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Asymptotic Behaviour of Pion–Pion Total Cross–Sections

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Abstract: We derive a sum rule which shows that the Froissart–Martin bound for the asymptotic behaviour of the $\pi\pi$ total cross sections at high energies, if modulated by the Lukaszuk–Martin coefficient of the leading $\log^2 s$ behaviour, cannot be an optimal bound in QCD. We next compute the total cross sections for $\pi^+\pi^-$, $\pi^\pm\pi^0$ and $\pi^0\pi^0$ scattering within the framework of the constituent chiral quark model (CχQM) in the limit of a large number of colours $\mathrm{N_c}$ and discuss their asymptotic behaviours. The same $\pi\pi$ cross sections are also discussed within the general framework of Large–$\mathrm{N_c}$ QCD and we show that it is possible to make an Ansatz for the isospin $I = 1$ and $I = 0$ spectrum which satisfy the Froissart–Martin bound with coefficients which, contrary to the Lukaszuk–Martin coefficient, are not singular in the chiral limit and have the correct Large–$\mathrm{N_c}$ counting. We finally propose a simple phenomenological model which matches the low energy behaviours of the $\sigma^{\mathrm{total}}_{\pi^\pm\pi^0}(s)$ cross section predicted by the CχQM with the high energy behaviour predicted by the Large–$\mathrm{N_c}$ Ansatz. The magnitude of these cross sections at very high energies is of the order of those observed for the pp and $p\bar{p}$ scattering total cross sections.

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I Introduction

The Froissart–Martin bound for the asymptotic behaviour of total cross sections has played a major role in the history of strong interactions. Using the Mandelstam representation, Froissart showed [1] that the total cross section $\sigma_{AB}^{\rm total}(s)$ for the scattering of two hadronic particles A and B with center of mass energy $\sqrt{s}$ cannot grow faster than

$$\sigma_{AB}^{\rm total}(s) \underset{s\to\infty}{\leq} C \log^2 \frac{s}{s_0}\,, \tag{1.1}$$

with C and s_0 unknown constants. The rigorous proof of this bound from axiomatic quantum field theory was later shown by Martin [2] and, quite remarkably, an explicit form for the coefficient C was first derived by Lukaszuk and Martin in ref. [3] with the result

$$C = \frac{4\pi}{t_0}\,, \tag{1.2}$$

where t_0 denotes the lowest mass squared singularity in the t–channel of the scattering process[1]. When applied to pion–pion scattering the Froissart–Martin–Lukaszuk bound (FML–bound) states that

$$\sigma_{\pi\pi}^{\rm tot}(s) \underset{s\to\infty}{\leq} \frac{\pi}{m_\pi^2} \log^2 \frac{s}{s_0}\,. \tag{1.3}$$

[1] With some assumptions, the normalization s_0 for *averaged* total cross sections has also been recently derived in ref. [4].

The presence of a factor $1/m_\pi^2$ in the r.h.s. of eq. (1.3) has recently been questioned by two of us [5]: *what happens to the FML–bound in QCD in the chiral limit when the pions, the Nambu–Goldstone states of the chiral–SU(2) flavour symmetry of QCD, become massless?* Does the FML–bound become irrelevant in this limit as eq. (1.3) seems to indicate? As pointed out in ref. [5], the usual derivation of the FML–bound from the rigorous principles of axiomatic quantum field theory does not take into account the fact that the underlying dynamics of the strong interactions has the property of spontaneous chiral symmetry breaking. In fact, it implicitly assumes a realization of the hadronic spectrum *à la* Wigner–Weyl without Nambu–Goldstone particles, in which case, the coefficient of the $\log^2 s$ in eq. (1.3) is perhaps not so surprising; but what is then the correct coefficient in QCD? In the next section we derive a sum rule which clearly shows that the FML–bound cannot be the optimal bound in QCD.

Another question which was also raised in ref. [5] is: *what becomes of the FML–bound in the QCD Large–N_c limit?* The Large–$\mathrm{N_c}$ counting rules fix $\sigma_{\pi\pi}^{\mathrm{tot}}(s)$ to be of $\mathcal{O}\left(1/\mathrm{N_c}\right)$, while the r.h.s. of eq. (1.3) appears to be of $\mathcal{O}(1)$. We propose here, as a very modest step towards an answer to these fundamental questions, to examine them first in the simple Constituent Chiral Quark Model [6] (CχQM) and then within the more general framework of Large–$\mathrm{N_c}$ QCD properties.

We have organized this paper in the following way. In the next section we reproduce basic properties of the elastic pion–pion scattering amplitudes which are necessary to derive the sum rule which we have mentioned above. The effective Lagrangian of the CχQM is described in section III as well as the results of the calculation of the total $\pi\pi$ annihilation cross sections in this model. We also show that the results obtained for these cross sections satisfy the sum rules which, within the model, fix some of the $\mathcal{O}(p^4)$ low–energy constants of the chiral Lagrangian. Section IV is then dedicated to the pion–pion total cross sections in the QCD Large–$\mathrm{N_c}$ limit where we reconsider in more detail some of the issues discussed in ref. [5] and where we propose a model which matches the low energy behaviours of the $\sigma_{\pi\pi}^{\mathrm{total}}(s)$ cross sections predicted by the CχQM with the high energy behaviours predicted by a simple Large–$\mathrm{N_c}$ Ansatz. Phenomenological comments and conclusions are summarized in Section V.

II Pion-Pion Amplitudes and Sum Rules

In full generality, elastic $\pi\pi$ scattering in the isospin symmetry limit is described by a single invariant Lorentz amplitude $A(s,t,u)$ [2].

$$\begin{aligned}&\langle\pi^d(p_4)\pi^c(p_3)\ \mathrm{out}|\pi^a(p_1)\pi^b(p_2)\ \mathrm{in}\rangle = \\ &\mathbf{1} + i(2\pi)^4\delta^4(p_3+p_4-p_1-p_2)\left\{\delta^{ab}\delta^{cd}A(s,t,u)+\delta^{ac}\delta^{bd}A(t,u,s)+\delta^{ad}\delta^{bc}A(u,s,t)\right\}\ ,\end{aligned} \tag{2.1}$$

[2] For a modern review see ref. [7].

where a, b, c, d denote the 1,2,3 components of the adjoint representation of the pion fields in $SU(2)$ and s, t and u the usual Mandelstam variables constrained by

$$s + t + u = 4m_\pi^2 \,. \tag{2.2}$$

It is convenient to work with the three s–channel isospin components $\mathbf{T} = (T^0, T^1, T^2)$ of the amplitudes in eq. (2.1):

$$\begin{aligned} T^0(s,t) &= 3A(s,t,u) + A(t,u,s) + A(u,s,t)\,, \\ T^1(s,t) &= A(t,u,s) - A(u,s,t)\,, \\ T^2(s,t) &= A(t,u,s) + A(u,s,t)\,. \end{aligned} \tag{2.3}$$

These isospin amplitudes obey fixed-t dispersion relations. They are the so–called Roy equations [8] which we shall consider at $t = 0$ and in the chiral limit where $m_\pi \to 0$. The linear combination of the isospin amplitudes which diagonalize the crossing matrix in the Roy equations are then:

$$\begin{aligned} F_1(s,0) &= -\frac{1}{6}\, T^0(s,0) - \frac{1}{4}\, T^1(s,0) + \frac{5}{12}\, T^2(s,0)\,, \\ F_2(s,0) &= +\frac{1}{6}\, T^0(s,0) + \frac{1}{4}\, T^1(s,0) + \frac{7}{12}\, T^2(s,0)\,, \\ F_3(s,0) &= -\frac{1}{6}\, T^0(s,0) + \frac{3}{4}\, T^1(s,0) + \frac{5}{12}\, T^2(s,0)\,. \end{aligned} \tag{2.4}$$

The amplitudes F_2 and F_3 obey the same dispersion relation:

$$\mathrm{Re}\ F_{2,3}(s,0) = s^2 \int_0^\infty \frac{\mathrm{d}s'^2}{s'^2} \frac{1}{s'^2 - s^2} \frac{1}{\pi} \mathrm{Im}\ F_{2,3}(s',0)\,, \tag{2.5}$$

and Re $F_{2,3}(s,0)$ are even under $s \leftrightarrow -s$, while the amplitude $F_1(s,0)$ obeys the dispersion relation:

$$\mathrm{Re}\ F_1(s,0) = -\frac{s}{f_\pi^2} + 2s^3 \int_0^\infty \frac{\mathrm{d}s'}{s'^2} \frac{1}{s'^2 - s^2} \frac{1}{\pi} \mathrm{Im}\ F_1(s',0)\,, \tag{2.6}$$

and Re $F_1(s,0)$ is odd under $s \leftrightarrow -s$. Indeed, one can check that there is no contribution of $\mathcal{O}(s^2)$ to Re $F_1(s,0)$ in Chiral Perturbation Theory (χPT), while the contributions of that order from χPT to the $F_2(s,0)$ and $F_3(s,0)$ amplitudes are [9]:

$$\mathrm{Re}\ F_2(s,0) \underset{s\to 0}{=} \frac{s^2}{f_\pi^4} \left[2l_1^{\mathrm{r}}(\mu) + 3l_2^{\mathrm{r}}(\mu) + \frac{1}{12\pi^2} \left(\log \frac{\mu^2}{s} + \frac{25}{24} \right) \right] + \mathcal{O}(s^4)\,, \tag{2.7}$$

$$\mathrm{Re}\ F_3(s,0) \underset{s\to 0}{=} \frac{s^2}{f_\pi^4} \left[-2l_1^{\mathrm{r}}(\mu) + l_2^{\mathrm{r}}(\mu) + \frac{1}{96\pi^2} \right] + \mathcal{O}(s^4)\,. \tag{2.8}$$

These χPT one–loop results are renormalization μ–scale independent and well defined in the chiral limit. The μ–scale dependence of the chiral log cancels with the μ–dependence in the renormalized constants and the combination $-2l_1^{\rm r}(\mu)+l_2^{\rm r}(\mu)$ is μ–scale independent. The relation between the chiral $SU(2)$ $l_i^{\rm r}$ constants and the more conventional $L_i^{\rm r}$ constants of the chiral $SU(3)$ Lagrangian is as follows [10]:

$$l_1^{\rm r}(\mu) = 4L_1^{\rm r}(\mu)+2L_3-\frac{1}{96\pi^2}\frac{1}{8}\left(\log\frac{M_K^2}{\mu^2}+1\right)\,, \tag{2.9}$$

$$l_2^{\rm r}(\mu) = 4L_2^{\rm r}(\mu)-\frac{1}{96\pi^2}\frac{1}{4}\left(\log\frac{M_K^2}{\mu^2}+1\right)\,, \tag{2.10}$$

where here, kaon particles have been treated as massive and integrated out, hence the dependence on their mass M_K.

The optical theorem relates the amplitudes Im $F_i(s,0)$ to the total $\pi\pi$ cross sections as follows (massless pions):

$$\begin{aligned}
{\rm Im}\ F_1(s,0) &= \frac{1}{2}\left[s\ \sigma^{\rm tot}_{\pi^+\pi^+}-s\ \sigma^{\rm tot}_{\pi^+\pi^-}\right]\,,\\
{\rm Im}\ F_2(s,0) &= \frac{1}{2}\left[s\ \sigma^{\rm tot}_{\pi^+\pi^+}+s\ \sigma^{\rm tot}_{\pi^+\pi^-}\right]=\frac{1}{2}\left[s\ \sigma^{\rm tot}_{\pi^\pm\pi^0}+s\ \sigma^{\rm tot}_{\pi^0\pi^0}\right]\,,\\
{\rm Im}\ F_3(s,0) &= \frac{1}{2}\left[3s\ \sigma^{\rm tot}_{\pi^\pm\pi^0}-s\ \sigma^{\rm tot}_{\pi^0\pi^0}\right]\,.
\end{aligned} \tag{2.11}$$

II.1 The Lukaszuk–Martin Bound

We are now in the position to explain what is the problem with the Lukaszuk–Martin coefficient [3] of the FML–bound in eq. (1.3) as well as with the recent averaged bound derived in ref. [4]. For that, let us consider the sum of eqs. (2.5) at small s, say $s=m_\pi^2$ to be precise, which we rewrite in the form of a convenient sum rule:

$$\frac{\pi}{f_\pi^4}\left[l_2^{\rm r}(\mu)+\frac{1}{48\pi^2}\left(\log\frac{\mu^2}{m_\pi^2}+\frac{7}{6}\right)\right]+\mathcal{O}\left(m_\pi^4\right)=\int_{4m_\pi^2}^{\infty}\frac{{\rm d}s'}{s'^2-m_\pi^4}\ \sigma^{\rm tot}_{\pi^\pm\pi^0}(s')\,, \tag{2.12}$$

where on the l.h.s. we have used eqs. (2.7), (2.8) and on the r.h.s. the fact that

$${\rm Im}\ F_2(s',0)+{\rm Im}\ F_3(s',0)=2s'\ \sigma^{\rm tot}_{\pi^\pm\pi^0}(s')\,. \tag{2.13}$$

We recall that the l.h.s. does not depend on the choice of the renormalization scale μ. Using the FML–bound we can write the integral in the r.h.s. as follows:

$$\int_{4m_\pi^2}^{\infty} \frac{\mathrm{d}s'}{s'^2 - m_\pi^4}\, \sigma^{\rm tot}_{\pi^\pm\pi^0}(s') \leq \int_{4m_\pi^2}^{s_0} \frac{\mathrm{d}s'}{s'^2 - m_\pi^4}\, \sigma^{\rm tot}_{\pi^\pm\pi^0}(s') + \frac{\pi}{m_\pi^2}\frac{2}{s_0}\left[1 + \frac{1}{27}\frac{m_\pi^4}{s_0^2} + \mathcal{O}\left(\frac{m_\pi^4}{s_0^2}\right)^2\right], \tag{2.14}$$

where s_0 is the finite threshold where the asymptotic behaviour sets in, the same s_0 which appears as the normalization of the $\log^2 s$ in the FML bound in Eq. (1.3). Both terms in the r.h.s. of this inequality are obviously positive. Inserting this inequality in the r.h.s. of the sum rule in eq. (2.12) shows that the l.h.s., which goes as $\log m_\pi^2$ in the chiral limit, is bounded by a quantity which diverges as $1/m_\pi^2$ in the same limit. We therefore conclude that, if the Froissart bound applies to $\pi\pi$ scattering in QCD, the Lukaszuk–Martin coefficient of the leading $\log^2 s$ term cannot be the optimal one.

We wish to emphasize that the derivation of this result follows from very general properties of $\pi\pi$ amplitudes in QCD and, so far, we have not consider the Large–$\mathrm{N_c}$ approximation.

III The Constituent Chiral Quark Model (CχQM)

Historically, the model in question emerged as an attempt to reconcile the successes of phenomenological quark models, like e.g. the De Rújula-Georgi-Glashow model [11], with QCD. The Lagrangian proposed by Manohar and Georgi (MG) is an effective field theory which incorporates the interactions of the low–lying pseudoscalar particles of the hadronic spectrum i.e., the Nambu-Goldstone modes of the spontaneously broken chiral symmetry [12], with chirally rotated quark fields $Q = (U, D, S)$. These quarks have become massive due to the phenomenon of spontaneous chiral symmetry breaking (SχSB). Their mass, however, ($M_Q \sim \frac{1}{3}M_{\rm nucleon}$) has nothing to do with the masses of the u, d, s quark fields in the QCD Lagrangian which explicitly break chiral symmetry and are known to be much smaller (see e.g. the recent review article in ref. [13]). The constituent quark fields may also have gluonic interactions but, since the Goldstone modes are already in the Lagrangian, the color–SU(3) coupling constant is then supposed to be no longer running and relatively small. The hope is that such an effective Lagrangian encodes the essential degrees of freedom to describe Hadron Physics at energies below the chiral symmetry breaking scale but still above the confinement regime.

It is fair to say, however, that in spite of some efforts (see e.g. refs. [14–16] and references therein), it has not been possible to establish the approximations at which the MG–Lagrangian could be derived from the underlying QCD theory. It can be shown to be a particular case of the Extended Nambu Jona-Lasinio (ENJL) Model [17, 18], but this only transfers the problem

of its derivation from first principles to another level where, in any case, the basic question remains so far unanswered.

An interesting observation made by Weinberg [15] is the fact that in the limit of a large number of colors $\mathrm{N_c}$ the CχQM becomes a renormalizable theory. A subsequent observation along the same line was made in ref. [19] where it is shown that the number of counterterms which in the Large–$\mathrm{N_c}$ limit have to be added to the primitive Manohar–Georgi Lagrangian, is minimized for the choice $g_A = 1$. In that respect, it has also been shown [19, 20] that there is a class of observables governed by integrals of specific QCD Green's functions which, for $g_A = 1$, have rather good matching to their short–distance behaviour. Interesting examples are the Hadronic Vacuum Polarization, the Hadronic Light–by–Light Scattering and the Hadronic Electroweak contributions to the anomalous magnetic moment of the muon, which have been recently discussed within the framework of the CχQM in ref. [20].

The effective Lagrangian in question is the following:

$$\begin{aligned}\mathcal{L}_{\mathrm{C\chi QM}}(x)=&\underbrace{i\bar{Q}\gamma^{\mu}\left(\partial_{\mu}+\Gamma_{\mu}+iG_{\mu}\right)Q+\frac{i}{2}g_A\ \bar{Q}\gamma^{\mu}\gamma_5\xi_{\mu}Q-M_Q\bar{Q}Q}_{M-G}-\frac{1}{2}\bar{Q}\left(\Sigma-\gamma_5\Delta\ \right)Q\\&+\underbrace{\frac{1}{4}{f_{\pi}}^2\mathrm{tr}\left[D_{\mu}UD^{\mu}U^{\dagger}}_{M-G}+U^{\dagger}\chi+\chi^{\dagger}U\right]-\underbrace{\frac{1}{4}\sum_{a=1}^{8}G^{(a)}_{\mu\nu}G^{(a)\mu\nu}}_{M-G}+e^2C\ \mathrm{tr}(Q_RUQ_LU^{\dagger})\\&+L_5\ \mathrm{tr}D_{\mu}U^{\dagger}D^{\mu}U(\chi^{\dagger}U+U^{\dagger}\chi)+L_8\ \mathrm{tr}(U\chi\dagger U\chi\dagger+U^{\dagger}\chi U^{\dagger}\chi)\,.\end{aligned}\tag{3.1}$$

The underbraced terms are those of the MG–Lagrangian, but in the presence of external $SU(3)$ vector $v_\mu(x)$ and axial-vector $a_\mu(x)$ sources. The field matrix $U(x)$ denotes the 3×3 unitary matrix in the flavour space which collects the Nambu-Goldstone fields and which under chiral rotations transforms as $U \to V_R U V_L^\dagger$. The vector field matrix $D_\mu U$ is the covariant derivative of U:

$$D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu\,, \quad l_\mu = v_\mu - a_\mu\,, \quad r_\mu = v_\mu + a_\mu\,, \tag{3.2}$$

and, with $U = \xi\xi$,

$$\Gamma_\mu = \frac{1}{2}\left[\xi^\dagger(\partial_\mu - i r_\mu)\xi + \xi(\partial_\mu - i l_\mu)\xi^\dagger\right]\,, \quad \xi_\mu = i\left[\xi^\dagger(\partial_\mu - i r_\mu)\xi - \xi(\partial_\mu - i l_\mu)\xi^\dagger\right]\,. \tag{3.3}$$

The gluon field matrix in the fundamental representation of color $SU(3)$ is $G_\mu(x)$ and $G^{(a)}_{\mu\nu}(x)$ its corresponding gluon field strength tensor. The presence of external scalar $s(x)$ and pseudoscalar $p(x)$ sources induces the extra terms proportional to

$$\chi = 2B[s(x) + ip(x)]\,, \tag{3.4}$$

where B, like f_π, are order parameters not fixed by the model. When the $s(x)$ and $p(x)$ sources are frozen to the up, down, and strange light quark masses of the QCD Lagrangian,

$$\chi = 2B\mathcal{M}\,, \quad \text{with} \quad \mathcal{M} = \mathrm{diag}(m_u\,, m_d\,, m_s)\,, \tag{3.5}$$

and then

$$\Sigma = \xi^\dagger \mathcal{M} \xi^\dagger + \xi \mathcal{M}^\dagger \xi \,, \quad \Delta = \xi^\dagger \mathcal{M} \xi^\dagger - \xi \mathcal{M}^\dagger \xi \,. \tag{3.6}$$

With the axial coupling fixed to $g_A = 1$, the extra couplings L_5 and L_8 are the only terms which are needed to absorb the ultraviolet (UV) divergences when the constituent quark fields $Q(x)$ are integrated out[3]. If one wants to consider the case where photons are also integrated out then, to leading order in the chiral expansion and in the electric charge coupling e, the last term in the second line is also required to absorb further UV–divergences. Loops involving pion fields are subleading in the $1/\mathrm{N_c}$–expansion and hence, following the observation of Weinberg in ref. [15], the Lagrangian in eq. (3.1), when considered within the framework of the large–$\mathrm{N_c}$ limit, is a renormalizable Lagrangian.

The $\pi\pi$ total cross sections in the CχQM are then simply given by the corresponding $\pi\pi \to Q\bar{Q}$ cross sections[4]. When restricted to chiral $SU(2)$ and in the chiral limit where $m_u = m_d = 0$, but still keeping the value of g_A free, the terms of the interaction Lagrangian which are needed for the evaluation of these cross sections are $[Q = (U, D)]$:

$$\begin{aligned}
\mathcal{L}_{\rm int}(x) \doteq & -i\frac{g_A}{\sqrt{2}f_\pi}\left[\bar{U}\gamma^\mu\gamma_5 D\ \partial_\mu\pi^+ + \bar{D}\gamma^\mu\gamma_5 U\ \partial_\mu\pi^- + \frac{1}{\sqrt{2}}\left(\bar{U}\gamma^\mu\gamma_5 U - \bar{D}\gamma^\mu\gamma_5 D\right)\ \partial_\mu\pi^0\right] \\
& +i\frac{1}{4f_\pi^2}\left\{\bar{U}\gamma^\mu U\ \left(\pi^+\partial_\mu\pi^- - \pi^-\partial_\mu\pi^+\right) + \bar{D}\gamma^\mu D\ \left(\pi^-\partial_\mu\pi^+ - \pi^+\partial_\mu\pi^-\right)\right\} \\
& +i\frac{\sqrt{2}}{4f_\pi^2}\left\{\bar{U}\gamma^\mu D\ (\pi^0\partial_\mu\pi^+ - \pi^+\partial_\mu\pi^0) - \bar{D}\gamma^\mu U\ (\pi^0\partial_\mu\pi^- - \pi^-\partial_\mu\pi^0)\right\} \\
& -M_Q \sum_{Q=U,D} \bar{Q}Q \,.
\end{aligned} \tag{3.7}$$

III.1 Calculation of the Pion-Pion Total Cross Sections

We fix the kinematics as follows:

$$\pi(k) + \pi(k') \to Q(p) + \bar{Q}(p') \,. \tag{3.8}$$

In the center of mass system:

$$\begin{aligned}
k\ :\ \left(\tfrac{\sqrt{s}}{2}\,, +\tfrac{\sqrt{s}}{2}\,, 0\,, 0\right)\,, \quad p\ :\ \left(\frac{\sqrt{s}}{2}\,, +\frac{\sqrt{s-4M_Q^2}}{2}\cos\theta\,, +\frac{\sqrt{s-4M_Q^2}}{2}\sin\theta\,, 0\right)\ ; \\
k'\ :\ \left(\tfrac{\sqrt{s}}{2}\,, -\tfrac{\sqrt{s}}{2}\,, 0\,, 0\right)\,, \quad p'\ :\ \left(\frac{\sqrt{s}}{2}\,, -\frac{\sqrt{s-4M_Q^2}}{2}\cos\theta\,, -\frac{\sqrt{s-4M_Q^2}}{2}\sin\theta\,, 0\right)\ .
\end{aligned} \tag{3.9}$$

[3] We disregard couplings involving external fields alone to lowest order in the chiral expansion.

[4] For the sake of simplicity we omit in this first analysis the contribution of gluon interactions in the CχQM.

The total cross sections for massless pions are then given by

$$\sigma^{\rm tot}_{\pi\pi}(s) = \frac{1}{32\pi s}\sqrt{1-\frac{4M_Q^2}{s}}\int_{-1}^{+1}{\rm d}(\cos\theta)\left|T(\pi\pi\to Q\bar{Q})\right|^2(s,\cos\theta)\,, \tag{3.10}$$

with $\left|T(\pi\pi\to Q\bar{Q})\right|^2(s,\cos\theta)$ receiving contributions from the relevant terms shown in the interaction Lagrangian in eq. (3.7).

III.1.1 The $\pi^+\pi^-$ Total Cross Section

The various contributions to this cross section come from the following sources:

- Terms proportional to $\frac{g_A^4}{f_\pi^4}N_c n_f$.

  They come from the squared amplitudes $\pi^+\pi^-\to U\bar{U}$ and $\pi^+\pi^-\to D\bar{D}$ generated by the terms in the first line in eq. (3.7). They give a contribution

$$\frac{g_A^4}{f_\pi^4}N_c n_f \Rightarrow \frac{N_c n_f}{192\pi}\frac{g_A^4}{f_\pi^4}\sqrt{1-\frac{4M_Q^2}{s}}\left(s+2M_Q^2\right)\,. \tag{3.11}$$

- Terms proportional to $\frac{1}{f_\pi^4}N_c n_f$.

  They come from the squared amplitudes $\pi^+\pi^-\to U\bar{U}$ and $\pi^+\pi^-\to D\bar{D}$ generated by the terms in the second line of eq. (3.7). They give a contribution

$$\frac{1}{f_\pi^4}N_c n_f \Rightarrow \frac{N_c n_f}{192\pi}\frac{1}{f_\pi^4}\sqrt{1-\frac{4M_Q^2}{s}}\left(s+2M_Q^2\right)\,. \tag{3.12}$$

- Terms proportional to $\frac{g_A^2}{f_\pi^4}N_c n_f$

  They come from the the interference of the amplitudes $\pi^+\pi^-\to U\bar{U}$ and $\pi^+\pi^-\to D\bar{D}$ in the first and second lines of eq. (3.7). They give a contribution

$$\frac{g_A^2}{f_\pi^4}N_c n_f \Rightarrow -\frac{N_c n_f}{192\pi}\frac{g_A^2}{f_\pi^4}\sqrt{1-\frac{4M_Q^2}{s}}2\left(s-10M_Q^2\right)\,. \tag{3.13}$$

The overall contribution to the $\pi^+\pi^-$ total cross section is then

$$\sigma^{\rm tot}_{\pi^+\pi^-}(s) = \frac{N_c n_f}{192\pi}\frac{M_Q^2}{f_\pi^4}\sqrt{1-\frac{4M_Q^2}{s}}\left[(1-g_A^2)^2\,\frac{s}{M_Q^2}+2(1+10g_A^2+g_A^4)\right]\,. \tag{3.14}$$

We therefore find that there is a leading contribution of $\mathcal{O}(s)$ to this cross section which, however, vanishes for $g_A=1$. In other words, the $\pi^+\pi^-$ scattering total cross section in the CχQM violates the Froissart bound unless $g_A=1$.

Equation (3.14), for $g_A = 1$, reduces to

$$\sigma^{\rm tot}_{\pi^+\pi^-}(s) = \frac{N_c n_f}{32\pi}\frac{4M_Q^2}{f_\pi^4}\sqrt{1-\frac{4M_Q^2}{s}}\,, \tag{3.15}$$

and, therefore, for large–s:

$$\sigma^{\rm tot}_{\pi^+\pi^-}(s) \underset{s\to\infty}{\sim} \frac{N_c n_f}{32\pi}\frac{4M_Q^2}{f_\pi^4}\,. \tag{3.16}$$

We then conclude that the leading contribution to the $\pi^+\pi^-$ scattering total cross section for large s, in the CχQM and provided $g_A = 1$ with neglect of gluon corrections, goes to a constant. There is no factor $1/m_\pi^2$ in this bound and it satisfies the Large–N$_{\rm c}$ counting rules; but it does not saturate the FM–bound.

III.1.2 The $\pi^\pm\pi^0$ Total Cross Sections

The various contributions to $\sigma^{\rm tot}_{\pi^\pm\pi^0}(s)$ come from the following sources:

- Terms proportional to $\frac{g_A^4}{f_\pi^4}N_c n_f$.

  They come from the squared amplitudes generated by the terms in the first line in eq. (3.7). They give a contribution

$$\frac{g_A^4}{f_\pi^4}N_c n_f \Rightarrow \frac{N_c n_f}{192\pi}\frac{g_A^4}{f_\pi^4}\left\{\sqrt{1-\frac{4M_Q^2}{s}}\left(s-11M_Q^2\right)+48\frac{M_Q^4}{s}\log\frac{1+\sqrt{1-\frac{4M_Q^2}{s}}}{1-\sqrt{1-\frac{4M_Q^2}{s}}}\right\}\,. \tag{3.17}$$

- Terms proportional to $\frac{1}{f_\pi^4}N_c n_f$.

  They come from the squared amplitudes generated by the terms in the second line of eq. (3.7). They give a contribution

$$\frac{1}{f_\pi^4}N_c n_f \Rightarrow \frac{N_c n_f}{192\pi}\frac{1}{f_\pi^4}\sqrt{1-\frac{4M_Q^2}{s}}\left(s-2M_Q^2\right)\,. \tag{3.18}$$

- Terms proportional to $\frac{g_A^2}{f_\pi^4}N_c n_f$

  They come from the the interference of amplitudes in the first and second lines of eq. (3.7). They give a contribution

$$\frac{g_A^2}{f_\pi^4}N_c n_f \Rightarrow -\frac{N_c n_f}{192\pi}\frac{g_A^2}{f_\pi^4}\sqrt{1-\frac{4M_Q^2}{s}}\left(2s-11M_Q^2\right)\,. \tag{3.19}$$

The overall contribution is then

$$\sigma^{\rm tot}_{\pi^\pm\pi^0}(s) = \frac{N_c n_f}{192\pi}\frac{M_Q^2}{f_\pi^4}\left\{\sqrt{1-\frac{4M_Q^2}{s}}\left[\frac{s}{M_Q^2}(1-g_A^2)\left[(1-g_A^2)-2(1+11g_A^2)\right]\right]\right.$$
$$\left. + 12 g_A^4 \frac{4M_Q^2}{s}\log\frac{1+\sqrt{1-\frac{4M_Q^2}{s}}}{1-\sqrt{1-\frac{4M_Q^2}{s}}}\right\}. \tag{3.20}$$

We also find that there is a leading contributions of $\mathcal{O}(s)$ to the total cross section which, like in the $\pi^+\pi^-$ case, vanishes for $g_A = 1$. Therefore, the $\pi^\pm\pi^0$ scattering total cross section in the CχQM also violates the Froissart bound unless $g_A = 1$.

Equation (3.20), for $g_A = 1$, reduces to

$$\sigma^{\rm tot}_{\pi^\pm\pi^0}(s) = \frac{N_c n_f}{16\pi}\frac{4M_Q^2}{f_\pi^4}\frac{M_Q^2}{s}\log\frac{1+\sqrt{1-\frac{4M_Q^2}{s}}}{1-\sqrt{1-\frac{4M_Q^2}{s}}}, \tag{3.21}$$

which asymptotically, for large–s, goes as:

$$\sigma^{\rm tot}_{\pi^\pm\pi^0}(s) \underset{s\to\infty}{\sim} \frac{N_c n_f}{16\pi}\frac{4M_Q^2}{f_\pi^4}\,\frac{M_Q^2}{s}\,\log\frac{s}{M_Q^2}, \tag{3.22}$$

and is subleading when compared to the corresponding asymptotic behaviour of $\pi^+\pi^-$ in eq. (3.16).

III.1.3 The $\pi^0\pi^0$ Total Cross Section

The only term in the interaction Lagrangian in eq. (3.7) which contributes to this process is the one in the last term of the first line. Therefore, the contribution to $\sigma^{\rm tot}_{\pi^0\pi^0}(s)$ only comes from:

- Terms proportional to $\frac{g_A^4}{f_\pi^4}N_c n_f$.

The overall contribution is then

$$\sigma^{\rm tot}_{\pi^0\pi^0}(s) = \frac{N_c n_f}{32\pi}\frac{4M_Q^2}{f_\pi^4}\,g_A^4\,\left\{\sqrt{1-\frac{4M_Q^2}{s}} - 2\frac{M_Q^2}{s}\log\frac{1+\sqrt{1-\frac{4M_Q^2}{s}}}{1-\sqrt{1-\frac{4M_Q^2}{s}}}\right\}. \tag{3.23}$$

and contrary to the previous $\sigma^{\rm tot}_{\pi^+\pi^-}(s)$ and $\sigma^{\rm tot}_{\pi^\pm\pi^0}(s)$ cases one does not need to fix $g_A = 1$ to respect the Froissart bound.

Asymptotically, for large–s and with $g_A = 1$, it goes as:

$$\sigma^{\rm tot}_{\pi^0\pi^0}(s) \underset{s\to\infty}{\sim} \frac{N_c n_f}{32\pi}\frac{4M_Q^2}{f_\pi^4}\left[1-2\frac{M_Q^2}{s}\,\log\frac{s}{M_Q^2}\right], \tag{3.24}$$

i.e. as a constant.

The total $\pi\pi$ cross sections calculated above, with $g_A = 1$, are plotted in figure 1 below. The higher curve corresponds to $\sigma^{\rm total}_{\pi^+\pi^-}(s)$, the middle curve to $\sigma^{\rm total}_{\pi^0\pi^0}(s)$ and the low curve to $\sigma^{\rm total}_{\pi^\pm\pi^0}(s)$. The cross sections are given in millibarn units and they are plotted versus s, the total center of mass squared energy, in GeV^2. The constituent quark mass which we have used is the center value $M_Q = 240$ MeV of the result $M_Q = (240 \pm 10)$ MeV obtained in ref. [5]. Asymptotically, $\sigma^{\rm tot}_{\pi^+\pi^-}(s)$ and $\sigma^{\rm tot}_{\pi^0\pi^0}(s)$ have the same constant behaviour:

$$\sigma^{\rm tot}_{\pi\pi}(s) \underset{s\to\infty}{\sim} \frac{N_c n_f}{32\pi}\frac{4M_Q^2}{f_\pi^4}\,, \quad \text{for} \quad \pi^+\pi^- \quad \text{and} \quad \pi^0\pi^0\,, \tag{3.25}$$

while $\sigma^{\rm tot}_{\pi^\pm\pi^0}(s)$ falls as the difference between $\sigma^{\rm tot}_{\pi^+\pi^-}(s)$ and $\sigma^{\rm tot}_{\pi^0\pi^0}(s)$ (see eq. (3.22)) and it is subleading. We therefore find that the $\sigma^{\rm tot}_{\pi\pi}(s)$ cross sections in the CχQM satisfy the Froissart–Martin bound, but they do not saturate it. Very likely this is a drawback of the model, which ceases to be reliable at center of mass energy values of the order $\sqrt{s} \sim 2$ GeV. It seems plausible that the rescattering of the constituent quarks in the presence of a gluonic background may be at the origin of an asymptotic black disc like behaviour, similar to the one observed in pp and $p\bar{p}$ scattering [21–23], which provides the Froissart–Martin $\log^2 s$ enhancement.

For a discussion of the present experimental situation, phenomenological models and future prospects concerning $\sigma^{\rm tot}_{\pi\pi}(s)$ cross sections at high energies see e.g. refs. [24, 25].

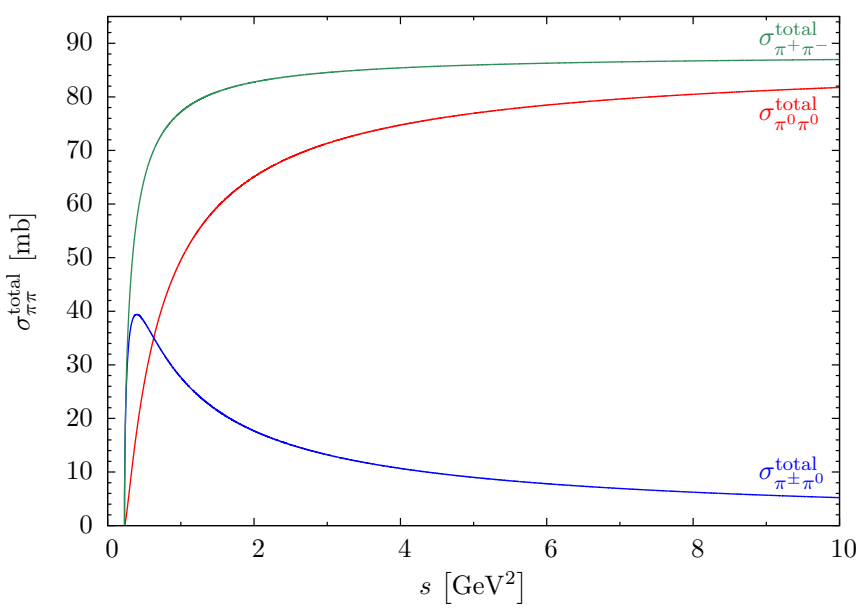

Figure 1. Total $\pi\pi$ cross sections in millibarns versus s in GeV^2.

III.2 Sum Rules in the CχQM

In the CχQM the total $\pi^+\pi^+$ cross section vanishes and therefore Im $T^2(s,0)=0$, which implies:

$$\text{Im } F_1(s,0) = -\text{Im } F_2(s,0)\,, \tag{3.26}$$

and also the relations (with $g_A=1$):

$$\text{Im } F_2(s,0) = \frac{s}{2}\left[\sigma^{\text{tot}}_{\pi^\pm\pi^0}+\sigma^{\text{tot}}_{\pi^0\pi^0}\right](s) = \frac{s}{2}\sigma^{\text{tot}}_{\pi^+\pi^-}(s) \tag{3.27}$$

$$= \frac{s}{2}\frac{N_c n_f}{32\pi}\frac{4M_Q^2}{f_\pi^4}\sqrt{1-\frac{4M_Q^2}{s}}\,, \tag{3.28}$$

and

$$\text{Im } F_3(s,0) = \frac{s}{2}\left[3\sigma^{\text{tot}}_{\pi^\pm\pi^0}-\sigma^{\text{tot}}_{\pi^0\pi^0}\right](s) \tag{3.29}$$

$$= \frac{s}{2}\frac{N_c n_f}{32\pi}\frac{4M_Q^2}{f_\pi^4}\left\{-\sqrt{1-\frac{4M_Q^2}{s}}+8\frac{M_Q^2}{s}\log\frac{1+\sqrt{1-\frac{4M_Q^2}{s}}}{1-\sqrt{1-\frac{4M_Q^2}{s}}}\right\}. \tag{3.30}$$

Inserting these expressions in the r.h.s. of the dispersion relations in eq. (2.5) results in the sum rules:

$$\frac{2}{\pi}\int_{4M_Q^2}^{\infty}\frac{ds'}{s'^3}\text{Im } F_{2,3}(s',0) = \frac{1}{f_\pi^4}\frac{1}{3}\frac{N_c n_f}{16\pi^2}\,, \tag{3.31}$$

which for $n_f=2$ reproduce the CχQM results for Re $F_{2,3}(s,0)$ in eqs. (2.7) and (2.8), with $g_A=1$, and where [14]:

$$2l_1+3l_2 = -2l_1+l_2 = \frac{2}{3}\frac{N_c}{16\pi^2}\,. \tag{3.32}$$

In fact, the chiral expansions of the Re $F_{1,2,3}(s,0)$ amplitudes in the CχQM, evaluated with the dispersion relations in eqs. (2.5) and (2.6) are:

$$\text{Re } F_1(s,0) \underset{s\to 0}{=} -\frac{s}{f_\pi^2}\left(1+\frac{N_c n_f}{240\pi^2}\frac{s^2}{f_\pi^2 M_Q^2}+\mathcal{O}(s^4/M_Q^8)\right)\,, \tag{3.33}$$

$$\text{Re } F_2(s,0) \underset{s\to 0}{=} \frac{s^2}{f_\pi^4}\frac{N_c n_f}{48\pi^2}\left(1+\frac{1}{70}\frac{s^2}{M_Q^4}+\mathcal{O}(s^4/M_Q^8)\right)\,, \tag{3.34}$$

$$\text{Re } F_3(s,0) \underset{s\to 0}{=} \frac{s^2}{f_\pi^4}\frac{N_c n_f}{48\pi^2}\left(1+\frac{1}{35}\frac{s^2}{M_Q^4}+\mathcal{O}(s^4/M_Q^8)\right)\,. \tag{3.35}$$

IV Pion-Pion Cross Sections in Large-N_cQCD

Let us now consider the pion–pion amplitudes within the framework of Large–N_c QCD. In this limit the Im $T^I(s,0)$ isospin amplitudes are composed of an infinite set of narrow states:

$$\frac{1}{\pi}\text{Im } T^I(s,0) = \sum_{n=0}^{\infty} |F_{I,n}|^2 \, \delta\left(s - M_{I,n}^2\right) \, , \quad I = 0,1 \, . \tag{4.1}$$

The spacing of the narrow states, explored in Lattice Large–N_c QCD simulations [26], is compatible with the Regge growth of the leading trajectories[5] ($n = 0,1,2,3,\dots$):

$$M_{I=1,n}^2 = \frac{1}{\alpha_1}\left(2n + 1 - \alpha_0\right) \quad \text{and} \quad M_{I=0,n}^2 = \frac{1}{\alpha_1}\left(2n + 2 - \alpha_0\right) \, , \tag{4.2}$$

and, to a good approximation, with α_1 and α_0 as given by the Veneziano model [28]:

$$\alpha_1 \simeq \frac{1}{2M_\rho^2} \quad \text{and} \quad \alpha_0 \simeq 1/2 \, . \tag{4.3}$$

With $M_\rho = 770$ MeV, this choice corresponds to:

$$M_\sigma = 1334 \text{ MeV} \sim f_2(1270) \quad \text{and} \quad \Lambda_\rho = \Lambda_\sigma = 2M_\rho \, . \tag{4.4}$$

In the absence of exotic trajectories and hence no poles with $I = 2$, the optical theorem relates the imaginary parts of the forward isospin amplitudes to the total $\pi\pi$ cross sections as follows:

$$\text{Im } T^1(s,0) = \frac{s}{2}\left[\sigma^{\text{total}}_{\pi^+\pi^-}(s) + 3\sigma^{\text{total}}_{\pi^\pm\pi^0}(s) - \sigma^{\text{total}}_{\pi^0\pi^0}(s)\right] = \frac{s}{2}\, 4\, \sigma^{\text{total}}_{\pi^\pm\pi^0}(s) \, . \tag{4.5}$$

$$\text{Im } T^0(s,0) = \frac{s}{2}\, 6\, \sigma^{\text{total}}_{\pi^0\pi^0}(s) \, . \tag{4.6}$$

Unfortunately, there is no information on the values of the residues $|F_{I,n}|$. As shown in ref. [5], demanding that the cross sections which define Im $T^{I=1}(s,0)$ and Im $T^{I=0}(s,0)$ in eqs. (4.5) and (4.6) grow asymptotically as the Froissart bound, requires the couplings $|F_{I,n}|^2$ to grow like $n \log^2 n$ as $n \to \infty$. This suggests as a possible set of Large–N_c Im $T^I(s,0)$ amplitudes the simple Ansatz:

$$\frac{1}{\pi}\text{Im } T^I(s,0) = \text{C}_\text{I} \sum_{n=0}^{\infty} \left(M_I^2 + n\Lambda_I^2\right) \log^2\left(\frac{M_I^2}{\Lambda_I^2} + n\right) \delta(s - M_I^2 - n\Lambda_I^2) \, , \tag{4.7}$$

where:

$$\Lambda_{I=1} = \Lambda_\rho \, , M_{I=1} = M_\rho \quad \text{and} \quad \Lambda_{I=0} = \Lambda_\sigma \, , M_{I=0} = M_\sigma \, , \tag{4.8}$$

[5] See e.g. ref. [27].

and the $\mathrm{C_I}$ are dimensionless constants. This is essentially the same Ansatz which was considered in ref. [5]. There is, however, an important extrapolation which one is making in assuming such an Ansatz, namely the fact that the $n \log^2 n$ pattern of the residues $|F_{I,n}|$ is valid not only at very high energies but also at low energies. Because of that extrapolation, the corresponding values that one obtains for the coefficients of the $\log^2 s$ behaviour of the total $\sigma_{\pi\pi}^{\rm total}$ cross sections are likely to be a gross overestimation. In what follows, we propose a more elaborated Large–$\mathrm{N_c}$ Ansatz where the residues $|F_{I,n}|$ start growing like n, as suggested by the Veneziano model, and only from a threshold s_0 onwards, where asymptotics sets in, are they modified by terms which grow as $n \log^2 n$. More precisely, we suggest the following Large–$\mathrm{N_c}$ Ansatz:

$$\frac{1}{\pi}\mathrm{Im}\ T^I(s,0) = \sum_{n=0}^{\infty} \left(M_I^2 + n\Lambda_I^2\right) \left[\mathrm{C_I^V} + \mathrm{C_I^F}\ \theta(n-N) \log^2\left(\frac{M_I^2}{\Lambda_I^2} + n\right)\right] \delta(s - M_I^2 - n\Lambda_I^2)\,, \tag{4.9}$$

with N a sufficiently large integer so as to match the asymptotic threshold

$$s_0 = M_I^2 + N\Lambda_I^2\,, \tag{4.10}$$

and the mass parameters $\Lambda_{I=1} = \Lambda_\rho\,, M_{I=1} = M_\rho$ and $\Lambda_{I=0} = \Lambda_\sigma\,, M_{I=0} = M_\sigma$ the same as in eq. (4.8). Equation (4.9) can also be written in the more convenient form:

$$\begin{aligned}\frac{1}{\pi}\mathrm{Im}\ T^I(s,0) &= \mathrm{C_I^V} \sum_{n=0}^{\infty} \left(M_I^2 + n\Lambda_I^2\right) \delta(s - M_I^2 - n\Lambda_I^2) \\ &\quad + \mathrm{C_I^F} \sum_{n=0}^{\infty} (s_0 + n\Lambda_I^2) \log^2\left(\frac{s_0}{\Lambda_I^2} + n\right) \delta(s - s_0 - n\Lambda_I^2)\,. \end{aligned} \tag{4.11}$$

The Mellin transforms of infinite sums like the ones in eq. (4.11) have a close analytic form:

$$\Sigma^I(\xi) = \int_0^\infty \mathrm{d}\left(\frac{s}{\Lambda_I^2}\right)\left(\frac{s}{\Lambda_I^2}\right)^{\xi-1} \frac{1}{\pi}\mathrm{Im}\ T^I(s,0) \tag{4.12}$$

$$= \mathrm{C_I^V}\ \zeta\left(-\xi, \frac{M_I^2}{\Lambda_I^2}\right) + \mathrm{C_I^F}\ \frac{d^2}{d\xi^2}\ \zeta\left(-\xi, \frac{s_0^2}{\Lambda_I^2}\right)\,, \tag{4.13}$$

where $\zeta\left(-\xi, \frac{M_I^2}{\Lambda_I^2}\right)$ is the Hurwitz function, a generalization of the Riemann zeta function, defined by the series:

$$\zeta(\xi, v) = \sum_{n=0}^{\infty} \frac{1}{(n+v)^\xi}, \quad v \neq -1, -2, -3, \ldots \tag{4.14}$$

and its analytic continuation. For $v = 1$ it reduces to the Riemann zeta function. The asymptotic behaviour of $\frac{1}{\pi}\text{Im } T^I(s,0)$ for $s \to \infty$ is then governed by the inverse Mellin transform

$$\frac{1}{\pi}\text{Im } T^I(s,0) = \frac{1}{2\pi i}\int_{c_\xi - i\infty}^{c_\xi + i\infty} \mathrm{d}\xi \left(\frac{s}{\Lambda_I^2}\right)^{-\xi} \left[\mathrm{C}_\mathrm{I}^\mathrm{V}\ \zeta\left(-\xi, \frac{M_I^2}{\Lambda_I^2}\right) + \mathrm{C}_\mathrm{I}^\mathrm{F}\ \frac{d^2}{d\xi^2}\ \zeta\left(-\xi, \frac{s_0^2}{\Lambda_I^2}\right)\right]\ ; \quad (4.15)$$

and more precisely, by the residues of the triple pole of the $\mathrm{C}_\mathrm{I}^\mathrm{F}$ term and the single pole of the $\mathrm{C}_\mathrm{I}^\mathrm{V}$ term of $\Sigma^I(\xi)$ at $\xi = -1$. This fixes the asymptotic behaviours of the total $\pi^\pm\pi^0$ and $\pi^0\pi^0$ cross sections in the r.h.s. of eqs. (4.5) and (4.6) as follows:

$$\sigma^{\text{total}}_{\pi^\pm\pi^0}(s) \underset{s\to\infty}{\sim} \frac{\pi}{2\Lambda_\rho^2}\left(\mathrm{C}_1^\mathrm{F}\log^2\frac{s}{s_0} + \mathrm{C}_1^\mathrm{V}\right), \quad (4.16)$$

$$\sigma^{\text{total}}_{\pi^0\pi^0}(s) \underset{s\to\infty}{\sim} \frac{\pi}{3\Lambda_\rho^2}\left(\mathrm{C}_0^\mathrm{F}\log^2\frac{s}{s_0} + \mathrm{C}_0^\mathrm{V}\right). \quad (4.17)$$

In order to learn something about the constants $\mathrm{C}_\mathrm{I}^{\mathrm{V,F}}$ one can use information from the low energy behaviour of Re $T^I(s,0)$ using a Mellin–Barnes representation of the dispersion relations in eq. (2.5). This results in the expression [5] :

$$\text{Re } T^I(s,0) = \frac{1}{2\pi i}\int_{c_\xi - i\infty}^{c_\xi + i\infty} \mathrm{d}\xi \left(\frac{s}{\Lambda_I^2}\right)^{2-\xi} \Gamma(\xi)\Gamma(1-\xi)\left[1 + \frac{\pi}{\Gamma(\frac{1}{2}+\xi)\Gamma(\frac{1}{2}-\xi)}\right]\Sigma^I(\xi-2)\,, \quad (4.18)$$

where $\Sigma^I(\xi-2)$ is the same Mellin transform as the one defined in eq. (4.12). The leading low energy behaviour of Re $T^I(s,0)$ is then governed by the leading singularity of the integrand in the r.h.s. of eq. (4.18) at the left of the fundamental strip $c_\xi = \text{Re } \xi \in\ ]0,1[$, i.e. at $\xi = 0$, with the result

$$\text{Re } T^I(s,0) \underset{s\to 0}{\sim} 2\frac{s^2}{\Lambda_I^4}\left[\mathrm{C}_\mathrm{I}^\mathrm{V}\zeta\left(2, \frac{M_I^2}{\Lambda_I^2}\right) + \mathrm{C}_\mathrm{I}^\mathrm{F}\zeta''\left(2, \frac{s_0}{\Lambda_I^2}\right)\right]. \quad (4.19)$$

This has to match the χPT expressions in eqs. (2.7) and (2.8) restricted to their Large–$\mathrm{N_c}$ limit form i.e.,

$$\text{Re } T^1(s,0) = \text{Re } F_2(s,0) + \text{Re } F_3(s,0) \underset{s\to 0}{\sim} \frac{s^2}{f_\pi^4}4l_2(M_\rho)\,, \quad (4.20)$$

$$\text{Re } T^0(s,0) = \frac{3}{2}\left[3\text{Re } F_2(s,0) - \text{Re } F_3(s,0)\right] \underset{s\to 0}{\sim} \frac{s^2}{f_\pi^4}12\left[l_1(M_\rho) + l_2(M_\rho)\right]. \quad (4.21)$$

This matching gives a linear constraint between $\mathrm{C}_\mathrm{I}^\mathrm{V}$, $\mathrm{C}_\mathrm{I}^\mathrm{F}$ and the low–energy constants l_2 and $l_1 + l_2$:

$$32 l_2 \, \frac{M_\rho^4}{f_\pi^4} = \left[\mathrm{C}_1^{\mathrm{V}} \zeta\left(2, \frac{M_\rho^2}{\Lambda_\rho^2}\right) + \mathrm{C}_1^{\mathrm{F}} \zeta''\left(2, \frac{s_0}{\Lambda_\rho^2}\right) \right] , \tag{4.22}$$

$$96 \, (l_1 + l_2) \, \frac{M_\rho^4}{f_\pi^4} = \left[\mathrm{C}_0^{\mathrm{V}} \zeta\left(2, \frac{M_\sigma^2}{\Lambda_\sigma^2}\right) + \mathrm{C}_1^{\mathrm{F}} \zeta''\left(2, \frac{s_0}{\Lambda_\sigma^2}\right) \right] . \tag{4.23}$$

The behaviour of $\sigma^{\mathrm{total}}_{\pi^\pm\pi^0}(s)$ and $\sigma^{\mathrm{total}}_{\pi^0\pi^0}(s)$ in eqs. (4.16) and (4.17) at the onset of the asymptotic threshold $s = s_0$ fixes the values of $\mathrm{C}_1^{\mathrm{V}}$ and $\mathrm{C}_0^{\mathrm{V}}$. We propose to identify this onset with the one of the asymptotic behaviours of $\sigma^{\mathrm{total}}_{\pi^\pm\pi^0}(s)$ and $\sigma^{\mathrm{total}}_{\pi^0\pi^0}(s)$ in the CχQM, evaluated in eqs. (3.22) and (3.24). This results in the values:

$$\mathrm{C}_1^{\mathrm{V}} = 2\Lambda_\rho^2 \frac{N_c n_f}{16\pi^2 f_\pi^2} \frac{4M_Q^2}{f_\pi^2} \, \frac{M_Q^2}{s_0} \, \log \frac{s_0}{M_Q^2} , \tag{4.24}$$

$$\mathrm{C}_0^{\mathrm{V}} = 2\Lambda_\rho^2 \frac{N_c n_f}{16\pi^2 f_\pi^2} \frac{16M_Q^2}{f_\pi^2} . \tag{4.25}$$

Given input values for s_0 and l_2, $l_1 + l_2$, the four coefficients $\mathrm{C}_1^{\mathrm{V}}$, $\mathrm{C}_1^{\mathrm{F}}$ $\mathrm{C}_0^{\mathrm{V}}$, $\mathrm{C}_0^{\mathrm{F}}$ are then fixed. Obviously the onset s_0 has to be larger than M_ρ^2 in the $I = 1$ channel and larger than M_σ^2 in the $I = 0$. Provided one can find solutions to these constraints, the result is then a phenomenological proposal which matches the low–energy behaviour of the the total cross sections $\sigma^{\mathrm{total}}_{\pi^\pm\pi^0}(s)$ and $\sigma^{\mathrm{total}}_{\pi^0\pi^0}(s)$, as predicted by the C$\chi$QM, with their high–energy behaviour predicted by the Large–$\mathrm{N_c}$ Ansatz which we have described above. Numerical solutions to this proposal are discussed in the next section.

V Numerical Results and Conclusions

Several remarks concerning the issues discussed in the previous sections are in order.

- *First, with regards to the χPT low energy constants l_1 and l_2.*

  A recent phenomenological determination of the l_1 and l_2 constants, renormalized at the ρ–mass scale, gives [29]:

  $$l_1^{\mathrm{r}}(M_\rho) = (-5.2 \pm 0.5) \times 10^{-3} \quad \text{and} \quad l_2^{\mathrm{r}}(M_\rho) = (4.0 \pm 1.5) \times 10^{-3} . \tag{5.1}$$

  For comparison, the values predicted by the low resonance saturation of the l_1 and l_2 constants are [30, 31]:

  $$l_2 = \frac{1}{2} \frac{f_\pi^2}{M_\rho^2} \simeq 6.6 \times 10^{-3} , \quad \text{and} \quad l_1 + l_2 = \frac{1}{4} \frac{|c_d|^2}{M_{f_0}^2} \simeq 0.3 \times 10^{-3} . \tag{5.2}$$

These are the values where l_2 is saturated by the ρ, with $f_\pi \simeq 0.088$ GeV i.e. the chiral limit value [32], and $l_1 + l_2$ is saturated by the $f_0(983)$, with $|c_d| \simeq 3.2 \times 10^{-2}$ GeV and $M_{f_0} = 0.983$ GeV [30]. Within errors, these values compare rather well with the phenomenological determinations above. It is, however, far from clear that the $f_0(983)$ is a particle one should identify with a Large–$\mathrm{N_c}$ $\bar{q}q$ state [33]. We also recall that in the CχQM (see eqs. (3.32) above) one finds:

$$l_2 = \frac{1}{3}\frac{N_c}{16\pi^2} = 6.3 \times 10^{-3} \quad \text{and} \quad l_1 + l_2 = \frac{1}{6}\frac{N_c}{16\pi^2} = 3.2 \times 10^{-3}\,. \tag{5.3}$$

The l_2 values of the CχQM and the ρ saturation approximation compare very well. Unfortunately, the combination of low energy constants $l_1 + l_2$ determined from experiments has a large error: $l_1 + l_2 = (-1.2 \pm 1.6) \times 10^{-3}$, which makes it difficult to extract useful information on the coefficients $\mathrm{C_0^V}$ and $\mathrm{C_0^F}$ using this phenomenological input.

- *The size of* $\zeta\left(2, \frac{M_I^2}{\Lambda_I^2}\right)$ *and* $\zeta''\left(2, \frac{s_0^2}{\Lambda_I^2}\right)$ *in the r.h.s. of eq. (4.19).*

  Figure 2 below shows the shape of the factor $\zeta\left(2, \frac{M_I^2}{\Lambda_I^2}\right)$ for the range $0.2 \leq \frac{M_I^2}{\Lambda_I^2} \leq 1$. The particular values we are interested in are ($\Lambda_\rho = \Lambda_\sigma = 2M_\rho$):

$$\zeta\left(2, \frac{M_\rho^2}{\Lambda_\rho^2} = \frac{1}{4}\right) = 17.2 \quad \text{and} \quad \zeta\left(2, \frac{M_\sigma^2}{\Lambda_\rho^2} = 0.75\right) = 2.54\,. \tag{5.4}$$

  Figure 3 shows the shape of the factor $\zeta''\left(2, \frac{s_0}{\Lambda_\rho^2}\right)$ for the range $1 \leq \frac{s_0}{M_\rho^2} \leq 25$. The particular values we shall be using below are:

$$\zeta''\left(2, \frac{(2.537M_\rho)^2}{4M_\rho^2} = 1.61\right) = 2.01 \quad \text{and} \quad \zeta''\left(2, \frac{(3.27M_\sigma)^2}{4M_\rho^2} = 8.02\right) = 1.34\,. \tag{5.5}$$

- *Fixing the constants* $\mathrm{C_1^V}$ *and* $\mathrm{C_1^F}$ *and* $\sigma^{\mathrm{total}}_{\pi^\pm\pi^0}$.

  As a numerical example concerning the $I = 1$ channel we show in figure 4 the total cross section $\sigma^{\mathrm{total}}_{\pi^\pm\pi^0}$ in millibarns versus $\sqrt{s}$ in GeV when l_2 is fixed to the center value of the phenomenological determination in eq. (5.1) and the choice of s_0 is tuned to $\sqrt{s_0} = 2.537M_\rho$ so as to obtain a slope of the $\log^2 \frac{s}{s_0}$ term in eq. (4.16) of the order of what is experimentally observed in the asymptotic behaviour of the total pp total cross section [21–23]. The fact that this relatively simple model of Large–$\mathrm{N_c}$ QCD we are considering can accommodate for such a solution with a reasonable value for s_0 which matches the asymptotic behaviour of the CχQM is quite a remarkable fact.

- *Fixing the constants* $\mathrm{C_0^V}$ *and* $\mathrm{C_0^F}$ *and* $\sigma^{\mathrm{total}}_{\pi^\pm\pi^0}$.

  As already mentioned, the phenomenological determination of $l_1 + l_2$ has, unfortunately, a rather large error. On the other hand, the low resonance saturation value given in

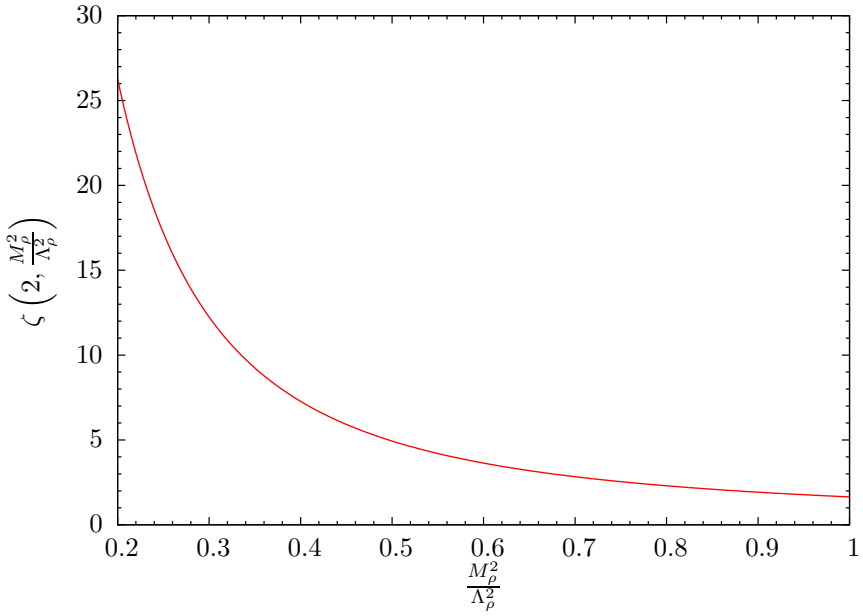

Figure 2. Shape of the factor $\zeta\left(2, \frac{M_I^2}{\Lambda_I^2}\right)$ in the r.h.s. of eqs. (4.19).

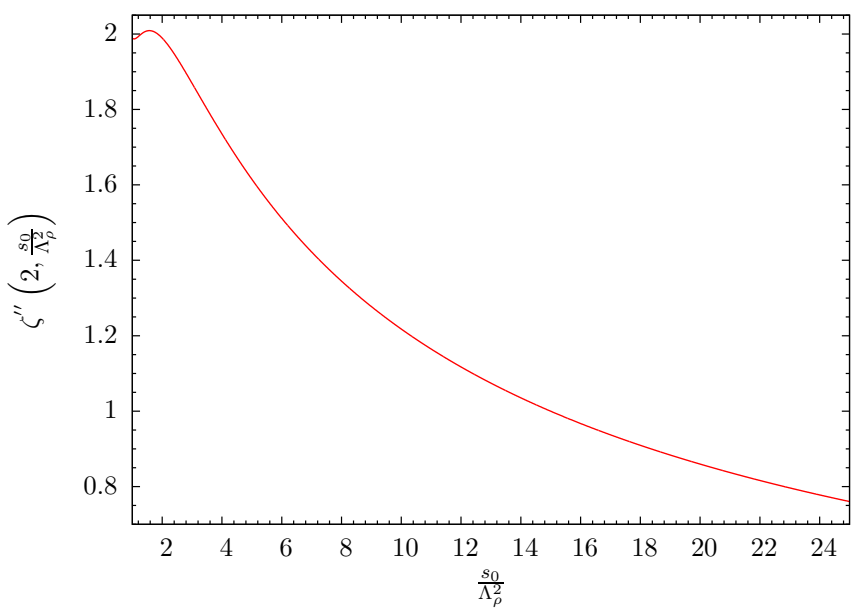

Figure 3. Shape of the factor $\zeta''\left(2, \frac{s_0}{\Lambda_\rho^2}\right)$ in the r.h.s. of eqs. (4.19).

eq. (5.2), because of the questionable input of the scalar mass, is not reliable. The best option is to use the CχQM determination in eq. (5.3), keeping in mind that this is a model–dependent input. Figure 5 shows the corresponding total cross section $\sigma_{\pi^0\pi^0}^{\text{total}}$ in

millibarns versus $\sqrt{s}$ in GeV for a choice of s_0 tuned to $\sqrt{s_0} = 3.27 M_\sigma$ so as to obtain a slope of the $\log^2 \frac{s}{s_0}$ term in eq. (4.17) of the order of what is experimentally observed in the asymptotic behaviour of the total pp total cross section [21–23]. In spite of the fact that this is only an illustrative example of a possible Large–$\mathrm{N_c}$ Ansatz, it is quite remarkable that we can find a solution which respects the constraints discussed above with rather reasonable input values.

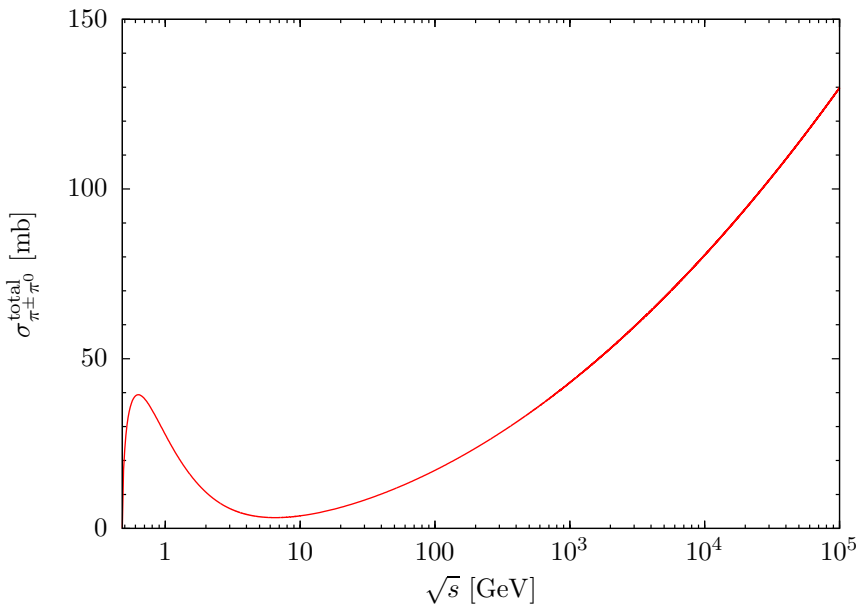

Figure 4. $\sigma^{\rm total}_{\pi^\pm\pi^0}$ in millibarns versus $\sqrt{s}$ in GeV.

The main conclusion of this paper is the observation which we have discussed in Section II.1, namely the fact that -if the Froissart bound applies in QCD to $\pi\pi$ total cross sections- then the Lukaszuk-Martin coefficient π/m_π^2 of the $\log^2 s$ asymptotic behaviour cannot be the optimal one. The Lukaszuk-Martin coefficient violates both the QCD chiral behaviour and the QCD Large–$\mathrm{N_c}$ counting rules.

Assuming that a Froissart–like $\log^2 s$ behaviour does apply to the total $\pi\pi$ scattering cross sections in QCD, we have then shown that it is possible to construct Large–$\mathrm{N_c}$ Ansatz which reproduces this behaviour with coefficients which are finite in the chiral limit and have the correct $\mathcal{O}(1/\mathrm{N_c})$ counting in the Large–$\mathrm{N_c}$ limit. We have constructed total cross sections for $\pi^\pm\pi^0$ and $\pi^0\pi^0$ scattering which match the low–energy behaviour predicted by the CχQM discussed in Section III with the Large–$\mathrm{N_c}$ Ansatz discussed in section IV. They are shown in Figs. 4 and 5 above. The order of magnitude of these cross sections at very high energies are similar to the total pp and $p\bar{p}$ scattering cross sections which are experimentally

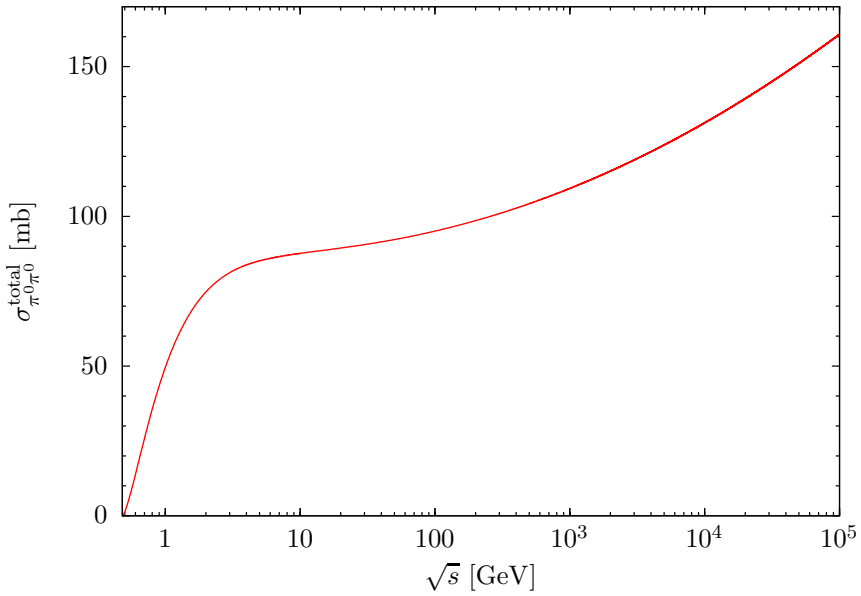

Figure 5. $\sigma^{\rm total}_{\pi^0\pi^0}$ in millibarns versus $\sqrt{s}$ in GeV.

observed [21–23].

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