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^J \mathcal{F}_j^k (\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}).$$

Bearing in mind that  $2\delta x_j = \delta x_{j-1/2} + \delta x_{j+1/2}$ , we then combine the two following expressions of mass fluxes (see (9)-(ii))

$$\mathcal{F}_j^k = \rho_{j\pm 1/2}^k V_j^k \mp R_j^{k,\pm} \text{ with } R_j^{k,\pm} = \mathcal{F}^\pm(\rho_{j+1/2}^k, V_j^k) - \mathcal{F}^\pm(\rho_{j-1/2}^k, V_j^k)$$

to write

$$\mathcal{F}_j^k = \left[ \frac{\delta x_{j-1/2}}{2\delta x_j} \rho_{j-1/2}^k V_j^k + \frac{\delta x_{j+1/2}}{2\delta x_j} \rho_{j+1/2}^k V_j^k \right] + \left[ \frac{\delta x_{j-1/2}}{2\delta x_j} R_j^{k,-} - \frac{\delta x_{j+1/2}}{2\delta x_j} R_j^{k,+} \right].$$

This expression of the mass fluxes leads to  $T_2 = -T_{2,1} - T_{2,2}$  with

$$T_{2,1} = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^J \frac{1}{2} \left[ \delta x_{j-1/2} \rho_{j-1/2}^k + \delta x_{j+1/2} \rho_{j+1/2}^k \right] V_j^k \frac{\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}}{\delta x_j},$$

$$T_{2,2} = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^J \frac{1}{2} \left[ \delta x_{j-1/2} R_j^{k,-} - \delta x_{j+1/2} R_j^{k,+} \right] V_j^k \frac{\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}}{\delta x_j}.$$

We now observe that, for  $k \in \{0, \dots, N-1\}$ ,  $j \in \{2, \dots, J\}$ ,

$$\int_{t^k}^{t^{k+1}} \int_{x_{j-1/2}}^{x_j} \rho_\delta^{(0)} V_\delta \partial_x \varphi_\tau \, dx \, dt = \delta t^k \frac{\delta x_{j-1/2}}{2} \left[ \rho_{j-1/2}^k V_j^k \frac{\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}}{\delta x_j} \right],$$

and

$$\int_{t^k}^{t^{k+1}} \int_{x_j}^{x_{j+1/2}} \rho_\delta^{(0)} V_\delta \partial_x \varphi_\tau \, dx \, dt = \delta t^k \frac{\delta x_{j+1/2}}{2} \left[ \rho_{j+1/2}^k V_j^k \frac{\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}}{\delta x_j} \right].$$

Summing these equalities yields

$$T_{2,1} = \int_0^T \int_{x_{3/2}}^{x_{J+1/2}} \rho_\delta^{(0)} V_\delta \partial_x \varphi_\tau \, dx \, dt = \int_0^T \int_0^L \rho_\delta^{(0)} V_\delta \partial_x \varphi_\tau \, dx \, dt,$$

since  $\partial_x \varphi_\tau(x, \cdot) \equiv 0$  for  $x \in [0, x_{3/2}] \cup [x_{J+1/2}, L]$ .

With (15), we pass to the limit in  $T_1$  and  $T_{2,1}$ . We prove that  $(\bar{\rho}, \bar{V})$  satisfies the mass conservation equation (16) by showing that  $T_{2,2} \rightarrow 0$  since

$$|T_{2,2}| \leq C_{\partial_\rho \mathcal{F}^\pm} |\partial_x \varphi|_{L^\infty} \|V_\delta\|_{\infty, \tau^*} \|\rho_\delta\|_{1;BV, \tau} \delta x \lesssim \delta x.$$

*Momentum balance.* We multiply (6) by  $\varphi_j^{k+1}$  and sum for  $0 \leq k \leq N-1$  and  $2 \leq j \leq J$  to obtain  $T_3 + T_4 + T_5 = 0$  with

$$T_3 = \sum_{k=0}^{N-1} \sum_{j=2}^J \delta x_j (\rho_j^{k+1} V_j^{k+1} - \rho_j^k V_j^k) \varphi_j^{k+1},$$

$$T_4 = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^J (\mathcal{G}_{j+1/2}^k - \mathcal{G}_{j-1/2}^k) \varphi_j^{k+1}, \quad T_5 = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^J (\pi_{j+1/2}^{k+1/2} - \pi_{j-1/2}^{k+1/2}) \varphi_j^{k+1}.$$

For  $T_3$ , integrating by part w.r.t time yields

$$T_3 = - \sum_{k=0}^{N-1} \sum_{j=2}^J \delta x_j \rho_j^k V_j^k (\varphi_j^{k+1} - \varphi_j^k) - \sum_{j=2}^J \delta x_j \rho_j^0 V_j^0 \varphi_j^0.$$

Next, we observe that

$$\begin{aligned} \int_{t^k}^{t^{k+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} \rho_\delta^{(0)} V_\delta \partial_t^\star \varphi \tau^\star dx dt &= V_j^k \frac{\varphi_j^{k+1} - \varphi_j^k}{\delta t^k} \int_{t^k}^{t^{k+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} \rho_\delta^{(0)} dx dt \\ &= \delta x_j \rho_j^k V_j^k (\varphi_j^{k+1} - \varphi_j^k). \end{aligned}$$

Summing these equalities for  $k \in \{0, \dots, N-1\}$ ,  $j \in \{2, \dots, J\}$  yields

$$\begin{aligned} T_3 &= - \int_0^T \int_{x_{3/2}}^{x_{J+1/2}} \rho_\delta^{(0)} V_\delta \partial_t^\star \varphi \tau^\star dx dt - \int_{x_{3/2}}^{x_{J+1/2}} \rho_\delta^{(0)}(x, 0) V_\delta(x, 0) \varphi \tau^\star(x, 0) dx, \\ &= - \int_0^T \int_0^L \rho_\delta^{(0)} V_\delta \partial_t^\star \varphi \tau^\star dx dt - \int_0^L \rho_\delta^{(0)}(x, 0) V_\delta(x, 0) \varphi \tau^\star(x, 0) dx. \end{aligned}$$

For  $T_4$  and  $T_5$ , we first integrate by part w.r.t space and obtain

$$T_4 = - \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J \mathcal{G}_{j+1/2}^k (\varphi_{j+1}^{k+1} - \varphi_j^{k+1}), \quad T_5 = - \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J \pi_{j+1/2}^{k+1/2} (\varphi_{j+1}^{k+1} - \varphi_j^{k+1}).$$

We then use the following expression of the momentum flux

$$\begin{cases} \mathcal{G}_{j+1/2}^k &= \frac{1}{2} \rho_{j+1/2}^k \left[ (V_j^k)^2 + (V_{j+1}^k)^2 \right] + Q_{j+1/2}^k, \\ Q_{j+1/2}^k &= -\frac{1}{2} V_j^k R_j^{k,+} + \frac{1}{2} V_{j+1}^k R_{j+1}^{k,-} \\ &\quad - \frac{1}{2} (V_{j+1}^k - V_j^k) \left[ \mathcal{F}^+(\rho_{j+1/2}^k, V_{j+1}^k) - \mathcal{F}^-(\rho_{j+1/2}^k, V_j^k) \right]. \end{cases}$$

to write  $T_4 = -T_{4,1} - T_{4,2}$  with

$$\begin{aligned} T_{4,1} &= \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J \frac{1}{2} \rho_{j+1/2}^k \left[ (V_j^k)^2 + (V_{j+1}^k)^2 \right] (\varphi_{j+1}^{k+1} - \varphi_j^{k+1}) \\ T_{4,2} &= \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J Q_{j+1/2}^k (\varphi_{j+1}^{k+1} - \varphi_j^{k+1}). \end{aligned}$$

Next, we observe that

$$\int_{t^k}^{t^{k+1}} \int_{x_j}^{x_{j+1}} \rho_\delta^{(0)} (V_\delta)^2 \partial_x^\star \varphi \tau^\star dx dt = \delta t^k \rho_{j+1/2}^k \frac{(V_j^k)^2 + (V_{j+1}^k)^2}{2} (\varphi_{j+1}^{k+1} - \varphi_j^{k+1})$$

and, consequently, summing for  $k \in \{0, \dots, N-1\}$ ,  $j \in \{1, \dots, J\}$ ,

$$T_{4,1} = \int_0^T \int_0^L \rho_\delta^{(0)} (V_\delta)^2 \partial_x^\star \varphi \tau^\star dx dt.$$

Similarly, for  $T_5$ , we get

$$T_5 = - \int_0^T \int_0^L \pi_\delta \partial_x^\star \varphi \tau^\star dx dt.$$

With (15), we pass to the limit in  $T_3$ ,  $T_{4,1}$  and  $T_5$ . We obtain that  $(\bar{\rho}, \bar{V})$  satisfies the momentum balance equation (17) by showing that  $T_{4,2} \rightarrow 0$ . Indeed, we have

$$|T_{4,2}| \leq \|V_\delta\|_{\infty, \tau^\star} |\partial_x \varphi|_{L^\infty} \left( C_{\partial_\rho \mathcal{F}^\pm} \|\rho_\delta\|_{1;BV, \tau} + C_{\mathcal{F}^\pm} \|V_\delta\|_{1;BV, \tau^\star} \right) \delta x \lesssim \delta x.$$

*Entropy inequality.* We now assume that  $\varphi \geqslant 0$ .

- Kinetic energy. We multiply (13) by  $\delta t^k \varphi_j^{k+1}$  and sum for  $0 \leqslant k \leqslant N-1$  and  $2 \leqslant j \leqslant J$ . We obtain to get  $T_6 + T_7 + T_8 \leqslant 0$  with

$$T_6 = \sum_{k=0}^{N-1} \sum_{j=2}^J \delta x_j [E_{K,j}^{k+1} - E_{K,j}^k] \varphi_j^{k+1}, \quad T_7 = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^J [\Gamma_{j+1/2}^k - \Gamma_{j-1/2}^k] \varphi_j^{k+1},$$

$$T_8 = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^J [\pi_{j+1/2}^{k+1/2} - \pi_{j-1/2}^{k+1/2}] V_j^{k+1} \varphi_j^{k+1} + \sum_{k=0}^{N-1} \sum_{j=2}^J D_j^k \varphi_j^{k+1}.$$

Integrating by part w.r.t. time yields

$$T_6 = - \sum_{k=0}^{N-1} \sum_{j=2}^J \delta x_j E_{K,j}^k [\varphi_j^{k+1} - \varphi_j^k] - \sum_{j=1}^J \delta x_j E_{K,j}^0 \varphi_j^0$$

$$= - \int_0^T \int_0^L \frac{1}{2} \rho_\delta^{(0)} (V_\delta)^2 \partial_t^\star \varphi_{\mathcal{T}^\star} dx dt - \int_0^L \frac{1}{2} \rho_\delta^{(0)} (x, 0) (V_\delta(x, 0))^2 \varphi_{\mathcal{T}^\star}(x, 0) dx.$$

For  $T_7$ , we write  $\Gamma_{j+1/2}^k = \frac{1}{4} \rho_{j+1/2}^k [(V_j^k)^3 + (V_{j+1}^k)^3] + \frac{1}{4} S_{j+1/2}^k$  where

$$S_{j+1/2}^k = V_j^k V_{j+1}^k [R_{j+1}^{k,-} - R_j^{k,+}] + (V_{j+1}^k - V_j^k)^2 [\mathcal{F}_j^{k,|\cdot|} + \mathcal{F}_{j+1}^{k,|\cdot|} - \rho_{j+1/2}^k (V_j^k + V_{j+1}^k)].$$

Integration by part w.r.t space leads to  $T_7 = -T_{7,1} - T_{7,2}$  with

$$T_{7,1} = \int_0^T \int_0^L \frac{1}{2} \rho_\delta^{(0)} (V_\delta)^3 \partial_x^\star \varphi_{\mathcal{T}^\star} dx dt, \quad T_{7,2} = \frac{1}{4} \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J S_{j+1/2}^k [\varphi_{j+1}^{k+1} - \varphi_j^{k+1}].$$

Finally  $|T_{7,2}| \lesssim \delta x$  since it is dominated by

$$\delta x |\partial_x \varphi|_{L^\infty} \|V_\delta\|_{\infty, \mathcal{T}^\star} \left[ \frac{C_{\partial_\rho \mathcal{F}^\pm}}{2} \|V_\delta\|_{\infty, \mathcal{T}^\star} \|\rho_\delta\|_{1; \text{BV}, \mathcal{T}}$$

$$+ (2C_{\mathcal{F}^\pm} + \|V_\delta\|_{\infty, \mathcal{T}^\star} \|\rho_\delta\|_{\infty, \mathcal{T}}) \|V_\delta\|_{1; \text{BV}, \mathcal{T}} \right].$$

- Internal energy. Multiply (12) by  $\delta t^k \varphi_{j+1/2}^{k+1}$  and sum for  $0 \leqslant k \leqslant N-1$  and  $1 \leqslant j \leqslant J$  to get  $T_9 + T_{10} + T_{11} \leqslant 0$  with

$$T_9 = \sum_{k=0}^{N-1} \sum_{j=1}^J \delta x_{j+1/2} [e_{j+1/2}^{k+1} - e_{j+1/2}^k] \varphi_{j+1/2}^{k+1}, \quad T_{10} = \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J [\bar{G}_{j+1}^k - \bar{G}_j^k] \varphi_{j+1/2}^{k+1},$$

$$T_{11} = \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J \pi_{j+1/2}^{k+1/2} (V_{j+1}^{k+1} - V_j^{k+1}) \varphi_{j+1/2}^{k+1} - \sum_{k=0}^{N-1} \sum_{j=1}^J D_j^k \varphi_{j+1/2}^{k+1}.$$

Owing to integration by part w.r.t. time, we get

$$T_9 = - \sum_{k=0}^{N-1} \sum_{j=1}^J \delta x_{j+1/2} \Phi(\rho_{j+1/2}^k) (\varphi_{j+1/2}^{k+1} - \varphi_{j+1/2}^k) - \sum_{j=1}^J \delta x_{j+1/2} \Phi(\rho_{j+1/2}^0) \varphi_{j+1/2}^0,$$

$$= - \int_0^T \int_0^L \Phi(\rho_\delta^{(0)}) \partial_t^\star \varphi_{\mathcal{T}} dx dt - \int_0^L \Phi(\rho_\delta^{(0)}(x, 0)) \varphi_{\mathcal{T}}(x, 0) dx.$$

For  $T_{10}$ , we rewrite the flux as follows

$$\begin{cases} \bar{G}_j^k = \frac{1}{2\delta x_j} [\delta x_{j-1/2} \Phi(\rho_{j-1/2}^k) + \delta x_{j+1/2} \Phi(\rho_{j+1/2}^k)] V_j^k + U_{1,j}^k + U_{2,j}^k + U_{3,j}^k, \\ U_{1,j}^k = e_{j-1/2}^k (V_j^{k+1} - V_j^k), \quad U_{2,j}^k = -\frac{\delta x_{j+1/2}}{2\delta x_j} [e_{j+1/2}^k - e_{j-1/2}^k] V_j^k, \\ U_{3,j}^k = -\frac{\delta x_{j-1/2}}{2\delta t^k} [\bar{\Phi}(\overline{\rho_{j-1/2}^{k+1}}) - \bar{\Phi}(\rho_{j-1/2}^k)]. \end{cases}$$

It leads to  $T_{10} = -T_{10,0} - T_{10,1} - T_{10,2} - T_{10,3}$  with

$$\begin{cases} T_{10,0} = \int_0^T \int_0^L \Phi(\rho_\delta^{(0)}) V_\delta \partial_x \varphi_\tau \, dx \, dt, \\ T_{10,i} = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^J U_{i,j}^k (\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}), \quad i = 1, 2, 3. \end{cases}$$

The term  $T_{10,1}$  can be bounded as follows

$$|T_{10,1}| \leq C_{\Phi,\rho} |\partial_x \varphi| \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^J \delta x_j \rho_{j-1/2}^k |V_j^{k+1} - V_j^k|. \quad (18)$$

Since  $a \leq \min(a, b) + |b - a|$ , we get  $\rho_{j-1/2}^k \leq \rho_j^k + |\rho_{j+1/2}^k - \rho_{j-1/2}^k|$ . This leads to

$$|T_{10,1}| \leq C_{\Phi,\rho} |\partial_x \varphi|_{L^\infty} \left( \underbrace{\sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^J \delta x_j \rho_j^k |V_j^{k+1} - V_j^k|}_{:= T^*} + 2 \|V_\delta\|_{\infty, \tau^*} \|\rho_\delta\|_{1;BV, \tau} \delta x \right).$$

Writing  $\rho_j^k = \rho_j^{k+1} - (\rho_j^{k+1} - \rho_j^k)$  and using the Cauchy-Schwarz inequality yields

$$T^* \leq 2 \left( TL \|\rho_\delta\|_{\infty, \tau} \right)^{1/2} \left( \delta t \sum_{k=0}^{N-1} \sum_{j=2}^J D_j^k \right)^{1/2} + 2 \|V_\delta\|_{\infty, \tau^*} \|\rho_\delta\|_{1;BV, \tau} \delta t \lesssim \delta t^{1/2}.$$

It finally leads to  $|T_{10,1}| \lesssim \delta t^{1/2} + \delta x$ . The term  $T_{10,2}$  can be bounded as follows

$$|T_{10,2}| \leq C_{\Phi'} \|V_\delta\|_{\infty, \tau^*} |\partial_x \varphi|_{L^\infty} \|\rho_\delta\|_{1;BV, \tau} \delta x \lesssim \delta x.$$

We now turn to  $T_{10,3}$ . We remark that

$$|\overline{\rho_{j-1/2}^{k+1}} - \rho_{j-1/2}^k| \leq \frac{2\delta t^k}{\delta x_{j-1/2}} \left( C_{\partial_\rho \mathcal{F}^\pm} |\rho_{j+1/2}^k - \rho_{j-1/2}^k| + \rho_{j-1/2}^k |V_j^{k+1} - V_j^k| \right).$$

Hence, using the same bound as for  $T_{10,1}$  yields

$$|T_{10,3}| \leq C_{\bar{\Phi}'} |\partial_x \varphi|_{L^\infty} \left( \left( C_{\partial_\rho \mathcal{F}^\pm} + 2 \|V_\delta\|_{\infty, \tau^*} \right) \|\rho_\delta\|_{1;BV, \tau} \delta x + T^* \right) \lesssim \delta t^{1/2} + \delta x.$$

- Pressure terms. It remains to get the limit of  $T_8 + T_{11} = -T_{12,0} - T_{12,1} - T_{12,2} - T_{12,3}$

with

$$\begin{aligned} T_{12,0} &= \int_0^T \int_0^L \pi_\delta V_\delta \partial_x^\star \varphi \tau^\star \, dx \, dt, \quad T_{12,1} = \sum_{k=0}^{N-1} \sum_{j=2}^J D_j^k (\varphi_{j+1/2}^{k+1} - \varphi_j^{k+1}), \\ T_{12,2} &= \frac{1}{2} \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J \pi_{j+1/2}^{k+1/2} (V_j^{k+1} - V_j^k + V_{j+1}^{k+1} - V_{j+1}^k) (\varphi_{j+1}^{k+1} - \varphi_j^{k+1}), \\ T_{12,3} &= -\frac{1}{2} \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J \pi_{j+1/2}^{k+1/2} (V_{j+1}^{k+1} - V_j^{k+1}) (2\varphi_{j+1/2}^{k+1} - \varphi_{j+1}^{k+1} - \varphi_j^{k+1}). \end{aligned}$$

We bound  $T_{12,1}$  and  $T_{12,3}$  as follows

$$T_{12,1} \leq \frac{|\partial_x \varphi|_{L^\infty}}{2} \left( \sum_{k=0}^{N-1} \sum_{j=2}^J D_j^k \right) \delta x \lesssim \delta x, \quad T_{12,3} \leq \frac{C_\pi}{4} |\partial_{xx} \varphi|_{L^\infty} \|V_\delta\|_{1;\text{BV},\tau^\star} (\delta x)^2 \lesssim (\delta x)^2.$$

Note that  $|\pi_{j+1/2}^{k+1/2}| \leq (C_{\Phi'} + C_{\Phi,\rho}) \rho_{j+1/2}^k$ . It readily leads to

$$|T_{12,2}| \leq (C_{\Phi'} + C_{\Phi,\rho}) |\partial_x \varphi|_{L^\infty} T^\star \lesssim \delta t^{\frac{1}{2}}.$$

With (15), we pass to the limit in  $T_6$ ,  $T_{7,1}$ ,  $T_9$ ,  $T_{10,0}$  and  $T_{12,0}$ . We arrive at (3) since the other terms tend to 0.  $\blacksquare$

## References

- [1] F. Berthelin, T. Goudon, and S. Minjeaud. Kinetic schemes on staggered grids for barotropic euler models: entropy-stability analysis. 2014. in revision.
- [2] R. Herbin, J.-C. Latché, and T. T. Nguyen. Explicit staggered schemes for the compressible Euler equations. *ESAIM:Proc.*, 40:83–102, 2013.