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SCALAR CONSERVATION LAWS WITH ROUGH (STOCHASTIC) FLUXES; THE SPATIALLY DEPENDENT CASE

PIERRE-LOUIS LIONS¹, BENOÎT PERTHAME² AND PANAGIOTIS E. SOUGANIDIS^{3,4}

ABSTRACT. We develop a pathwise theory for scalar conservation laws with spatially dependent fluxes and quasilinear multiplicative “rough path” dependence, a special case being spatially dependent stochastic conservation laws with quasilinear stochastic dependence. Following some of the ideas of a previous note where we considered spatially independent fluxes, we introduce the notion of pathwise stochastic entropy solutions, which is closed with respect to the local uniform limits of paths, and prove that, in the presence of a single “rough path”, it is well posed, that is we establish existence, uniqueness and continuous dependence in the form of (pathwise) L^1 -contraction. Our approach is motivated by the theory of stochastic viscosity solutions, which was introduced and developed by two of the authors, to study fully nonlinear first- and second-order stochastic pde with multiplicative noise. This theory relies on special test functions constructed by inverting locally the flow of the stochastic characteristics. For conservation laws this is best implemented at the level of the kinetic formulation which we follow here.

Key words. Stochastic differential equations, stochastic conservation laws, stochastic entropy condition, kinetic formulation, dissipative solutions, rough paths.

Mathematics Subject Classification 35L65; 35R60; 60H15; 81S30

1. INTRODUCTION

We continue the development of the theory of pathwise stochastic entropy solutions weak for scalar conservation laws in \mathbb{R}^N with quasilinear multiplicative “rough path” dependence by considering spatially dependence fluxes and a single rough path. In our previous note [15] we studied spatially independent fluxes and multiple paths.

Our approach is based on the concepts and methods introduced in [15] as well as by Lions and Souganidis [18, 19] and extended by the same authors in [23, 21, 22, 20] for the theory of pathwise stochastic viscosity solution of fully nonlinear first- and second-order stochastic pde including stochastic Hamilton-Jacobi equations. One of the fundamental tools of this theory is the class of test functions constructed by inverting locally, and at the level of test functions, the flow of the characteristics corresponding to the stochastic first-order part of the equation and smooth initial data. Such approach is best implemented for conservation laws using the kinetic formulation which we follow here.

It is important to remark that throughout the paper we use the term “rough path” for a non differentiable time dependent function and we do not make the connection with the Lyons [24] theory of rough paths since we are only considering a single path. The Lyons theory may become more relevant in the study of problems with multiple paths. We also note that to keep statements shorter, we write ssel for scalar stochastic conservation laws any time we refer equations of the type of study here.

Let

$$(1) \quad \mathbf{A} = (A_1, \dots, A_N) \in C^2(\mathbb{R}^N \times \mathbb{R}; \mathbb{R}^N)$$

be the flux and consider a single continuous “rough path”

$$(2) \quad W \in C([0, \infty); \mathbb{R}) \text{ with } W(0) = 0,$$

which may be, for example, a Brownian motion.

We are interested in the sscl

$$(3) \quad \begin{cases} du + \operatorname{div} \mathbf{A}(x, u) \circ dW = 0 & \text{in } \mathbb{R}^N \times (0, \infty), \\ u = u^0 & \text{on } \mathbb{R}^N \times \{0\}. \end{cases}$$

Note that $u(x, t) = v(x, W(t))$, where v solves a time-homogeneous equation, is formally a solution but it can not be an entropy solution because this change of time is incompatible with the formation of shocks.

Throughout the paper we adopt the notation and terminology of stochastic calculus. In general du denotes some kind of time differential, while, in the case that W is Brownian, it is the usual stochastic differential. Similarly in the general setting \circ does not have any particular meaning and can be ignored, while in the stochastic setting it denotes the Stratonovich differential. The need to use the latter stems from the fact that we are developing a theory which is closed (stable) on paths in the local uniform topology. It is well known that ordinary differential equations with time dependence converging local uniformly to Brownian paths give rise to stochastic differential equations with Stratonovich differential. That a pathwise theory is more appropriate to study (3) is also justified from the fact that in the actual stochastic case, taking expectations leads, in view of the properties of the Ito calculus, to terms that are not possible to handle by the available estimates. We refer to [15] for an extended discussion of this point.

Our interest in sscl is twofold. Given the theory of stochastic viscosity solutions and the connection between conservation laws and Hamilton-Jacobi equations when $N = 1$, it is very natural from the mathematical point of view to ask whether there is such a theory for the former. The second reason is that sscl like (3) arise naturally as models in the theory of mean field games developed by Lasry and Lions (see [10], [11], [12]).

A concrete example concerns that asymptotic behavior, as $L \rightarrow \infty$, of the law of the solution of the system

$$dX_t^i = \sigma(X_t^i, \frac{1}{L-1} \sum_{j \neq i} \delta_{X_t^j}) \circ d\mathbf{W}_t \text{ with } i = 1, \dots, L,$$

where δ_y denotes the Dirac mass at y , $\mathbf{W} = (W^1, \dots, W^N)$ is a N -dimensional Brownian motion and the $N \times N$ matrix σ is Lipschitz continuous on $\mathbb{R}^N \times \mathcal{P}(\mathbb{R}^N)$, where $\mathcal{P}(X)$ is the space of probability measures on the metric space X endowed with the usual 2-Wasserstein metric.

It turns out that, as $L \rightarrow \infty$, the law, $\mathcal{L}(X_t^1, \dots, X_t^L)$, of the (X_t^1, \dots, X_t^L) converges in the sense of measures to some $\pi_t \in \mathcal{P}(\mathcal{P}(\mathbb{R}^N))$, which evolves in time according to

$$\int U(m) d\pi_t(m) = E[U(m_t)] \text{ for all } U \in C(\mathcal{P}(\mathbb{R}^N)),$$

where E is the expectation with respect to the probability space associated with \mathbf{W} and, for σ^* the adjoint of σ , m_t solves the sscl

$$dm = - \sum_{i=1}^N (\sigma^*(x, m)m)_{x_i} \circ dW^i.$$

If, instead of (2), we assume that $\mathbf{W} \in C^1((0, \infty); \mathbb{R}^N)$, then (3) is a “classical” problem with a well known theory; see, for example, the books of Dafermos [2] and Serre [28]. The solution can develop singularities in the form of shocks (discontinuities). Hence it is necessary to consider entropy solutions which, although not regular, satisfy the L^1 -contraction property established by Kruzkov that yields uniqueness. Entropy solutions, which are based on certain inequalities, cannot be used when \mathbf{W} is not smooth.

There are several challenges when trying to extend the analysis of [15] to spatially dependent fluxes. The underlying idea is the same, namely to use the characteristics at the kinetic level to “eliminate the bad stochastic terms”. In the spatially independent setting the characteristics can be solved explicitly and, as a result, it is possible to keep track of all the cancellations that are taking place and, hence, obtain rather strong estimates. In the setting we are considering here, that is when the flux depends on the space variable, there are no explicit solution of the system of the stochastic characteristics. As a result the calculations are not transparent and it becomes necessary to employ more complicated arguments to study the problem.

We remark that recently Debussche and Vovelle [5] (see also Feng and Nualart [7], Chen, Ding and Karlsen [1], Debussche, Hofmanova and Vovelle [4] and Hofmanova [8], [9]) put forward a theory of weak entropy solutions of scalar conservation laws with Ito-type semilinear and not quasilinear stochastic dependence. Such problems do not appear amenable to a pathwise theory. Our results do not cover the equations studied in [5, 7] and vice versa. We refer to section 6 of [15] for a discussion of these issues.

Organization of the paper. The paper is organized as follows: In section 2 we state the assumptions, we review briefly some facts about entropy solutions and their kinetic formulation and we introduce the notion of the pathwise solutions. In section 3 we summarize some of the tools we use and prove a preliminary result. Sections 4 and 5 are devoted respectively to the proofs of the contraction and intrinsic uniqueness of the stochastic entropy solutions and their existence.

2. THE KINETIC FORMULATION AND PATHWISE STOCHASTIC ENTROPY SOLUTIONS

We consider the sscl (3). We write

$$\operatorname{div} \mathbf{A}(x, u) = \operatorname{div}_x \mathbf{A}(x, u) + \operatorname{div}_u \mathbf{A}(x, u),$$

where

$$\operatorname{div}_x \mathbf{A}(x, u) = \sum_{i=1}^N A_{i,x_i}(x, u) \quad \text{and} \quad \operatorname{div}_u \mathbf{A}(x, u) = \sum_{i=1}^N A_{i,u}(x, u) u_{x_i},$$

and introduce the notation

$$(4) \quad \begin{cases} \mathbf{a}(x, \xi) = (a_1(x, \xi), \dots, a_N(x, \xi)) := \mathbf{A}_u(x, \xi) = (A_{1,u}(x, \xi), \dots, A_{N,u}(x, \xi)) \\ \text{and } b(x, \xi) = \operatorname{div}_x \mathbf{A}(x, \xi). \end{cases}$$

Assumptions. We summarize here the main assumptions we need in the paper. In addition to (1), we assume that

$$(5) \quad b(x, 0) = 0,$$

and

$$(6) \quad \partial_j a_i, \partial_u a_i, \partial_j b, \partial_u b \in L^\infty(\mathbb{R}^N \times \mathbb{R}) \text{ for } 1 \leq i, j \leq N.$$

The theory we are developing deals with solutions in $(L^1 \cap L^\infty)(\mathbb{R}^N)$. As a result it is not necessary to introduce additional requirements on the flux, which would be needed if we were dealing with only L^1 -solutions.

Entropy solutions for smooth paths and some estimates. If $W \in C^1([0, \infty); \mathbb{R})$ and u, v solve (3), then, the classical entropy inequalities yield

$$\partial_t |u - v| + \operatorname{div} [[\mathbf{A}(x, u) - \mathbf{A}(x, v)] \operatorname{sgn}(u - v)] \circ dW \leq 0,$$

and thus

$$(7) \quad \frac{d}{dt} \int_{\mathbb{R}^N} |(u - v)(x, t)| dx \leq 0.$$

Since, in view of (5), $v \equiv 0$ is a solution, the contraction yield the a priori L^1 estimate

$$(8) \quad \|u(t)\|_1 \leq \|u^0\|_1.$$

On the other hand L^∞ estimates, which are also known as invariant regions, of the form

$$(9) \quad \|u(t)\|_\infty \leq C(t)$$

for space dependent fluxes are not as general and require some additional work and stronger assumptions. We give a general statement in section 5.

Here we mention some special cases which do not fall in the general theory. A classical problem for applications is when there exist two ordered bounded steady states, that is smooth and bounded k_\pm such that $k_- \leq k_+$ and $\operatorname{div} A(x, k_\pm(x)) = 0$. If $k_- \leq u^0 \leq k_+$, then we find the estimate

$$\|u(t)\|_\infty \leq \max(\|k_-\|_\infty, \|k_+\|_\infty).$$

For instance when $N = 1$ and $A(x, u) = c(x)u^2$ with $c(x) \geq c_m > 0$, one can choose $k_\pm(x) = \pm \lambda_\pm c(x)^{-1/2}$. When $N = 2$ and $A(x, u) = {}^\perp DV(x)B(u)$, one can choose constants. Another example, which is a model for multiphase flow in a porous medium, is $A(x, u) = V(x)u(1 - u)$ and $k_- = 0$, $k_+ = 1$ give the physical invariant region.

The kinetic formulation for smooth paths. We review here the basic concepts of the kinetic theory of scalar conservation laws and we show that it allows to define a change of variable along the “kinetic” characteristics which turns out to be a very convenient tool for the study of sscl. Note that, as it was done on [15], when the flux is spatially independent it was possible to obtain a global in time change of variables. Recall that for the conservation laws in the physical space the characteristics are only defined for short times (before crossing) and the method is not so convenient. Such a conclusion was also drawn in [5] but for a different reason. There the kinetic setting keeps better track of the entropy dissipation due to the noise.

Although we use the notation of the introduction, in the first this section we assume that

$$(10) \quad W \in C^1((0, \infty); \mathbb{R}),$$

in which case du stands for the usual derivative and \circ is the usual multiplication and, hence, should be ignored.

The entropy inequalities (see, for example, [2, 28]), which yield (7) and guarantee the uniqueness of the entropy solutions, are

$$(11) \quad \begin{cases} dS(u) + [\operatorname{div}_u(\mathbf{A}^S(x, u)) + S'(u)[\operatorname{div}_x \mathbf{A}](x, u) - B^S(x, u)] \circ dW \leq 0 & \text{in } \mathbb{R}^N \times (0, \infty), \\ S(u) = S(u^0) & \text{on } \mathbb{R}^N \times \{0\}, \end{cases}$$

for all C^2 -convex functions S and entropy fluxes \mathbf{A}^S and forcing terms \mathbf{B}^S defined by

$$\mathbf{A}^S(x, u)_u = \mathbf{A}_u(x, u)S'(u) \quad \text{and} \quad \mathbf{B}^S(x, u)_u = \operatorname{div} \mathbf{A}_u(x, u)S'(u).$$

It is by now well established that the simplest way to handle conservation laws is through their kinetic formulation developed in a series of papers – see Perthame and Tadmor [27], Lions, Perthame and Tadmor [16], Perthame [25, 26], and Lions, Perthame and Souganidis [13], [14]. The basic idea is to write a linear equation on the nonlinear function $\chi : \mathbb{R}^N \times \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$ given by

$$(12) \quad \chi(x, \xi, t) = \chi(u(x, t), \xi) = \begin{cases} +1 & \text{if } 0 \leq \xi \leq u(x, t), \\ -1 & \text{if } u(x, t) \leq \xi \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The kinetic formulation says (see, for example, Dalibard [3]) that, for $u^0 \in (L^1 \cap L^\infty)(\mathbb{R}^N)$, the set of entropy inequalities (11) for all convex entropies S can be replaced by the following equation which is supposed to be satisfied in the sense of distributions:

$$(13) \quad \begin{cases} d\chi + \operatorname{div}_x[\mathbf{a}(x, \xi)\chi] \circ dW - \operatorname{div}_\xi[b(x, \xi)\chi] \circ dW = \partial_\xi m dt & \text{in } \mathbb{R}^N \times \mathbb{R} \times (0, \infty), \\ \chi = \chi(u^0(\cdot), \cdot) & \text{on } \mathbb{R}^N \times \mathbb{R} \times \{0\}, \end{cases}$$

where

$$(14) \quad \begin{cases} m \text{ is a nonnegative measure in } \mathbb{R}^N \times \mathbb{R} \times (0, \infty) \text{ such that,} \\ \sup_{\xi \in \mathbb{R}} \int_0^\infty \int_{\mathbb{R}^N} m(x, \xi, t) dx dt \leq \|u^0\|_1 \text{ and } m(x, \xi, t) = 0 \text{ for } |\xi| \geq \|u(t)\|_\infty. \end{cases}$$

We remark that the theory developed in [3] for classical conservation laws does not require $u^0 \in L^\infty$. Instead it needs an additional condition for the flux and an extra term in the inequality for the measure m . For problems with smooth time dependence, this extra term will involve the total variation of the path. Here we need estimates which are independent of the total variation, since the solutions of the sctl will arise as the limit of the solutions of problems with smooth paths. As a result we work with $u^0 \in (L^1 \cap L^\infty)(\mathbb{R}^N)$.

Notice that the transport equation for χ has a Hamiltonian type structure and is equivalently written in the strong form

$$(15) \quad d\chi + \mathbf{a}(x, \xi) \cdot D_x \chi \circ dW - b(x, \xi) D_\xi \chi \circ dW = \partial_\xi m dt.$$

For $\rho^0 \in C_b^\infty(\mathbb{R}^N \times \mathbb{R})$ and $t_0 \geq 0$, let $\rho(\cdot, \cdot, \cdot; t_0)$ be the solution of the linear stochastic pde

$$(16) \quad \begin{cases} d\rho + [\mathbf{a}(x, \xi) \cdot D_x \rho - b(x, \xi) D_\xi \rho] \circ dW = 0 & \text{in } \mathbb{R}^N \times \mathbb{R} \times (t_0, \infty), \\ \rho = \rho^0 & \text{on } \mathbb{R}^N \times \mathbb{R} \times \{t_0\}. \end{cases}$$

Since W is a single path, it is immediate that

$$\rho(x, \xi, t; t_0) = \hat{\rho}(x, \xi, W(t) - W(t_0)),$$

where, in view of the assumptions above, $\hat{\rho}$ is the smooth solution of

$$(17) \quad \begin{cases} \hat{\rho}_t + \mathbf{a}(x, \xi) \cdot D_x \hat{\rho} - b(x, \xi) D_\xi \hat{\rho} = 0 & \text{in } \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}, \\ \hat{\rho} = \rho^0 & \text{on } \mathbb{R}^N \times \mathbb{R} \times \{0\}. \end{cases}$$

Next for $\rho^0 \in C_b^\infty(\mathbb{R}^N \times \mathbb{R})$, $t_0 \geq 0$ and $y, \eta \in \mathbb{R}^N$ we consider the solution $\rho(\cdot, \cdot, y, \eta, \cdot; t_0)$ of (16) with initial datum, at $t = t_0$, $\rho^0(\cdot - y, \cdot - \eta)$ and introduce the ‘‘convolution’’ along characteristics given by

$$(18) \quad \rho \star \chi(y, \eta, t; t_0) := \int \rho(x, \xi, y, \eta, t; t_0) \chi(x, \xi, t) dx d\xi.$$

Combining (15) and (16) we find that, in the sense of distributions,

$$(19) \quad \frac{d}{dt} \rho \star \chi(y, \eta, t; t_0) = - \int \partial_\xi \rho(x, \xi, y, \eta, t; t_0) m(x, \xi, t) dx d\xi.$$

It follows that for all almost every $t \geq 0$ and $t_0 \geq 0$

$$(20) \quad \rho \star \chi(y, \eta, t) = \rho \star \chi^0(y, \eta, 0, t_0) - \int_0^{t_0} \int \partial_\xi \rho(x, \xi, y, \eta, t) m(x, \xi, s) dx d\xi ds.$$

The continuity of $u(t)$ at $t = 0$, as stated later, makes that this formula holds for $t_0 = 0$ and for all almost every $t \geq 0$.

Note that although the regularity of the path was used to derive (19) and (20), the actual statements do not need it and make sense for paths which are only continuous. Notice also that (19) and (20) are equivalent to the kinetic formulation when the measure m satisfies (14).

Pathwise stochastic entropy solutions. Neither the notion of entropy solution nor the kinetic formulation can be used to study (3), since both involve either inequalities or quantities with sign which do not make sense for equations/expressions with, in principle, are nowhere differentiable functions. We refer to [21, 22, 18, 19] for a general discussion about the difficulties encountered when attempting to use the classical weak solution approaches to study fully nonlinear stochastic pde.

Motivated by our previous work [15] as well as the theory of stochastic viscosity solutions ([21, 22, 18, 19]) we use (19) to introduce next the notion of pathwise stochastic entropy solutions for SSCL. Recall, as remarked previously, that the statement of (19) does not rely on any regularity of the paths.

We have:

Definition 2.1. *Assume (1), (2), (5) and (6). Then $u \in (L^1 \cap L^\infty)(\mathbb{R}^N \times (0, T))$, for all $T > 0$, is a pathwise stochastic entropy solution to (3), if there exists a nonnegative bounded measure m on $\mathbb{R}^N \times \mathbb{R} \times (0, \infty)$ satisfying (14) such that (19) and (20) hold for all ρ 's given by (18) with $\rho^0 \in C_b^\infty(\mathbb{R}^N \times \mathbb{R})$, almost all $t_0 \geq 0$ and almost everywhere in $t \geq t_0$.*

We show in the next two sections that the pathwise stochastic entropy solutions exist and satisfy a contraction in L^1 and, hence, are stable and unique.

3. SOME TECHNICAL RESULTS

We discuss here the class of test functions we use in the paper, recall the method of characteristics which provide (explicit) representation for the solutions to (16) and (17) and show what they imply for the special solutions we are considering, and, finally, state and prove a technical result.

The class of solutions. We use a family $(\rho_\varepsilon)_{\varepsilon>0}$ of test functions solving (16) with initial datum, at t_0 , $(\rho_\varepsilon^0)_{\varepsilon>0}$ that separates space, $x \in \mathbb{R}^d$, and velocity, $\xi \in \mathbb{R}$. A natural choice is convolution approximations to Dirac masses, that is we choose

$$(21) \quad \rho_\varepsilon^0(x - y, \xi - \eta) = \rho_\varepsilon^s(x - y)\rho_\varepsilon^v(\xi - \eta),$$

where, as $\varepsilon \rightarrow 0$ and in the sense of distributions,

$$(22) \quad \rho_\varepsilon^s \xrightarrow{\varepsilon \rightarrow 0} \delta(x - y) \text{ and } \rho_\varepsilon^v \xrightarrow{\varepsilon \rightarrow 0} \delta(\xi - \eta);$$

here we use the superscripts s and v to signify whether the initial data approximates the space or the velocity variables. Typical choices for ρ_ε^s and ρ_ε^v are

$$\rho_\varepsilon^s(x) = \varepsilon^{-N} \rho^s(x/\varepsilon) \text{ and } \rho_\varepsilon^v(\xi) = \varepsilon^{-1} \rho^v(\xi/\varepsilon),$$

for some smooth functions ρ^s and ρ^v with compact support of diameter 1 and such that $0 \leq \rho^s, \rho^v \leq 1$, $\rho^s(0) = \rho^v(0) = 1$ and $\int_{\mathbb{R}^N} \rho^s = \int_{\mathbb{R}} \rho^v = 1$.

The characteristics. Depending on the context we use the forward and backwards characteristics of (17) to construct the smoothing kernels ρ_ε and $\hat{\rho}_\varepsilon$, that is the solutions to (16) and (17) respectively.

Using the forward characteristics

$$(23) \quad \begin{cases} \dot{Y}_{(y,\eta)}(s) = \mathbf{a}(Y_{(y,\eta)}(s), \zeta_{(y,\eta)}(s)), & Y_{(y,\eta)}(0) = y, \\ \dot{\zeta}_{(y,\eta)}(s) = -b(Y_{(y,\eta)}(s), \zeta_{(y,\eta)}(s)), & \zeta_{(y,\eta)}(0) = \eta, \end{cases}$$

we generate the smoothing kernels $\hat{\rho}_\varepsilon$ and ρ_ε that satisfy, in the sense of distributions,

$$(24) \quad \hat{\rho}_\varepsilon(x, \xi, y, \eta, s) \xrightarrow{\varepsilon \rightarrow 0} \delta(x - Y_{(y,\eta)}(s))\delta(\xi - \zeta_{(y,\eta)}(s))$$

and

$$(25) \quad \rho_\varepsilon(x, \xi, y, \eta, t) \xrightarrow{\varepsilon \rightarrow 0} \delta(x - Y_{(y,\eta)}(W(t) - W(t_0)))\delta(\xi - \zeta_{(y,\eta)}(W(t) - W(t_0))).$$

We may also use the backward characteristics, that is the solution to

$$(26) \quad \begin{cases} \dot{X}_{(t,x,\xi)}(s) = -\mathbf{a}(X_{(t,x,\xi)}(s), \Xi_{(t,x,\xi)}(s)), & X_{(t,x,\xi)}(t) = x, \\ \dot{\Xi}_{(t,x,\xi)}(s) = b(X_{(t,x,\xi)}(s), \Xi_{(s,x,\xi)}(s)), & \Xi_{(t,x,\xi)}(t) = \xi. \end{cases}$$

and write

$$\hat{\rho}_\varepsilon(x, \xi, y, \eta, t) = \rho_\varepsilon^0(X_{(t,x,\xi)}(0) - y, \Xi_{(t,x,\xi)}(0) - \eta).$$

With an abuse of notation we will write $(X(0), \Xi(0))$ for $(X_{(W(t)-W(t_0),x,\xi)}(0), \Xi_{(W(t)-W(t_0),x,\xi)}(0))$ and

$$\rho_\varepsilon(x, \xi, y, \eta, t) = \rho_\varepsilon^s(X(0) - y)\rho_\varepsilon^v(\Xi(0) - \eta).$$

A technical result. We present now a technical fact which we will use in the next two sections. It concerns the behavior, as $\varepsilon \rightarrow 0$, of

$$(27) \quad \bar{q}_\varepsilon(x, \xi, t; t_0) := \int_{\mathbb{R}^N} \int_{\mathbb{R}} q_\varepsilon(x, \xi, y, \eta, t; t_0) dy d\eta,$$

where, for each $y \in \mathbb{R}^N$ and $\eta \in \mathbb{R}$, $q_\varepsilon(x, \xi, y, \eta, t; t_0)$ is the solution of (16) starting with initial datum

$$(28) \quad q_\varepsilon(x, \xi, y, \eta, t_0; t_0) := -\frac{1}{2}\rho_\varepsilon^s(x - y)\rho_\varepsilon^v(-\eta) + \rho_\varepsilon^s(x - y) \int_{-\infty}^{\xi} \rho_\varepsilon^v(\bar{\xi} - \eta)\rho_\varepsilon^v(-\eta) d\bar{\xi};$$

it is, of course, immediate that $\bar{q}(\cdot, \cdot, \cdot; t_0)$ is the solution to (16) with initial datum

$$\bar{q}_\varepsilon(x, \xi, t_0; t_0) = -\frac{1}{2} + \int_{-\infty}^{\xi} \rho_\varepsilon^v(\bar{\xi} - \eta) \rho_\varepsilon^v(-\eta) d\bar{\xi} d\eta.$$

The result is:

Lemma 3.1. *Assume (1), (2), (5) and (6) and, for $t_0 \geq 0$ and $\varepsilon > 0$, let $\bar{q}_\varepsilon(\cdot, \cdot, \cdot; t_0)$ be given by (27) where, for each $y \in \mathbb{R}^N$ and $\eta \in \mathbb{R}$, $q_\varepsilon(\cdot, \cdot, y, \eta, \cdot; t_0)$ is the solution of (16) satisfying (28). As $\varepsilon \rightarrow 0$, for all times t and a.e. in (x, ξ) , $\bar{q}_\varepsilon(x, \xi, t; t_0) \rightarrow \frac{1}{2} \text{sign}(\xi)$ and, thus, in $L_{loc}^p(\mathbb{R}^N \times \mathbb{R})$ for all $p \in [1, \infty)$.*

Proof. In view of the relationship between the solutions to (16) and (17), it suffices to prove the claim for the solution \hat{q}_ε of (17) with $\hat{q}_\varepsilon(x, \xi, y, \eta, 0)$ as in (28).

For each fixed t , in view of the discussion in the previous paragraph, we have

$$\begin{aligned} \int_{\mathbb{R}^N} \int_{\mathbb{R}} \hat{q}_\varepsilon(x, \xi, y, \eta, t) dy d\eta &= \int_{\mathbb{R}^N} \int_{\mathbb{R}} \hat{q}_\varepsilon(X_{(t,x,\xi)}(0), \Xi_{(t,x,\xi)}(0), y, \eta, 0) dy d\eta \\ &= -\frac{1}{2} + \int_{\mathbb{R}} \int_{-\infty}^{\Xi_{(t,x,\xi)}(0)} \rho_\varepsilon^v(\bar{\xi} - \eta) \rho_\varepsilon^v(-\eta) d\bar{\xi} d\eta \\ &\xrightarrow{\varepsilon \rightarrow 0} -\frac{1}{2} + \mathbf{1}_{\{\Xi_{(t,x,\xi)}(0) > 0\}} = -\frac{1}{2} + \mathbf{1}_{\{\xi > 0\}}, \end{aligned}$$

the last equality being a consequence of (5) and the uniqueness of the system of the characteristics. \square

4. THE STABILITY AND UNIQUENESS OF PATHWISE STOCHASTIC ENTROPY SOLUTIONS

The result. The first result in this note, which is the L^1 contraction property and, hence, the intrinsic uniqueness of the pathwise stochastic entropy solutions, is stated next.

Theorem 4.1. *Assume (1), (2), (5), (6) and $u^0 \in (L^1 \cap L^\infty)(\mathbb{R}^N)$. There exists at most one pathwise stochastic entropy solution which is continuous at $t = 0$ with values in $L^1(\mathbb{R}^N)$ $u \in L^\infty((0, T); L^1 \cap L^\infty(\mathbb{R}^N))$ for all $T > 0$, to (3). In addition any two pathwise stochastic entropy solutions u_1, u_2 satisfy, for almost all $t > 0$, the “contraction” property*

$$(29) \quad \|u_2(\cdot, t) - u_1(\cdot, t)\|_1 \leq \|u_2^0 - u_1^0\|_1.$$

The proof. When W is smooth, the proof of (29) is based on considering, for two solutions u_1 and u_2 and $\chi^{(i)}(x, \xi, t) = \chi(u_i(x, t); \xi)$,

$$F(t) := \int [|\chi^{(1)}(x, \xi, t)| + |\chi^{(2)}(x, \xi, t)| - 2\chi^{(1)}(x, \xi, t)\chi^{(2)}(x, \xi, t)] dx d\xi = \|u_1(t) - u_2(t)\|_1$$

and showing that $dF/dt \leq 0$, which yields the contraction property

$$F(t_2) \leq F(t_1) \text{ for almost all } 0 \leq t_1 \leq t_2.$$

This is, however, not possible for the paths we are considering in this note and instead we need to modify the integrands above in order to make use of the defining properties of stochastic entropy solutions. In particular we need to replace $|\chi(u_i(x, t), \xi)| = \text{sign}(\xi) \chi(u_i(x, t), \xi)$ by the “convolution” along characteristics (18) for an appropriate choice of ρ , which is going to be the $q_\varepsilon(\cdot, \cdot, \cdot; t_0)$ in (27). The reason we need to use t_0 is that the approximation creates errors which can be controlled by the oscillations of the path W . It is therefore necessary to discretize in time and, hence, to use q_ε ’s starting at different times and to add the errors.

We present now the

The proof of Theorem 4.1. Since the proof is long and technical we divide it in several steps some of which, although formally true, require arguments similar to the ones used for Lemma 3.1 to be justified.

The general set up. Fix $T > 0$. We begin with a regularization of the functional F above. For $t_0 \geq 0$, $\varepsilon > 0$ small, for $t \geq t_0$ and q_ε as in Lemma 3.1, let

$$F_{t_0, \varepsilon}(t) := \int [q_\varepsilon \star \chi^{(1)}(y, \eta, t) + q_\varepsilon \star \chi^{(2)}(y, \eta, t) - 2\rho_\varepsilon \star \chi^{(1)}(y, \eta, t) \rho_\varepsilon \star \chi^{(2)}(y, \eta, t)] dy d\eta.$$

It follows from the a.e. continuity in time of the F as well as (25) that, for almost every $t \in [0, T]$ as $\varepsilon \rightarrow 0$,

$$(30) \quad F_{t_0, \varepsilon}(t) \rightarrow F(t).$$

For $h > 0$, let $\omega(h)$ denote the oscillation of the path W over time intervals of size h , that is

$$(31) \quad \omega(h) = \sup_{0 \leq s \leq h, 0 \leq t \leq T} |W(t+s) - W(t)|.$$

Fix s, t so that $0 \leq s \leq t \leq T$ and let $\Delta = \{s = t_0 \leq t_1 \leq \dots \leq t_M = t\}$ be a partition of $[s, t]$ with mesh size $h = t_{i+1} - t_i$ and such that (30) holds for all $t_i \in \Delta$. The conclusion follows if we show that there exist some $C > 0$ and, for each $i = 0, \dots, M-1$, $A_{t_i} = A_{t_i}(h, \varepsilon) > 0$ and $C_i > 0$ such that

$$(32) \quad F_{t_i, \varepsilon}(t_{i+1}) - F_{t_i, \varepsilon}(t_i) \leq C_i A_{t_i}(h, \varepsilon), \quad A_{t_i}(h, \varepsilon) \leq C\omega(h) \quad \text{and} \quad \sum_{i=0}^{M-1} C_i \leq C.$$

Indeed if (32) holds, then

$$\begin{aligned} F(t) - F(s) &= \sum_{i=0}^{M-1} [F(t_{i+1}) - F(t_i)] = \lim_{\varepsilon \rightarrow 0} \sum_{i=0}^{M-1} [F_{t_i, \varepsilon}(t_{i+1}) - F_{t_i, \varepsilon}(t_i)] \\ &\leq \limsup_{\varepsilon \rightarrow 0} \sum_{i=0}^{M-1} A_{t_i}(h, \varepsilon) \leq C\omega(h). \end{aligned}$$

In view of (14), to prove (32) it suffices to show that, for each $t_0 \geq 0$ and $h, \varepsilon > 0$, there exists $C > 0$ and $A = A(h, \varepsilon) > 0$ such that $A(h, \varepsilon) \leq C\omega(h)$ and

$$F_{t_0, \varepsilon}(t_0 + h) - F_{t_0, \varepsilon}(t_0) \leq A(h, \varepsilon) \sup_{\xi \in \mathbb{R}} \int_{t_0}^{t_0+h} \int_{\mathbb{R}^N} [m^{(1)}(x, \xi, t) + m^{(2)}(x, \xi, t)] dx dt,$$

and for the last claim it is enough to obtain a suitable upper bound for $dF_{t_0, \varepsilon}/dt$ in $(t_0, t_0 + h)$. Notice that this can be written only for continuity times of the right hand side, which include $t_0 = 0$ and this is enough for our purpose.

Since the necessary upper bound for $dF_{t_0, \varepsilon}/dt$ requires some long and tedious calculations, we divide the computations and estimates in two parts. The first is about the derivative of the product term and the second about the single terms, which we group together. In what follows, to keep the notation simple, we assume that $t_0 = 0$, write F_ε instead of $F_{0, \varepsilon}$ and omit the dependence of the solutions of (16) and (17) on the initial time.

The product term in $\frac{d}{dt}F_\varepsilon$. Using the continuity of the u_i 's in time we find

$$(33) \quad \begin{cases} -\frac{d}{dt}[\rho_\varepsilon \star \chi^{(1)}(y, \eta, t)\rho_\varepsilon \star \chi^{(2)}(y, \eta, t)] = \\ \int \left[\partial_\xi \rho_\varepsilon(x, \xi, y, \eta, t)m^{(1)}(x, \xi, t)\rho_\varepsilon(x', \xi', y, \eta, t)\chi^{(2)}(x', \xi', t) + \right. \\ \left. \int \rho_\varepsilon(x, \xi, y, \eta, t)\chi^{(1)}(x, \xi, t)\partial_{\xi'} \rho_\varepsilon(x', \xi', y, \eta, t)m^{(2)}(x', \xi', t) \right] dx d\xi dx' d\xi'. \end{cases}$$

We focus next on the first term in the right hand side above since the other is handled similarly.

The classical proof of uniqueness for smooth paths uses $\rho^s(x-y)\rho^s(x'-y)\rho^v(\xi-\eta)\rho^v(\xi'-\eta)$ in place of $\rho(x, \xi, y, \eta, t)\rho_\varepsilon(x', \xi', y, \eta, t)$. As a result, instead of $\partial_\xi \rho(x, \xi, y, \eta, t)\rho_\varepsilon(x', \xi', y, \eta, t)$, the integrand has $\partial_\xi \rho_\varepsilon^v(\xi-\eta)\rho_\varepsilon^v(\xi'-\eta')$, and, hence, it is easy to interchange ξ and ξ' derivatives to obtain cancellations. This is, however, not the case for ‘‘rough’’ paths and we need to introduce the appropriate derivatives, a fact that gives rise to additional terms and error terms that need to be approximated.

After integrating with respect to y and η , introducing the needed terms and integrating by parts, we find (note that the integrals below are in $dy d\eta dx d\xi dx' d\xi'$ but for notational simplicity we omit this dependence):

$$\begin{aligned} & \int m^{(1)}(x, \xi, t) \chi^{(2)}(x', \xi', t) \frac{\partial}{\partial \xi} \rho_\varepsilon(x, y, \xi, \eta, t) \rho_\varepsilon(x', \xi', y, \eta) \\ &= - \int m^{(1)}(x, \xi, t) \chi^{(2)}(x', \xi', t) \rho_\varepsilon(x, \xi, y, \eta, t) \frac{\partial}{\partial \xi'} \rho_\varepsilon(x', \xi', y, \eta, t) + \text{Err}_\varepsilon^{(1,1)}(t) \\ &= \int m^{(1)}(x, \xi, t) \rho_\varepsilon(x, \xi, y, \eta, t) \rho_\varepsilon(x', \xi', y, \eta, t) [\delta(\xi') - \delta(\xi' - u_2(x', t))] + \text{Err}_\varepsilon^{(1,1)}(t) \\ &\leq \int m^{(1)}(x, \xi, t) \rho_\varepsilon(x, \xi, y, \eta, t) \rho_\varepsilon^v(-\eta) dy d\eta dx d\xi + \text{Err}_\varepsilon^{(1,1)}(t) + \text{Err}_\varepsilon^{(2,1)}(t). \end{aligned}$$

Repeating this calculation for the other term in the right hand side of (33) and after adding the two inequalities, we get

$$\begin{aligned} & -\frac{d}{dt} [\rho_\varepsilon \star \chi^{(1)}(y, \eta, t) \rho_\varepsilon \star \chi^{(2)}(y, \eta, t)] \\ & \leq \int [m^{(1)}(x, \xi, t) + m^{(2)}(x, \xi, t)] \rho_\varepsilon(x, \xi, y, \eta, t) \rho_\varepsilon^v(-\eta) dy d\eta dx d\xi + \text{Err}_\varepsilon^{(1)}(t) + \text{Err}_\varepsilon^{(2)}(t), \end{aligned}$$

with

$$(34) \quad \begin{aligned} \text{Err}_\varepsilon^{(1)}(t) &:= \int [m^{(1)}(x, \xi, t) \chi^{(2)}(x', \xi', t) + m^{(2)}(x', \xi', t) \chi^{(1)}(x, \xi, t)] \\ & [\partial_\xi \rho_\varepsilon(x, \xi, y, \eta, t) \rho_\varepsilon(x', \xi', y, \eta, t) + \rho_\varepsilon(x, \xi, y, \eta, t) \partial_{\xi'} \rho_\varepsilon(x', \xi', y, \eta)] dy d\eta dx d\xi dx' d\xi', \end{aligned}$$

and

$$(35) \quad \begin{aligned} \text{Err}_\varepsilon^{(2)}(t) &:= \int m^{(1)}(x, \xi, t) \rho_\varepsilon(x, \xi, y, \eta, t) [\int \rho_\varepsilon(x', 0, y, \eta, t) dx' - \rho_\varepsilon^v(-\eta)] dy d\eta dx d\xi + \\ & \int m^{(2)}(x', \xi', t) \rho_\varepsilon(x', \xi', y, \eta, t) [\int \rho_\varepsilon(x, 0, y, \eta, t) dx - \rho_\varepsilon^v(-\eta)] dy d\eta dx' d\xi'. \end{aligned}$$

The other terms in $\frac{d}{dt}F_\varepsilon$. Since, for each y and η , $q_\varepsilon(x, \xi, y, \eta, t)$ solves (16), for $i = 1, 2$ we have

$$\frac{d}{dt} q_\varepsilon \star \chi^{(i)}(y, \eta, t) = - \int \partial_\xi q_\varepsilon(x, \xi, y, \eta, t) m^{(i)}(x, \xi, t) dx d\xi.$$

The upper bound $\frac{d}{dt}F_\varepsilon$. Combining the previous two steps, we conclude that, for $0 \leq t \leq h$,

$$(36) \quad \frac{d}{dt} F_\varepsilon(t) \leq \text{Err}_\varepsilon^{(1)}(t) + \text{Err}_\varepsilon^{(2)}(t) + \text{Err}_\varepsilon^{(3)}(t),$$

with

$$(37) \quad \text{Err}_\varepsilon^{(3)}(t) := \int [m^{(1)}(x, \xi, t) + m^{(2)}(x, \xi, t)] [\rho_\varepsilon(x, \xi, y, \eta, t) \rho_\varepsilon^v(-\eta) - \partial_\xi q_\varepsilon(x, \xi, y, \eta, t)] dy dx d\xi d\eta.$$

The estimate of the error terms. We follow the same strategy to estimate the three error terms. For $0 \leq \tau \leq h$ and $i = 1, 2, 3$, we observe that

$$\int_0^\tau |\text{Err}_\varepsilon^{(i)}(t)| dt \leq A^{(i)}(\Delta t, \varepsilon) \sup_{\xi \in \mathbb{R}} \int_0^\tau \int [m^{(1)}(x, \xi, t) + m^{(2)}(x, \xi, t)] dx dt,$$

where

$$A^{(i)}(h, \varepsilon) := \int \sup_{0 \leq t \leq h, x \in \mathbb{R}^N} B_\varepsilon^{(i)}(x, \xi, t) d\xi,$$

and

$$B_\varepsilon^{(1)} := \int \left| \int [\partial_\xi \rho_\varepsilon(x, \xi, y, \eta, t) \rho_\varepsilon(x', \xi', y, \eta, t) + \rho_\varepsilon(x, \xi, y, \eta, t) \partial_\xi' \rho_\varepsilon(x', \xi', y, \eta)] dy d\eta \right| dx' d\xi',$$

$$B_\varepsilon^{(2)} := \int \rho_\varepsilon(x, \xi, y, \eta, t) \left| \int \rho_\varepsilon(x', 0, y, \eta, t) dx' - \rho_\varepsilon^v(-\eta) \right| dy d\eta,$$

and

$$B_\varepsilon^{(3)} := \left| \int \left[\int \rho_\varepsilon(x, \xi, y, \eta, t) \rho_\varepsilon^v(-\eta) d\eta - \int \partial_\xi q_\varepsilon(x, \xi, y, \eta, t) d\eta \right] dy \right|.$$

Note that, either by construction or by elementary property of the convolutions,

$$A^{(1)}(0, \varepsilon) = A^{(2)}(0, \varepsilon) = A^{(3)}(0, \varepsilon) = 0.$$

We show next, that there exists $C > 0$ such that, for each $i = 1, 2, 3$,

$$(38) \quad A^{(i)}(h, \varepsilon) \leq C\omega(h).$$

Since it is the simplest, we begin with estimate for $B_\varepsilon^{(2)}$ and then study the ones for $B_\varepsilon^{(1)}$ and $B_\varepsilon^{(3)}$.

The estimate of $B_\varepsilon^{(2)}$. Recall $\rho_\varepsilon(x', 0, y, \eta, t; 0) = \hat{\rho}(x', 0, y, \eta, W(t))$. Since $b(x, 0) = 0$, $\hat{\rho}(x', 0, y, \eta, s)$ solves

$$\begin{cases} \partial_s \hat{\rho}_\varepsilon(x', 0, y, \eta, s) + a(x', 0) D_{x'} \hat{\rho}_\varepsilon(x', 0, y, \eta, s) = 0 & \text{in } \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}, \\ \hat{\rho}_\varepsilon(x', 0, y, \eta, 0) = \rho_\varepsilon^s(x' - y) \rho_\varepsilon^v(-\eta). \end{cases}$$

Let Q be defined by

$$\int \hat{\rho}_\varepsilon(x', 0, y, \eta, s) dx' = \rho_\varepsilon^v(-\eta) Q(y, s).$$

It is then immediate that

$$\begin{cases} \partial_s Q(y, s) = - \int_{x'} a(x', 0) \nabla_{x'} \hat{\rho}_\varepsilon(x', 0, y, \eta, s) dx' = \int_{x'} \text{div}_x a(x', 0) \hat{\rho}_\varepsilon(x', 0, y, \eta, s) dx', \\ Q(y, 0) \equiv 1. \end{cases}$$

As a result we find, for some $C > 0$,

$$|\partial_s Q(y, s)| \leq C \int_{\mathbb{R}^N} \hat{\rho}_\varepsilon(x', 0, y, \eta, s) dx' = CQ(y, s),$$

and, thus,

$$|Q(y, s) - 1| \leq Cs.$$

Inserting this inequality in the definition of $B_\varepsilon^{(2)}$, we find

$$B_\varepsilon^{(2)} = \int \rho_\varepsilon(x, \xi, y, \eta, t) \rho_\varepsilon^v(-\eta) |Q(y, W(t)) - 1| dy d\eta \leq C\omega(h) \int \rho_\varepsilon(x, \xi, y, \eta, t) \rho_\varepsilon^v(-\eta) dy d\eta.$$

We claim that, for some other $C > 0$,

$$(39) \quad \int \sup_{(x,t) \in \mathbb{R}^N \times (0,h)} B_\varepsilon^{(2)}(x, \xi, t) d\xi \leq C\omega(h).$$

The formal reason is that $p_\varepsilon =: \int \rho_\varepsilon(x, \xi, y, \eta, t) \rho_\varepsilon^v(-\eta) dy d\eta$ solves (16) with initial datum

$$\int \rho_\varepsilon^s(x - y) dy \int \rho_\varepsilon^v(\xi - \eta) \rho_\varepsilon^v(-\eta) d\eta = \int \rho_\varepsilon^v(\xi - \eta) \rho_\varepsilon^v(-\eta) d\eta \approx \delta(\xi).$$

Next we make the above rigorous. As before, since $p_\varepsilon(x, \xi, t) = \hat{p}_\varepsilon(x, y, W(t))$, we work with \hat{p}_ε which solves (17) and $\hat{p}_\varepsilon(x, \xi, 0) = \int \rho_\varepsilon^v(\xi - \eta) \rho_\varepsilon^v(-\eta) d\eta$.

The fact that ρ_ε^v is supported in $[-\varepsilon, \varepsilon]$ implies that

$$\hat{p}_\varepsilon(\cdot, \cdot, 0) \text{ is supported in } \mathbb{R}^N \times [-2\varepsilon, 2\varepsilon] \text{ and } \hat{p}_\varepsilon(\cdot, \xi, 0) \leq \varepsilon^{-2}(2\varepsilon - |\xi|)_+.$$

In particular, since, for $t \in (0, h)$, $\hat{p}_\varepsilon(x, \xi, t) = \int \rho_\varepsilon^v(\Xi(t, x, \xi)(0) - \eta) \rho_\varepsilon^v(-\eta) d\eta$, we must have

$$\Xi(t, x, \xi)(0) \in [-2\varepsilon, 2\varepsilon] \text{ for all } (x, t) \in \mathbb{R}^N \times (0, h).$$

This It follows from the ode satisfied by Ξ and (5) that $|\dot{\Xi}| \leq C\varepsilon$ for some $C > 0$. This gives some other uniform $C > 0$ such that, for all $(x, t) \in \mathbb{R}^N \times (0, h)$,

$$\hat{p}_\varepsilon(x, \xi, t) \leq \varepsilon^{-2}(C\varepsilon - |\xi|)_+.$$

Integrating with respect to ξ yields the claim.

The estimate of $B_\varepsilon^{(1)}$. To prove (38) for $A_\varepsilon^{(1)}$ we observe that the properties of ρ_ε^s and ρ_ε^v give the following sequence of equalities:

$$\begin{aligned} & \int [\partial_\xi \rho_\varepsilon(x, y, \xi, \eta, t) \rho_\varepsilon(x', \xi', y, \eta) + \rho_\varepsilon(x, y, \xi, \eta, t) \partial_\xi \rho_\varepsilon(x', \xi', y, \eta)] dy d\eta \\ &= \int [\partial_\xi X \cdot D\rho_\varepsilon^s(X - y) \rho_\varepsilon^v(\Xi - \eta) \rho_\varepsilon^s(X' - y) \rho_\varepsilon^s(\Xi' - \eta) + \rho_\varepsilon^s(X - y) \rho_\varepsilon^s(\Xi - \eta) \partial_\xi X' \cdot D\rho_\varepsilon^s(X' - y) \rho_\varepsilon^v(\Xi' - \eta)] dy d\eta \\ &+ \int [\rho_\varepsilon^s(X - y) \partial_\xi \Xi \cdot D\rho_\varepsilon^v(\Xi - \eta) \rho_\varepsilon^s(X' - y) \rho_\varepsilon^s(\Xi' - \eta) + \rho_\varepsilon^s(X - y) \rho_\varepsilon^s(\Xi - \eta) \rho_\varepsilon^s(X' - y) \partial_\xi \Xi' \cdot D\rho_\varepsilon^v(\Xi' - \eta)] dy d\eta. \end{aligned}$$

The integrands in the two terms behave similarly, hence, here, we study the first one. To this end, we first notice that $\partial_\xi X$ been independent of y and η can be factored out of the integral, while

$$\begin{aligned} & \int_{\mathbb{R}^N} [D\rho_\varepsilon^s(X - y) \rho_\varepsilon^v(\Xi - \eta) \rho_\varepsilon^s(X' - y) \rho_\varepsilon^s(\Xi' - \eta) + \rho_\varepsilon^s(X - y) \rho_\varepsilon^s(\Xi - \eta) D\rho_\varepsilon^s(X' - y) \rho_\varepsilon^v(\Xi' - \eta)] dy = \\ &= - \int_{\mathbb{R}^N} D_y [\rho_\varepsilon^s(X - y) \rho_\varepsilon^s(X' - y)] \rho_\varepsilon^v(\Xi - \eta) \rho_\varepsilon^s(\Xi' - \eta) dy = 0. \end{aligned}$$

Therefore the term we need to estimate is

$$[\partial_\xi X - \partial_\xi X'] \cdot \int D\rho_\varepsilon^s(X - y) \rho_\varepsilon^v(\Xi - \eta) \rho_\varepsilon^s(X' - y) \rho_\varepsilon^s(\Xi' - \eta) dy d\eta,$$

and, in norm, is controlled by

$$|\partial_\xi X - \partial_\xi X'| \int |D\rho_\varepsilon^s(X - y)| \rho_\varepsilon^v(\Xi - \eta) \rho_\varepsilon^s(X' - y) \rho_\varepsilon^s(\Xi' - \eta) dy d\eta.$$

It follows that

$$A(h, \varepsilon) \leq \sup_{\mathcal{A}} |\partial_\xi X - \partial_\xi X'| \int |D\rho_\varepsilon^s(X - y)| \rho_\varepsilon^v(\Xi - \eta) \rho_\varepsilon^s(X' - y) \rho_\varepsilon^s(\Xi' - \eta) dy d\eta dx' d\xi',$$

where

$$\mathcal{A} =: \{(x, \xi, t) \in \mathbb{R}^N \times \mathbb{R} \times (0, h] : |X(0) - X'(0)| \leq C\varepsilon, |\Xi(0) - \Xi'(0)| \leq C\varepsilon\}.$$

Since $dx' d\xi' = dX' d\xi'$, there exists some $C > 0$ so that

$$\int |D\rho_\varepsilon^s(X - y)| \rho_\varepsilon^v(\Xi - \eta) \rho_\varepsilon^s(X' - y) \rho_\varepsilon^s(\Xi' - \eta) dy d\eta dx' d\xi' = \int |D\rho_\varepsilon^s(X - y)| \rho_\varepsilon^v(\Xi - \eta) dy d\eta \leq C/\varepsilon,$$

and the estimate is reduced to showing that

$$\sup_{\mathcal{A}} |\partial_\xi X - \partial_\xi X'| C/\varepsilon \leq \omega(h).$$

The last inequality follows from a cancellation property proved in [17], which yields that the characteristics by (26) satisfy, with variable $s = W(t)$,

$$|\partial_\xi X(s) - \partial_\xi X'(s)| \leq Cs\varepsilon,$$

whenever the data at $s = 0$ are ε close.

The estimate of $B_\varepsilon^{(3)}$. Let $p(x, \xi, t) := \int q_\varepsilon(x, \xi, y, \eta, t) d\eta dy$ and $r(x, \xi, t) = \partial_\xi p(x, \xi, t)$. As before, $p(x, \xi, t) = \hat{p}(x, \xi, W(t))$ and $r(x, \xi, t) = \hat{r}(x, \xi, W(t))$ with \hat{p} and \hat{r} satisfying

$$\hat{p}_t + \mathbf{a}(x, \xi) \cdot D_x \hat{p} - b(x, \xi) D_\xi \hat{p} = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R} \times (0, \infty),$$

and

$$\hat{r}_t + \mathbf{a}(x, \xi) \dot{D}_x \hat{r} - b(x, \xi) D_\xi \hat{r} = \mathbf{a}_\xi(x, \xi) D_x \hat{p} + b(x, \xi) D_\xi \hat{p} \quad \text{in } \mathbb{R}^N \times \mathbb{R} \times (0, \infty),$$

while

$$\hat{r}(x, \xi, 0) = \int \rho_\varepsilon^v(\xi - \eta) \rho_\varepsilon^v(-\eta) d\eta \approx \delta(\xi).$$

The main observation is again that, since $R_\varepsilon(\xi) \approx \delta(\xi)$, in view of (5), \hat{r} stays localized near $\xi = 0$ and this allows to control the right hand side.

Indeed let \hat{R}_ε be the solution to (17) with initial datum (at $t = 0$) $\hat{r}(x, \xi, 0)$. Then we may rewrite

$$B_\varepsilon^{(3)} = \left| \int \sup_{x, t} \Delta r(x, \xi, t) d\xi \right|$$

where $\Delta r(x, \xi, t) = \widehat{\Delta} r(x, \xi, W(t))$ and $\widehat{\Delta} r(x, \xi, t) = \hat{R}_\varepsilon(x, \xi, t) - r(x, \xi, t)$ solves

$$\begin{cases} \widehat{\Delta} r_t + \mathbf{a}(x, \xi) \cdot D_x \widehat{\Delta} r - b(x, \xi) D_\xi \widehat{\Delta} r = \mathbf{a}_\xi(x, \xi) \cdot D_x \hat{p} + b_\xi(x, \xi) D_\xi \hat{p} & \text{in } \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}, \\ \widehat{\Delta} r(x, \xi, 0) = 0. \end{cases}$$

To estimate the right hand side we argue as follows. As in the previous step, we observe that the support of $p(x, \xi, t)$ in ξ remains in an interval of diameter $C\varepsilon$. Since (5) gives $b_x(x, 0) = 0$, Gronwall's lemma applied to

$\dot{\Xi}_x(s) = -b_x(X(s), \Xi(s)) X_x(s) - b_\xi(X(s), \Xi(s)) \Xi_x(s) = O(\varepsilon) + b_\xi((X(s), \Xi(s))) \Xi_x(s)$ with $\Xi_x(0) = 0$, yields

$$|\Xi_x(s)| \leq C\varepsilon s.$$

Finally, in view of the representation formula of \hat{p} , using arguments similar to the ones employed for the other two error estimates, we find that $D_x \hat{p}(x, \xi, t) = \hat{r}_\varepsilon(X(0), \Xi(0), 0) \Xi_x(0) = O(\varepsilon s)/\varepsilon = O(s)$ and $D_\xi \hat{p}(x, \xi, t) = \hat{r}_\varepsilon(X(0), \Xi(0), 0) \Xi_\xi(0) = O(1)$. □

5. THE EXISTENCE OF STOCHASTIC ENTROPY SOLUTIONS

The general strategy of the existence. We construct pathwise stochastic entropy solutions as the limit of solutions to (3) with smooth paths. Indeed we consider a family of approximate problems using smooth local uniform approximations W_ε of W , that is paths $W_\varepsilon \in C^1(\mathbb{R})$ such that, as $\varepsilon \rightarrow 0$ and for every $T > 0$, $W_\varepsilon \rightarrow W$ uniformly on $[0, T]$. The conservation law (3) with W_ε in place of W has a unique entropy solution u_ε and its kinetic formulation is

$$(40) \quad \begin{cases} \partial_t \chi_\varepsilon + \operatorname{div}_x[\mathbf{a}(x, \xi) \chi_\varepsilon] W'_\varepsilon - \operatorname{div}_\xi[b(x, \xi) \chi_\varepsilon] W'_\varepsilon = \partial_\xi m_\varepsilon & \text{in } \mathbb{R}^N \times \mathbb{R} \times (0, \infty), \\ \chi_\varepsilon = \chi(u^0(\cdot), \cdot) & \text{on } \mathbb{R}^N \times \mathbb{R} \times \{0\}, \end{cases}$$

where the measure m_ε satisfies (14) and $\chi_\varepsilon(x, \xi, t) = \chi(u_\varepsilon(x, t), \xi)$ with the notation (12).

As it was already discussed in section 2, a fairly general theory in L^1 for (3) with smooth paths was developed in [3]. To prove, however, that the u_ε 's converge, as $\varepsilon \rightarrow 0$, to a pathwise stochastic entropy solution we need to work in $L^1 \cap L^\infty$ and to obtain the, uniform in ε , a priori estimates

$$(41) \quad \|u_\varepsilon(t)\|_1 \leq \|u^0\|_1 \quad \text{and} \quad \|u_\varepsilon(t)\|_\infty \leq K(t),$$

for K such that $\sup_{t \in [0, T]} K(t) \leq K_T < \infty$ for each $T > 0$.

Once (41) is established then the bounds (14) on the measures m_ε are also uniform in ε .

As mentioned earlier, the L^1 bound always holds true. But the L^∞ estimate is more difficult to establish. Some examples were given in section 2 and a new argument, that uses (6), is presented at the end of this section.

The limiting process. In what follows we fix a final time $T > 0$ and work on $(0, T)$. We follow the construction by weak limits proposed in Perthame [25, 26] which is based on the kinetic formulation and is an alternative to the construction by Young measures in Di Perna [6]. With the above bounds, we can extract subsequences such, as $\varepsilon \rightarrow 0$,

$$u_\varepsilon \rightharpoonup u, \quad \chi_\varepsilon \rightharpoonup f \quad \text{and} \quad m_\varepsilon \rightharpoonup m \quad \text{in } L^\infty(\text{resp. } M^1) \text{ weak-}\star$$

with u , f and m also satisfying (14) and (41).

In addition some elementary structural properties of the nonlinear function χ in ξ pass to the weak limit and give

$$(42) \quad \operatorname{sgn} f(x, \xi, t) = \operatorname{sgn}(\xi), \quad |f(x, \xi, t)| \leq 1 \quad \text{and} \quad \partial_\xi f(x, \xi, t) = \delta(\xi) - \nu(x, \xi, t),$$

where $\nu(x, \xi, t) \geq 0$ is the Young measure associated with the weak- \star limit of u_ε .

Passing to the weak limit in the definition of pathwise stochastic entropy solutions for (40) and using test function $\rho_\varepsilon(x, \xi, t) = \hat{\rho}(x, \xi, W_\varepsilon(t) - W_\varepsilon(t_0))$ we also get

$$(43) \quad \begin{cases} df + \operatorname{div}_x[\mathbf{a}(x, \xi) f] \circ dW(t) - \operatorname{div}_\xi[b(x, \xi) f] \circ dW(t) = \partial_\xi m dt & \text{in } \mathbb{R}^N \times \mathbb{R} \times (0, \infty), \\ f = \chi(u^0(\cdot), \cdot) & \text{on } \mathbb{R}^N \times \mathbb{R} \times \{0\}. \end{cases}$$

The definition of pathwise solutions apply to this equation following the lines of Section 2.

The existence result. The existence theorem is:

Theorem 5.1. *Assume (1), (2), (5), (6) and $u^0 \in (L^1 \cap L^\infty)(\mathbb{R}^N)$. There exists pathwise stochastic entropy solution of (43) satisfying (14), (41) and (42), it is such that $f(x, \xi, t) = \chi(u(x, t), \xi)$, for $u(x, t) = \int_{\mathbb{R}} f(x, \xi, t) d\xi$ with $u \in L^\infty((0, T); L^1 \cap L^\infty(\mathbb{R}^N))$ for all $T > 0$ and $u(t)$ is continuous at $t = 0$.*

Proof. The existence of u_ε has been already discussed and the L^∞ bound is proved below. Therefore, we may pass to the limit as indicated above and obtain the pathwise stochastic entropy solution f of (43). It remains to prove that $f(x, \xi, t) = \chi(u(x, t), \xi)$.

Consider the functional

$$G(t) := \int [|f(x, \xi, t)| - f^2(x, \xi, t)] dx d\xi$$

in place of $F(\cdot)$ in Section 4 and we note that $G(0) = 0$ because $f(x, \xi, t = 0) = \chi(x, \xi, 0)$. Following the same proof, we find that $G(t) \leq G(0) = 0$ and since $|f| \leq 1$ we conclude that $G \equiv 0$ and thus f takes only the values 0 or 1. In other words, thanks to (42), f is an indicator function $\chi(u(x, t), \xi)$.

The continuity at time $t = 0$ follows from a similar procedure, see [26], Prop. 4.1.7. For a sequence $t_n \rightarrow 0$, there is a weak limit $g(x, \xi)$, satisfying (42), of $\chi(x, \xi, t_n)$ and by the definition of pathwise stochastic solutions $g \leq \chi(u^0(x), \xi)$. This means that $u(t_n)$ converges to u^0 strongly. \square

The L^∞ bound. For $u^0 \in L^\infty(\mathbb{R}^N)$ and $M = \|u^0\|_\infty$, we build, using the method of characteristics, a local in time smooth solution U of

$$(44) \quad \begin{cases} U_t + \operatorname{div} \mathbf{A}(x, U) = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}, \\ U = M & \text{on } \mathbb{R} \times \{0\}. \end{cases}$$

We show below that this smooth solution exists in $\mathbb{R}^N \times [-\tau, \tau]$ for some $\tau > 0$ which does not depend on the size of M . It then follows from the contraction property that $|u_\varepsilon(x, t)| \leq U(x, W_\varepsilon(t))$ as long as $W_\varepsilon \in [-\tau, \tau]$. In view of the uniform continuity of the paths, this last statement holds for $t \in [0, \tau^*]$ with τ^* depending only on τ and the modulus of continuity of the paths. Then we iterate the argument departing from the constant $\sup_{s \in [-\tau, \tau]} \|U(\cdot, s)\|_\infty$ and built the smooth large solution on $(\tau, 2\tau)$, $(2\tau, 3\tau)$, etc.. After a finite number of steps of order $(T/\tau^* + 1)$, we reach the final time and obtain a uniform bound.

To construct the smooth solution on $[-\tau, \tau]$ we argue as follows. Departing from a constant $U(x, 0) = M$, the smooth solution of (44) is built by the method of characteristics (see (23)) as long as the latter do not intersect when departing from $Y(0) = x$ and $\zeta(0) = M$ and this is possible as long as $\partial_x Y(t)$ is invertible. Notice that $(\partial_x Y, \partial_x \zeta)$ solves a system of differential equations (the linearization of (23) along (Y, ζ)) with coefficients which, in view (6), are uniformly bounded independently of M . Since $\det \partial_x Y(0) = 1$, the matrix $\partial_x Y(t)$ remains invertible for all $t \in [-\tau, \tau]$ for some uniform $\tau > 0$. Because the solution is smooth, it generates a solution $U(x, W(t))$ for the stochastic equation (3) on the time interval $(0, \tau^*)$ with $\tau^* > 0$ defined in the previous paragraph.

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