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STABLE DETERMINATION OF TWO COEFFICIENTS IN A DISSIPATIVE WAVE EQUATION FROM BOUNDARY MEASUREMENTS

KAÏS AMMARI AND MOURAD CHOULLI

ABSTRACT. We are concerned with the inverse problem of determining both the potential and the damping coefficient in a dissipative wave equation from boundary measurements. We establish stability estimates of logarithmic type when the measurements are given by the operator who maps the initial conditions to Neumann boundary trace of the solution of the corresponding initial-boundary value problem. We build a method combining an observability inequality together with a spectral decomposition.

Keywords: Damping coefficient, potential, dissipative wave equation, boundary measurements, boundary observability, initial-to-boundary operator.

MSC: 93C25, 93B07, 93C20, 35R30.

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1. INTRODUCTION

We consider the following initial-boundary value problem (abbreviated to IBVP in the sequel) for the wave equation:

$$(1.1) \quad \begin{cases} \partial_t^2 u - \Delta u + q(x)u + a(x)\partial_t u = 0 & \text{in } \Omega \times (0, \tau), \\ u = 0 & \text{in } \partial\Omega \times (0, \tau), \\ u(\cdot, 0) = u_0, \partial_t u(\cdot, 0) = u_1, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$ and $\tau > 0$.

In all of this text, the coefficients q and a are assumed to be real-valued.

Under the assumptions that $q, a \in L^\infty(\Omega)$, for each $\tau > 0$ and $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the IBVP (1.1) has a unique solution $u_{q,a} \in C([0, \tau], H_0^1(\Omega))$ such that $\partial_t u_{q,a} \in C([0, \tau], L^2(\Omega))$ (e.g. for instance pages 699-702 of [5]). On the other hand, by a classical energy estimate, we have

$$\|u_{q,a}\|_{C([0,\tau],H_0^1(\Omega))} + \|\partial_t u_{q,a}\|_{C([0,\tau],L^2(\Omega))} \leq C(\|u_0\|_{1,2} + \|u_1\|_0).$$

Here and henceforth, $\|\cdot\|_p$ and $\|\cdot\|_{s,p}$, $1 \leq p \leq \infty$, $s \in \mathbb{R}$, denote respectively the usual L^p -norm and the $W^{s,p}$ -norm.

We note that the constant C above is a non decreasing function of $\|q\|_\infty + \|a\|_\infty$.

Now, since $u_{q,a}$ coincides with the solution of the IBVP (1.1) in which $-q(x)u_{q,a} - a(x)\partial_t u_{q,a}$ is seen as a right-hand side, we can apply Theorem 2.1 in [15] to deduce that $\partial_\nu u_{q,a}$, the derivative of the $u_{q,a}$ in the

direction of ν , the unit outward normal vector to $\partial\Omega$, belongs to $L^2(\partial\Omega \times (0, \tau))$. In addition, the mapping

$$(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega) \longrightarrow \partial_\nu u_{q,a} \in L^2(\partial\Omega \times (0, \tau))$$

defines a bounded operator.

Let Γ be an open non empty subset of $\partial\Omega$ and $\Sigma = \Gamma \times (0, \tau)$. To $q, a \in L^\infty(\Omega)$, we associate the initial-to-boundary (abbreviated to IB in the following) operator $\Lambda_{q,a}$ defined by

$$\Lambda_{q,a} : (u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega) \longrightarrow \partial_\nu u_{q,a}|_\Sigma \in L^2(\Sigma).$$

Clearly, from the above discussion, $\Lambda_{q,a} \in \mathcal{B}(H_0^1(\Omega) \times L^2(\Omega), L^2(\Sigma))$.

We also consider two partial IB operators: Λ_q and $\tilde{\Lambda}_{q,a}$ given by

$$\Lambda_q(u_0) = \Lambda_{q,0}(u_0, 0), \quad \tilde{\Lambda}_{q,a}(u_1) = \Lambda_{q,a}(0, u_1).$$

Therefore, $\Lambda_q \in \mathcal{B}(H_0^1(\Omega), L^2(\Sigma))$ and $\tilde{\Lambda}_{q,a} \in \mathcal{B}(L^2(\Omega), L^2(\Sigma))$.

Next, we see that $\partial_t u$ is the solution of the IBVP (1.1) corresponding to the initial conditions u_1 and $\Delta u_0 - qu_0 - au_1$. Repeating for $\partial_t u$ the same arguments as for u , we get that

$$\Lambda_{q,a} \in \mathcal{B}([H_0^1(\Omega) \cap H^2(\Omega)] \times H_0^1(\Omega), H^1((0, \tau), L^2(\Gamma))).$$

Consequently,

$$\begin{aligned} \Lambda_q &\in \mathcal{B}(H_0^1(\Omega) \cap H^2(\Omega), H^1((0, \tau), L^2(\Gamma))), \\ \tilde{\Lambda}_{q,a} &\in \mathcal{B}(H_0^1(\Omega), H^1((0, \tau), L^2(\Gamma))). \end{aligned}$$

We are interested to the stability issue for the inverse problem consisting in the determination of both the potential q and the damping coefficient a , appearing in the IBVP (1.1), from the IB map $\Lambda_{q,a}$. We succeed in proving logarithmic stability estimates of determining q from Λ_q , a from $\tilde{\Lambda}_{q,a}$ and (q, a) from $\Lambda_{q,a}$.

We introduce the unbounded operators, defined on $H_0^1(\Omega) \times L^2(\Omega)$,

$$\mathcal{A}_0 = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \quad D(\mathcal{A}_0) = [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$$

and $\mathcal{A} = \mathcal{A}_{q,a} = \mathcal{A}_0 + \mathcal{B}$ with $D(\mathcal{A}) = D(\mathcal{A}_0)$, where

$$\mathcal{B} = \mathcal{B}_{q,a} = \begin{pmatrix} 0 & 0 \\ -q & -a \end{pmatrix}.$$

Let

$$\mathcal{C} : D(\mathcal{A}_0) \rightarrow L^2(\Gamma) : (\varphi, \psi) \longrightarrow \partial_\nu \varphi.$$

Since we deal with the wave equation, it is necessary to make assumptions on Γ and τ in order to guarantee that our system is observable. To this end, we assume that Γ is chosen in such a way that there is τ_0 such that the pair $(\mathcal{A}, \mathcal{C})$ is exactly observable for any $\tau \geq \tau_0$. The definition of exact observability will be given in the next section in an abstract framework.

We give sufficient conditions ensuring that the pair $(\mathcal{A}, \mathcal{C})$ is exactly observable. We fix $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ and we set

$$\Gamma_0 = \{x \in \partial\Omega; \nu(x) \cdot (x - x_0) > 0\} \quad \text{and} \quad d = \max_{x \in \overline{\Omega}} |x - x_0|.$$

Let us assume that $\Gamma \supset \Gamma_0$. Following Theorem 7.2.3 in page 233 of [22], $(\mathcal{A}_0, \mathcal{C})$ is exactly observable with $\tau \geq \tau_0 = 2d$. In light of Theorem 7.3.2 in page 235 of [22] and the remark following it, we conclude that $(\mathcal{A}, \mathcal{C})$ is also exactly observable for $\tau \geq \tau_0$, again with $\tau_0 = 2d$.

We mention that sharp sufficient conditions on Γ and τ_0 was given in a work by Bardos, Lebeau and Rauch [6].

Henceforth, $\tau \geq \tau_0$ is fixed and, for sake of simplicity, all operator norms will denoted by $\|\cdot\|$. Also, we use the notation B_p (resp. $B_{s,p}$) for the unit ball of $L^p(\Omega)$ (resp. $W^{s,p}(\Omega)$).

In the present work, we aim to prove the following theorem.

Theorem 1.1. *Let $0 \leq q_0 \in L^\infty(\Omega)$, there is a constant $\delta > 0$ so that*

$$(1.2) \quad \|q - q_0\|_2 \leq C \left| \ln \left(C^{-1} \|\Lambda_{q_0} - \Lambda_q\| \right) \right|^{-1/2}, \quad q \in q_0 + \delta B_{1,\infty},$$

and for any $M > 0$,

$$(1.3) \quad \|a\|_2 \leq C \left| \ln \left(C^{-1} \|\tilde{\Lambda}_{q_0,a} - \tilde{\Lambda}_{q_0,0}\| \right) \right|^{-1/2}, \quad a \in [\delta B_\infty] \cap [MB_{1,2}],$$

$$(1.4) \quad \|q - q_0\|_2 + \|a\|_0 \leq C \left| \ln \left(C^{-1} \|\Lambda_{q,a} - \Lambda_{q_0,0}\| \right) \right|^{-1/2}, \quad q \in q_0 + \delta B_{1,\infty}, \quad a \in [\delta B_\infty] \cap [MB_{1,2}].$$

Here, C is a generic constant not depending on q and a .

This theorem gives only stability estimates at zero damping coefficient. The difficulty of stability estimates at a non zero damping coefficient is related to the fact that the spectral analysis of the operator \mathcal{A} is not easier in that case. We observe that, contrary to case where $a = 0$, this operator is no longer skew-adjoint. We detail a result on the stability estimate at a non zero damping coefficient in a separate section.

The problem of determining the potential in a wave equation from the so-called Dirichlet-to-Neumann (usually abbreviated to DN) map was initiated by Rakesh and Symes [17] (see also [8] and [11]). They prove that the potential can be recovered uniquely from the DN map provided that the length of the time interval is larger than the diameter of the space domain. The key point in their method is the construction of special solutions, called beam solutions. An extension was obtained by Bellassoued, Choulli and Yamamoto [2] in the case of a partial DN map by a method built on the quantification of the continuation of the solution of the wave equation from partial Cauchy data. In the case of the complete DN map, a sharp uniqueness result was proved by the so-called boundary control method. More details on this method can be found for instance in [4] and [14]. Also, when the DN map is given on the whole lateral boundary, Sun [19] establishes Hölder stability estimates. Most recently, Bao and Yun [3] improve the result of [19]. Specifically, they prove a nearly Lipschitz stability estimate. We refer to the introduction of [2] for a short overview of inverse problems related to the wave equation.

This text is organized as follows. In Section 2, we consider the inverse source problem for exactly observable systems in an abstract framework. This material is necessary in establishing stability estimates for the determination of the potential and the damping coefficient appearing in the IBVP (1.1). The Section 3 is devoted to the proof of Theorem 1.1. We give in Section 4 a sufficient condition that guarantee the existence of Riesz basis consisting in eigenfunctions of the operator \mathcal{A} , provided that the L^∞ -norm of a is small. From this spectral decomposition we show that we can get a variant of Theorem 1.1 at a non zero damping coefficient. In Section 5, we explain how our approach can be adapted to other exactly observable systems. Precisely, we study the Euler-Bernoulli plate and Schrödinger equations in a rectangle of \mathbb{R}^2 . The possible adaptation of our method to the heat equation is discussed in Section 6. Due to the fact that the heat equation is not exactly observable but only observable at final time, we obtain a stability estimate only when we perturb the zero order term by a finite dimensional subspace.

2. INVERSE SOURCE PROBLEM: AN ABSTRACT FRAMEWORK

Let H be a Hilbert space and $A : D(A) \subset H \rightarrow H$ be the generator of continuous semigroup $T(t)$. An operator $C \in \mathcal{B}(D(A), Y)$, Y a Hilbert space that we identify with its dual space, is called an admissible observation for $T(t)$ if for some (and hence for all) $\tau > 0$, the operator $\Psi \in \mathcal{B}(D(A), L^2((0, \tau), Y))$ given by

$$(\Psi x)(t) = CT(t)x, \quad t \in [0, \tau], \quad x \in D(A),$$

has a bounded extension to H .

We introduce the notion of exact observability of the system

$$(2.1) \quad z'(t) = Az(t), \quad z(0) = x,$$

$$(2.2) \quad y(t) = Cz(t),$$

where C is an admissible observation for $T(t)$. Following the usual definition, the pair (A, C) is said exactly observable at time $\tau > 0$ if there is a constant κ such that the solution (z, y) of (2.1) and (2.2) satisfies

$$\int_0^\tau \|y(t)\|_Y^2 dt \geq \kappa^2 \|x\|_X^2, \quad x \in D(A).$$

Or equivalently

$$(2.3) \quad \int_0^\tau \|(\Psi x)(t)\|_Y^2 dt \geq \kappa^2 \|x\|_X^2, \quad x \in D(A).$$

We consider the Cauchy problem

$$(2.4) \quad z'(t) = Az(t) + \lambda(t)x, \quad z(0) = 0,$$

and we set

$$(2.5) \quad y(t) = Cz(t), \quad t \in [0, \tau].$$

By Duhamel's formula, we have

$$(2.6) \quad y(t) = \int_0^t \lambda(t-s)CT(s)x ds = \int_0^t \lambda(t-s)(\Psi x)(s) ds.$$

Let

$$H_\ell^1((0, \tau), Y) = \{u \in H^1((0, \tau), Y); u(0) = 0\}.$$

We define the operator $S : L^2((0, \tau), Y) \rightarrow H_\ell^1((0, \tau), Y)$ by

$$(2.7) \quad (Sh)(t) = \int_0^t \lambda(t-s)h(s) ds.$$

If $E = S\Psi$, we observe that (2.6) takes the form

$$y(t) = (Ex)(t).$$

Theorem 2.1. *We assume that (A, C) is exactly observable and $\lambda \in H^1((0, T))$ satisfies $\lambda(0) = 1$. Then E is one-to-one from H onto $H_\ell^1((0, \tau), Y)$ and*

$$(2.8) \quad (\kappa/\sqrt{2})e^{-2\tau\|\lambda'\|_{L^2((0, \tau))}^2} \|x\|_X \leq \|Ex\|_{H_\ell^1((0, \tau), Y)}, \quad x \in X.$$

Proof. First, taking the derivative with respect to t of each side of the integral equation

$$\int_0^t \lambda(t-s)\varphi(s) ds = \psi(t),$$

we get a Volterra equation of second kind

$$\varphi(t) + \int_0^t \lambda'(t-s)\varphi(s) ds = \psi'(t).$$

Mimicking the proof of Theorem 2 in page 33 of [13], we obtain that this integral equation has a unique solution $\varphi \in L^2((0, \tau), Y)$ and

$$\begin{aligned} \|\varphi\|_{L^2((0, \tau), Y)} &\leq C\|\psi'\|_{L^2((0, \tau), Y)} \\ &\leq C\|\psi\|_{H_\ell^1((0, \tau), Y)}. \end{aligned}$$

Here $C = C(\lambda)$ is a constant.

Next, we estimate the constant C above. From the elementary convexity inequality $(a+b)^2 \leq 2(a^2 + b^2)$, we deduce

$$\|\varphi(t)\|_Y^2 \leq 2 \left(\int_0^t |\lambda'(t-s)| \|\varphi(s)\|_Y ds \right)^2 + 2\|\psi'(t)\|_Y^2.$$

Hence

$$\|\varphi(t)\|_Y^2 \leq 2\|\lambda'\|_{L^2((0, \tau))}^2 \int_0^t \|\varphi(s)\|_Y^2 ds + 2\|\psi'(t)\|_Y^2$$

by the Cauchy-Schwarz's inequality. Therefore, using Gronwall's lemma, we obtain in a straightforward manner that

$$\|\varphi\|_{L^2((0,\tau),Y)} \leq \sqrt{2}e^{2\tau\|\lambda'\|_{L^2((0,\tau))}^2} \|\psi'\|_{L^2((0,\tau),Y)}$$

and then

$$\|\varphi\|_{L^2((0,\tau),Y)} \leq \sqrt{2}e^{2\tau\|\lambda'\|_{L^2((0,\tau))}^2} \|S\varphi\|_{H_\ell^1((0,\tau),Y)}.$$

In light of (2.3), we end up getting

$$\|Ex\|_{H_\ell^1((0,\tau),Y)} \geq (\kappa/\sqrt{2})e^{-2\tau\|\lambda'\|_{L^2((0,\tau))}^2} \|x\|_X.$$

□

Let us note that the condition $\lambda(0) = 1$ can be relaxed. It is enough to assume that $\lambda(0) \neq 0$. In that case κ has to be changed to $|\lambda(0)|^{-1}\kappa$.

We need a variant of Theorem 2.1. Let (A, C) be as in Theorem 2.1. As a consequence of Proposition 6.3.3 in page 189 of [22], we obtain that there is $\delta > 0$ such that for any $P \in \mathcal{B}(H)$ satisfying $\|P\| \leq \delta$, $(A + P, C)$ is exactly observable with $\kappa(P + A) \geq \kappa/2$.

We define E^P similarly to E by replacing A by $A + P$.

Theorem 2.2. *We assume that (A, C) is exactly observable and $\lambda \in H^1((0, T))$ satisfies $\lambda(0) = 1$. There is $\delta > 0$ such that, for any $P \in \mathcal{B}(H)$ satisfying $\|P\| \leq \delta$, E^P is one-to-one from H onto $H_\ell^1((0, \tau), Y)$ and*

$$(2.9) \quad (\kappa/(2\sqrt{2}))e^{-2\tau\|\lambda'\|_{L^2((0,\tau))}^2} \|x\|_X \leq \|E^P x\|_{H_\ell^1((0,\tau),Y)}, \quad x \in X.$$

We apply the preceding theorem to the following IBVP for the wave equation:

$$(2.10) \quad \begin{cases} \partial_t^2 u - \Delta u + q(x)u + a(x)\partial_t u = \lambda(t)f(x) & \text{in } \Omega \times (0, \tau), \\ u = 0 & \text{in } \partial\Omega \times (0, \tau), \\ u(\cdot, 0) = 0, \quad \partial_t u(\cdot, 0) = 0. \end{cases}$$

We recall that

$$\mathcal{A}_0 = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \quad D(\mathcal{A}_0) = [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$$

and $\mathcal{A} = \mathcal{A}_{q,a} = \mathcal{A}_0 + \mathcal{B}_{q,a}$ with $D(\mathcal{A}) = D(\mathcal{A}_0)$, where

$$\mathcal{B}_{q,a} = \begin{pmatrix} 0 & 0 \\ -q & -a \end{pmatrix}.$$

Also

$$\mathcal{C} : D(\mathcal{A}_0) \rightarrow L^2(\Gamma) : (\varphi, \psi) \longrightarrow \partial_\nu \varphi.$$

We fix $q_0, a_0 \in L^\infty(\Omega)$ and we assume that $(\mathcal{A}_{q_0, a_0}, \mathcal{C})$ is exactly observable with constant κ . This is for instance the case when $\Gamma \supset \Gamma_0 = \{x \in \partial\Omega; \nu(x) \cdot (x - x_0) > 0\}$ (e.g. Theorem 1.2 in page 141 of [10]¹).

Corollary 2.1. *There is $\delta > 0$ such that, for any $q \in q_0 + \delta B_{1,\infty}$ and $a \in a_0 + \delta B_\infty$, we have*

$$\|f\|_2 \leq (2\sqrt{2}/\kappa)e^{2\tau\|\lambda'\|_{L^2((0,\tau))}^2} \|\partial_\nu u_f\|_{H^1((0,\tau), L^2(\Gamma))},$$

where u_f is the solution of the IBVP (2.10).

This result gives a Lipschitz stability estimate for the inverse problem consisting in the determination of the source term f from the boundary data $\partial_\nu u_f|_\Sigma$, when λ is supposed to be known.

¹We notice that from the proof of this theorem it is not possible to extract the dependance of κ on q_0 and a_0 .

3. PROOF OF THEOREM 1.1

Proof of Theorem 1.1. Let (λ_k) and (ϕ_k) be respectively the sequence of Dirichlet eigenvalues of $-\Delta + q_0$, counted according to their multiplicity, and the corresponding eigenfunctions. We assume that the sequence (ϕ_k) forms an orthonormal basis of $L^2(\Omega)$.

We recall that by the min-max principle, we have the following two-sided estimates

$$(3.1) \quad c^{-1}k^{2/n} \leq \lambda_k \leq ck^{2/n}.$$

Here, the constant $c > 1$ depends only on Ω and q_0 .

Let u_q be the solution of the IBVP (1.1) corresponding to q , $a = 0$, $u_0 = \phi_k$ and $u_1 = 0$. Taking into account that $u_{q_0} = \cos(t\sqrt{\lambda_k})\phi_k$ is the solution of the IBVP (1.1) corresponding to $q = q_0$, $a = 0$, $u_0 = \phi_k$ and $u_1 = 0$, we see that $u = u_q - u_{q_0}$ is the solution of the IBVP

$$(3.2) \quad \begin{cases} \partial_t^2 u - \Delta u + qu = -(q - q_0) \cos(t\sqrt{\lambda_k})\phi_k & \text{in } \Omega \times (0, \tau), \\ u = 0 & \text{on } \partial\Omega \times (0, \tau), \\ u(\cdot, 0) = 0, \partial_t u(\cdot, 0) = 0. \end{cases}$$

In the remaining part of this proof, C is a generic constant not depending on k .

Let δ be as in Corollary 2.1. If $q \in q_0 + \delta B_{1,\infty}$, we get by applying Corollary 2.1:

$$\|(q - q_0)\phi_k\|_2 \leq Ce^{C\lambda_k} \|\partial_\nu u\|_{H^1((0,\tau), L^2(\Gamma))}.$$

Since $|(q - q_0, \phi_k)| \leq |\Omega|^{1/2} \|(q - q_0)\phi_k\|_{L^2(\Omega)}$ by the Cauchy-Schwarz's inequality, the last inequality entails

$$|(q - q_0, \phi_k)| \leq Ce^{C\lambda_k} \|\partial_\nu u\|_{H^1((0,\tau), L^2(\Gamma))}.$$

Or $\partial_\nu u = (\Lambda_{q_0} - \Lambda_q)\phi_k$. Therefore

$$(3.3) \quad |(q - q_0, \phi_k)|^2 \leq Ce^{C\lambda_k} \|\Lambda_{q_0} - \Lambda_q\|^2.$$

Let $\lambda \geq \lambda_1$ and $N = N(\lambda)$ be the smallest integer satisfying $\lambda_N \leq \lambda < \lambda_{N+1}$. Then

$$\begin{aligned} \|q - q_0\|_2^2 &= \sum_k |(q - q_0, \phi_k)|^2 \\ &= \sum_{k \leq N} |(q - q_0, \phi_k)|^2 + \sum_{k > N} |(q - q_0, \phi_k)|^2 \\ &\leq \sum_{k \leq N} |(q - q_0, \phi_k)|^2 + \frac{1}{\lambda} \sum_{k > N} \lambda_k |(q - q_0, \phi_k)|^2 \\ &\leq \sum_{k \leq N} |(q - q_0, \phi_k)|^2 + \frac{C\delta^2}{\lambda}. \end{aligned}$$

Here we used the fact that $(\sum_{k \geq 1} (1 + \lambda_k)(\cdot, \varphi_k)^2)^{1/2}$ defines an equivalent norm on $H^1(\Omega)$.

In light of (3.3), we get

$$\|q - q_0\|_2^2 \leq CN e^{C\lambda} \|\Lambda_{q_0} - \Lambda_q\|^2 + \frac{C\delta^2}{\lambda}.$$

Or by (3.1), $N \leq C\lambda^{n/2}$. Hence

$$\|q - q_0\|_2^2 \leq Ce^{C\lambda} \|\Lambda_{q_0} - \Lambda_q\|^2 + \frac{C\delta^2}{\lambda}.$$

We minimize with respect to λ , we obtain that there is $\delta_0 > 0$ such that if $\|\Lambda_{q_0} - \Lambda_q\| \leq \delta_0$, then

$$\|q - q_0\|_2 \leq C |\ln(C^{-1} \|\Lambda_{q_0} - \Lambda_q\|)|^{-1/2}.$$

Estimate (1.2) follows then from the continuity of the mapping $q \in L^\infty(\Omega) \rightarrow \Lambda_q \in \mathcal{B}(H_0^1(\Omega), L^2(\Sigma))$.

We proceed similarly for proving (1.3). In the actual case we have to replace the previous u_{q_0} by $u_{q_0} = \lambda_k^{-1} \sin(t\sqrt{\lambda_k})\phi_k$, corresponding to the initial conditions $u_0 = 0$ and $u_1 = \phi_k$. Therefore in place of (3.2), we have

$$(3.4) \quad \begin{cases} \partial_t^2 u - \Delta u = -a \cos(t\sqrt{\lambda_k})\phi_k, & \text{in } \Omega \times (0, \tau), \\ u = 0 & \text{on } \partial\Omega \times (0, \tau), \\ u(\cdot, 0) = 0, \partial_t u(\cdot, 0) = 0. \end{cases}$$

We continue the proof as in the preceding case. We get

$$|(a, \phi_k)|^2 \leq C e^{C\lambda_k} \|\tilde{\Lambda}_{q_0, a} - \tilde{\Lambda}_{q_0, 0}\|.$$

We finish the proof as above.

We complete the proof by showing how we proceed for proving (1.4). Taking into account that the solution corresponding to $q = q_0$, $a = 0$, $u_0 = \phi_k$ and $u_1 = i\lambda_k\phi_k$ is $u_{q_0} = e^{i\sqrt{\lambda_k}t}\phi_k$, in place of (3.2) we have the following IBVP

$$(3.5) \quad \begin{cases} \partial_t^2 u - \Delta u + qu = -[(q - q_0) + i\sqrt{\lambda_k}a]e^{i\sqrt{\lambda_k}t}\phi_k, & \text{in } \Omega \times (0, \tau), \\ u = 0 & \text{on } \partial\Omega \times (0, \tau), \\ u(\cdot, 0) = 0, \partial_t u(\cdot, 0) = 0. \end{cases}$$

Here again we can argue as in the proof (1.2). We find

$$|(\varphi, q - q_0) + i\sqrt{\lambda_k}(\varphi, a)|^2 \leq C e^{C\lambda_k} \|\Lambda_{q, a} - \Lambda_{q_0, 0}\|^2,$$

entailing

$$\begin{aligned} |(\varphi, q - q_0)|^2 &\leq C e^{C\lambda_k} \|\Lambda_{q, a} - \Lambda_{q_0, 0}\|^2, \\ |(\varphi, a)|^2 &\leq C e^{C\lambda_k} \|\Lambda_{q, a} - \Lambda_{q_0, 0}\|^2. \end{aligned}$$

We end up getting (1.4) by mimicking the rest of the proof of the estimate (1.2). \square

4. STABILITY ESTIMATES AROUND A NON ZERO DAMPING COEFFICIENT

For sake of simplicity, we assume that the potential q is equal to zero. But the analysis carried out in the present section is still applicable for any non negative bounded potential.

In the sequel $0 < \lambda_1 < \lambda_2 \leq \dots \lambda_k \leq \dots$ denote the sequence of eigenvalues, counted according to their multiplicity, of $-\Delta$ under Dirichlet boundary condition.

We introduced in the first section the unbounded operators, defined on $H_0^1(\Omega) \times L^2(\Omega)$,

$$\mathcal{A}_0 = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \quad D(\mathcal{A}_0) = [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$$

and $\mathcal{A}_a = \mathcal{A}_0 + \mathcal{B}_a$ with $D(\mathcal{A}) = D(\mathcal{A}_0)$, where

$$\mathcal{B}_a = \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix}.$$

We know that $-i\mathcal{A}_0$ is self-adjoint operator whose spectrum is reduced to the sequence $(\sqrt{\lambda_k})$. We make the assumption that Ω is chosen in such a way that the following gap condition holds true²:

$$(4.1) \quad \sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq d > 0, \quad k \geq 1.$$

We set

$$\kappa = \sum_{k \geq 1} \frac{1}{(2k+1)^2} \quad \text{and} \quad \alpha = \frac{d}{2\sqrt{2(1+\kappa)}}.$$

In light of Theorem 2 and Lemma 10 in [18], we can state the following result.

²An example of such a domain will be given in Section 5.

Theorem 4.1. *Under the assumption*

$$\rho := \|a_0\|_\infty < \alpha,$$

the spectrum of \mathcal{A}_{a_0} consists in a sequence $(i\mu_k)$ such that, for any $\delta \in (0, 1 - \rho^2/\alpha^2)$, there is an integer \tilde{k} such that

$$|i\mu_k - i\sqrt{\lambda_k}| \leq \bar{\alpha} = \bar{\alpha}(a) := \frac{\rho d}{\sqrt{4\rho^2 + d^2\delta}}, \quad k \geq \tilde{k}.$$

In addition, $H_0^1(\Omega) \times L^2(\Omega)$ admits a Riesz basis $(\phi_k) = \left(\begin{pmatrix} \varphi_k \\ i\mu_k \varphi_k \end{pmatrix} \right)$, each ϕ_k is an eigenfunction corresponding to $i\mu_k$.

We pick a_0 as in the preceding theorem and let $(\phi_k) = \left(\begin{pmatrix} \varphi_k \\ i\mu_k \varphi_k \end{pmatrix} \right)$ the corresponding Riesz basis consisting of the eigenfunctions of the operator \mathcal{A}_{a_0} . It is straightforward to check that $u_{a_0} = e^{i\mu_k t} \varphi_k$ is the solution of the IBVP (1.1) with $q = 0$, $a = a_0$, $\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \phi_k$. If u_a is the solution of the IBVP (1.1), then $u = u_a - u_{a_0}$ is the solution of the IBVP

$$(4.2) \quad \begin{cases} \partial_t^2 u - \Delta u + a(x)\partial_t u = (a_0 - a)i\mu_k e^{i\mu_k t} \varphi_k & \text{in } \Omega \times (0, \tau), \\ u = 0 & \text{on } \partial\Omega \times (0, \tau), \\ u(\cdot, 0) = 0, \quad \partial_t u(\cdot, 0) = 0. \end{cases}$$

We observe that μ_k are not necessarily real, but asymptotically, $|\mu_k|$ behaves like $\sqrt{\lambda_k}$. Indeed, for fixed δ as in the statement of Theorem 4.1, we have for some integer \tilde{k}

$$\begin{aligned} |e^{i\mu_k t}| &\leq e^{|\mu_k - i\sqrt{\lambda_k}|t} |e^{i\sqrt{\lambda_k} t}| \leq e^{\bar{\alpha}\tau}, \quad k \geq \tilde{k}, \\ |\mu_k| &\leq \sqrt{\lambda_k} + \bar{\alpha}, \quad k \geq \tilde{k}. \end{aligned}$$

With the help of these estimates, we can proceed as in the previous section to get, where $\psi_k = i\mu_k \varphi_k$,

$$(4.3) \quad |(a - a_0, \psi_k)|^2 = \left| \left\langle \begin{pmatrix} 0 \\ a - a_0 \end{pmatrix}, \phi_k \right\rangle \right|^2 \leq C e^{C\lambda_k} \|\Lambda_a - \Lambda_{a_0}\|^2.$$

Now since the biorthogonal basis to (ϕ_k) is also a Riesz basis, we can apply Proposition 2.5.2 in page 37 of [22] to deduce that, where m is some constant,

$$(4.4) \quad m^2 \|a - a_0\|_2^2 = m^2 \left\| \begin{pmatrix} 0 \\ a - a_0 \end{pmatrix} \right\|_{H_0^1(\Omega) \times L^2(\Omega)}^2 \leq \sum_{k \geq 1} \left| \left\langle \begin{pmatrix} 0 \\ a - a_0 \end{pmatrix}, \phi_k \right\rangle \right|^2.$$

In light of (4.3) and (4.4), we have

$$(4.5) \quad \begin{aligned} m^2 \|a - a_0\|_2^2 &\leq C N e^{C\lambda} \|\Lambda_a - \Lambda_{a_0}\|^2 + \frac{1}{\lambda} \sum_{k > N} \lambda_k |(a - a_0, \psi_k)|^2 \\ &\leq C N e^{C\lambda} \|\Lambda_a - \Lambda_{a_0}\|^2 + \frac{1}{\lambda} \sum_{k \geq 1} \lambda_k |(a - a_0, \psi_k)|^2. \end{aligned}$$

Here $\lambda \geq \lambda_1$ and $N = N(\lambda)$ be the smallest integer satisfying $\lambda_N \leq \lambda < \lambda_{N+1}$.

We note that we cannot pursue the proof similarly to that of (1.2) because (ψ_k) is not necessarily an orthonormal basis of $L^2(\Omega)$. So instead of the boundedness of $a - a_0$ in $H^1(\Omega)$, we can only assume the following estimate, where $M > 0$ is fixed,

$$(4.6) \quad \sum_{k \geq 1} \lambda_k |(a - a_0, \psi_k)|^2 \leq M.$$

Under the assumption (4.6), (4.5) entails

$$m^2 \|a - a_0\|_2^2 \leq C e^{C\lambda} \|\Lambda_a - \Lambda_{a_0}\|^2 + \frac{M}{\lambda}.$$

The same minimization argument used in the proof of (1.2) (see Section 3) yields: there are constants $C > 0$ and $\delta > 0$ so that

$$\|a - a_0\|_2 \leq C \left| \ln \left(C^{-1} \|\tilde{\Lambda}_a - \tilde{\Lambda}_{a_0}\| \right) \right|^{-1/2}, \quad a \in a_0 + \delta B_\infty \text{ and (4.6) holds.}$$

5. EXTENSION TO EULER-BERNOULLI PLATE AND SCHRÖDINGER EQUATIONS

For sake of clarity, we limit ourselves to the case where Ω is a rectangle of the form $\Omega = (0, a) \times (0, b)$. However, we believe that the analysis we carry out in the sequel is extendable to other domains.

Following [1], we introduce the following space

$$\begin{aligned} H_{1/2} &= H_0^1(\Omega), \\ H_{3/2} &= \{h \in H^3(\Omega); h = \Delta h = 0 \text{ on } \partial\Omega\}, \\ H_{5/2} &= \{h \in H^5(\Omega); h = \Delta h = \Delta^2 h = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

The natural norm of H_s will be denoted by $\|\cdot\|_s$, $s = j + 1/2$, $j = 0, 1, 2$.

On $\mathcal{H} = H_{3/2} \times H_{1/2}$, we introduce the unbounded operator \mathcal{A} given by

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -(-\Delta)^2 & 0 \end{pmatrix}, \quad D(\mathcal{A}) = H_{5/2} \times H_{3/2}.$$

Henceforth, Γ denotes an open subset of $\partial\Omega$ containing both a horizontal and a vertical segment of non zero length. We define $C : H_1 = H^2(\Omega) \cap H_0^1(\Omega) \rightarrow L^2(\Gamma)$ by

$$Cw = \partial_\nu w$$

and we set $\mathcal{C} = [0 \ C]$, that we consider as an element of $\mathcal{B}(\mathcal{D}(\mathcal{A}), L^2(\Gamma))$.

We consider the following IBVP for the Euler-Bernoulli plate equation

$$(5.1) \quad \begin{cases} \partial_t^2 u + \Delta^2 u = 0 & \text{in } \Omega \times (0, \tau), \\ u = \Delta u = 0 & \text{on } \partial\Omega \times (0, \tau), \\ u(\cdot, 0) = u_0, \quad \partial_t u(\cdot, 0) = u_1. \end{cases}$$

From Theorem 3.2 in [1], for any $\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in H_{5/2} \times H_{3/2}$ the IBVP (5.1) has a unique solution $u \in C([0, \tau], H_{5/2}) \cap C^1([0, \tau], H_{3/2})$. Moreover, $(\mathcal{A}, \mathcal{C})$ is exactly observable and there is a constant κ such that

$$(5.2) \quad \kappa^2 (\|u_0\|_{3/2}^2 + \|u_1\|_{1/2}^2) \leq \|\partial_\nu u\|_{H^1((0, \tau), L^2(\Gamma))}^2.$$

Here the constant κ is independent on u_0 and u_1 .

Let \mathcal{B}_a be the operator, where $a = a(x)$,

$$\mathcal{B}_a = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}.$$

This operator is bounded on \mathcal{H} whenever $a \in W^{1, \infty}(\Omega)$. Therefore, bearing in mind that $\mathcal{A} + \mathcal{B}_a$ generates a continuous semigroup, the IBVP

$$(5.3) \quad \begin{cases} \partial_t^2 u + \Delta^2 u + a(x) \partial_t u = 0 & \text{in } \Omega \times (0, \tau), \\ u = \Delta u = 0 & \text{on } \partial\Omega \times (0, \tau), \\ u(\cdot, 0) = u_0, \quad \partial_t u(\cdot, 0) = u_1, \end{cases}$$

has a unique solution $u \in C([0, T], H_{5/2}) \cap C^1([0, T], H_{3/2})$, for any $\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in H_{5/2} \times H_{3/2}$. Moreover, the same perturbation argument used in the proof of Theorem 2.2 enables us to show that $(\mathcal{A} + \mathcal{B}_a, \mathcal{C})$ is exactly

observable with constant $\tilde{\kappa}^2 \geq \kappa^2/2$ provided the operator norm of \mathcal{B}_a is sufficiently small. That is, there is $\delta > 0$ such that for any $\mathcal{B}_a \in \mathcal{B}(\mathcal{H})$ with $\|\mathcal{B}_a\| \leq \delta$, we have

$$(5.4) \quad (1/2)\kappa^2(\|u_0\|_{3/2}^2 + \|u_1\|_{1/2}^2) \leq \|\partial_\nu u\|_{H^1((0,\tau),L^2(\Gamma))}^2.$$

It is known that \mathcal{A} is skew-adjoint and its spectrum is given by $\sigma(\mathcal{A}) = \{i\lambda_{k\ell}; k, \ell \in \mathbb{N}^*\}$ (here $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$), where

$$\lambda_{k\ell} = \pi^2 \left(\frac{k^2}{a^2} + \frac{\ell^2}{b^2} \right).$$

We recall that a finite or infinite sequence of real numbers is said to be non resonant if every nontrivial rational linear combination of finitely many of its elements is different from zero.

In the sequel we assume that $(a/\pi, b/\pi)$ are non resonant. In light of the fact that the sequence $(\lambda_{k\ell})$ represents the sequence of the Dirichlet-Laplacian on $\Omega = (0, a) \times (0, b)$, these eigenvalues are simple as it is noticed in Proposition 5 of [16]. Hence

$$\min(\lambda_{(k+1)\ell} - \lambda_{k\ell}, \lambda_{k(\ell+1)} - \lambda_{k\ell}) \geq \pi^2 \min(1/a^2, 1/b^2) = d.$$

Henceforth the eigenvalues of \mathcal{A} are relabeled as $i\lambda_k$.

We set $\mathcal{A}_a = \mathcal{A} + \mathcal{B}_a$. We have similarly to Theorem 4.1,

Theorem 5.1. *Let α and κ be as in Section 4. Then, under the assumption*

$$\rho := \|a_0\|_{1,\infty} < \alpha,$$

the spectrum of \mathcal{A}_{a_0} consists in a sequence $(i\mu_k)$ such that, for any $\delta \in (0, 1 - \rho^2/\alpha^2)$, there is an integer \tilde{k} such that

$$|i\mu_k - i\lambda_k| \leq \bar{\alpha} = \bar{\alpha}(a) := \frac{\rho d}{\sqrt{4\rho^2 + d^2\delta}}, \quad k \geq \tilde{k}.$$

In addition, \mathcal{H} admits a Riesz basis $(\phi_k) = \left(\left(\begin{array}{c} \varphi_k \\ i\mu_k \varphi_k \end{array} \right) \right)$, each ϕ_k is an eigenfunction corresponding to μ_k .

A similar analysis to that carried out after Theorem 4.1 leads in the present case to the following stability estimate: let $M > 0$; there are constants $C > 0$ and $\delta > 0$ so that

$$\|a - a_0\|_0 \leq C \left| \ln \left(C^{-1} \|\tilde{\Lambda}_a - \tilde{\Lambda}_{a_0}\| \right) \right|^{-1/4},$$

if $a \in a_0 + \delta B_{1,\infty}$ and

$$\sum_{k \geq 1} \lambda_k |(a - a_0, \psi_k)|^2 \leq M,$$

where $\psi_k = i\mu_k \varphi_k$.

With the same assumptions as those in the present section, we know from Theorem 1.4 in [20] that the Schrödinger equation is exactly observable for the same boundary observation as above. We will not develop here the analysis of the inverse problem corresponding to the Schrödinger equation. It is quite similar to that for Euler-Bernoulli plate equation. We leave to the interested reader to write down the details.

6. THE CASE OF THE HEAT EQUATION

We consider the following IBVP for the heat equation

$$(6.1) \quad \begin{cases} \partial_t u - \Delta u + q(x)u = 0 & \text{in } \Omega \times (0, \tau), \\ u = 0 & \text{on } \partial\Omega \times (0, \tau), \\ u(\cdot, 0) = u_0. \end{cases}$$

From Theorem 1.43 in page 27 of [9], for any $q \in L^\infty(\Omega)$ and $u_0 \in H_0^1(\Omega)$, the IBVP has a unique solution $u_q = u_q(u_0) \in H^{2,1}(\Omega \times (0, \tau))$ ³ and, for any $M > 0$,

$$\|u_q\|_{H^{2,1}(Q)} \leq C\|u_0\|_{1,2},$$

where the constant $C = C(M)$ is independent on q , $\|q\|_\infty \leq M$.

Let Γ be a nonempty open subset of $\partial\Omega$. According to the trace theorem in page 26 of [9], we can say that the following IB map

$$\Lambda_q : u_0 \in H_0^1(\Omega) \longrightarrow \partial_\nu u_q(u_0) \in L^2(\Gamma \times (0, \tau))$$

is bounded.

Without loss of generality, in the rest of this subsection we always assume that $q \geq 0$. In fact, substituting u by $ue^{-\|q\|_\infty t}$, we see that q in (6.1) is changed to $q + \|q\|_\infty$.

Next, we fix $q_0 \in L^\infty(\Omega)$ satisfying $0 \leq q_0$. Let then $0 < \lambda_1 < \lambda_2 \dots \leq \lambda_k \rightarrow +\infty$ be the sequence of eigenvalues, counted according to their multiplicity, of $-\Delta + q_0$ under Dirichlet boundary condition. The orthonormal basis consisting in the corresponding eigenfunctions is denoted by (φ_k) .

Let $q \in L^\infty(\Omega)$ satisfying $\|q\|_\infty \leq M$. We pick an integer k . Taking into account that $u_{q_0}(\varphi_k) = e^{-\lambda_k t} \varphi_k$, we obtain that $u = u_q(\varphi_k) - u_{q_0}(\varphi_k)$ is the solution of the IBVP

$$(6.2) \quad \begin{cases} \partial_t u - \Delta u + q(x)u = (q_0 - q)e^{-\lambda_k t} \varphi_k & \text{in } \Omega \times (0, \tau), \\ u = 0 & \text{on } \partial\Omega \times (0, \tau), \\ u(\cdot, 0) = 0. \end{cases}$$

We set $f = (q - q_0)\varphi_k$ and $\lambda(t) = e^{-\lambda_k t}$. Therefore (6.2) becomes

$$(6.3) \quad \begin{cases} \partial_t u - \Delta u + q(x)u = \lambda(t)f(x) & \text{in } \Omega \times (0, \tau), \\ u = 0 & \text{on } \partial\Omega \times (0, \tau), \\ u(\cdot, 0) = 0. \end{cases}$$

It is straightforward to check that

$$(6.4) \quad u(x, t) = \int_0^t \lambda(t-s)v(x, s),$$

where v is the solution of

$$\begin{cases} \partial_t v - \Delta v + q(x)v = 0 & \text{in } \Omega \times (0, \tau), \\ v = 0 & \text{on } \partial\Omega \times (0, \tau), \\ v(\cdot, 0) = f. \end{cases}$$

In light of the Carleman estimate in Theorem 3.4 in page 165 of [9], we can extend Proposition 3.5 in page 170 of [9] in order to get the following final time observability inequality

$$(6.5) \quad \|v(\cdot, \tau)\|_{H_0^1(\Omega)} \leq C\|\partial_\nu v\|_{L^2(\Gamma \times (0, \tau))}.$$

Here C is a constant depending on M but not on q .

By the continuity of trace operator $w \in H^{2,1}(\Omega \times (0, \tau)) \rightarrow \partial_\nu w|_{\Gamma \times (0, \tau)} \in L^2(\Gamma \times (0, \tau))$, we get from (6.4)

$$\partial_\nu u(x, t)|_{\Gamma \times (0, \tau)} = \int_0^t \lambda(t-s)\partial_\nu v(x, s)|_{\Gamma \times (0, \tau)}.$$

We proceed as in the beginning of the proof of Theorem 2.1 to deduce the following estimate

$$(6.6) \quad \|\partial_\nu v\|_{L^2(\Gamma \times (0, \tau))} \leq \sqrt{2}e^{2\tau^2 \lambda_k^2} \|\partial_\nu u\|_{H^1((0, \tau), L^2(\Gamma))}.$$

On the other hand

$$(6.7) \quad v(x, t) = \sum_{\ell \geq 1} e^{-\lambda_\ell t} (f, \varphi_\ell) \varphi_\ell.$$

³We recall that $H^{2,1}(\Omega \times (0, \tau)) = L^2((0, \tau), H^2(\Omega)) \cap H^1((0, \tau), L^2(\Omega))$.

Hence

$$\|v(\cdot, \tau)\|_2^2 = \sum_{\ell \geq 1} e^{-2\lambda_\ell \tau} |(f, \varphi_\ell)|^2.$$

Arguing as in Section 3, we get, for any $\lambda \geq \lambda_1$ and $N = N(\lambda)$ satisfying $\lambda_N \leq \lambda < \lambda_{N+1}$,

$$(6.8) \quad \|f\|_2^2 \leq e^{2\lambda_k \tau} \|v(\cdot, \tau)\|_2^2 + \frac{1}{\lambda^2} \sum_{\ell > N} \lambda_\ell^2 |(f, \varphi_\ell)|^2.$$

By Green's formula, we obtain

$$\lambda_\ell (f, \varphi_\ell) = - \int_{\Omega} \Delta(q - q_0) \varphi_\ell \varphi_k dx + 2 \int_{\Omega} \nabla(q - q_0) \cdot \nabla \varphi_k \varphi_\ell dx + \lambda_k (f, \varphi_\ell).$$

Therefore, under the assumption that $q - q_0 \in W^{2,\infty}(\Omega)$ and $\|q - q_0\|_{2,\infty} \leq M$,

$$\lambda_\ell |(f, \varphi_\ell)| \leq (1 + \sqrt{\lambda_k})M + \lambda_k |(f, \varphi_\ell)|.$$

This estimate in (6.8) yields

$$\begin{aligned} \|f\|_2^2 &\leq e^{2\lambda\tau} \|v(\cdot, \tau)\|_2^2 + \frac{2(1 + \lambda_k)M^2 + \lambda_k^2}{\lambda^2} \sum_{\ell > N} |(f, \varphi_\ell)|^2 \\ &\leq e^{2\lambda\tau} \|v(\cdot, \tau)\|_2^2 + \frac{2(1 + \lambda_k)M^2 + \lambda_k^2}{\lambda^2} \|f\|_2^2 \\ &\leq e^{2\lambda\tau} \|v(\cdot, \tau)\|_2^2 + \frac{C\lambda_k^2}{\lambda^2} \|f\|_2^2 \\ &\leq e^{2\lambda\tau} \|v(\cdot, \tau)\|_2^2 + \frac{C\lambda_k^2}{\lambda^2}. \end{aligned}$$

This inequality together with (6.6) imply

$$(6.9) \quad \|f\|_2^2 \leq \sqrt{2} e^{2\tau^2 \lambda_k^2 + 2\lambda\tau} \|\partial_\nu u\|_{H^1((0,\tau), L^2(\Gamma))}^2 + \frac{C\lambda_k^2}{\lambda^2}.$$

Now the usual way consists in minimizing the right hand side of the above inequality. This argument is possible only if

$$\frac{\lambda_k^2 e^{-2\tau^2 \lambda_k^2}}{\|\partial_\nu u\|_{H^1((0,\tau), L^2(\Gamma))}^2} \gg 1.$$

But this estimate does not guarantee that $\|\partial_\nu u\|_{H^1((0,\tau), L^2(\Gamma))}$ can be chosen arbitrarily small uniformly in k . However, the minimization argument works if we perturb q_0 by a finite dimensional space. That what we will discuss now.

Let $m > 0$ be a given integer and $E_m = \text{span}\{\varphi_1, \dots, \varphi_m\}$. Since $|(q - q_0, \varphi_k)|^2 \leq |\Omega| \|(q - q_0)\varphi_k\|_2^2$ by the Cauchy-Schwarz's inequality, we get from (6.9)

$$\|q - q_0\|_2^2 = \sum_{k=1}^m |(q - q_0, \varphi_k)|^2 \leq C_m \left(e^{2\lambda\tau} \|\partial_\nu u\|_{H^1((0,\tau), L^2(\Gamma))}^2 + \frac{1}{\lambda^2} \right),$$

for some constant C_m depending on m . We notice that from the remark above, C_m surely blows-up when $m \rightarrow +\infty$.

Minimizing with respect to λ the right hand side of the inequality above, we get

$$\|q - q_0\|_2 \leq C_m \left| \ln \left(\|\partial_\nu u\|_{H^1((0,\tau), L^2(\Gamma))} \right) \right|^{-1/4},$$

provided that $\|\partial_\nu u\|_{H^1((0,\tau), L^2(\Gamma))}$ is sufficiently small. We observe that a simple continuity argument shows that $\|\partial_\nu u\|_{H^1((0,\tau), L^2(\Gamma))}$ is small whenever $\|q - q_0\|_\infty$ is small. If $\Lambda_q^m = \Lambda_q|_{E_m}$, we end up getting

$$\|q - q_0\|_2 \leq C_m \left| \ln \left(\|\Lambda_q^m - \Lambda_{q_0}^m\| \right) \right|^{-1/4},$$

if $\|q - q_0\|_\infty$ is sufficiently small.

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