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# NOTES ON LIEB-THIRRING TYPE INEQUALITY FOR A COMPLEX PERTURBATION OF FRACTIONAL SCHRÖDINGER OPERATOR

CLÉMENT DUBUISSON

ABSTRACT. We prove Lieb-Thirring type inequalities for a complex perturbation of fractional Schrödinger operator using either Borichev-Golinskii-Kupin Theorem or a Theorem from [Ha].

## 1. INTRODUCTION

In the spirit of [Du] and [DeHaKa], we prove a Lieb-Thirring inequality for a complex perturbation of some positive power of the free Laplacian operator.

We define for  $s > 0$  the fractional Laplacian  $(-\Delta)^s$  with the help of the functional calculus applied to the nonnegative self-adjoint operator  $-\Delta$ . Then  $(-\Delta)^s$  is essentially self-adjoint on  $\mathcal{C}_c^\infty(\mathbb{R}^d; \mathbb{C}^n)$  and the domain of its closure is the fractional Sobolev space  $W^{2,2s}(\mathbb{R}^d, \mathbb{C}^n) := \{f, \int_{\mathbb{R}^d} (1 + |\zeta|^2)^s \|\widehat{f}(\zeta)\|^2 d\zeta < +\infty\}$ , where  $\widehat{f}$  is the Fourier transform of  $f$ . By the spectral mapping theorem the spectrum of  $(-\Delta)^s$  is  $[0; +\infty[$ .

Let us denote by  $H_0 := (-\Delta)^s$ ,  $s > 0$  and consider the perturbed operator

$$(1.1) \quad H = H_0 + V$$

where  $V$  is the operator of multiplication by  $V$ . We suppose it to be a relatively Schatten-von Neumann perturbation of  $H_0$ , *i.e.*,  $\text{dom}(H_0) \subset \text{dom}(V)$ , and

$$(1.2) \quad V(\lambda - H_0)^{-1} \in \mathbf{S}_p,$$

for one  $\lambda \in \mathbb{C} \setminus \sigma(H_0)$  (and hence for all these  $\lambda$ 's).

The potential  $V$  belongs to  $L^p(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C})) = \left\{ V : \int_{\mathbb{R}^d} \|V(x)\|_F^p dx < +\infty \right\}$ ,

where  $\|\cdot\|_F$  is the *Frobenius* norm,

$$\|V(x)\|_F = \left( \sum_{i,j=1,\dots,n} |(V(x))_{i,j}|^2 \right)^{1/2}.$$

When  $V \in L^p(\mathbb{R}^d; \mathcal{M}_{n,n}(\mathbb{C}))$  and  $p > \frac{d}{2s}$  we see that hypothesis (1.2) is fulfilled (see Proposition 2.4).

Let us introduce some general definition and notation. For a (possibly unbounded) operator  $A$  on a separable Hilbert space  $\mathcal{H}$ , the spectrum, the essential spectrum, and the discrete spectrum of  $A$  will be denoted respectively by  $\sigma(A)$ ,  $\sigma_{\text{ess}}(A)$ , and  $\sigma_d(A)$ . Here the discrete spectrum is the set of all eigenvalues which are discrete points of the spectrum whose corresponding eigenspaces (or

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rootspaces) are finite dimensional. Then the essential spectrum of  $A$  is  $\sigma(A) \setminus \sigma_d(A)$ . For more details, see [ReSi1, subsection VII.3] or [DeHaKa1, p.5]. We denote by  $\mathbf{S}_p$ ,  $p \geq 1$  the Schatten-von Neumann class of compact operators, see section 2.3 for further references on the subject.

By Weyl's theorem on essential spectrum (see [ReSi4, Theorem XIII.14])

$$\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0) = \sigma(H_0) = [0, +\infty[.$$

The main results of the article are the following Lieb-Thirring type inequalities. They are obtained with the help of Theorem 2.1.

**Theorem 1.1.** *Let  $H$  be the operator defined by (1.1) and  $V \in L^p(\mathbb{R}^d; \mathcal{M}_{n,n}(\mathbb{C}))$ ,  $p > \frac{d}{2s}$ .*

*For  $0 < s \leq \frac{d}{2}$  and  $\tau > 0$  small enough, we have the following Lieb-Thirring bound*

$$(1.3) \quad \sum_{\lambda \in \sigma(H)} \frac{d(\lambda, \sigma(H_0))^{p+\tau}}{|\lambda|^{\min\{\frac{p+\tau}{2}; \frac{d}{2s}\}} (1 + |\lambda|)^{\frac{p}{2} + \max\{p; \frac{d}{s}\} - \frac{d}{2s} + 2\tau}} \leq C \cdot a^\gamma \cdot \|V\|_{L^p}^p,$$

where  $\gamma = \frac{p+\tau}{2} + \max\{\frac{p+\tau}{2}; \frac{d}{s}\}$  and  $C$  is a constant depending on  $n, d$ , and  $p$ .

For  $s > \frac{d}{2}$ ,  $p \geq 1$  and  $\tau$  small enough, we have

$$(1.4) \quad \sum_{\lambda \in \sigma(H)} \frac{d(\lambda, \sigma(H_0))^{p - \frac{d}{2s} + 1 + \tau}}{|\lambda|^\alpha (1 + |\lambda|)^\beta} \leq C \cdot a^\gamma \cdot \|V\|_{L^p}^p,$$

where

- (1)  $\alpha = \frac{1}{2} + \frac{1}{2} \min\{p - \frac{d}{2s} + \tau; 1\}$ ,
- (2)  $\beta = p - \frac{d}{2s} + 1 + \frac{3\tau}{2} + \frac{1}{2} \max\{p - \frac{d}{2s} - 1; 0; \frac{3d}{2s} - p - 1\}$ ,
- (3)  $\gamma = \frac{1}{2}(p + \frac{d}{2s} + 1 + \tau)$ ,

and  $C$  is a constant depending on  $n, d$ , and  $p$ .

The following theorem is our second result. It is proved with the help of Theorem 5.1.

**Theorem 1.2.** *Let  $H$  be as above and  $V \in L^p(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C}))$ ,  $p > \max\{1; \frac{d}{2s}\}$  with  $s > 0$ . Then the following holds*

$$(1.5) \quad \sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^p}{(1 + |\lambda|)^{2p}} \leq C \cdot a^{2p + \frac{d}{2s}} \cdot \|V\|_{L^p}^p,$$

where the constant  $C$  depends on  $n, d, p$  and  $s$ .

For the self-adjoint case, we refer to Theorem 2.1 in [FrLiSe]. In this case,  $\sigma_d(K)$  lies on the negative real half-axis and  $0 < s < \min\{1; \frac{d}{2}\}$ . In our notation formula (5.11) from [FrLiSe] says

$$(1.6) \quad \sum_{\lambda \in \sigma_d(K)} |\lambda|^{p - \frac{d}{2s}} \leq C_{p,d} \|V_-\|_{L^p}^p,$$

where  $p > \frac{d}{2s}$  and  $V_- = \min\{V, 0\}$ . As in [Du], but this time for  $0 < s < \min\{1; \frac{d}{2}\}$ , a simple computation proves that (1.3) and (1.5) are weaker than (1.6).

To compare Theorem 1.1 and Theorem 1.2, we can use the same computations as above or, as in [DeHaKa] or [Du], we may consider a sequence of eigenvalues

$(\lambda_n)_\mathbb{N}, \lambda_n \in \sigma_d(H)$ , converging to  $\lambda^* \in \sigma_{\text{ess}}(H) = \mathbb{R}^+$  and compare the bounds we obtain from (1.3), (1.4), and (1.5). We see that neither Theorem 1.1 nor Theorem 1.2 take the advantage over the other. However, it seems that Theorem 1.1 gives locally better estimate at the edges of  $\sigma_{\text{ess}}(H)$  than Theorem 1.2 and that Theorem 1.2 gives locally better estimate in  $]0; +\infty[$ .

Some precisions on the notation. Constants will be generic, that is, they may change from one relation to another. They will be usually denoted by  $C$ . For two strictly positive functions  $f, g$  defined on a domain  $\Omega$  of the complex plane  $\mathbb{C}$ , we write  $f(\lambda) \approx g(\lambda)$  if there are constants  $C_1, C_2 > 0$  so that  $C_1 f(\lambda) \leq g(\lambda) \leq C_2 f(\lambda)$  for all  $\lambda \in \Omega$ . And we write  $f(\lambda) \lesssim g(\lambda)$  if there is a positive constant  $C$  such that  $f(\lambda) \leq C \cdot g(\lambda)$  for  $\lambda \in \Omega$ . The choice of the domain  $\Omega$  will be clear from the context.

We conclude the introduction with few words on the structure of the paper. We recall some known results and give references in section 2. Section 3 is devoted to certain bounds on the resolvent of  $H_0$ . In section 4 we prove Theorem 1.1 and in section 5 we deal with Theorem 5.1 and its applications to the fractional Laplacian and a Klein-Gordon operator.

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## 2. PRELIMINARIES

**2.1. Theorem of Borichev-Golinskii-Kupin.** The following theorem, proved in [BoGoKu, Theorem 0.2], gives a bound on the zeros of a holomorphic function on the unit disc  $\mathbb{D} = \{|z| < 1\}$  in terms of its growth towards the boundary  $\mathbb{T} := \{|z| = 1\}$ .

**Theorem 2.1.** *Let  $h$  be a holomorphic function on  $\mathbb{D}$  with  $h(0) = 1$ . Assume that  $h$  satisfies a bound of the form*

$$|h(z)| \leq \exp \left( \frac{K}{(1 - |z|)^\alpha} \prod_{j=1}^N \frac{1}{|z - \zeta_j|^{\beta_j}} \right),$$

where  $|\zeta_j| = 1$  and  $\alpha, \beta_j \geq 0, j = 1, \dots, N$ .

Then for any  $\tau > 0$  the zeros of  $h$  satisfy the inequality

$$\sum_{h(z)=0} (1 - |z|)^{\alpha+1+\tau} \prod_{j=1}^N |z - \zeta_j|^{(\beta_j-1+\tau)_+} \leq C \cdot K,$$

where  $C$  depends on  $\alpha, \beta_j, \zeta_j$  and  $\tau$ .

Above,  $x_+ = \max\{x, 0\}$ .

**2.2. Conformal mappings.** Let  $\varphi_a$  be a map sending the unit disc to the resolvent set of the operator  $H_0, \rho(H_0) = \mathbb{C} \setminus \mathbb{R}^+$ . It is defined by :

$$(2.1) \quad \varphi_a : z \mapsto \lambda := -a \left( \frac{z+1}{z-1} \right)^2,$$

where  $a > 0$ , and the inverse map defined in  $\bar{\mathbb{C}} \setminus \mathbb{R}^+$  is

$$(2.2) \quad \varphi_a^{-1} : \lambda \mapsto z := \frac{\sqrt{\lambda} - i\sqrt{a}}{\sqrt{\lambda} + i\sqrt{a}}.$$

Later in the paper, we will have to compare the distance from  $\lambda = \varphi_a(z)$  to  $\partial\rho(H_0) = \mathbb{R}^+$ , the boundary of  $\rho(H_0)$ , and the distance from  $z$  to  $\partial\mathbb{D} = \mathbb{T}$ . This kind of results are called distortion theorems. We note by  $d(z, A) := \inf_{w \in A} |z - w|$ , the distance between  $z$  and  $A$ .

The following technical proposition is a direct application of Koebe distortion theorem [Po, Corollary 1.4] to the map  $\varphi_a$ , so the proof is omitted.

**Proposition 2.2** (From  $\mathbb{C} \setminus \mathbb{R}^+$  to  $\mathbb{D}$ ). *We have*

$$a \cdot d(z, \mathbb{T}) \frac{|z+1|}{|z-1|^3} \leq d(\lambda, \mathbb{R}^+) \leq 8a \cdot d(z, \mathbb{T}) \frac{|1+z|}{|1-z|^3}.$$

**Proposition 2.3** (From  $\mathbb{D}$  to  $\mathbb{C} \setminus \mathbb{R}^+$ ). *The following relation holds true*

$$\frac{1}{2\sqrt{a}} \cdot \frac{d(\lambda, \mathbb{R}^+)}{\sqrt{|\lambda|(1+|\lambda|)}} \leq d(z, \mathbb{T}) \leq 4\sqrt{a} \frac{d(\lambda, \mathbb{R}^+)}{\sqrt{|\lambda|(1+|\lambda|)}}.$$

*Proof.* By simple computations we obtain

$$(2.3) \quad |1+z| = \frac{2\sqrt{|\lambda|}}{|\sqrt{\lambda} + i\sqrt{a}|} \text{ and } |1-z| = \frac{2\sqrt{a}}{|\sqrt{\lambda} + i\sqrt{a}|}.$$

And moreover

$$|\lambda| + 1 \leq |\lambda| + a = |\sqrt{\lambda} + i\sqrt{a}|^2 \leq \left(|\sqrt{\lambda}| + |\sqrt{a}|\right)^2 \leq 2a(1 + |\lambda|)$$

since  $a \geq 1$  (see the beginning of section 4). Using Proposition 2.2 we find the inequalities.  $\square$

**2.3. Schatten classes and determinants.** One can find the definitions and properties of Schatten classes and determinants related to these classes in [DeHaKa] or [Du] for instance. For detailed discussion and proofs, see the monographs by Gohberg-Krein [GoKr] and Simon [Si].

Let us introduce the following operator defined by

$$(2.4) \quad F(\lambda) := (\lambda + a)(a + H)^{-1}V(\lambda - H_0)^{-1},$$

where  $a$  is large enough in order that  $(a + H)$  is invertible. From the next proposition 2.4 we deduced that  $V(\lambda - H_0)^{-1} \in \mathbf{S}_p$  for  $\lambda \in \rho(H_0)$  and  $p > \frac{d}{2s}$  since  $V \in L^p(\mathbb{R}^d, \mathcal{M}_n(\mathbb{C}))$ .

Therefore, for  $\lambda \in \rho(H_0)$ ,  $F(\lambda) \in \mathbf{S}_p$ . The holomorphic function of interest is

$$(2.5) \quad f(\lambda) := \det_{[p]}(\text{Id} - F(\lambda)).$$

We proved in [Du, subsection 4.2] the following proposition.

**Proposition 2.4.** *Let  $V \in L^p(\mathbb{R}^d; \mathbb{C}^n)$ ,  $p > \frac{d}{2s}$ , and  $\lambda \in \rho(H_0)$ .*

*Then  $V(\lambda - H_0)^{-1} \in \mathbf{S}_p$ , and*

$$\|V(\lambda - H_0)^{-1}\|_{\mathbf{S}_p}^p \leq (2\pi)^{-d} \|V\|_{L^p}^p \cdot \|(\lambda - |x|^{2s})^{-1}\|_{L^p}^p.$$

## 3. BOUND ON THE RESOLVENT

In this section, we bound the expression  $\|(\lambda - |x|^{2s})^{-1}\|_{L^p}$  appearing in Proposition 2.4. We will repeatedly use the fact that, for  $\alpha > 0$ , and  $a, b > 0$

$$(a + b)^\alpha \approx a^\alpha + b^\alpha.$$

**Proposition 3.1.** *Let  $\lambda = \lambda_0 + i\lambda_1 \in \mathbb{C} \setminus \mathbb{R}^+$  and  $p > \frac{d}{2s}$ .*

*For  $0 < s \leq \frac{d}{2}$ , we have*

$$(3.1) \quad \left\| (\lambda - |x|^{2s})^{-1} \right\|_{L^p}^p \leq C_1 \frac{|\lambda|^{\frac{d}{2s}-1}}{d(\lambda, \sigma(H_0))^{p-1}},$$

*and for  $s > \frac{d}{2}$ , we have*

$$(3.2) \quad \left\| (\lambda - |x|^{2s})^{-1} \right\|_{L^p}^p \leq \frac{C_2}{d(\lambda, \sigma(H_0))^{p-\frac{d}{2s}}},$$

*where  $C_1$  and  $C_2$  are constants depending on  $p, d, n$  and  $s$ .*

*Proof.* After a change of variable, we are led to bound

$$(3.3) \quad I = \int_0^{+\infty} \frac{r^{d-1}}{|r^{2s} - \lambda|^p} dr = \int_0^{+\infty} \frac{r^{d-1}}{|(r^{2s} - \lambda_0)^2 + \lambda_1^2|^{\frac{p}{2}}} dr.$$

We assume first that  $\lambda_0 < 0$ , that is  $d(\lambda, \sigma(H_0)) = |\lambda|$ . In (3.3), we use  $(r^{2s} - \lambda_0)^2 \geq r^{4s} + \lambda_0^2$ , and make the change of variable  $t = \frac{r^{2s}}{|\lambda|}$ , hence

$$(3.4) \quad \begin{aligned} I &\leq \int_0^{+\infty} \frac{r^{d-1}}{(r^{4s} + |\lambda|^2)^{\frac{p}{2}}} dr \\ &\leq \frac{1}{2s} \cdot \frac{|\lambda|^{\frac{d}{2s}}}{|\lambda|^p} \int_0^{+\infty} \frac{t^{\frac{d}{2s}-1}}{(t^2 + 1)^{\frac{p}{2}}} dt \\ &\leq C \cdot \frac{|\lambda|^{\frac{d}{2s}}}{|\lambda|^p}. \end{aligned}$$

The integral in (3.4) is convergent since  $p > \frac{d}{2s} > 0$ .

For the rest of the proof we have  $\lambda_0 \geq 0$ . In (3.3), we make the change of variable  $t = \frac{r^{2s} - \lambda_0}{\lambda_1}$ . Hence,

$$(3.5) \quad \begin{aligned} I &= \frac{1}{2s\lambda_1^{p-1}} \int_{-\frac{\lambda_0}{\lambda_1}}^{+\infty} \frac{(\lambda_1 t + \lambda_0)^{\frac{d-1}{2s}} (\lambda_1 t + \lambda_0)^{\frac{1}{2s}-1}}{(t^2 + 1)^{\frac{p}{2}}} dt \\ &\approx \frac{1}{\lambda_1^{p-1}} \int_{-\frac{\lambda_0}{\lambda_1}}^{+\infty} \frac{(\lambda_1 t + \lambda_0)^{\frac{d}{2s}-1}}{(t^2 + 1)^{\frac{p}{2}}} dt. \end{aligned}$$

If  $s = \frac{d}{2}$ , then the bound is obvious.

We suppose that  $0 < s < \frac{d}{2}$  and  $\lambda_1 > 0$ , so  $-\frac{\lambda_0}{\lambda_1} < 0$ .

$$\begin{aligned} \int_{-\frac{\lambda_0}{\lambda_1}}^{+\infty} \frac{(\lambda_1 t + \lambda_0)^{\frac{d}{2s}-1}}{(t^2 + 1)^{\frac{p}{2}}} dt &= \int_{-\frac{\lambda_0}{\lambda_1}}^0 \frac{(\lambda_1 t + \lambda_0)^{\frac{d}{2s}-1}}{(t^2 + 1)^{\frac{p}{2}}} dt \\ &\quad + \int_0^{+\infty} \frac{(\lambda_1 t + \lambda_0)^{\frac{d}{2s}-1}}{(t^2 + 1)^{\frac{p}{2}}} dt. \end{aligned}$$

In the first integral of the right-hand side of the previous equality, we have  $\lambda_1 t + \lambda_0 \leq \lambda_0$ , and in the second one, we have  $(\lambda_1 t + \lambda_0)^{\frac{d}{2s}-1} \lesssim (\lambda_1 t)^{\frac{d}{2s}-1} + \lambda_0^{\frac{d}{2s}-1}$ . Then

$$\begin{aligned} I &\lesssim \frac{1}{\lambda_1^{p-1}} \left[ \lambda_0^\alpha \int_{-\frac{\lambda_0}{\lambda_1}}^0 \frac{1}{(t^2 + 1)^{\frac{p}{2}}} dt + \right. \\ &\quad \left. + \lambda_1^\alpha \int_0^{+\infty} \frac{t^\alpha}{(t^2 + 1)^{\frac{p}{2}}} dt + \lambda_0^\alpha \int_0^{+\infty} \frac{1}{(t^2 + 1)^{\frac{p}{2}}} dt \right] \\ &\lesssim \frac{1}{\lambda_1^{p-1}} [\lambda_0^\alpha + \lambda_1^\alpha], \end{aligned}$$

hence,  $I \leq C \cdot \frac{|\lambda|^{\frac{d}{2s}-1}}{d(\lambda, \sigma(H_0))^{p-1}}$ .

For  $\lambda_1 < 0$ , we take the absolute value of  $\lambda_1 t + \lambda_0$ . This finishes the case  $s \leq \frac{d}{2}$ .

We now assume that  $s > \frac{d}{2}$  and again  $\lambda_1 > 0$ . From (3.5), we know that

$$I \approx \frac{1}{\lambda_1^{p-1}} \int_{-\frac{\lambda_0}{\lambda_1}}^{+\infty} \frac{(\lambda_1 t + \lambda_0)^\alpha}{(t^2 + 1)^{\frac{p}{2}}} dt.$$

Take care that  $-1 < \alpha = \frac{d}{2s} - 1 < 0$ . Making the change of variable  $u = t + \frac{\lambda_0}{\lambda_1}$ , we obtain

$$\begin{aligned} \int_{-\frac{\lambda_0}{\lambda_1}}^{+\infty} \frac{(\lambda_1 t + \lambda_0)^\alpha}{(t^2 + 1)^{\frac{p}{2}}} dt &= \lambda_1^\alpha \int_{-\frac{\lambda_0}{\lambda_1}}^{+\infty} \frac{(t + \frac{\lambda_0}{\lambda_1})^\alpha}{(t^2 + 1)^{\frac{p}{2}}} dt \\ &= \lambda_1^\alpha \int_0^{+\infty} \frac{u^\alpha}{\left( (u - \frac{\lambda_0}{\lambda_1})^2 + 1 \right)^{\frac{p}{2}}} du. \end{aligned}$$

Then we bound the last integral in the following way :

$$\begin{aligned} \int_0^{+\infty} \frac{u^\alpha}{\left( (u - \frac{\lambda_0}{\lambda_1})^2 + 1 \right)^{\frac{p}{2}}} du &\leq \int_0^1 u^\alpha du + \int_1^{+\infty} \frac{1}{\left( (u - \frac{\lambda_0}{\lambda_1})^2 + 1 \right)^{\frac{p}{2}}} du \\ &\leq \int_0^1 u^\alpha du + \int_{\mathbb{R}} \frac{1}{(u^2 + 1)^{\frac{p}{2}}} du. \end{aligned}$$

Indeed, for the first bound, we use that  $(u - \frac{\lambda_0}{\lambda_1})^2 + 1 \geq 1$  when  $0 \leq u \leq 1$ , and  $u^\alpha \geq 1$  when  $u \geq 1$  since  $\alpha < 0$ . Hence,  $I \leq C \cdot \frac{\lambda_1^\alpha}{\lambda_1^{p-1}}$ . This concludes the proof.  $\square$

## 4. PROOF OF THE MAIN RESULT

Recall from (2.5) that  $f(\lambda) = \det_{[p]}(\text{Id} - F(\lambda))$ , with

$$F(\lambda) := (\lambda + a)(a + H)^{-1}V(\lambda - H_0)^{-1} \in \mathbf{S}_p.$$

We have by the property of the regularized determinant

$$|f(\lambda)| \leq \exp\left(\Gamma_p \|(\lambda + a)(a + H)^{-1}V(\lambda - H_0)^{-1}\|_{\mathbf{S}_p}^p\right).$$

As in [Du, Lemma 5.1], we can prove that, for  $p > \frac{d}{2s}$  and  $a$  large enough, we have

$$\|(a + H)^{-1}\| \leq 1.$$

Applying Propositions 2.4, we get to

$$(4.1) \quad \log |f(\lambda)| \leq C \|V\|_{L^p}^p |\lambda + a|^p \|(\lambda - |x|^{2s})^{-1}\|_{L^p}^p$$

for  $\lambda \in \mathbb{C} \setminus \mathbb{R}^+$  and  $C$  depends on  $p, d$  and  $n$ .

4.1. **Case**  $0 < s \leq \frac{d}{2}$ . Using (3.1), we have

$$\log |f(\lambda)| \leq C \|V\|_{L^p}^p \frac{|\lambda + a|^p |\lambda|^{\frac{d}{2s}-1}}{d(\lambda, \sigma(H_0))^{p-1}}.$$

and  $C$  depends only on  $d, n$ , and  $p$ .

We now have to go in  $\mathbb{D}$  in order to apply Theorem 2.1. That is, recalling definitions, we consider the function  $g(z) = f \circ \varphi_a(z)$ ; it is trivially holomorphic on  $\mathbb{D}$ . By definition (2.1) we have  $|\lambda + a| = \frac{4a|z|}{|1-z|^2}$ . By the previous relation, (4.1), and Proposition 2.2, we obtain

$$\begin{aligned} \log |g(z)| &\leq C \cdot \|V\|_{L^p}^p \frac{a^{\frac{d}{2s}} |z|^p}{d(z, \mathbb{T})^{p-1} |1+z|^{p-\frac{d}{s}+1} |1-z|^{\frac{d}{s}-p+1}} \\ &\leq C \cdot \|V\|_{L^p}^p \frac{a^{\frac{d}{2s}}}{d(z, \mathbb{T})^{p-1} |1+z|^{p-\frac{d}{s}+1} |1-z|^{\frac{d}{s}-p+1}}. \end{aligned}$$

We apply Theorem 2.1, hence for  $\tau > 0$ , we have

$$(4.2) \quad \sum_{g(z)=0} (1-|z|)^{p+\tau} |z-1|^{\frac{d}{s}-p+\tau} |z+1|^{(p-\frac{d}{s}+\tau)_+} \leq C \cdot a^{\frac{d}{2s}} \cdot \|V\|_{L^p}^p,$$

where  $C$  depends on  $d, n, p, s$ , and  $\tau$ .

Consider first the case  $\frac{d}{2s} < p < \frac{d}{s}$ . Hence, for  $0 < \tau < \frac{d}{s} - p$ , (4.2) becomes

$$(4.3) \quad \sum_{g(z)=0} (1-|z|)^{p+\tau} |z-1|^{\frac{d}{s}-p+\tau} \leq C \cdot a^{\frac{d}{2s}} \cdot \|V\|_{L^p}^p,$$

where  $C$  depends on  $d, n, p, s$ , and  $\tau$ .

In the second case, we have  $p = \frac{d}{s}$ . We obtain

$$(4.4) \quad \sum_{g(z)=0} (1-|z|)^{p+\tau} |z-1|^\tau |1+z|^\tau \leq C \cdot a^{\frac{d}{2s}} \cdot \|V\|_{L^p}^p,$$

where  $C$  depends on  $d, n, p, s$ , and  $\tau$ .

The third and last case is  $\frac{d}{s} < p$ , and (4.2) becomes, for  $0 < \tau < p - \frac{d}{s}$ ,

$$(4.5) \quad \sum_{g(z)=0} (1 - |z|)^{p+\tau} |z + 1|^{p - \frac{d}{s} + \tau} \leq C \cdot a^{\frac{d}{2s}} \cdot \|V\|_{L^p}^p,$$

where  $C$  depends on  $d, n, p, s$ , and  $\tau$ .

The last step of the proof consists in transferring relations (4.3), (4.4), and (4.5) back to  $\rho(H_0) = \mathbb{C} \setminus \mathbb{R}^+$ . Remind that we have by Proposition 2.3

$$1 - |z| = d(z, \mathbb{T}) \geq \frac{1}{2\sqrt{a}} \cdot \frac{d(\lambda, \sigma(H_0))}{|\lambda|^{1/2}(1 + |\lambda|)},$$

$$|1 + z|^2 \geq \frac{4|\lambda|}{a(1 + |\lambda|)}, \quad \text{and} \quad |1 - z|^2 \geq \frac{4}{1 + |\lambda|}.$$

From equation (4.3) we get

$$(1 - |z|)^{p+\tau} |z - 1|^{\frac{d}{s} - p + \tau} \geq C \cdot \frac{d(\lambda, \sigma(H_0))^{p+\tau} \cdot a^{-\frac{p+\tau}{2}}}{|\lambda|^{(p+\tau)/2} (1 + |\lambda|)^{\frac{d}{2s} + \frac{p}{2} + \frac{3\tau}{2}}}.$$

From equation (4.4) we obtain

$$(1 - |z|)^{p+\tau} |z - 1|^\tau |z + 1|^\tau \geq C \cdot \frac{d(\lambda, \sigma(H_0))^{p+\tau} \cdot a^{-\frac{p}{2} - \tau}}{|\lambda|^{\frac{p}{2}} (1 + |\lambda|)^{p+2\tau}}.$$

From equation (4.5) we get

$$(1 - |z|)^{p+\tau} |z + 1|^{p - \frac{d}{s} + \tau} \geq C \cdot \frac{d(\lambda, \sigma(H_0))^{p+\tau} \cdot a^{\frac{d}{2s} - (p+\tau)}}{|\lambda|^{\frac{d}{2s}} (1 + |\lambda|)^{\frac{3p}{2} - \frac{d}{2s} + \frac{3\tau}{2}}}.$$

The case  $0 < s \leq \frac{d}{2}$  is done.

**4.2. Case  $s > \frac{d}{2}$ .** Using (3.2), (4.1) becomes

$$\log |f(\lambda)| \leq C \|V\|_{L^p}^p \frac{|\lambda + a|^p}{d(\lambda, \sigma(H_0))^{p - \frac{d}{s}}}.$$

with  $C$  depending only on  $d, n$ , and  $p$ .

Then we have

$$\log |g(z)| \lesssim a^{\frac{d}{2s}} \cdot \|V\|_{L^p}^p \frac{1}{d(z, \mathbb{T})^{p - \frac{d}{2s}} |1 - z|^{\frac{3d}{2s} - p} |1 + z|^{p - \frac{d}{2s}}}.$$

Applying Theorem 2.1, we obtain

$$\sum_{g(z)=0} (1 - |z|)^{p - \frac{d}{2s} + 1 + \tau} |z - 1|^{(\frac{3d}{2s} - p - 1 + \tau)_+} |z + 1|^{(p - \frac{d}{2s} - 1 + \tau)_+} \lesssim a^{\frac{d}{2s}} \cdot \|V\|_{L^p}^p.$$

The possible cases with respect to  $p$  and  $\frac{d}{2s}$  are clear from the following drawing (Figure 1). The x-axis represents  $p$  and the y-axis represents  $\frac{d}{2s}$ . There is 3 lines of the equations  $y = 1$ ,  $x - y - 1 = 0$ , and  $-x + 3y - 1 = 0$ .

Consequently, there are three cases to consider :

- case 1. The part 1 :  $p - \frac{d}{2s} - 1 \geq 0$  and  $\frac{3d}{2s} - p - 1 \leq 0$ .
- case 2. The part 2 :  $p - \frac{d}{2s} - 1 \leq 0$  and  $\frac{3d}{2s} - p - 1 \leq 0$ .
- case 3. The part 3 :  $p - \frac{d}{2s} - 1 \leq 0$  and  $\frac{3d}{2s} - p - 1 \geq 0$ .

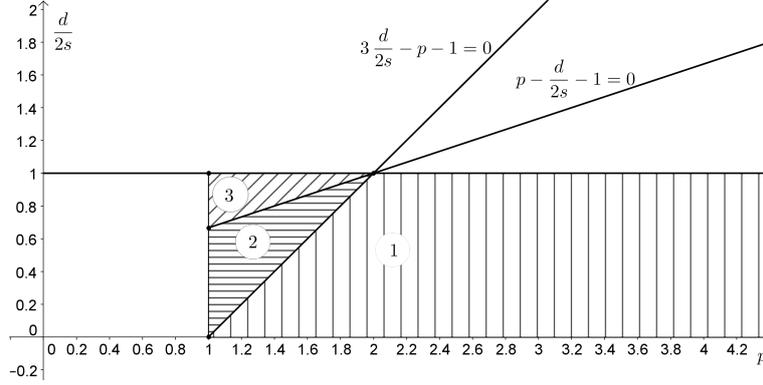


FIGURE 1. The different cases

The computations are similar to the case  $s \leq \frac{d}{2}$ , so they are omitted. In the first case we find

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p - \frac{d}{2s} + 1 + \tau}}{|\lambda|(1 + |\lambda|)^{\frac{3p}{2} - \frac{3d}{4s} + \frac{1}{2} + \frac{3\tau}{2}}} \lesssim a^{p+\tau} \cdot \|V\|_{L^p}^p.$$

In the second case, we have

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p - \frac{d}{2s} + 1 + \tau}}{|\lambda|^{\frac{p}{2} - \frac{d}{4s} + \frac{1}{2} + \frac{\tau}{2}} (1 + |\lambda|)^{p - \frac{d}{2s} + 1 + \tau}} \lesssim a^{\frac{p+1+\tau}{2} + \frac{d}{4s}} \cdot \|V\|_{L^p}^p.$$

In the last case, we have

$$\sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^{p - \frac{d}{2s} + 1 + \tau}}{|\lambda|^{\frac{p}{2} - \frac{d}{4s} + \frac{1}{2} + \frac{\tau}{2}} (1 + |\lambda|)^{\frac{p}{2} + \frac{d}{4s} + \frac{1}{2} + \frac{3\tau}{2}}} \lesssim a^{\frac{p+1+\tau}{2} + \frac{d}{4s}} \cdot \|V\|_{L^p}^p.$$

The proof of the Theorem 1.1 is finished.  $\square$

## 5. LIEB-THIRRING BOUND USING A THEOREM FROM [Ha]

**5.1. Theorem and conformal mapping.** The following theorem is the main ingredient to obtain result (1.5). It is proved in [Ha].

**Theorem 5.1.** *Let  $A \in \mathcal{L}(\mathcal{H})$  be self-adjoint and let  $B \in \mathcal{L}(\mathcal{H})$  such that  $B - A \in \mathcal{S}_p(\mathcal{H})$  for some  $p > 1$ . Then the following holds:*

$$\sum_{\lambda \in \sigma_d(B)} d(\lambda, \sigma(A))^p \leq C_p \|B - A\|_{\mathcal{S}_p}^p,$$

where  $C_p$  depends only on  $p$ .

For explicit constants  $C_p$  see [Ha].

Since we will apply Theorem 5.1 to the resolvents of the operators, we need to know how the distance to the spectrum behaves by some conformal mappings. We introduce the following map  $g : \mathbb{C} \setminus [0, +\infty[ \rightarrow \mathbb{C} \setminus [-\frac{1}{a}, 0[$  define by

$$g(\lambda) = \frac{-1}{a + \lambda}.$$

Recall that  $a \geq 1$ . For the sequel we denote respectively by  $\lambda$  and  $\mu$  the variables in  $\mathbb{C} \setminus \mathbb{R}^+$  and  $\mathbb{C} \setminus [-\frac{1}{a}, 0[$ .

The following technical proposition concerns the distortion between the spectrum of  $H_0$  and the spectrum of the resolvent of  $H$ .

**Proposition 5.2.** *The distortion between  $\mathbb{C} \setminus [-\frac{1}{a}, 0[$  and  $\mathbb{C} \setminus \mathbb{R}^+$  is given by*

$$d\left(g(\lambda), \left[-\frac{1}{a}, 0\right]\right) \geq \frac{1}{2a^{3p}} \cdot \frac{d(\lambda, \mathbb{R}^+)}{|1 + \lambda| \cdot (1 + |\lambda|)}.$$

*Proof.* We compute the distortion for the function  $\tilde{g} : \mathbb{C} \setminus \{[a; +\infty[ \} \rightarrow \mathbb{C} \setminus \{]0; \frac{1}{a}]\}$  define by  $\tilde{g}(\lambda) = \frac{1}{\lambda}$  and then compose by the translation  $T : \lambda \mapsto \lambda + a$ , that is  $g = -\tilde{g} \circ T$ .

After technical computations, we find

$$\frac{1}{2a} \frac{d(\lambda, [a; +\infty[)}{|\lambda| \cdot (1 + |\lambda|)} \leq d\left(\tilde{g}(\lambda), \left]0; \frac{1}{a}\right]\right) \leq 2(1 + a) \frac{d(\lambda, [a; +\infty[)}{|\lambda| \cdot (1 + |\lambda|)}.$$

The claimed inequality follows.  $\square$

**5.2. Proof of Theorem 1.2.** Set  $A = (-a - H_0)^{-1}$  and  $B = (-a - H)^{-1}$  with  $a$  large enough in order that  $A$  and  $B$  exist. We have  $B - A = BV A \in \mathbf{S}_p$ , hence we can apply Theorem 5.1. This gives

$$(5.1) \quad \sum_{\mu \in \sigma_d(B)} d(\mu, \sigma(A))^p \leq C_p \|B - A\|_{\mathbf{S}_p}^p.$$

Now we bound the right-hand side of (5.1),

$$(5.2) \quad \begin{aligned} \|B - A\|_{\mathbf{S}_p}^p &\leq C \|(-a - H)^{-1}\| \cdot \|V\|_{L^p}^p \cdot \|(-a - |x|^{2s})^{-1}\|_{L^p}^p \\ &\leq C \cdot |a|^{\frac{d}{2s} - p} \cdot \|V\|_{L^p}^p. \end{aligned}$$

Then  $\mu = (-a - \lambda)^{-1} = g(\lambda) \in \sigma_d(B)$  if and only if  $\lambda \in \sigma_d(H)$ , hence

$$(5.3) \quad \begin{aligned} \sum_{\mu \in \sigma_d(B)} d(\mu, \sigma(A))^p &= \sum_{\{g(\lambda), \lambda \in \sigma_d(H)\}} d(g(\lambda), \sigma(A))^p \\ &\gtrsim \frac{1}{a^p} \sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^p}{|\lambda + a|^p (1 + |\lambda + a|)^p} \\ &\gtrsim \frac{1}{a^{3p}} \cdot \sum_{\lambda \in \sigma_d(H)} \frac{d(\lambda, \sigma(H_0))^p}{(1 + |\lambda|)^{2p}}, \end{aligned}$$

since for  $a \geq 1$ ,  $|\lambda + a| \leq a(1 + |\lambda|)$ .

The claim of the theorem follows from (5.2) and (5.3).  $\square$

In [Du] we also were interested to the so-called Klein-Gordon operator, given, for  $m > 0$  and  $n \geq 1$ , by

$$K_m = \sqrt{-\Delta_{\mathbb{R}^d} + m^2} \times \text{Id}_{\mathbb{C}^n}.$$

We know that  $\sigma(K_m) = [m, +\infty[$ .

**Corollary 5.3.** *Let  $K = K_m + V$  be a perturbation of the operator  $K_m$ ,  $m > 0$  and  $V \in L^p(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C}))$ ,  $p > d$ .*

$$(5.4) \quad \sum_{\lambda \in \sigma_d(K)} \frac{d(\lambda, \sigma(K_m))^p}{(1 + |\lambda|)^{2p}} \leq C \cdot \frac{1 + (a + m)^{d-1}}{(a + m)^{2p-1}} \cdot \|V\|_{L^p}^p,$$

where the constant  $C$  depends on  $n$ ,  $d$ , and  $p$ .

*Proof.* The hypothesis of Theorem 5.1 are fulfilled with  $A = (-a - K_m)^{-1}$  and  $B = (-a - K)^{-1}$  for  $a$  large enough. We change  $a$  in  $a + m$  in Proposition 5.2 and bound  $\|V(-a - K_m)^{-1}\|_{\mathfrak{S}_p}$  with the results in section 4 of [Du].  $\square$

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