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# Gradient Schemes for Stokes problem

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## Abstract

We provide a framework which encompasses a large family of conforming and nonconforming numerical schemes, for the approximation of the steady state incompressible Stokes equations with homogeneous Dirichlet's boundary conditions. Three examples (Taylor-Hood, extended MAC and Crouzeix-Raviart schemes) are shown to enter into this framework. The convergence of the scheme is proved by compactness arguments, thanks to estimates on the discrete solution that allow to prove the weak convergence to the unique continuous solution of the problem. Then strong convergence results are obtained thanks to the limit problem. An error estimate result is provided, applying on solutions with low regularity.

## 1 Incompressible steady Stokes problem

We consider the incompressible steady Stokes problem:

$$\begin{cases} \eta \bar{u} - \Delta \bar{u} + \nabla \bar{p} &= f & \text{in } \Omega \\ \operatorname{div} \bar{u} &= 0 & \text{in } \Omega \\ \bar{u} &= 0 & \text{on } \partial\Omega \end{cases} \quad (1)$$

where  $\bar{u}$  represents the velocity field and  $\bar{p}$  the pressure, under the following hypotheses (called Hypotheses  $H$  in the following):  $\Omega$  an open bounded Lipschitz domain of  $\mathbb{R}^d$  with  $d = 2$  or  $3$ ,  $f \in L^2(\Omega)^d$  and  $\eta \in [0, +\infty)$ .

### Definition 1.1

Under Hypotheses  $(H)$ , denoting, for  $(\xi^{(i)}, \chi^{(i)})_{i=1,\dots,d}$  with  $\xi^{(i)}, \chi^{(i)} \in \mathbb{R}^d$ , by  $\xi : \chi = \sum_{i=1}^d \xi^{(i)} \cdot \chi^{(i)}$ ,  $(\bar{u}, \bar{p})$  is called a weak solution to (1) if

$$\begin{cases} \bar{u} \in H_0^1(\Omega)^d, \bar{p} \in L_0^2(\Omega) \text{ where } L_0^2(\Omega) = \{q \in L^2(\Omega), \int_{\Omega} q dx = 0\}, \\ \eta \int_{\Omega} \bar{u} \cdot \bar{v} dx + \int_{\Omega} \nabla \bar{u} : \nabla \bar{v} dx - \int_{\Omega} \bar{p} \operatorname{div} \bar{v} dx = \int_{\Omega} f \cdot \bar{v} dx, \forall \bar{v} \in H_0^1(\Omega)^d, \\ \operatorname{div}(\bar{u}) = 0 \text{ a.e. in } \Omega. \end{cases} \quad (2)$$

**Theorem 1.1 (Existence and uniqueness [10])** Under Hypotheses  $(H)$ , there exists one and only one weak solution  $(\bar{u}, \bar{p})$  to Problem (1) in the sense of Definition 1.1.

The aim of this paper is to provide a theoretical framework which includes several useful schemes for the Stokes (and the Navier-Stokes) problems. This framework is given in Section 2, as an extension of the notion of gradient schemes provided for scalar elliptic problems [3, 4, 5, 7, 6]. Among the schemes which are included in this framework, we briefly present in Section 3 the Taylor-Hood scheme, an extended version [1] of the Marker-And-Cell (MAC) scheme [8, 9, 11] and the Crouzeix-Raviart scheme [2] (these three schemes are useful in many industrial applications). In Section 4, we provide the convergence result for this general framework, followed by an error estimate result, also providing a proof for the convergence of the general scheme, but needing slightly more regularity than the convergence result.

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## 2 Gradient scheme

**Definition 2.1 (Gradient Discretisation for the Stokes problem)** A gradient discretisation  $D$  for the Stokes problem with homogeneous Dirichlet's boundary conditions, is defined by  $D = (X_{D,0}, \Pi_D, \nabla_D, Y_D, \chi_D, \text{div}_D)$ , with:

1.  $X_{D,0}$  is a vector space on  $\mathbb{R}$  with finite dimension.
2.  $Y_D$  is a vector space on  $\mathbb{R}$  with finite dimension.
3. The linear mapping  $\Pi_D : X_{D,0} \rightarrow L^2(\Omega)^d$  is the reconstruction of the approximate velocity field.
4. The linear mapping  $\chi_D : Y_D \rightarrow L^2(\Omega)$  is the reconstruction of the approximate pressure, and must be chosen such that  $\|\chi_D \cdot\|_{L^2(\Omega)}$  is a norm on  $Y_D$ . We then denote  $Y_{D,0} = \{q \in Y_D, \int_{\Omega} \chi_D q dx = 0\}$ .
5. The linear mapping  $\nabla_D : X_{D,0} \rightarrow L^2(\Omega)^{d \times d}$  is the discrete gradient operator. It must be chosen such that  $\|\cdot\|_D := \|\nabla_D \cdot\|_{L^2(\Omega)^{d \times d}}$  is a norm on  $X_{D,0}$ .
6. The linear mapping  $\text{div}_D : X_{D,0} \rightarrow L^2(\Omega)$  is the discrete divergence operator.

**Remark 2.1 (Boundary conditions)** The definition of  $\|\cdot\|_D$  depends on the considered boundary conditions. Here for simplicity we only consider homogeneous Dirichlet's boundary conditions (leading to the notation  $X_{D,0}$ ) but other can easily be addressed.

**Definition 2.2 (Coercivity)** Let  $D$  be a discretisation in the sense of definition 2.1. Let  $C_D$  and  $\beta_D$  be defined by

$$C_D = \max_{v \in X_{D,0}, \|v\|_D=1} \|\Pi_D v\|_{L^2(\Omega)} + \max_{v \in X_{D,0}, \|v\|_D=1} \|\text{div}_D v\|_{L^2(\Omega)},$$

$$\beta_D = \min \left\{ \max_{v \in X_{D,0}, \|v\|_D=1} \int_{\Omega} \chi_D q \text{div}_D v \, dx, q \in Y_{D,0} \text{ such that } \|\chi_D q\|_{L^2(\Omega)} = 1 \right\}.$$

A sequence  $(D_m)_{m \in \mathbb{N}}$  of gradient discretisation is said to be **coercive** if there exist  $C_P \in \mathbb{R}_+$  such that  $C_{D_m} \leq C_P$  (discrete Poincaré inequality and control of discrete divergence) and if there exists  $\beta \in (0, +\infty)$  such that  $\beta_{D_m} \geq \beta$  (discrete LBB condition), for all  $m \in \mathbb{N}$ .

**Definition 2.3 (Consistency)** Let  $D$  be a gradient discretisation in the sense of definition 2.1, and let  $I_D : H_0^1(\Omega)^d \mapsto X_{D,0}$ ,  $S_D : H_0^1(\Omega)^d \rightarrow [0, +\infty)$ ,  $\tilde{I}_D : L_0^2(\Omega) \mapsto Y_{D,0}$  and  $\tilde{S}_D : L_0^2(\Omega) \rightarrow [0, +\infty)$  be defined by

$$\begin{aligned} \forall \varphi \in H_0^1(\Omega)^d, \forall v \in X_{D,0}, \\ E_D(v, \varphi) &= \|\Pi_D v - \varphi\|_{L^2(\Omega)^d} + \|\nabla_D v - \nabla \varphi\|_{L^2(\Omega)^{d \times d}} + \|\text{div}_D v - \text{div} \varphi\|_{L^2(\Omega)}, \\ \forall \varphi \in H_0^1(\Omega)^d, I_D(\varphi) &= \underset{v \in X_{D,0}}{\text{argmin}} E_D(v, \varphi), \quad S_D(\varphi) = E_D(I_D(\varphi), \varphi), \end{aligned}$$

and

$$\forall \psi \in L_0^2(\Omega), \quad \tilde{I}_D(\psi) = \underset{z \in Y_{D,0}}{\text{argmin}} \|\chi_D z - \psi\|_{L^2(\Omega)}, \quad \tilde{S}_D(\psi) = \|\chi_D \tilde{I}_D(\psi) - \psi\|_{L^2(\Omega)}.$$

A sequence  $(D_m)_{m \in \mathbb{N}}$  of gradient discretisation is said to be **consistent** if, for all  $\varphi \in H_0^1(\Omega)^d$ ,  $S_{D_m}(\varphi)$  tends to 0 when  $m \rightarrow \infty$  and for all  $\psi \in L_0^2(\Omega)$ ,  $\tilde{S}_{D_m}(\psi)$  tends to 0 as  $m \rightarrow \infty$ .

**Definition 2.4 (Limit-conformity)** Let  $D$  be a gradient discretisation in the sense of definition 2.1, and let  $W_D : H_{\text{div}}(\Omega)^d \rightarrow [0, +\infty)$  and  $\widetilde{W}_D : H^1(\Omega) \rightarrow [0, +\infty)$  be respectively defined by

$$\begin{aligned} \forall \varphi \in H_{\text{div}}(\Omega)^d, W_D(\varphi) &= \max_{v \in X_{D,0}, \|v\|_D=1} \left( \int_{\Omega} (\nabla_D v : \varphi + \Pi_D v \cdot \text{div} \varphi) dx \right), \\ \forall \psi \in H^1(\Omega), \widetilde{W}_D(\psi) &= \max_{v \in X_{D,0}, \|v\|_D=1} \left( \int_{\Omega} (\Pi_D v \cdot \nabla \psi + \psi \text{div}_D v) dx \right). \end{aligned}$$

A sequence  $(D_m)_{m \in \mathbb{N}}$  of gradient discretisation is said to be **limit-conforming** if, for all  $\varphi \in H_{\text{div}}(\Omega)^d$ ,  $W_{D_m}(\varphi)$  tends to 0 when  $m \rightarrow \infty$  and if, for all  $\psi \in H^1(\Omega)$ ,  $\widetilde{W}_{D_m}(\psi)$  tends to 0 as  $m \rightarrow \infty$ .

Under Hypotheses (H), let  $D$  be a gradient discretisation of  $\Omega$  in the sense of definition 2.1. The gradient scheme for the approximation of Problem (1) is given by

$$\begin{cases} (u, p) \in X_{D,0} \times Y_{D,0}, \quad \forall v \in X_{D,0}, \\ \eta \int_{\Omega} \Pi_D u \cdot \Pi_D v dx + \int_{\Omega} \nabla_D u : \nabla_D v dx - \int_{\Omega} \chi_D p \operatorname{div}_D v dx = \int_{\Omega} f \cdot \Pi_D v dx, \\ \int_{\Omega} \operatorname{div}_D u \chi_D q dx = 0, \quad \forall q \in Y_{D,0}. \end{cases} \quad (3)$$

### 3 Examples of gradient schemes for the Stokes problem

#### Conforming Taylor-Hood scheme

For this example,  $X_{D,0}$  (resp.  $Y_D$ ) is the vector space of the degrees of freedom for the velocity (resp. the pressure) in the Taylor-Hood element,  $\Pi_D$  and  $\chi_D$  are obtained through the finite element basis functions, and we define the conforming operators  $\nabla_D = \nabla \circ \Pi_D$  and  $\operatorname{div}_D = \operatorname{div} \circ \Pi_D$  (this implies that  $W_D$  and  $\tilde{W}_D$  are identically null).

#### The extended MAC scheme for non conforming meshes

This example is detailed in [1]. We consider 2D or 3D meshes of  $\Omega$ , which are such that all internal faces have their normal vector parallel to one of the basis vector  $e^{(k)}$  of the space  $\mathbb{R}^d$ , for some  $k = 1, \dots, d$  (see an example at left part of Figure 1). Note that on the other hand, the external faces need not be aligned with the axes: they are only assumed to be planar. Hence curved boundaries may be meshed with such grids, by using local refinement close to the boundaries, such as in Figure 1. These meshes are used for the approximation of the pressure and of the divergence operator. Then the gradient scheme is defined as follows.

1.  $X_{D,0}$  is the vector space on  $\mathbb{R}$  of all families of normal velocities to all internal edges of the mesh (see the left part of Figure 1).
2.  $Y_D$  is the vector space on  $\mathbb{R}$  of all families of values in the cells of the mesh.
3. The linear mapping  $\Pi_D : X_{D,0} \rightarrow L^2(\Omega)^d$  is the reconstruction of the approximate velocity field, defined as the piecewise constant value of each component of the velocity in the corresponding Voronoi cells (the right part of Figure 1 presents the grid for the horizontal velocity).
4. The linear mapping  $\chi_D : Y_D \rightarrow L^2(\Omega)$  is the piecewise constant reconstruction of the approximate pressure in the pressure mesh.
5. The linear mapping  $\nabla_D : X_{D,0} \rightarrow L^2(\Omega)^{d \times d}$  is the discrete gradient operator, obtained as the gradient of the  $P^1$  reconstruction of each component of the velocity in the corresponding triangular grid (the medium part of Figure 1 shows this triangular grid for the horizontal velocity, joining the barycenters of the vertical edges of the pressure grid).
6. The linear mapping  $\operatorname{div}_D : X_{D,0} \rightarrow L^2(\Omega)$  is the discrete divergence operator, simply computed as the piecewise constant value in the cells of the pressure grid obtained through the balance of the normal velocities integrated over the faces of the mesh.

#### The Crouzeix-Raviart scheme

We consider 2D or 3D simplicial meshes of  $\Omega$  (triangles in 2D, tetrahedra in 3D). Then the Crouzeix-Raviart scheme [2] can be defined as a gradient scheme by the following way:

1.  $X_{D,0}$  is the vector space on  $\mathbb{R}$  of all families of vectors of  $\mathbb{R}^d$  at the center of all internal faces of the mesh.
2.  $Y_D$  is the vector space on  $\mathbb{R}$  of all families of values in the simplices.
3. The linear mapping  $\Pi_D : X_{D,0} \rightarrow L^2(\Omega)^d$  is the nonconforming piecewise affine reconstruction of each component of the velocity.
4. The linear mapping  $\chi_D : Y_D \rightarrow L^2(\Omega)$  is the piecewise constant reconstruction in the simplices.
5. The linear mapping  $\nabla_D : X_{D,0} \rightarrow L^2(\Omega)^{d \times d}$  is the so-called “broken gradient” of the velocity, defined as the piecewise constant field of the gradient of the affine components of the velocity in all the simplices.

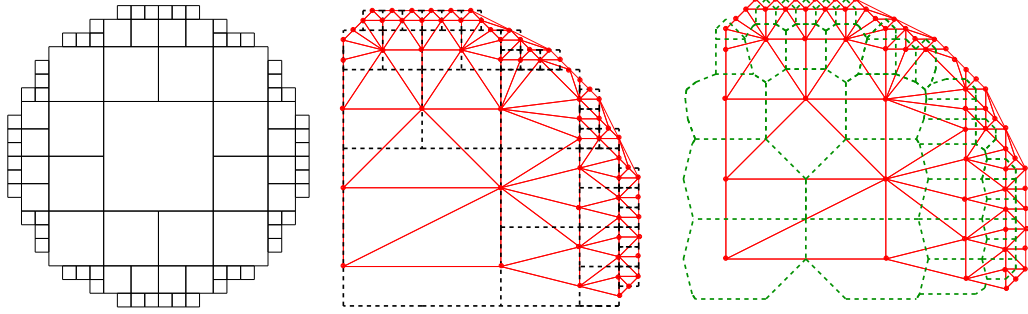


Figure 1: Left: pressure grid. Middle: Zoom on the top right triangular velocity grid, used for the gradient reconstruction of the horizontal velocity (pressure grid recalled by discontinuous lines). Right: Voronoï cells used for the velocity reconstruction.

6. The linear mapping  $\text{div}_D : X_{D,0} \rightarrow L^2(\Omega)$  is the discrete divergence operator, simply computed as the piecewise constant value in the cells of the pressure grid obtained through the balance of the normal velocities integrated over the faces of the mesh.

## 4 Convergence results

**Lemma 4.1 (Estimates)** *Under Hypotheses (H), Let  $D$  a gradient discretisation of  $\Omega$  in the sense of definition 2.1 such that  $\beta_D > 0$  (see Definition 2.2). Let  $(u, p)$  be a solution of (3). Then, there exists  $C_1 \geq 0$ , only depending on  $\Omega$ , et  $d$ ,  $\eta$ , and any  $C \geq C_D + \frac{1}{\beta_D}$  such that:*

$$\|u\|_D \leq C_1 \|f\|_{L^2(\Omega)^d} \text{ and } \|\chi_D p\|_{L^2(\Omega)} \leq C_1 \|f\|_{L^2(\Omega)^d}. \quad (4)$$

As an immediate consequence, there exists one and only one  $(u, p)$ , solution to (3).

**Proof.** One first set  $v = u$  in (3). This immediately provides the left part of (4), thanks to the Cauchy-Schwarz inequality and to Definition 2.2. Then one selects some  $v \in X_{D,0}$  such that  $\|v\|_D = 1$  and  $\beta_D \|\chi_D p\|_{L^2(\Omega)} \leq \int_{\Omega} \chi_D p \text{div}_D v \, dx$ . Choosing this  $v$  in (3) leads to the right part of (4). The existence and uniqueness of  $(u, p)$  results from the fact that it is the solution of a square linear system, with kernel reduced to  $(0, 0)$ .  $\square$

**Theorem 4.1 (Convergence of the scheme)** *Under Hypotheses (H), Let  $(\bar{u}, \bar{p})$  the unique weak solution of the incompressible steady Stokes problem (1) in the sense of definition 1.1 and let  $(D^{(m)})_{m \in \mathbb{N}}$  be a sequence of gradient discretisation on  $\Omega$  in the sense of definition 2.1, which is consistent, limit-conforming and coercive in the sense of the above definitions. Let  $(u_m, p_m)$  be the unique solution of the scheme (3) for  $D = D_m$ . Then, as  $m \rightarrow \infty$ ,*

- $\Pi_{D^{(m)}} u_m$  converges to  $\bar{u}$  in  $L^2(\Omega)^d$ ,
- $\nabla_{D^{(m)}} u_m$  converges to  $\nabla \bar{u}$  in  $L^2(\Omega)^{d \times d}$ ,
- $\chi_{D^{(m)}} p_m$  converges to  $\bar{p}$  in  $L^2(\Omega)$ .

**Proof.** In the following proof, we use simplified notations for the integrals for shortness reasons, and we replace all indices  $D^{(m)}$  by  $m$ , hence denoting by  $D^{(m)} = (X_m, \Pi_m, \nabla_m, Y_m, \chi_m, \text{div}_m)$ , and the values provided by Definition 2.2 are denoted by  $C_m$  and  $\beta_m \geq \beta > 0$ . We first observe that, thanks to Lemma 4.1 and to the limit-conformity property, up to a subsequence, there exists  $\bar{u} \in H_0^1(\Omega)^d$  and  $\bar{p} \in L_0^2(\Omega)$  such that weak convergence properties hold for the discrete reconstructions of the approximate velocity, its approximate gradient, its approximate divergence, and the approximate pressure. Then, for any test function  $\bar{v} \in H_0^1(\Omega)^d$ , we set in (3)  $v = I_m \bar{v}$  and  $q = \tilde{I}_m(\text{div} \bar{u})$  (we have  $\text{div} \bar{u} \in L_0^2(\Omega)$  since  $\int_{\Omega} \text{div} \bar{u} = \int_{\partial \Omega} \bar{u} \cdot n = 0$ , see Definition 2.4). Then we pass to the limit  $m \rightarrow \infty$

in the scheme. We get, by weak/strong convergence, that  $\int_{\Omega} (\operatorname{div} \bar{u})^2 = 0$  and that (2) holds. This proves that  $(\bar{u}, \bar{p})$  is the unique weak solution of the incompressible steady Stokes problem (1). The uniqueness of the limit shows that the whole sequence converges.

Passing to the limit in the scheme with  $v = u_m$  shows the convergence of the norm of the discrete velocity gradient  $\nabla_m u_m$  to its continuous counterpart  $\nabla \bar{u}$ . This shows the strong convergence of the gradient. The coercivity property and interpolation of the limit shows that the reconstruction of the velocity  $\Pi_m u_m$  is strongly convergent. Let us now turn to the convergence of the approximate pressure in  $L^2(\Omega)$ . We select  $v_m \in X_m$  such that  $\|v_m\|_{D(m)} = 1$  and

$$\beta \|\chi_m(\tilde{I}_m \bar{p} - p_m)\|_{L^2(\Omega)} \leq \int_{\Omega} \chi_m(\tilde{I}_m \bar{p} - p_m) \operatorname{div}_m v_m.$$

Letting  $v = v_m$  in the scheme, we get

$$\eta \int_{\Omega} \Pi_m u_m \cdot \Pi_m v_m + \int_{\Omega} \nabla_m u_m : \nabla_m v_m - \int_{\Omega} \chi_m p_m \operatorname{div}_m v_m = \int_{\Omega} f \cdot \Pi_m v_m.$$

Combining the two above relations and using the triangle inequality, we deduce

$$\begin{aligned} \beta \|\bar{p} - \chi_m p_m\|_{L^2(\Omega)} &\leq \beta \|\chi_m \tilde{I}_m \bar{p} - \bar{p}\|_{L^2(\Omega)} + \int_{\Omega} f \cdot \Pi_m v_m + \int_{\Omega} \chi_m \tilde{I}_m \bar{p} \operatorname{div}_m v_m \\ &\quad - \eta \int_{\Omega} \Pi_m u_m \cdot \Pi_m v_m - \int_{\Omega} \nabla_m u_m : \nabla_m v_m. \end{aligned}$$

Up to the extraction of a subsequence, we may assume that there exists  $\bar{v} \in H_0^1(\Omega)^d$  such that the following weak convergences hold in  $L^2$ :  $\Pi_m v_m$  to  $\bar{v}$ ,  $\nabla_m v_m$  to  $\nabla \bar{v}$  and  $\operatorname{div}_m v_m$  to  $\operatorname{div} \bar{v}$ . Using the (already proved) strong convergence properties for the velocity, we may now pass to the limit  $m \rightarrow \infty$ , since all integrals involve weak/strong convergence properties. We get

$$\beta \limsup_{m \rightarrow \infty} \|\bar{p} - \chi_m p_m\|_{L^2(\Omega)} \leq \int_{\Omega} f \cdot \bar{v} + \int_{\Omega} \bar{p} \operatorname{div} \bar{v} - \eta \int_{\Omega} \bar{u} \cdot \bar{v} - \int_{\Omega} \nabla \bar{u} : \nabla \bar{v}.$$

It now suffices to use the fact that we already proved that  $(\bar{u}, \bar{p})$  is a weak solution to the Stokes equation. We then get that the right hand side of the previous inequality vanishes, which shows the convergence in  $L^2$  for this subsequence. Using a standard uniqueness argument, we deduce that the whole sequence converges.  $\square$

The following error estimate needs more regularity hypotheses than that which have been done for the above convergence theorem.

**Theorem 4.2** *Under Hypotheses (H), Let  $(\bar{u}, \bar{p})$  be the unique solution of the incompressible steady Stokes problem (1) in the sense of definition 1.1 such that  $\bar{p} \in H^1(\Omega)$  (which implies that  $\nabla \bar{u} \in H_{\operatorname{div}}(\Omega)^d$ ). Let  $D$  be a gradient discretisation on  $\Omega$  in the sense of definition 2.1 such that  $\beta_D > 0$  (see Definition 2.2). Let  $(u, p) \in V_D$  be the unique solution of the scheme (3). Then there exists  $C_e$ , which increasingly depends on only  $\eta$ ,  $C_D$  and  $\frac{1}{\beta_D}$ , such that there holds*

$$\|\bar{u} - \Pi_D u\|_D + \|\bar{p} - \chi_D p\|_{L^2(\Omega)} \leq C_e \left( W_D(\nabla \bar{u}) + \widetilde{W}_D(\bar{p}) + S_D(\bar{u}) + \widetilde{S}_D(\bar{p}) \right).$$

**Proof.** Under the hypotheses of the theorem, since  $-\Delta \bar{u} = -\operatorname{div}(\nabla \bar{u}) = f - \nabla \bar{p} - \eta \bar{u}$  a.e., we get that  $\nabla \bar{u} \in H_{\operatorname{div}}(\Omega)^d$ . Using this expression in  $W_D(\nabla \bar{u})$  and using (3), we can write, for any  $v \in X_{D,0}$ ,

$$\begin{aligned} \int_{\Omega} (\eta(\bar{u} - \Pi_D u_D) \cdot \Pi_D v + (\nabla \bar{u} - \nabla_D u_D) : \nabla_D v + (\chi_D p_D \operatorname{div}_D v + \nabla \bar{p} \cdot \Pi_D v)) dx \\ \leq W_D(\nabla \bar{u}) \|v\|_D. \end{aligned}$$

Then, introducing the expression of  $\widetilde{W}_D(\bar{p})$ , we get

$$\begin{aligned} \int_{\Omega} (\eta(\bar{u} - \Pi_D u_D) \cdot \Pi_D v + (\nabla \bar{u} - \nabla_D u_D) : \nabla_D v + (\chi_D p_D - \bar{p}) \operatorname{div}_D v) dx \\ \leq (W_D(\nabla \bar{u}) + \widetilde{W}_D(\bar{p})) \|v\|_D. \end{aligned}$$

We now use the values  $I_D \bar{u}$  and  $\tilde{I}_D \bar{p}$  introduced in Definition 2.3, and we denote by  $\varepsilon_D(\bar{u}, \bar{p}) = W_D(\nabla \bar{u}) + \tilde{W}_D(\bar{p}) + S_D(\bar{u}) + \tilde{S}_D(\bar{p})$ . We can then write

$$\begin{aligned} & \int_{\Omega} (\eta(\Pi_D I_D \bar{u} - \Pi_D u_D) \cdot \Pi_D v + (\nabla_D I_D \bar{u} - \nabla_D u_D) : \nabla_D v) dx \\ & + \int_{\Omega} (\chi_D p_D - \chi_D \tilde{I}_D \bar{p}) \operatorname{div}_D v \leq (1 + \eta) \varepsilon_D(\bar{u}, \bar{p}) \|v\|_D. \end{aligned} \quad (5)$$

Thanks to Definition 2.2, let us now take  $v \in X_{D,0}$  such that  $\|v\|_D = 1$  and

$$\int_{\Omega} \chi_D (p_D - \tilde{I}_D \bar{p}) \operatorname{div}_D v \, dx \geq \beta_D \|\chi_D (p_D - \tilde{I}_D \bar{p})\|_{L^2(\Omega)}.$$

We then get, from (5), and using Definition 2.2,

$$\|\chi_D (p_D - \tilde{I}_D \bar{p})\|_{L^2(\Omega)} \leq \frac{1 + \eta}{\beta_D} \varepsilon_D(\bar{u}, \bar{p}) + \frac{1 + \eta C_D}{\beta_D} \|I_D \bar{u} - u_D\|_D. \quad (6)$$

Now setting  $v = I_D \bar{u} - u_D$  in (5) and using  $\int_{\Omega} \operatorname{div}_D u_D \, \chi_D q = 0$  for all  $q \in Y_D$ , we can write

$$\|I_D \bar{u} - u_D\|_D^2 + \int_{\Omega} \chi_D (p_D - \tilde{I}_D \bar{p}) \operatorname{div}_D I_D \bar{u} \, dx \leq (1 + \eta) \varepsilon_D(\bar{u}, \bar{p}) \|I_D \bar{u} - u_D\|_D,$$

which implies

$$\|I_D \bar{u} - u_D\|_D^2 \leq (1 + \eta) \varepsilon_D(\bar{u}, \bar{p}) \|I_D \bar{u} - u_D\|_D + S_D(\bar{u}) \|\chi_D (p_D - \tilde{I}_D \bar{p})\|_{L^2(\Omega)}.$$

Thanks to (6) and to the inequality  $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ , the above inequality yields the existence of  $C_1$ , increasing function of  $1/\beta_D$ ,  $C_D$  and  $\eta$ , such that  $\|I_D \bar{u} - u_D\|_D \leq C_1 \varepsilon_D(\bar{u}, \bar{p})$ . The conclusion follows, thanks to the definitions of  $I_D \bar{u}$  and  $\tilde{I}_D \bar{p}$ , to Definition 2.2, to the triangle inequality and to the use of (6).  $\square$

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