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Consistency analysis of a 1D Finite Volume scheme for barotropic Euler models

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Abstract

This work is concerned with the consistency study of a 1D (staggered kinetic) finite volume scheme for barotropic Euler models. We prove a Lax-Wendroff-like statement: the limit of a converging (and uniformly bounded) sequence of stepwise constant functions defined from the scheme is a weak entropic-solution of the system of conservation laws.

1 Introduction

The model. This work is concerned with the consistency study of a (staggered kinetic) Finite Volume (FV) scheme for barotropic Euler models

$$\partial_t \rho + \partial_x(\rho V) = 0, \quad (1)$$

$$\partial_t(\rho V) + \partial_x(\rho V^2 + p(\rho)) = 0. \quad (2)$$

The unknowns are the density ρ and the velocity V . The pressure ($\rho \mapsto p(\rho)$) is assumed to be $\mathcal{C}^2([0, \infty))$ with $p(\rho) > 0$, $p'(\rho) > 0$, $p''(\rho) \geq 0$, $\forall \rho > 0$. Thus, the sound speed $c : \rho \mapsto \sqrt{p'(\rho)}$ is well defined and is an increasing function.

We consider the problem (1)-(2) on the bounded domain $(0, L) \times [0, T]$ with the boundary conditions $V(0, t) = 0 = V(L, t)$, $\forall t > 0$ and the initial conditions $\rho(x, 0) = \rho_0(x)$, $V(x, 0) = V_0(x)$, $\forall x \in (0, L)$ with $\rho_0, V_0 \in L^\infty(0, L)$.

Let $\Phi : \rho > 0 \mapsto \Phi(\rho)$ such that $\rho \Phi'(\rho) - \Phi(\rho) = p(\rho)$, $\forall \rho > 0$. The quantity $\mathcal{S} = \frac{1}{2}\rho|V|^2 + \Phi(\rho)$ is an entropy of the system: entropy solutions to (1)-(2) are required to satisfy: for any $\varphi \in \mathcal{C}_c^\infty((0, L) \times [0, T])$ such that $\varphi \geq 0$,

$$-\int_0^T \int_0^L [\mathcal{S} \partial_t \varphi + (\mathcal{S} + p(\rho)) V \partial_x \varphi](x, t) dx dt - \int_0^L \mathcal{S}_0(x) \varphi(x, 0) dx \leq 0. \quad (3)$$

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The meshes. We consider a set of $J + 1$ points $0 = x_1 < x_2 < \dots < x_J < x_{J+1} = L$. The x_j are the edges of the so-called primal mesh \mathcal{T} . We set $\delta x_{j+1/2} = x_{j+1} - x_j$. The centers of the primal cells, $x_{j+1/2} = (x_j + x_{j+1})/2$ for $j \in \{1, \dots, J\}$, realize the dual mesh \mathcal{T}^* . We set $\delta x_j = (\delta x_{j-1/2} + \delta x_{j+1/2})/2$ for $j \in \{2, \dots, J-1\}$ and $\delta x = \text{size}(\mathcal{T}) = \max_j \delta x_{j+1/2}$. The adaptive time step is δt^k and we set $\delta t = \max_k \delta t^k$.

The scheme. We analyze the scheme introduced in [1]. It works on staggered grids: the densities, $\rho_{j+1/2}$, $j \in \{1, \dots, J\}$, are evaluated at centers whereas the velocities, V_j , $j \in \{1, \dots, J+1\}$, are evaluated at edges. We set, for $j \in \{1, \dots, J\}$ and $i \in \{2, \dots, J\}$

$$\rho_{j+1/2}^0 = \frac{1}{\delta x_{j+1/2}} \int_{x_j}^{x_{j+1}} \rho_0(x) dx, \quad V_i^0 = \frac{1}{\delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} V_0(x) dx. \quad (4)$$

The density is first updated with a FV approximation on the primal mesh

$$\delta x_{j+1/2} (\rho_{j+1/2}^{k+1} - \rho_{j+1/2}^k) + \delta t^k (\mathcal{F}_{j+1}^k - \mathcal{F}_j^k) = 0, \quad \forall j \in \{1, \dots, J\}. \quad (5)$$

Then, the velocity is updated with a FV approximation on the dual mesh:

$$\delta x_j (\rho_j^{k+1} V_j^{k+1} - \rho_j^k V_j^k) + \delta t^k (\mathcal{G}_{j+1/2}^k - \mathcal{G}_{j-1/2}^k + \pi_{j+1/2}^{k+1/2} - \pi_{j-1/2}^{k+1/2}) = 0, \quad (6)$$

for $j \in \{2, \dots, J\}$, while $V_1^{k+1} = V_{J+1}^{k+1} = 0$. The density on the edges ρ_j^k is defined by

$$2\delta x_j \rho_j^k = \delta x_{j+1/2} \rho_{j+1/2}^k + \delta x_{j-1/2} \rho_{j-1/2}^k, \quad \forall j \in \{2, \dots, J\}.$$

The definition of the fluxes relies on the kinetic framework. We refer the reader to [1] for details. Let us introduce the two following functions \mathcal{F}^+ and \mathcal{F}^-

$$\mathcal{F}^\pm(\rho, V) = \frac{\rho}{2c(\rho)} \int_{\xi \geq 0} \xi \mathbb{1}_{|\xi - V| \leq c(\rho)} d\xi.$$

We adopt the following formulas for mass fluxes: $\mathcal{F}_1^k = \mathcal{F}_{J+1}^k = 0$,

$$\mathcal{F}_j^k = \mathcal{F}^+(\rho_{j-1/2}^k, V_j^k) + \mathcal{F}^-(\rho_{j+1/2}^k, V_j^k), \quad \forall j \in \{2, \dots, J\}, \quad (7)$$

and, for momentum fluxes: $\mathcal{G}_{3/2}^k = \frac{V_2^k}{2} \mathcal{F}^-(\rho_{5/2}^k, V_2^k)$, $\mathcal{G}_{J+1/2}^k = \frac{V_J^k}{2} \mathcal{F}^+(\rho_{J-1/2}^k, V_J^k)$,

$$\begin{aligned} \mathcal{G}_{j+1/2}^k &= \frac{V_j^k}{2} (\mathcal{F}^+(\rho_{j-1/2}^k, V_j^k) + \mathcal{F}^+(\rho_{j+1/2}^k, V_{j+1}^k)) \\ &+ \frac{V_{j+1}^k}{2} (\mathcal{F}^-(\rho_{j+1/2}^k, V_j^k) + \mathcal{F}^-(\rho_{j+3/2}^k, V_{j+1}^k)), \quad \forall j \in \{2, \dots, J-1\}. \end{aligned} \quad (8)$$

The discrete pressure gradient combines a space centered scheme and time semi implicit discretization, namely it uses

$$\pi_{j+1/2}^{k+1/2} = \rho_{j+1/2}^k \Phi'(\rho_{j+1/2}^{k+1}) - \Phi(\rho_{j+1/2}^k).$$

Properties of the scheme. The analysis is driven by the shapes of the functions \mathcal{F}^\pm , see [1, Lemma 3.2]. Here, we shall use the following properties

- (i) Smoothness: $(\rho, V) \in (0, \infty) \times \mathbb{R} \mapsto \mathcal{F}^\pm(\rho, V)$ are of class C^1 ,
- (ii) Consistency: $\mathcal{F}^+(\rho, V) + \mathcal{F}^-(\rho, V) = \rho V, \quad \forall V \in \mathbb{R}, \forall \rho \geq 0.$

Under CFL conditions, see [1], the scheme preserves the positivity of the discrete density and discrete kinetic and internal energies evolution equations hold.

Lemma 1.1 *Let $N \in \mathbb{N}$. Assume $\min_i (\rho_{i+1/2}^0) > 0$. For all $k \in \{0, \dots, N-1\}$, there exists $\mathcal{V}^k > 0$, which depends only on the state (ρ^k, V^k) , such that if*

$$\frac{\delta t^k}{\min_j (\delta x_{j+1/2})} \mathcal{V}^k \leq 1, \quad (10)$$

then, $\min_i (\rho_{i+1/2}^k) > 0, \forall k \in \{0, \dots, N\}$ and

$$0 \leq \sum_{k=0}^{N-1} \sum_{j=2}^J D_j^k \leq C, \quad \text{with } D_j^k = \frac{1}{4} \delta x_j \rho_j^{k+1} (V_j^{k+1} - V_j^k)^2, \quad (11)$$

$$\frac{\delta x_{j+1/2}}{\delta t^k} \left[e_{j+1/2}^{k+1} - e_{j+1/2}^k \right] + \overline{G}_{j+1}^k - \overline{G}_j^k + \pi_{j+1/2}^{k+1/2} \left[V_{j+1}^{k+1} - V_j^{k+1} \right] \leq \frac{D_j^k}{\delta t^k}, \quad (12)$$

$$\frac{\delta x_j}{\delta t^k} \left[E_{K,j}^{k+1} - E_{K,j}^k \right] + \Gamma_{j+1/2}^k - \Gamma_{j-1/2}^k + \left[\pi_{j+1/2}^{k+1/2} - \pi_{j-1/2}^{k+1/2} \right] V_j^{k+1} + \frac{D_j^k}{\delta t^k} \leq 0, \quad (13)$$

where $E_{K,j}^k = \frac{1}{2} \rho_j^k (V_j^k)^2$ and $e_{j+1/2}^k = \Phi(\rho_{j+1/2}^k)$ are the kinetic and internal energies. The fluxes are defined by $\overline{G}_1^k = \overline{G}_{J+1}^k = 0$ and

$$\begin{aligned} \overline{G}_j^k &= \Phi(\rho_{j-1/2}^k) V_j^{k+1} - \frac{\delta x_{j-1/2}}{2\delta t^k} \left[\overline{\Phi}(\overline{\rho_{j-1/2}^{k+1}}) - \overline{\Phi}(\rho_{j-1/2}^k) \right], \quad \forall j \in \{2, \dots, J\}, \\ \Gamma_{j+1/2}^k &= \frac{1}{2} V_j^k V_{j+1}^k \frac{\mathcal{F}_j^k + \mathcal{F}_{j+1}^k}{2} + \frac{1}{2} (V_j^k - V_{j+1}^k)^2 \frac{\mathcal{F}_j^{k,|\cdot|} + \mathcal{F}_{j+1}^{k,|\cdot|}}{2}, \quad \forall j \in \{1, \dots, J\}, \\ \overline{\rho_{j-1/2}^{k+1}} &= \rho_{j-1/2}^k - \frac{2\delta t^k}{\delta x_{j-1/2}} \left(\mathcal{F}^-(\rho_{j+1/2}^k, V_j^k) - \mathcal{F}^-(\rho_{j-1/2}^k, V_j^k) - \rho_{j-1/2}^k (V_j^{k+1} - V_j^k) \right), \end{aligned}$$

and $\mathcal{F}_1^{k,|\cdot|} = \mathcal{F}_{J+1}^{k,|\cdot|} = 0$, $\mathcal{F}_j^{k,|\cdot|} = \mathcal{F}^+(\rho_{j-1/2}^k, V_j^k) - \mathcal{F}^-(\rho_{j+1/2}^k, V_j^k)$, $\forall j \in \{2, \dots, J\}$. The function $\overline{\Phi}$ is a C^2 extension of the function Φ (see [1, Section 4.3]).

Results. As in [2], we prove a Lax-Wendroff-like statement: the limit of a converging (and uniformly bounded) sequence of stepwise constant functions defined from the scheme is a weak entropic-solution of the system of conservation laws.

2 Consistency analysis

Notation. Assuming that $\sum_{k=0}^{N-1} \delta t^k = T$, we define the reconstructions ($i = 0, 1$)

$$\rho_\delta^{(i)} = \sum_{k=0}^{N-1} \sum_{j=1}^J \rho_{j+1/2}^{k+i} \chi_{j+1/2}^{k+1/2}, \quad \pi_\delta = \sum_{k=0}^{N-1} \sum_{j=1}^J \pi_{j+1/2}^{k+1/2} \chi_{j+1/2}^{k+1/2}, \quad V_\delta = \sum_{k=0}^{N-1} \sum_{j=2}^J V_j^k \chi_j^{k+1/2},$$

where $\chi_j^{k+1/2} = \chi_{[x_{j-1/2}, x_{j+1/2}[\times [t^k, t^{k+1}[}$, $\chi_{j+1/2}^{k+1/2} = \chi_{[x_j, x_{j+1}[\times [t^k, t^{k+1}[}$.

We also introduce the following discrete norms

$$\begin{aligned} \|\rho_\delta\|_{\infty, \mathcal{T}} &= \max_{0 \leq k \leq N} \max_{1 \leq j \leq J} |\rho_{j+1/2}^k|, & \|V_\delta\|_{\infty, \mathcal{T}^*} &= \max_{0 \leq k \leq N} \max_{2 \leq j \leq J} |V_j^k|, \\ \|\rho_\delta\|_{1; \text{BV}, \mathcal{T}} &= \sum_{k=0}^N \delta t^k \sum_{j=2}^J |\rho_{j+1/2}^k - \rho_{j-1/2}^k|, & \|V_\delta\|_{1; \text{BV}, \mathcal{T}^*} &= \sum_{k=0}^N \delta t^k \sum_{j=1}^J |V_{j+1}^k - V_j^k|, \\ \|\rho_\delta\|_{\text{BV}; 1, \mathcal{T}} &= \sum_{j=1}^J \delta x_{j+1/2} \sum_{k=0}^{N-1} |\rho_{j+1/2}^{k+1} - \rho_{j+1/2}^k|. \end{aligned}$$

For $\varphi \in C_c^\infty((0, L) \times [0, T))$, we set $\varphi_{j+1/2}^k = \varphi(x_{j+1/2}, t^k)$ and $\varphi_j^k = \varphi(x_j, t^k)$. The interpolate $\varphi_\mathcal{T}$ of φ on the primal mesh and its discrete derivatives are defined by

$$\begin{aligned} \varphi_\mathcal{T}(\cdot, 0) &= \sum_{j=1}^J \varphi_{j+1/2}^0 \chi_{j+1/2}^{1/2}(\cdot, 0), & \varphi_\mathcal{T}(\cdot, t) &= \sum_{k=0}^{N-1} \sum_{j=1}^J \varphi_{i+1/2}^{k+1} \chi_{j+1/2}^{k+1/2}(\cdot, t), \quad \forall t > 0, \\ \bar{\partial}_t \varphi_\mathcal{T} &= \sum_{k=0}^{N-1} \sum_{j=1}^J \frac{\varphi_{j+1/2}^{k+1} - \varphi_{j+1/2}^k}{\delta t^k} \chi_{j+1/2}^{k+1/2}, & \bar{\partial}_x \varphi_\mathcal{T} &= \sum_{k=0}^{N-1} \sum_{j=2}^J \frac{\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}}{\delta x_j} \chi_j^{k+1/2}. \end{aligned}$$

Similarly, the interpolate $\varphi_{\mathcal{T}^*}$ of φ on \mathcal{T}^* and its discrete derivatives are given by

$$\begin{aligned} \varphi_{\mathcal{T}^*}(\cdot, 0) &= \sum_{j=2}^J \varphi_j^0 \chi_j^{1/2}(\cdot, 0), & \varphi_{\mathcal{T}^*}(\cdot, t) &= \sum_{k=0}^{N-1} \sum_{j=2}^J \varphi_j^{k+1} \chi_j^{k+1/2}(\cdot, t), \quad \forall t > 0, \\ \bar{\partial}_t^* \varphi_{\mathcal{T}^*} &= \sum_{k=0}^{N-1} \sum_{j=2}^J \frac{\varphi_j^{k+1} - \varphi_j^k}{\delta t^k} \chi_j^{k+1/2}, & \bar{\partial}_x^* \varphi_{\mathcal{T}^*} &= \sum_{k=0}^{N-1} \sum_{j=1}^J \frac{\varphi_{j+1}^{k+1} - \varphi_j^{k+1}}{\delta x_{j+1/2}} \chi_{j+1/2}^{k+1/2}. \end{aligned}$$

Assumptions. Let $(\mathcal{T}_m)_{m \geq 1}$ be a sequence of meshes s. t. $\text{size}(\mathcal{T}_m) \rightarrow 0$ and a family of time steps $(\delta t_m^k)_{k \geq 0, m \geq 1}$ verifying $\delta t_m \rightarrow 0$ and (10). Assume that there exists $N_m \in \mathbb{N}$ s. t. $\sum_{k=0}^{N_m-1} \delta t_m^k = T$. The scheme defines $(\rho_{\delta_m}^{(0)}, V_{\delta_m})_{m \geq 1}$. Suppose that

$$\|\rho_{\delta_m}^{(0)}\|_{\infty, \mathcal{T}} + \|V_{\delta_m}\|_{\infty, \mathcal{T}^*} \leq C_\infty, \quad \|\rho_{\delta_m}^{(0)}\|_{1; \text{BV}, \mathcal{T}} + \|V_{\delta_m}\|_{1; \text{BV}, \mathcal{T}^*} \leq C_{\text{BV}} \quad (14)$$

holds and, in the case $(\rho \mapsto \frac{\rho'(\rho)}{\rho}) \notin L_{\text{loc}}^1(0, \infty)$, $\|(\rho_{\delta_m}^{(0)})^{-1}\|_{\infty, \mathcal{T}} \leq C$. We assume that there exists $(\bar{\rho}, \bar{V}) \in L^\infty((0, T) \times (0, L))^2$ such that

$$(\rho_{\delta_m}^{(0)}, V_{\delta_m}) \longrightarrow (\bar{\rho}, \bar{V}) \text{ in } L^r((0, T) \times (0, L))^2, \quad 1 \leq r < \infty. \quad (15)$$

Main results. The uniform bounds imply that there exists constants such that

$$\begin{aligned} \sup_{0 \leq \rho, |V| \leq C_\infty} |\mathcal{A}(\rho, V)| &\leq C_{\mathcal{A}}, \quad \text{with } \mathcal{A} = \mathcal{F}^\pm, \partial_\rho \mathcal{F}^\pm \text{ and } \partial_V \mathcal{F}^\pm, \\ \sup_{0 \leq \rho \leq C_\infty + 4(C_\infty^2 + C_{\mathcal{F}^\pm})} |\mathcal{B}(\rho)| &\leq C_{\mathcal{B}}, \quad \text{with } \mathcal{B} = \Phi, \Phi', \text{ and } \bar{\Phi}'. \end{aligned}$$

Note also that we have $|\Phi(\rho_{j+1/2}^k)| \leq C_{\Phi, \rho} \rho_{j+1/2}^k$, $\forall j, k$. Furthermore, we show that $\|\rho_{\delta_m}^{(0)}\|_{\text{BV}; 1, \mathcal{T}} \leq C$ by using (5) which allows to dominate $\delta x_{j+1/2} |\rho_{j+1/2}^{k+1} - \rho_{j+1/2}^k|$ by

$$\delta t^k \left[C_{\partial_\rho, \mathcal{F}^\pm} (|\rho_{j+1/2}^k - \rho_{j-1/2}^k| + |\rho_{j+3/2}^k - \rho_{j+1/2}^k|) + 2C_{\partial_V, \mathcal{F}^\pm} |V_{j+1}^k - V_j^k| \right].$$

Consequently, $\rho_{\delta_m}^{(1)} \rightarrow \bar{\rho}$ and $\pi_{\delta_m} \rightarrow p(\bar{\rho})$ in $L^r((0, T) \times (0, L))$; with (4) and since $\rho_0, V_0 \in L^\infty(0, L)$, we get $\rho_{\delta_m}^{(0)}(\cdot, 0) \rightarrow \rho_0$ and $V_{\delta_m}(\cdot, 0) \rightarrow V_0$ in $L^r((0, L))$, $1 \leq r < \infty$.

Finally, in the sequel, when a function $\varphi \in \mathcal{C}_c^\infty((0, L) \times [0, T])$ is given, we assume that δt_m and δx_m are sufficiently small so that $\varphi(x, \cdot) \equiv 0$, $\forall x \in [0, x_{3/2}] \cup [x_{J+1/2}, L]$ and $\varphi(\cdot, t) \equiv 0$, $\forall t \in [t^{N-1}, t^N]$. Moreover, since φ is smooth, $\varphi_{\mathcal{T}_m}, \varphi_{\mathcal{T}_m^*} \rightarrow \varphi$, $\partial_t \varphi_{\mathcal{T}_m}, \partial_t^* \varphi_{\mathcal{T}_m^*} \rightarrow \partial_t \varphi$, and $\partial_x \varphi_{\mathcal{T}_m}, \partial_x^* \varphi_{\mathcal{T}_m^*} \rightarrow \partial_x \varphi$, in $L^r((0, T) \times (0, L))$, $1 \leq r \leq \infty$.

Theorem 2.1 *Assume (14) and (15). Then, $(\bar{\rho}, \bar{V})$ satisfies (1)-(2) in the distribution sense in $(\mathcal{C}_c^\infty((0, L) \times [0, T]))'$, that is*

$$-\int_0^T \int_0^L [\bar{\rho} \partial_t \varphi + \bar{\rho} \bar{V} \partial_x \varphi](x, t) dx dt - \int_0^L \rho_0(x) \varphi(x, 0) dx = 0, \quad (16)$$

$$-\int_0^T \int_0^L [\bar{\rho} \bar{V} \partial_t \varphi + (\bar{\rho} \bar{V}^2 + p(\bar{\rho})) \partial_x \varphi](x, t) dx dt - \int_0^L \rho_0(x) V_0(x) \varphi(x, 0) dx = 0. \quad (17)$$

Moreover, $(\bar{\rho}, \bar{V})$ satisfies the entropy inequality (3).

Proof. Let $\varphi \in \mathcal{C}_c^\infty((0, L) \times [0, T])$. For the sake of simplicity, the index m is dropped.

Mass balance. We multiply (5) by $\varphi_{j+1/2}^{k+1}$ and sum the results for $0 \leq k \leq N-1$ and $1 \leq j \leq J$ to obtain

$$\underbrace{\sum_{k=0}^{N-1} \sum_{j=1}^J \delta x_{j+1/2} (\rho_{j+1/2}^{k+1} - \rho_{j+1/2}^k) \varphi_{j+1/2}^{k+1}}_{:=T_1} + \underbrace{\sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J (\mathcal{F}_{j+1}^k - \mathcal{F}_j^k) \varphi_{j+1/2}^{k+1}}_{:=T_2} = 0.$$

For T_1 , since $\varphi_{j+1/2}^N = 0$, a discrete integration by part w.r.t. time yields

$$T_1 = - \sum_{k=0}^{N-1} \sum_{j=1}^J \delta x_{j+1/2} \rho_{j+1/2}^k (\varphi_{j+1/2}^{k+1} - \varphi_{j+1/2}^k) - \sum_{j=1}^J \delta x_{j+1/2} \rho_{j+1/2}^0 \varphi_{j+1/2}^0.$$

Noting that

$$\int_{t^k}^{t^{k+1}} \int_{x_j}^{x_{j+1}} \rho_\delta^{(0)} \partial_t \varphi_\tau dx dt = \delta x_{j+1/2} \rho_{j+1/2}^k (\varphi_{j+1/2}^{k+1} - \varphi_{j+1/2}^k)$$

for $k \in \{0, \dots, N-1\}$, $j \in \{1, \dots, J\}$, we get

$$T_1 = - \int_0^T \int_0^L \rho_\delta^{(0)} \partial_t \varphi_\tau dx dt - \int_0^L \rho_\delta^{(0)}(x, 0) \varphi_\tau(x, 0) dx.$$

For T_2 , by integrating by part w.r.t. space, we readily obtain

$$T_2 = - \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^J \mathcal{F}_j^k (\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}).$$

Bearing in mind that $2\delta x_j = \delta x_{j-1/2} + \delta x_{j+1/2}$, we then combine the two following expressions of mass fluxes (see (9)-(ii))

$$\mathcal{F}_j^k = \rho_{j\pm 1/2}^k V_j^k \mp R_j^{k,\pm} \text{ with } R_j^{k,\pm} = \mathcal{F}^\pm(\rho_{j+1/2}^k, V_j^k) - \mathcal{F}^\pm(\rho_{j-1/2}^k, V_j^k)$$

to write

$$\mathcal{F}_j^k = \left[\frac{\delta x_{j-1/2}}{2\delta x_j} \rho_{j-1/2}^k V_j^k + \frac{\delta x_{j+1/2}}{2\delta x_j} \rho_{j+1/2}^k V_j^k \right] + \left[\frac{\delta x_{j-1/2}}{2\delta x_j} R_j^{k,-} - \frac{\delta x_{j+1/2}}{2\delta x_j} R_j^{k,+} \right].$$

This expression of the mass fluxes leads to $T_2 = -T_{2,1} - T_{2,2}$ with

$$T_{2,1} = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^J \frac{1}{2} \left[\delta x_{j-1/2} \rho_{j-1/2}^k + \delta x_{j+1/2} \rho_{j+1/2}^k \right] V_j^k \frac{\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}}{\delta x_j},$$

$$T_{2,2} = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^J \frac{1}{2} \left[\delta x_{j-1/2} R_j^{k,-} - \delta x_{j+1/2} R_j^{k,+} \right] V_j^k \frac{\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}}{\delta x_j}.$$

We now observe that, for $k \in \{0, \dots, N-1\}$, $j \in \{2, \dots, J\}$,

$$\int_{t^k}^{t^{k+1}} \int_{x_{j-1/2}}^{x_j} \rho_\delta^{(0)} V_\delta \bar{\partial}_x \varphi_\tau \, dx \, dt = \delta t^k \frac{\delta x_{j-1/2}}{2} \left[\rho_{j-1/2}^k V_j^k \frac{\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}}{\delta x_j} \right],$$

and

$$\int_{t^k}^{t^{k+1}} \int_{x_j}^{x_{j+1/2}} \rho_\delta^{(0)} V_\delta \bar{\partial}_x \varphi_\tau \, dx \, dt = \delta t^k \frac{\delta x_{j+1/2}}{2} \left[\rho_{j+1/2}^k V_j^k \frac{\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}}{\delta x_j} \right].$$

Summing these equalities yields

$$T_{2,1} = \int_0^T \int_{x_{3/2}}^{x_{J+1/2}} \rho_\delta^{(0)} V_\delta \bar{\partial}_x \varphi_\tau \, dx \, dt = \int_0^T \int_0^L \rho_\delta^{(0)} V_\delta \bar{\partial}_x \varphi_\tau \, dx \, dt,$$

since $\bar{\partial}_x \varphi_\tau(x, \cdot) \equiv 0$ for $x \in [0, x_{3/2}] \cup [x_{J+1/2}, L]$.

With (15), we pass to the limit in T_1 and $T_{2,1}$. We prove that $(\bar{\rho}, \bar{V})$ satisfies the mass conservation equation (16) by showing that $T_{2,2} \rightarrow 0$ since

$$|T_{2,2}| \leq C_{\partial_\rho, \mathcal{F}^\pm} \|\partial_x \varphi\|_{L^\infty} \|V_\delta\|_{\infty, \mathcal{T}^*} \|\rho_\delta\|_{1; \text{BV}, \mathcal{T}} \delta x \lesssim \delta x.$$

Momentum balance. We multiply (6) by φ_j^{k+1} and sum for $0 \leq k \leq N-1$ and $2 \leq j \leq J$ to obtain $T_3 + T_4 + T_5 = 0$ with

$$T_3 = \sum_{k=0}^{N-1} \sum_{j=2}^J \delta x_j (\rho_j^{k+1} V_j^{k+1} - \rho_j^k V_j^k) \varphi_j^{k+1},$$

$$T_4 = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^J (\mathcal{G}_{j+1/2}^k - \mathcal{G}_{j-1/2}^k) \varphi_j^{k+1}, \quad T_5 = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^J (\pi_{j+1/2}^{k+1/2} - \pi_{j-1/2}^{k+1/2}) \varphi_j^{k+1}.$$

For T_3 , integrating by part w.r.t time yields

$$T_3 = - \sum_{k=0}^{N-1} \sum_{j=2}^J \delta x_j \rho_j^k V_j^k (\varphi_j^{k+1} - \varphi_j^k) - \sum_{j=2}^J \delta x_j \rho_j^0 V_j^0 \varphi_j^0.$$

Next, we observe that

$$\begin{aligned} \int_{t^k}^{t^{k+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} \rho_\delta^{(0)} V_\delta \bar{\partial}_t^* \varphi_{\mathcal{T}^*} dx dt &= V_j^k \frac{\varphi_j^{k+1} - \varphi_j^k}{\delta t^k} \int_{t^k}^{t^{k+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} \rho_\delta^{(0)} dx dt \\ &= \delta x_j \rho_j^k V_j^k (\varphi_j^{k+1} - \varphi_j^k). \end{aligned}$$

Summing these equalities for $k \in \{0, \dots, N-1\}$, $j \in \{2, \dots, J\}$ yields

$$\begin{aligned} T_3 &= - \int_0^T \int_{x_{3/2}}^{x_{J+1/2}} \rho_\delta^{(0)} V_\delta \bar{\partial}_t^* \varphi_{\mathcal{T}^*} dx dt - \int_{x_{3/2}}^{x_{J+1/2}} \rho_\delta^{(0)}(x, 0) V_\delta(x, 0) \varphi_{\mathcal{T}^*}(x, 0) dx, \\ &= - \int_0^T \int_0^L \rho_\delta^{(0)} V_\delta \bar{\partial}_t^* \varphi_{\mathcal{T}^*} dx dt - \int_0^L \rho_\delta^{(0)}(x, 0) V_\delta(x, 0) \varphi_{\mathcal{T}^*}(x, 0) dx. \end{aligned}$$

For T_4 and T_5 , we first integrate by part w.r.t space and obtain

$$T_4 = - \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J \mathcal{G}_{j+1/2}^k (\varphi_{j+1}^{k+1} - \varphi_j^{k+1}), \quad T_5 = - \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J \pi_{j+1/2}^{k+1/2} (\varphi_{j+1}^{k+1} - \varphi_j^{k+1}).$$

We then use the following expression of the momentum flux

$$\begin{cases} \mathcal{G}_{j+1/2}^k &= \frac{1}{2} \rho_{j+1/2}^k [(V_j^k)^2 + (V_{j+1}^k)^2] + Q_{j+1/2}^k, \\ Q_{j+1/2}^k &= -\frac{1}{2} V_j^k R_j^{k,+} + \frac{1}{2} V_{j+1}^k R_{j+1}^{k,-} \\ &\quad - \frac{1}{2} (V_{j+1}^k - V_j^k) [\mathcal{F}^+(\rho_{j+1/2}^k, V_{j+1}^k) - \mathcal{F}^-(\rho_{j+1/2}^k, V_j^k)]. \end{cases}$$

to write $T_4 = -T_{4,1} - T_{4,2}$ with

$$\begin{aligned} T_{4,1} &= \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J \frac{1}{2} \rho_{j+1/2}^k [(V_j^k)^2 + (V_{j+1}^k)^2] (\varphi_{j+1}^{k+1} - \varphi_j^{k+1}) \\ T_{4,2} &= \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J Q_{j+1/2}^k (\varphi_{j+1}^{k+1} - \varphi_j^{k+1}). \end{aligned}$$

Next, we observe that

$$\int_{t^k}^{t^{k+1}} \int_{x_j}^{x_{j+1}} \rho_\delta^{(0)} (V_\delta)^2 \bar{\partial}_x^* \varphi_{\mathcal{T}^*} dx dt = \delta t^k \rho_{j+1/2}^k \frac{(V_j^k)^2 + (V_{j+1}^k)^2}{2} (\varphi_{j+1}^{k+1} - \varphi_j^{k+1})$$

and, consequently, summing for $k \in \{0, \dots, N-1\}$, $j \in \{1, \dots, J\}$,

$$T_{4,1} = \int_0^T \int_0^L \rho_\delta^{(0)} (V_\delta)^2 \bar{\partial}_x^* \varphi_{\mathcal{T}^*} dx dt.$$

Similarly, for T_5 , we get

$$T_5 = - \int_0^T \int_0^L \pi_\delta \bar{\partial}_x^* \varphi_{\mathcal{T}^*} dx dt.$$

With (15), we pass to the limit in T_3 , $T_{4,1}$ and T_5 . We obtain that $(\bar{\rho}, \bar{V})$ satisfies the momentum balance equation (17) by showing that $T_{4,2} \rightarrow 0$. Indeed, we have

$$|T_{4,2}| \leq \|V_\delta\|_{\infty, \mathcal{T}^*} |\partial_x \varphi|_{L^\infty} (C_{\partial_\rho, \mathcal{F}^\pm} \|\rho_\delta\|_{1, \text{BV}, \mathcal{T}^*} + C_{\mathcal{F}^\pm} \|V_\delta\|_{1, \text{BV}, \mathcal{T}^*}) \delta x \lesssim \delta x.$$

Entropy inequality. We now assume that $\varphi \geq 0$.

• **Kinetic energy.** We multiply (13) by $\delta t^k \varphi_j^{k+1}$ and sum for $0 \leq k \leq N-1$ and $2 \leq j \leq J$. We obtain to get $T_6 + T_7 + T_8 \leq 0$ with

$$\begin{aligned} T_6 &= \sum_{k=0}^{N-1} \sum_{j=2}^J \delta x_j [E_{K,j}^{k+1} - E_{K,j}^k] \varphi_j^{k+1}, \quad T_7 = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^J [\Gamma_{j+1/2}^k - \Gamma_{j-1/2}^k] \varphi_j^{k+1}, \\ T_8 &= \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^J [\pi_{j+1/2}^{k+1/2} - \pi_{j-1/2}^{k+1/2}] V_j^{k+1} \varphi_j^{k+1} + \sum_{k=0}^{N-1} \sum_{j=2}^J D_j^k \varphi_j^{k+1}. \end{aligned}$$

Integrating by part w.r.t. time yields

$$\begin{aligned} T_6 &= - \sum_{k=0}^{N-1} \sum_{j=2}^J \delta x_j E_{K,j}^k [\varphi_j^{k+1} - \varphi_j^k] - \sum_{j=1}^J \delta x_j E_{K,j}^0 \varphi_j^0 \\ &= - \int_0^T \int_0^L \frac{1}{2} \rho_\delta^{(0)} (V_\delta)^2 \partial_t^* \varphi_{\mathcal{T}^*} dx dt - \int_0^L \frac{1}{2} \rho_\delta^{(0)} (x, 0) (V_\delta(x, 0))^2 \varphi_{\mathcal{T}^*}(x, 0) dx. \end{aligned}$$

For T_7 , we write $\Gamma_{j+1/2}^k = \frac{1}{4} \rho_{j+1/2}^k [(V_j^k)^3 + (V_{j+1}^k)^3] + \frac{1}{4} S_{j+1/2}^k$ where

$$S_{j+1/2}^k = V_j^k V_{j+1}^k [R_{j+1}^{k,-} - R_{j+1}^{k,+}] + (V_{j+1}^k - V_j^k)^2 [\mathcal{F}_j^{k,| \cdot |} + \mathcal{F}_{j+1}^{k,| \cdot |} - \rho_{j+1/2}^k (V_j^k + V_{j+1}^k)].$$

Integration by part w.r.t space leads to $T_7 = -T_{7,1} - T_{7,2}$ with

$$T_{7,1} = \int_0^T \int_0^L \frac{1}{2} \rho_\delta^{(0)} (V_\delta)^3 \partial_x^* \varphi_{\mathcal{T}^*} dx dt, \quad T_{7,2} = \frac{1}{4} \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J S_{j+1/2}^k [\varphi_{j+1}^{k+1} - \varphi_j^{k+1}].$$

Finally $|T_{7,2}| \lesssim \delta x$ since it is dominated by

$$\begin{aligned} \delta x |\partial_x \varphi|_{L^\infty} \|V_\delta\|_{\infty, \mathcal{T}^*} &\left[\frac{C_{\partial_\rho \mathcal{F}^\pm}}{2} \|V_\delta\|_{\infty, \mathcal{T}^*} \|\rho_\delta\|_{1; \text{BV}, \mathcal{T}} \right. \\ &\left. + (2C_{\mathcal{F}^\pm} + \|V_\delta\|_{\infty, \mathcal{T}^*} \|\rho_\delta\|_{\infty, \mathcal{T}}) \|V_\delta\|_{1; \text{BV}, \mathcal{T}^*} \right]. \end{aligned}$$

• **Internal energy.** Multiply (12) by $\delta t^k \varphi_{j+1/2}^{k+1}$ and sum for $0 \leq k \leq N-1$ and $1 \leq j \leq J$ to get $T_9 + T_{10} + T_{11} \leq 0$ with

$$\begin{aligned} T_9 &= \sum_{k=0}^{N-1} \sum_{j=1}^J \delta x_{j+1/2} [e_{j+1/2}^{k+1} - e_{j+1/2}^k] \varphi_{j+1/2}^{k+1}, \quad T_{10} = \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J [\bar{G}_{j+1}^k - \bar{G}_j^k] \varphi_{j+1/2}^{k+1}, \\ T_{11} &= \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J \pi_{j+1/2}^{k+1/2} (V_{j+1}^{k+1} - V_j^{k+1}) \varphi_{j+1/2}^{k+1} - \sum_{k=0}^{N-1} \sum_{j=1}^J D_j^k \varphi_{j+1/2}^{k+1}. \end{aligned}$$

Owing to integration by part w.r.t. time, we get

$$\begin{aligned} T_9 &= - \sum_{k=0}^{N-1} \sum_{j=1}^J \delta x_{j+1/2} \Phi(\rho_{j+1/2}^k) (\varphi_{j+1/2}^{k+1} - \varphi_{j+1/2}^k) - \sum_{j=1}^J \delta x_{j+1/2} \Phi(\rho_{j+1/2}^0) \varphi_{j+1/2}^0, \\ &= - \int_0^T \int_0^L \Phi(\rho_\delta^{(0)}) \partial_t \varphi_\tau dx dt - \int_0^L \Phi(\rho_\delta^{(0)}(x, 0)) \varphi_\tau(x, 0) dx. \end{aligned}$$

For T_{10} , we rewrite the flux as follows

$$\begin{cases} \bar{G}_j^k = \frac{1}{2\delta x_j} \left[\delta x_{j-1/2} \Phi(\rho_{j-1/2}^k) + \delta x_{j+1/2} \Phi(\rho_{j+1/2}^k) \right] V_j^k + U_{1,j}^k + U_{2,j}^k + U_{3,j}^k, \\ U_{1,j}^k = e_{j-1/2}^k (V_j^{k+1} - V_j^k), \quad U_{2,j}^k = -\frac{\delta x_{j+1/2}}{2\delta x_j} \left[e_{j+1/2}^k - e_{j-1/2}^k \right] V_j^k, \\ U_{3,j}^k = -\frac{\delta x_{j-1/2}}{2\delta t^k} \left[\bar{\Phi}(\overline{\rho_{j-1/2}^{k+1}}) - \bar{\Phi}(\rho_{j-1/2}^k) \right]. \end{cases}$$

It leads to $T_{10} = -T_{10,0} - T_{10,1} - T_{10,2} - T_{10,3}$ with

$$\begin{cases} T_{10,0} = \int_0^T \int_0^L \Phi(\rho_\delta^{(0)}) V_\delta \bar{\partial}_x \varphi_\tau \, dx \, dt, \\ T_{10,i} = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^J U_{i,j}^k (\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}), \quad i = 1, 2, 3. \end{cases}$$

The term $T_{10,1}$ can be bounded as follows

$$|T_{10,1}| \leq C_{\Phi, \rho} |\partial_x \varphi| \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^J \delta x_j \rho_{j-1/2}^k |V_j^{k+1} - V_j^k|. \quad (18)$$

Since $a \leq \min(a, b) + |b - a|$, we get $\rho_{j-1/2}^k \leq \rho_j^k + |\rho_{j+1/2}^k - \rho_{j-1/2}^k|$. This leads to

$$|T_{10,1}| \leq C_{\Phi, \rho} |\partial_x \varphi|_{L^\infty} \underbrace{\left(\sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^J \delta x_j \rho_j^k |V_j^{k+1} - V_j^k| + 2 \|V_\delta\|_{\infty, \mathcal{T}^*} \|\rho_\delta\|_{1; \text{BV}, \mathcal{T}} \delta x \right)}_{:= T^*}.$$

Writing $\rho_j^k = \rho_j^{k+1-} - (\rho_j^{k+1-} - \rho_j^k)$ and using the Cauchy-Schwarz inequality yields

$$T^* \leq 2 (TL \|\rho_\delta\|_{\infty, \mathcal{T}})^{1/2} \left(\delta t \sum_{k=0}^{N-1} \sum_{j=2}^J D_j^k \right)^{1/2} + 2 \|V_\delta\|_{\infty, \mathcal{T}^*} \|\rho_\delta\|_{\text{BV}, 1, \mathcal{T}} \delta t \lesssim \delta t^{1/2}.$$

It finally leads to $|T_{10,1}| \lesssim \delta t^{\frac{1}{2}} + \delta x$. The term $T_{10,2}$ can be bounded as follows

$$|T_{10,2}| \leq C_{\Phi'} \|V_\delta\|_{\infty, \mathcal{T}^*} |\partial_x \varphi|_{L^\infty} \|\rho_\delta\|_{1; \text{BV}, \mathcal{T}} \delta x \lesssim \delta x.$$

We now turn to $T_{10,3}$. We remark that

$$\left| \overline{\rho_{j-1/2}^{k+1}} - \rho_{j-1/2}^k \right| \leq \frac{2\delta t^k}{\delta x_{j-1/2}} \left(C_{\partial_\rho \mathcal{F}^\pm} |\rho_{j+1/2}^k - \rho_{j-1/2}^k| + \rho_{j-1/2}^k |V_j^{k+1} - V_j^k| \right).$$

Hence, using the same bound as for $T_{10,1}$ yields

$$|T_{10,3}| \leq C_{\bar{\Phi}'} |\partial_x \varphi|_{L^\infty} \left(\left(C_{\partial_\rho \mathcal{F}^\pm} + 2 \|V_\delta\|_{\infty, \mathcal{T}^*} \right) \|\rho_\delta\|_{1; \text{BV}, \mathcal{T}} \delta x + T^* \right) \lesssim \delta t^{1/2} + \delta x.$$

• Pressure terms. It remains to get the limit of $T_8 + T_{11} = -T_{12,0} - T_{12,1} - T_{12,2} - T_{12,3}$

with

$$\begin{aligned}
T_{12,0} &= \int_0^T \int_0^L \pi_\delta V_\delta \bar{\partial}_x^* \varphi_{\mathcal{T}^*} dx dt, & T_{12,1} &= \sum_{k=0}^{N-1} \sum_{j=2}^J D_j^k (\varphi_{j+1/2}^{k+1} - \varphi_j^{k+1}), \\
T_{12,2} &= \frac{1}{2} \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J \pi_{j+1/2}^{k+1/2} (V_j^{k+1} - V_j^k + V_{j+1}^{k+1} - V_{j+1}^k) (\varphi_{j+1}^{k+1} - \varphi_j^{k+1}), \\
T_{12,3} &= -\frac{1}{2} \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^J \pi_{j+1/2}^{k+1/2} (V_{j+1}^{k+1} - V_j^{k+1}) (2\varphi_{j+1/2}^{k+1} - \varphi_{j+1}^{k+1} - \varphi_j^{k+1}).
\end{aligned}$$

We bound $T_{12,1}$ and $T_{12,3}$ as follows

$$T_{12,1} \leq \frac{|\partial_x \varphi|_{L^\infty}}{2} \left(\sum_{k=0}^{N-1} \sum_{j=2}^J D_j^k \right) \delta x \lesssim \delta x, \quad T_{12,3} \leq \frac{C_\pi}{4} |\partial_{xx} \varphi|_{L^\infty} \|V_\delta\|_{1;\text{BV},\mathcal{T}^*} (\delta x)^2 \lesssim (\delta x)^2.$$

Note that $|\pi_{j+1/2}^{k+1/2}| \leq (C_{\Phi'} + C_{\Phi,\rho}) \rho_{j+1/2}^k$. It readily leads to

$$|T_{12,2}| \leq (C_{\Phi'} + C_{\Phi,\rho}) |\partial_x \varphi|_{L^\infty} T^* \lesssim \delta t^{\frac{1}{2}}.$$

With (15), we pass to the limit in $T_6, T_{7,1}, T_9, T_{10,0}$ and $T_{12,0}$. We arrive at (3) since the other terms tend to 0. \blacksquare

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