



HAL
open science

Derivations of negative degree on quasihomogeneous isolated complete intersection singularities.

Michel Granger, Mathias Schulze

► **To cite this version:**

Michel Granger, Mathias Schulze. Derivations of negative degree on quasihomogeneous isolated complete intersection singularities.. 2014. hal-00960092

HAL Id: hal-00960092

<https://hal.science/hal-00960092>

Preprint submitted on 17 Mar 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

DERIVATIONS OF NEGATIVE DEGREE ON QUASIHOMOGENEOUS ISOLATED COMPLETE INTERSECTION SINGULARITIES

MICHEL GRANGER AND MATHIAS SCHULZE

ABSTRACT. J. Wahl conjectured that every quasihomogeneous isolated normal singularity admits a positive grading for which there are no derivations of negative weighted degree. We confirm his conjecture for quasihomogeneous isolated complete intersection singularities of either order at least 3 or embedding dimension at most 5. For each embedding dimension larger than 5 (and each dimension larger than 3), we give a counter-example to Wahl's conjecture.

CONTENTS

Introduction	1
1. Graded analytic algebras	3
2. Negative derivations	5
3. ICIS of embedding dimension 5	9
4. Counter-examples	9
References	10

INTRODUCTION

By a singularity we mean a quotient A of a convergent power series ring over a valued field K of characteristic zero (see §1). We use the acronym *negative derivation* for a derivation of negative weighted degree on a quasihomogeneous singularity. The question of existence of such negative derivations has important consequences in rational homotopy theory (see [Mei82, Thm. A]) and in deformation theory (see [Wah82, Thm. 3.8]).

By a result of Kantor [Kan79], quasihomogeneous curve and hypersurface singularities do not admit any negative derivations. J. Wahl [Wah82, Thm. 2.4, Prop. 2.8] reached the same conclusion in (the much deeper) case of quasihomogeneous normal surface singularities. Motivated by his cohomological characterization of projective space in [Wah83a], he formulates the following conjecture in [Wah83b, Conj. 1.4].

Conjecture (Wahl). *Let R be a normal graded ring, with isolated singularity. Then there is a normal graded \bar{R} , with $\hat{R} \cong \hat{\bar{R}}$, so that \bar{R} has no derivations of negative weight.*

In case R is a graded normal locally complete intersection with isolated singularity, \hat{R} becomes a quasihomogeneous normal isolated complete intersection singularity (ICIS) and Wahl's conjecture can be rephrased as follows (see Lemma 5 and Remark 7).

Date: March 17, 2014.

1991 Mathematics Subject Classification. 13N15, 14M10 (Primary) 14H20 (Secondary).

Key words and phrases. complete intersection, derivation, singularity.

The research leading to these results has received funding from the People Programme (Marie Curie Actions) of the European Union's Seventh Framework Programme (FP7/2007-2013) under REA grant agreement n° PCIG12-GA-2012-334355.

Conjecture (Wahl, ICIS case). *Any quasihomogeneous normal ICIS has no negative derivations with respect to some positive grading.*

For quasihomogeneous normal ICIS, there is an explicit description of all derivations due to Kersken [Ker84]. Based on this description, we prove our main

Theorem 1. *For any quasihomogeneous normal ICIS of order at least 3 there are no negative derivations with respect to any positive grading.*

Proof. This follows from Corollary 12 and Proposition 16. \square

Our investigations lead to a family of counter-examples to Wahl's Conjecture. In order to describe it, we first fix our notation. A quasihomogeneous singularity can be represented as

$$(0.1) \quad A = P/\mathfrak{a}, \quad \mathfrak{a} = \langle g_1, \dots, g_t \rangle \subseteq K\langle\langle x_1, \dots, x_n \rangle\rangle =: P$$

where g_1, \dots, g_t are homogeneous polynomials of degree $p_i := \deg(g_i)$ with respect to weights $w_1, \dots, w_n \in \mathbb{Z}_+$ on the variables x_1, \dots, x_n (see §1). We order these weights and degrees decreasingly as

$$(0.2) \quad \begin{aligned} w_1 &\geq \dots \geq w_n > 0, \\ p_1 &\geq \dots \geq p_t. \end{aligned}$$

Example 2. Let $n \geq 6$ and pick $c_7, \dots, c_n \in K \setminus \{1\}$ pairwise different such that $c_i^9 + 1 \neq 0$ for all i . Assigning weights $8, 8, 5, 2, \dots, 2$ to the variables x_1, \dots, x_n , the equations

$$(0.3) \quad \begin{aligned} g_1 &:= x_1x_4 + x_2x_5 + x_3^2 - x_4^5 + \sum_{i=7}^n x_i^5 \\ g_2 &:= x_1x_5 + x_2x_6 + x_3^2 + x_6^5 + \sum_{i=7}^n c_i x_i^5 \end{aligned}$$

define a quasihomogeneous complete intersection A as in (0.1) with isolated singularity. On A there is a derivation

$$(0.4) \quad \eta := \begin{vmatrix} \partial_1 & \partial_2 & \partial_3 \\ x_4 & x_5 & 2x_3 \\ x_5 & x_6 & 2x_3 \end{vmatrix} = 2x_3(x_5 - x_6)\partial_1 - 2x_3(x_4 - x_5)\partial_2 + (x_4x_6 - x_5^2)\partial_3$$

of degree -1 . We work out the details of this example in §4.

We show that Example 2.8 gives a counter-example to the ICIS case of Wahl's conjecture of minimal embedding dimension $n = 6$.

Theorem 3. *Exactly up to embedding dimension 5, all quasihomogeneous ICIS have no negative derivations with respect to some positive grading.*

Proof. This follows from Kantor [Kan79], [Wah82, Thm. 2.4, Prop. 2.8], Proposition 18, Example 2 and Corollary 12. \square

As a consequence of our arguments we obtain a simple special case of the following conjecture due to S. Halperin.

Conjecture (Halperin). *On any graded zero-dimensional complete intersection there are no negative derivations.*

The following result bounds the degree of negative derivations (see also [Ale91, Prop.]). The bound does not require a complete intersection hypothesis and it is independent of further hypotheses as for instance in [Hau02, Thm. 2].

Proposition 4. *For any quasihomogeneous zero-dimensional singularity A as in (0.1) there are no derivations of degree strictly less than $p_n - p_1$. In particular, Halperin's conjecture holds true if $p_1 = p_n$.*

Proof. As A is assumed to be zero-dimensional, condition $\mathfrak{A}(k)$ on page 4 must hold true for all $k = 1, \dots, n$. Then the claim follows from Remark 14 and Lemma 15. \square

Acknowledgments. The second author would like to thank the LAREMA at the University of Angers for providing financial support and a pleasant working atmosphere during his research visit in February 2014.

1. GRADED ANALYTIC ALGEBRAS

Consider a (local) analytic algebra $A = (A, \mathfrak{m}_A)$ over a (possibly trivially) valued field K of characteristic zero. We assume in addition that A is non-regular and can be represented as a quotient $A = P/\mathfrak{a}$ of a convergent power series ring $P := K\langle\langle x_1, \dots, x_n \rangle\rangle \supseteq \mathfrak{a}$. In the sequel such an A will be referred to as a *singularity*. We choose n minimal such that $n = \text{embdim } A$ and set $d := \dim A$.

A K_+ -grading on A is given by a *diagonalizable derivation* $\chi \in \text{Der}_K A =: \Theta_A$ which means that \mathfrak{m}_A is generated by eigenvectors x_1, \dots, x_n (see [SW73, (2.2),(2.3)]). Such a derivation is also called an *Euler derivation*. We refer to w_1, \dots, w_n defined by $w_i := \chi(x_i)/x_i$ as the *eigenvalues of χ* . More generally, we call χ -eigenvectors $f \in A$ *homogeneous* and define their *degree* to be the corresponding eigenvalue denoted by $\deg(f) := \chi(f)/f \in k$. We denote by A_a the K -vector space of all such eigenvector $f \in A$ with $\deg(f) = a$. This defines a K -subalgebra

$$(1.1) \quad \bar{A} := \bigoplus_{a \in K} A_a \subset A \subset \hat{A}.$$

The derivation $\chi \in \Theta_A$ lifts to $\chi \in \Theta_P := \text{Der}_K P$ (see [SW73, (2.1)]). In particular, P is K_+ -graded and $\mathfrak{a} \subseteq P$ is a χ -invariant ideal and hence homogeneous (see [SW73, (2.4)]). Pick homogeneous $g_1, \dots, g_t \in \mathfrak{a}$ inducing a K -vector space basis of $\mathfrak{a}/\mathfrak{m}_A \mathfrak{a}$. Then $\mathfrak{a} = \langle g_1, \dots, g_t \rangle$ by Nakayama's Lemma. We set $p_i := \deg(g_i)$ ordered as in (0.2). To summarize, we can write A as in (0.1).

A K_+ -grading is called a *positive grading* if $w_i \in \mathbb{Z}_+$ for all $i = 1, \dots, n$ (see [SW73, §3, Def.]). We call A *quasihomogeneous* if it admits a positive grading. In this case, we shall always normalize χ to make the w_i coprime and order the variables according to (0.2). Positivity of weights enforces $g_i \in \bar{P} = K[x_1, \dots, x_n]$ and that

$$(1.2) \quad \bar{A} = \bigoplus_{i \geq 0} A_i = \bar{P}/\bar{\mathfrak{a}}, \quad \bar{\mathfrak{a}} = \langle g_1, \dots, g_t \rangle \subseteq K[x_1, \dots, x_n] = \bar{P},$$

is a (positively) graded-local k -algebra with completion

$$(1.3) \quad \hat{\bar{A}} = \hat{A}$$

and graded maximal ideal $\mathfrak{m}_{\bar{A}} = \bar{\mathfrak{m}}_A := \bigoplus_{i > 0} A_i$. The preceding discussion enables us to reformulate Wahl's Conjecture in the language of Scheja and Wiebe as follows.

Lemma 5. *The following supplementary structures on a singularity A are equivalent:*

- (1) *an Euler derivation χ on A with positive eigenvalues,*
- (2) *a positive grading on A ,*
- (3) *a positive grading on \hat{A} ,*
- (4) *a (positively) graded K -algebra \bar{A} such that $\hat{\bar{A}} = \hat{A}$.*

Proof. The equivalences of (1), (2), and (3) are due to Scheja and Wiebe (see [SW73, (2.2),(2.3)] and [SW77, (1.6)]). For the equivalence with (4), note that the obvious Euler derivation on a graded K -algebra \bar{A} lifts to an Euler derivation on the completion $\hat{\bar{A}} = \hat{A}$. The converse follows from (1.1), (1.2) and (1.3). \square

Let us assume now that A is an isolated complete intersection singularity (ICIS). We may then take g_1, \dots, g_t to be a regular sequence and $d + t = n$. The isolated singularity hypothesis can be expressed in terms of the Jacobian ideal

$$(1.4) \quad J_A := \left\langle \left| \frac{\partial g}{\partial x_\nu} \right| \mid |\nu| = t \right\rangle \trianglelefteq A$$

of A as follows.

Proposition 6. *A complete intersection singularity A is isolated if and only if J_A is \mathfrak{m}_A -primary. An analogous statement holds for \bar{A} .*

Proof. We denote by $\Omega_{A/k}^1$ the universally finite module of differentials of A over k . By the standard sequence

$$\mathfrak{a}/\mathfrak{a}^2 \longrightarrow A \otimes_P \Omega_{P/k}^1 \longrightarrow \Omega_{A/k}^1 \longrightarrow 0,$$

the Jacobian ideal J_A is the 0th Fitting ideal $F_A^0 \Omega_{A/k}^1$. By [SS72, (6.4),(6.9)], reducedness of A is equivalent to $\text{rk } \Omega_{A/k}^1 = d$ and $A_{\mathfrak{p}}$ is regular if and only if $\Omega_{A_{\mathfrak{p}}/k}^1$ is free. Hence, $A_{\mathfrak{p}}$ being regular is equivalent to $\mathfrak{p} \not\supseteq F_A^0 \Omega_{A/k}^1 = J_A$ by [BH93, Lem. 1.4.9]. In particular, A having an isolated singularity means exactly that A/J_A is supported at \mathfrak{m}_A and hence that J_A is \mathfrak{m}_A -primary as claimed. The analogous statement for \bar{A} is proved similarly. \square

Remark 7. Let A be a quasihomogeneous singularity. By (1.2),

$$(1.5) \quad J_{\bar{A}} := \bar{J}_A = \left\langle \left| \frac{\partial g}{\partial x_\nu} \right| \mid |\nu| = t \right\rangle \trianglelefteq \bar{A}$$

is the Jacobian ideal of \bar{A} defined analogous to (1.4). By (1.3), A is a complete intersection if and only if \bar{A} is locally a complete intersection (see [BH93, Def. 2.3.1, Ex. 2.3.21.(c)]). By Proposition 6, A is an ICIS if and only if J_A is \mathfrak{m}_A -primary. This is equivalent to $J_{\bar{A}}$ being $\mathfrak{m}_{\bar{A}}$ -primary. The latter is then equivalent to \bar{A} being locally a complete intersection with isolated singularity by (1.5) and Proposition 6. Complete intersections are Cohen–Macaulay and hence (S_2) so normality is equivalent to (R_1) by Serre’s Criterion (see [BH93, §2.3, Thm. 2.2.22]). Since $d = \dim A = \dim \bar{A}$ by (1.3) (see [BH93, Cor. 2.1.8]), normality for both A and \bar{A} reduces to $d \geq 2$.

Scheja and Wiebe [SW77, (3.1)] (see also [Sai71, Satz 1.3]) proved that any K_+ -graded ICIS is quasihomogeneous unless $t = 1$ and $g_1 \notin \mathfrak{m}_P^3$. Their starting point (see [SW77, (2.5)] and [Sai71, Lem. 1.5]) is that A being an ICIS implies, by Proposition 6, that for each $k = 1, \dots, n$ one of the following two conditions must hold true.

$\mathfrak{A}(k)$ For some $m \geq 2$ and $1 \leq j \leq t$, the monomial x_k^m occurs in g_j .

$\mathfrak{B}(k)$ For some pairwise different $1 \leq \nu_1, \dots, \nu_t \leq n$, each g_j contains a monomial $x_k^{m_j} x_{\nu_j}$ for some $m_j \geq 1$.

The following result gives numerical constraints for A to be a quasihomogeneous ICIS.

Lemma 8. *If A is a quasihomogeneous ICIS then*

$$(1.6) \quad p_1 + \dots + p_j \geq w_1 + \dots + w_j + j$$

for all $j = 1, \dots, t$.

Proof. We proceed by induction on j . Assume that $p_1 + \dots + p_{j-1} \geq w_1 + \dots + w_{j-1} + j - 1$ but $p_1 + \dots + p_j \leq w_1 + \dots + w_j + j - 1$. Then $p_j \leq w_j$ and hence $g_i = g_i(x_{j+1}, \dots, x_n)$ for all $i = j, \dots, n$. Then J_A maps to zero in

$$A/\langle x_{j+1}, \dots, x_n \rangle = K\langle\langle x_1, \dots, x_j \rangle\rangle/\langle g_1, \dots, g_{j-1} \rangle$$

and hence J_A cannot be \mathfrak{m}_A -primary as required by Proposition 6. \square

2. NEGATIVE DERIVATIONS

Let A be a quasihomogeneous singularity as in §1. The target of our investigations is the positively graded A -module $\Theta_A = \text{Der}_K A$ of K -linear derivations on A . More precisely, we are concerned with the question whether its negative part

$$\Theta_{A, < 0} = \Theta_{\bar{A}, < 0} = \bigoplus_{i < 0} \Theta_{A, i}$$

is trivial. A priori this condition depends on the choice of a grading. In Proposition 9 below, we shall prove the independence of this choice for a general singularity under a strong hypothesis satisfied in the ICIS case (see Corollary 12). To this end, we write (see [SW73, (2.1)])

$$(2.1) \quad \Theta_A = \Theta_{\mathfrak{a} \subset P} / \mathfrak{a} \Theta_P$$

as a quotient of a (k, P) -Lie algebra

$$\Theta_{\mathfrak{a} \subset P} := \{ \delta \in \Theta_P \mid \delta \mathfrak{a} \subset \mathfrak{a} \} \supseteq \mathfrak{a} \Theta_P$$

of logarithmic derivations along \mathfrak{a} by the (k, P) -Lie ideal $\mathfrak{a} \Theta_P$.

Proposition 9. *Let A be a quasihomogeneous singularity with positive grading given by χ and assume that*

$$(2.2) \quad \Theta_{\mathfrak{a} \subset P} = P\chi + \Theta'_P + \mathfrak{a} \Theta_P,$$

$$(2.3) \quad \text{for some } \Theta'_P \subset \mathfrak{m}_P^2 \Theta_P.$$

Then the condition $\Theta_{A, < 0} = 0$ and the p_1, \dots, p_t in (0.2) are independent of the chosen positive grading.

Proof. Consider a second positive grading with corresponding Euler derivation χ' (see Lemma 5). By (2.1) and (2.2), any $\delta \in \Theta_A$ lifts to an element of $\Theta_{\mathfrak{a} \subset P}$ of the form

$$(2.4) \quad \delta = c\chi + \delta_+, \quad \delta_+ = a\chi + \eta, \quad c \in K, \quad a \in \mathfrak{m}_P, \quad \eta \in \Theta'_P,$$

denoted by the same symbol. By (2.3) and the Leibniz rule,

$$(2.5) \quad \chi \mathfrak{m}_P^k \subset \mathfrak{m}_P^k, \quad \delta_+ \mathfrak{m}_P^k \subset \mathfrak{m}_P^{k+1}$$

for all $k \geq 1$. Specializing to $\delta = \chi$, this implies that $\chi_+ = 0$ and $\chi' = c\chi$ on $\mathfrak{m}_A / \mathfrak{m}_A^2 = \mathfrak{m}_P / \mathfrak{m}_P^2$ and hence $c = 1$ by the definition of a positive grading and our normalization of weights.

Using (2.1), we equip Θ_A with the decreasing \mathfrak{m}_P -adic filtration F^\bullet induced from Θ_P which is defined as follows

$$F^k \Theta_A = (\Theta_{\mathfrak{a} \subset P} \cap \mathfrak{m}_P^k \Theta_P) / (\mathfrak{a} \Theta_P \cap \mathfrak{m}_P^k \Theta_P).$$

Due to (2.3), (2.4) and (2.5) this is a filtration by (k, P) -Lie ideals and

$$\delta_+ F^k \Theta_A \subset F^{k+1} \Theta_A$$

for the adjoint action of δ_+ . Therefore, for any $k \geq 1$, the adjoint action of $\chi' = \chi + \chi_+$ on the truncation

$$F^{\leq k} \Theta_A := \Theta_A / F^{k+1} \Theta_A$$

is triangularizable with semisimple part equal to that of χ . Thus, χ' and χ have the same eigenvalues on $F^{\leq k} \Theta_A$ for any $k \geq 1$. The first claim then follows by choosing k sufficiently large. A similar argument yields the second claim. \square

For a Gorenstein singularity A , there is a natural way to produce elements of Θ_A . The A -submodule $\Theta'_A \subset \Theta_A$ of *trivial derivations* is by definition the image of the inclusion

$$(2.6) \quad \Omega_{A/K}^{d-1} \hookrightarrow \omega_{A/K}^{d-1} = \text{Hom}_A(\Omega_{A/K}^1, \omega_{A/K}^d) = \Theta_A \otimes_A \omega_{A/K}^d \cong \Theta_A.$$

We return to the case of an ICIS singularity A . For $1 \leq \nu_0 < \dots < \nu_t \leq n$ with complementary indices $1 \leq \mu_1 < \dots < \mu_{d-1} \leq n$, the lift to P of the image of $dx_{\mu_1} \wedge \dots \wedge dx_{\mu_{d-1}}$ can be written (up to sign) explicitly as

$$(2.7) \quad \delta_\nu := \begin{vmatrix} \partial_{\nu_0} & \cdots & \partial_{\nu_t} \\ \partial_{\nu_0} g_1 & \cdots & \partial_{\nu_t} g_1 \\ \vdots & & \vdots \\ \partial_{\nu_0} g_t & \cdots & \partial_{\nu_t} g_t \end{vmatrix}.$$

Note that

$$(2.8) \quad \deg \delta_\nu = p_1 + \dots + p_t - w_{\nu_0} - \dots - w_{\nu_t},$$

$$(2.9) \quad \delta_\nu g_j = 0$$

for all $j = 1, \dots, t$ and ν . Consider the P -module

$$(2.10) \quad \Theta'_P := \langle \delta_\nu \mid 1 \leq \nu_0 < \dots < \nu_t \leq n \rangle_P \subset \Theta_P.$$

The key to our investigations is the following result due to Kersken [Ker84, (5.2)]. From now on we assume in addition that A is quasihomogeneous and normal, that is, $\dim A \geq 2$.

Theorem 10 (Kersken). *Let A be a quasihomogeneous normal ICIS. Then the module Θ_A of K -linear derivations on A is generated by the Euler derivation χ and the trivial derivations Θ'_A .*

Although Kersken only states that Θ'_A is minimally generated by the δ_ν in (2.7), his arguments show that together with χ they form a minimal set of generators of Θ_A . We denote by $\mu(-)$ the minimal number of generators.

Corollary 11. *Let A be quasihomogeneous normal ICIS. Then Θ_A is minimally generated by the Euler derivation χ and the trivial derivations δ_ν in (2.7). In particular,*

$$\mu(\Theta_A) = \binom{n}{t+1} + 1.$$

Proof. Since the case $d = 2$ is covered by [Wah87, Prop. 1.12], we may assume that $d \geq 3$. In this case, the inclusion (2.6) fits into the following commutative diagram with exact

rows and columns (see [Ker84, Proof of (4.8)] or [Wah87, Prop. 1.7]).

$$(2.11) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & H_{\mathfrak{m}_A}^1(\Omega_{A/K}^d) & \xrightarrow[\cong]{\chi} & H_{\mathfrak{m}_A}^1(\Omega_{A/K}^{d-1}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \omega_{A/K}^d & \xrightarrow{\chi} & \omega_{A/K}^{d-1} & \xrightarrow{\chi} & \omega_{A/K}^{d-2} \\ & & \uparrow & & \uparrow & & \uparrow \cong \\ 0 & \longrightarrow & \Omega_{A/K}^d & \xrightarrow{\chi} & \Omega_{A/K}^{d-1} & \xrightarrow{\chi} & \Omega_{A/K}^{d-2} \\ & & & & \uparrow & & \uparrow \\ & & & & 0 & & 0 \end{array}$$

It follows that

$$\chi(\omega_{A/K}^{d-1}) \cong \chi(\Omega_{A/K}^{d-1}) \cong \Omega_{A/K}^{d-1} / \chi(\Omega_{A/K}^d)$$

where $\chi(\Omega_{A/K}^d) \subset \mathfrak{m}_A \Omega_{A/K}^{d-1}$ and hence

$$\mu(\chi(\Omega_{A/K}^{d-1})) = \mu(\Omega_{A/K}^{d-1}) = \mu(\Theta'_A).$$

Now the middle row of (2.11) yields an exact sequence

$$0 \longrightarrow A \xrightarrow{\chi} \Theta_A \longrightarrow \chi(\omega_{A/K}^{d-1}) \otimes (\omega_{A/K}^d)^{-1} \longrightarrow 0$$

Since $\chi \notin \mathfrak{m}_A \Theta_A$, the claim follows. \square

Note that Θ'_P in (2.10) satisfies (2.3) due to (2.7) unless $t = 1$ and $g_1 \notin \mathfrak{m}_P^3$. As a consequence of Proposition 9 and Theorem 10 we therefore obtain the following result. It is crucial for Example 2 to be a counter-example to Wahl's Conjecture.

Corollary 12. *Let A be a quasihomogeneous normal ICIS. Unless $t = 1$ and $g_1 \notin \mathfrak{m}_P^3$, the condition $\Theta_{A, < 0} = 0$ and the p_1, \dots, p_t in (0.2) are independent of the choice of a positive grading. \square*

We shall now derive numerical constraints for minimal negative trivial derivations. To this end, suppose that $0 \neq \eta \in \Theta_{A, < 0}$. For reasons of degree (see (0.2)), η can be written as

$$(2.12) \quad \eta = q_1 \partial_1 + \cdots + q_n \partial_n, \quad q_i = q_i(x_{i+1}, \dots, x_n)$$

By Theorem 10, we may assume that $\eta = \delta_\nu \neq 0$ is a trivial derivation as in (2.7). By (0.2) and (2.8), we may further assume that $\nu_i = i + 1$ for $i = 0, \dots, t$. Explicitly, we may write

$$(2.13) \quad q_i = (-1)^{i-1} \begin{vmatrix} \partial_1 g_1 & \cdots & \widehat{\partial_i g_1} & \cdots & \partial_{t+1} g_1 \\ \vdots & & \vdots & & \vdots \\ \partial_1 g_t & \cdots & \widehat{\partial_i g_t} & \cdots & \partial_{t+1} g_t \end{vmatrix}, \quad q_{t+2} = \cdots = q_n = 0.$$

Now (2.8) and (2.9) specialize to the following simple

Lemma 13. For η as in (2.12) with (2.13), we have

$$(2.14) \quad \eta g_j = 0$$

for all $j = 1, \dots, t$. If $\Theta_{A, <0} \neq 0$ for a quasihomogeneous normal ICIS then

$$(2.15) \quad p_1 + \dots + p_t < w_1 + \dots + w_{t+1}. \quad \square$$

Remark 14. For degree reasons (see (0.2)), the identity (2.14) holds true for any $\eta \in \Theta_{A, <p_t - p_1}$ and any quasihomogeneous singularity A as in (0.1).

We now link the conditions $\mathfrak{A}(k)$ and $\mathfrak{B}(k)$ from page 4 to the existence of a negative derivation as in (2.12).

Lemma 15. Assume that the identity (2.14) holds true for all $j = 1, \dots, t$. Then $\mathfrak{A}(k)$ implies $q_k = 0$ in (2.12) for a suitable choice of coordinates.

Proof. Pick $k \in \{1, \dots, t+1\}$ such that $\mathfrak{A}(k)$ holds. Then some g_j contains x_k^m , $m > 1$, and all other monomials in g_j contain only strictly lower powers of x_k by homogeneity. Let $t_{k,j} = t_{k,j}(x_1, \dots, \widehat{x}_k, \dots, x_n)$ denote the coefficient of x_k^{m-1} in g_j , and assume, without loss of generality, that the coefficient of x_k^m is $\frac{1}{m}$. Note that $t_{k,j}$ is independent of variables of weight larger than w_k . Expanding (2.14) with respect to the variable x_k and taking the terms involving x_k^{m-1} gives

$$q_k x_k^{m-1} = q_k \partial_k \left(\frac{1}{m} x_k^m \right) = - \sum_{i \neq k} q_i \partial_i (t_{k,j} x_k^{m-1}) = - \sum_{i \neq k} q_i \partial_i (t_{k,j}) x_k^{m-1}$$

and hence

$$(2.16) \quad \eta = \sum_{i \neq k} q_i \cdot (\partial_i - \partial_i(t_{k,j}) \partial_k).$$

The χ -homogeneous coordinate change

$$x'_i = \begin{cases} x_k + t_{k,j}, & \text{if } i = k, \\ x_i, & \text{else.} \end{cases}$$

replaces $\partial_i - \partial_i(t_{k,j}) \partial_k$ in (2.16) by ∂'_i , and thus q_k in (2.12) by 0. Iterating this process yields the claim. \square

Our main technical result is the following

Proposition 16. Let A be a quasihomogeneous normal ICIS such that $\Theta_{A, <0} \neq 0$. Then $\mathfrak{B}(k)$ holds for at least two indices $k \leq t+1$. Each such k satisfies $k \geq t-d+2$ and $g_k, \dots, g_t \notin \mathfrak{m}_P^3$.

Proof. By hypothesis and Lemma 15, $\mathfrak{B}(k)$ holds for some $k \leq t+1$ with $q_k \neq 0$. Assuming that k is unique, (2.9) reads $q_k \partial_k g_j = 0$ which would imply that g_j is independent of x_k for all $j = 1, \dots, t$. By the isolated singularity hypothesis, this is impossible.

Combining (1.6) and (2.15), we obtain

$$(2.17) \quad p_j + \dots + p_t + j \leq w_j + \dots + w_{t+1}$$

for all $j = 1, \dots, t$. Using (0.2), $\mathfrak{B}(k)$ and (2.17) for $j = k$, we compute

$$\begin{aligned} m_k w_k + \dots + m_t w_t &\leq (m_k + \dots + m_t) w_k \\ &= \deg(\partial_{\nu_k} g_k \cdots \partial_{\nu_t} g_t) \\ &= p_k + \dots + p_t - w_{\nu_k} - \dots - w_{\nu_t} \\ &\leq w_k + \dots + w_{t+1} - k - w_{\nu_k} - \dots - w_{\nu_t}. \end{aligned}$$

and hence

$$(m_k - 1)w_k + \cdots + (m_t - 1)w_t \leq w_{t+1} - k - w_{\nu_k} - \cdots - w_{\nu_t}.$$

By (0.2), this forces

$$(2.18) \quad \begin{aligned} m_k &= \cdots = m_t = 1, \\ w_{t+1} &\geq w_{\nu_k} + \cdots + w_{\nu_t} + k. \end{aligned}$$

In particular,

$$(2.19) \quad \nu_k, \dots, \nu_t \geq t + 2$$

and hence $k \geq t - d + 2$. □

3. ICIS OF EMBEDDING DIMENSION 5

Lemma 17. *Let A be a quasihomogeneous normal ICIS such that $\Theta_{A, <0} \neq 0$. Then $\mathfrak{A}(k_1)$ and $\mathfrak{B}(k_2)$ for $\{k_1, k_2\} = \{1, 2\}$ is impossible.*

Proof. Assuming the contrary, one of the g_j has a monomial divisible by $x_{k_1}^2$ by $\mathfrak{A}(k_1)$ and each of the g_j has a monomial divisible by x_{k_2} by $\mathfrak{B}(k_2)$. In particular,

$$p_1 + \cdots + p_t \geq 2w_{k_1} + (t - 1)w_{k_2} \geq w_1 + \cdots + w_{t+1}$$

contradicting (2.15). □

Proposition 18. *For any quasihomogeneous ICIS A as in (0.1) with $n = 5$ and $t = 2$, we have $\Theta_{A, <0} = 0$.*

Proof. Assume that $\Theta_{A, <0} \neq 0$. By Proposition 16 and Lemma 17, we must have $\mathfrak{B}(1)$ and $\mathfrak{B}(2)$. Using (0.2), (2.18), and (2.19), we may write

$$\begin{aligned} g_1 &= x_1x_4 + c_1x_2^jx_{k_1} + \cdots \\ g_2 &= x_1x_5 + c_2x_2x_{k_2} + \cdots \end{aligned}$$

with $\{k_1, k_2\} = \{4, 5\}$ and $c_1, c_2 \in K^*$. As in the proof of Lemma 17, the inequality (2.15) can only hold true if $j = 1$. In this case,

$$A/(J_A + \langle x_3, \dots, x_n \rangle) = K \langle \langle x_1, x_2 \rangle \rangle / \left\langle \left| \frac{\partial g}{\partial(x_4, x_5)} \right| \right\rangle.$$

for degree reasons (see (0.2)), and hence J_A is not \mathfrak{m}_A -primary. This contradicts to the isolated singularity hypothesis. □

4. COUNTER-EXAMPLES

Proof of Example 2. The sequence g is clearly regular and defines a complete intersection as in (0.1). Note that η in (0.4) agrees with $\eta = \delta_{1,2,3}$ in (2.12). Since $\deg(g_1) = 10 = \deg(g_2)$, (2.9) shows that η has negative degree $\deg \eta = -1$.

It remains to check that A has an isolated singularity, that is, the Jacobian ideal J_A from (1.4) is \mathfrak{m}_A -primary. To this end, we may assume that $K = \bar{K}$ which enables us to argue geometrically on the variety

$$\bar{X} := \text{Spec } \bar{A} \subset \mathbb{A}_K^n$$

with \bar{A} as in (1.2) using the Nullstellensatz.

The ideal J_A is the image in A of the Jacobian ideal $\bar{J}_g \subseteq \bar{P}$ of g generated by the 2×2 -minors

$$M_{i,j} := \left| \frac{\partial g}{\partial(x_i x_j)} \right|$$

of the Jacobian matrix of g which reads

$$\frac{\partial g}{\partial x} = \begin{pmatrix} x_4 & x_5 & 2x_3 & x_1 - 5x_4^4 & x_2 & 0 & 5x_7^4 & \cdots & 5x_n^4 \\ x_5 & x_6 & 2x_3 & 0 & x_1 & x_2 + 5x_6^4 & 5c_7x_7^4 & \cdots & 5c_nx_n^4 \end{pmatrix}.$$

With this notation we have to show that

$$\text{Sing } \bar{X} = V(g, \bar{J}_g) = \{0\}.$$

Due to those 2×2 -minors of $\frac{\partial g}{\partial x}$ which involve only the columns 3, 7, 8, 9, \dots , n , only one of components $x_3, x_7, x_8, x_9, \dots, x_n$ of any $x \in \text{Sing } \bar{X}$ can be non-zero. We may therefore reduce to the case $n \leq 7$.

Because of the 3rd column of $\frac{\partial g}{\partial x}$, we have $\bar{J}_g \cap K[x_1, \dots, x_6] \supseteq x_3I$ where

$$I := \langle x_4 - x_5, x_5 - x_6, x_1 - x_2, x_1 - 5x_4^4, x_2 + 5x_6^4 \rangle.$$

Note that $V(I)$ is the x_3 -axis which is not contained in $V(g)$. It follows that $\text{Sing } \bar{X} \cap V(x_7)$ is contained in the hyperplane $V(x_3)$. Similarly because of the 7th column of $\frac{\partial g}{\partial x}$ and setting $c := c_7$, we have $\bar{J}_g \cap K[x_1, \dots, \hat{x}_3, \dots, x_7] \supseteq x_7I'$ where

$$I' := \langle cx_4 - x_5, cx_5 - x_6, cx_2 - x_1, x_1 - 5x_4^4, x_2 + 5x_6^4 \rangle.$$

Using $c^9 + 1 \neq 0$, we find that $V(I')$ is the x_7 -axis and conclude $\text{Sing } \bar{X} \cap V(x_3) \subset V(x_7)$ as before. Summarizing the two cases, $\text{Sing } \bar{X}$ is in fact contained in $V(x_3, x_7)$.

Fix a point $(x_1, x_2, 0, x_4, x_5, x_6, 0) \in \text{Sing } \bar{X}$. Successively using the the equations

$$\begin{aligned} M_{1,2} &= x_4x_6 - x_5^2 = 0, \\ M_{2,5} &= x_1x_5 - x_2x_6 = 0, \\ g_2 &= x_1x_5 + x_2x_6 + x_6^5 = 0, \\ M_{4,5} &= x_1(x_1 - 5x_4^4) = 0, \\ M_{5,6} &= x_2(x_2 + 5x_6^4) = 0, \end{aligned}$$

we derive

$$x_4 = 0 \Rightarrow x_5 = 0 \Rightarrow x_2x_6 = 0 \Rightarrow x_6 = 0 \Rightarrow x_1 = x_2 = 0.$$

Similarly $x_6 = 0$ leaves no possibility except $x = 0$ and $x_5 = 0$ reduces to one of these two cases by $M_{1,2} = 0$.

Assume now that x_4, x_5, x_6 are all non zero. Then the minors $M_{1,5}, M_{2,4}, M_{2,5}, M_{2,6}$ give equations

$$x_1x_4 = x_2x_5, \quad x_1 = 5x_4^4, \quad x_1x_5 = x_2x_6, \quad x_2 = -5x_6^4.$$

Substituting into g , we obtain

$$g_1 = 2x_1x_4 - x_4^5 = 9x_4^5, \quad g_2 = 2x_2x_6 + x_6^5 = -9x_6^5$$

and hence $x_4 = x_6 = 0$ contradicting our assumption. \square

REFERENCES

- [Ale91] A. G. Aleksandrov, *Vector fields on a complete intersection*, Funktsional. Anal. i Prilozhen. **25** (1991), no. 4, 64–66. MR 1167721 (93d:14073) ([document](#))
- [BH93] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR MR1251956 (95h:13020) **1, 7**
- [Hau02] V. Hauschild, *Discriminants, resultants and a conjecture of S. Halperin*, Jahresber. Deutsch. Math.-Verein. **104** (2002), no. 1, 26–47. MR 1913265 (2003h:55018) ([document](#))

- [Kan79] Jean-Michel Kantor, *Rectificatif à la note: “Dérivations sur les singularités quasi-homogènes: cas des courbes”*, C. R. Acad. Sci. Paris Sér. A-B **288** (1979), no. 14, A697 (French). ([document](#))
- [Ker84] Masumi Kersken, *Reguläre Differentialformen*, Manuscripta Math. **46** (1984), no. 1-3, 1–25. MR 735512 (85j:14032) ([document](#)), [2](#), [2](#)
- [Mei82] W. Meier, *Rational universal fibrations and flag manifolds*, Math. Ann. **258** (1981/82), no. 3, 329–340. MR 649203 (83g:55009) ([document](#))
- [Sai71] Kyoji Saito, *Quasihomogene isolierte Singularitäten von Hyperflächen*, Invent. Math. **14** (1971), 123–142. MR 0294699 (45 #3767) [1](#)
- [SS72] Günter Scheja and Uwe Storch, *Differentielle Eigenschaften der Lokalisierungen analytischer Algebren*, Math. Ann. **197** (1972), 137–170. MR 0306172 (46 #5299) [1](#)
- [SW73] Günter Scheja and Hartmut Wiebe, *Über Derivationen von lokalen analytischen Algebren*, Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971), Academic Press, London, 1973, pp. 161–192. MR 0338461 (49 #3225) [1](#), [1](#), [1](#), [2](#)
- [SW77] ———, *Über Derivationen in isolierten Singularitäten auf vollständigen Durchschnitten*, Math. Ann. **225** (1977), no. 2, 161–171. MR 0508048 (58 #22649) [1](#), [1](#)
- [Wah83a] J. M. Wahl, *A cohomological characterization of \mathbf{P}^n* , Invent. Math. **72** (1983), no. 2, 315–322. MR 700774 (84h:14024) ([document](#))
- [Wah83b] Jonathan M. Wahl, *Derivations, automorphisms and deformations of quasihomogeneous singularities*, Singularities, Part 2 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, RI, 1983, pp. 613–624. MR 713285 (85g:14008) ([document](#))
- [Wah87] ———, *The Jacobian algebra of a graded Gorenstein singularity*, Duke Math. J. **55** (1987), no. 4, 843–871. MR 916123 (89a:14042) [2](#)
- [Wah82] ———, *Derivations of negative weight and nonsmoothability of certain singularities*, Math. Ann. **258** (1981/82), no. 4, 383–398. MR 650944 (84h:14043) ([document](#))

M. GRANGER, UNIVERSITÉ D’ANGERS, DÉPARTEMENT DE MATHÉMATIQUES, LAREMA, CNRS UMR N°6093, 2 BD LAVOISIER, 49045 ANGERS, FRANCE

E-mail address: granger@univ-angers.fr

M. SCHULZE, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KAISERSLAUTERN, 67663 KAISERSLAUTERN, GERMANY

E-mail address: mschulze@mathematik.uni-kl.de