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CYCLIC COVERS OF AFFINE \mathbb{T} -VARIETIES

CHARLIE PETITJEAN

ABSTRACT. We consider normal affine \mathbb{T} -varieties X endowed with an action of finite abelian group G commuting with the action of \mathbb{T} . For such varieties we establish the existence of G -equivariant geometrico-combinatorial presentations in the sense of Altmann and Hausen. As an application, we determine explicit presentations of the Koras-Russell threefolds as bi-cyclic covers of \mathbb{A}^3 equipped with a hyperbolic \mathbb{C}^* -action.

INTRODUCTION

Every algebraic action of the one dimensional torus $\mathbb{T} \simeq \mathbb{C}^*$ on a complex affine variety X is determined by a \mathbb{Z} -grading $A = \bigoplus_{m \in \mathbb{Z}} A_m$ of its coordinate ring A , the spaces A_m consisting of semi-invariant regular functions of weight m on X . One possible way to construct \mathbb{Z} -graded algebras, which was studied by Demazure [2], is to start with a variety Y and a \mathbb{Q} -divisor D on Y and to let $A = \bigoplus_{m \in \mathbb{Z}} \Gamma(Y, \mathcal{O}_Y(mD))$. For a well chosen pair (Y, D) , this algebra is finitely generated, corresponding to the ring of regular functions of an affine variety, $X = \mathbb{S}(Y, D)$ with a \mathbb{C}^* -action whose algebraic quotient is isomorphic to $\text{Spec}(\Gamma(Y, \mathcal{O}_Y))$. A slight variant of this construction [3] already enabled a complete description of \mathbb{C}^* -actions on normal surfaces X : namely they correspond to graded algebras of the form:

$$A = \bigoplus_{m < 0} \Gamma(Y, \mathcal{O}_Y(mD_-)) \oplus \Gamma(Y, \mathcal{O}_Y) \oplus \bigoplus_{m > 0} \Gamma(Y, \mathcal{O}_Y(mD_+)),$$

for suitably chosen triples (Y, D_+, D_-) consisting of a smooth curve Y and a pair of \mathbb{Q} -divisors D_+ and D_- on it.

Demazure's construction was generalized by Altmann and Hausen [1] to give a description of all normal affine varieties X equipped with an effective action of an algebraic torus $\mathbb{T} \simeq (\mathbb{C}^*)^k$, $k \geq 1$. Here the \mathbb{Z} -grading is replaced by a grading by the lattice $M \simeq \mathbb{Z}^k$ of characters of the torus, and the graded pieces are recovered from a datum consisting of a variety Y of dimension $\dim(X) - \dim(\mathbb{T})$ and a so-called polyhedral divisor \mathcal{D} on Y , a generalization of \mathbb{Q} -divisors for higher dimensional tori: \mathcal{D} can be considered as a collection of \mathbb{Q} -divisors $\mathcal{D}(u)$ parametrized by a "weight cone" $\sigma^\vee \cap M$, for which we have $A = \bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(u)))$. The \mathbb{T} -variety associated to a pair (Y, \mathcal{D}) is denoted by $\mathbb{S}(Y, \mathcal{D})$.

In this article, we consider affine \mathbb{T} -varieties X endowed with an additional action of a finite abelian group G commuting with the action of \mathbb{T} . The quotient $X' = X//G$ is again an affine \mathbb{T}' -variety for a torus $\mathbb{T}' \simeq \mathbb{T}$ obtained as a quotient of \mathbb{T} by an appropriate finite group, and our aim is to understand the relation between the presentations $X = \mathbb{S}(Y, \mathcal{D})$ of X and those of $X' = \mathbb{S}(Y', \mathcal{D}')$. A pair (Y, \mathcal{D}) such that $X = \mathbb{S}(Y, \mathcal{D})$ is not unique but we will show that it is always possible to choose a particular pair (Y, \mathcal{D}_G) consisting of a variety Y endowed with a G -action and a G -invariant polyhedral divisor \mathcal{D}_G such that X is $G \times \mathbb{T}$ equivariantly isomorphic to $\mathbb{S}(Y, \mathcal{D}_G)$. The G -invariant divisor \mathcal{D}_G corresponds in turn to a certain polyhedral divisor \mathcal{D}' on the quotient $Y//G$ with property that $X' = \mathbb{S}(Y//G, \mathcal{D}')$ as a \mathbb{T}' -variety.

More precisely, our main result reads as follows:

Theorem. *Let X be a \mathbb{T} -variety and let G be a finite abelian group acting on X such that the two actions commute. Then the following hold:*

- 1) *There exist a semi-projective variety Y endowed with an action of G and a G -invariant pp-divisor \mathcal{D}_G defined on Y such that X is $\mathbb{T} \times G$ equivariantly isomorphic to $\mathbb{S}(Y, \mathcal{D}_G)$.*
- 2) *Moreover $X//G$ is equivariantly isomorphic to the \mathbb{T}' -variety $\mathbb{S}(Y//G, \mathcal{D}')$ where \mathcal{D}' can be chosen such that $F_*(\mathcal{D}_G) = \varphi_G^*(\mathcal{D}')$, where $\varphi_G : Y \rightarrow Y//G$ denotes the quotient morphism and $F : M^\vee \rightarrow M'^\vee$ is a linear map induced by the inclusion between the character lattices M' of \mathbb{T}' and M of \mathbb{T} .*

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We then apply this result to determine presentations of a family of exotic affine spaces of dimension 3 with hyperbolic \mathbb{C}^* -actions: the Koras-Russell threefolds. We exploit the fact that these threefolds arise as equivariant bi-cyclic cover of the affine space \mathbb{A}^3 equipped with a hyperbolic \mathbb{C}^* -action.

The article is organized as follows. The first section is devoted to a short recollection on Altmann-Hausen representations, with a particular focus on the methods to construct pairs (Y, \mathcal{D}) corresponding to a given graded algebra. The main theorem above is then established in section two. Finally, explicit Altmann-Hausen representations of the Koras-Russell threefolds are determined in section three.

1. RECOLLECTION ON THE ALTMANN-HAUSEN REPRESENTATION

In this section, we introduce the correspondence between normal affine \mathbb{T} -varieties X and pairs (Y, \mathcal{D}) composed of a normal semi-projective variety Y and a so-called polyhedral divisor \mathcal{D} established by Altmann-Hausen [1]. In particular, for a given X , we summarize a construction of a corresponding Y and explain a method to determine a possible \mathcal{D} .

1.1. Normal affine \mathbb{T} -varieties. Let $N \simeq \mathbb{Z}^k$ be a lattice of rank k and let $M = \text{Hom}(N, \mathbb{Z})$ be its dual. A pointed convex polyhedral cone $\sigma \subseteq N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ is an intersection of finitely many closed linear half spaces in $N_{\mathbb{Q}}$ which does not contain any line. Its dual:

$$\sigma^{\vee} := \{v \in M_{\mathbb{Q}} \mid \forall u \in \sigma \langle u, v \rangle \geq 0\} \subseteq M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q},$$

consists of all linear forms on $M_{\mathbb{Q}}$ that are non-negative on σ . A polytope $\Pi \subset N_{\mathbb{Q}}$ is the convex hull of finitely many points in $N_{\mathbb{Q}}$, and a convex polyhedron $\Delta \subseteq N_{\mathbb{Q}}$ is the intersection of finitely many closed affine half spaces in $N_{\mathbb{Q}}$. Every polyhedron admits a decomposition: $\Delta = \Pi_{\Delta} + \sigma$, where Π_{Δ} is a polytope and σ is a pointed convex polyhedral cone, called the tail cone of Δ . The set of all polyhedra which admit the same tail cone is a semigroup with Minkowski addition, which we denote by $\text{Pol}_{\sigma}^+(N_{\mathbb{Q}})$.

Definition 1.1. A σ -tailed polyhedral divisor \mathcal{D} on an algebraic variety Y is a formal finite sum

$$\mathcal{D} = \sum \Delta_i \otimes D_i \in \text{Pol}_{\sigma}^+(N_{\mathbb{Q}}) \otimes_{\mathbb{Z}} \text{WDiv}(Y),$$

where D_i are prime divisors on Y and Δ_i are σ -polyhedra.

Every element $u \in \sigma^{\vee} \cap M$ determines a map $\text{Pol}_{\sigma}^+(N_{\mathbb{Q}}) \otimes_{\mathbb{Z}} \text{WDiv}(Y) \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \text{WDiv}(Y)$ which associates to $\mathcal{D} = \sum \Delta_i \otimes D_i$ the Weil \mathbb{Q} -divisor $\mathcal{D}(u) = \sum \min_{v \in \Delta_i} \langle u, v \rangle D_i$ on Y .

Given a Weil \mathbb{Q} -divisor D and a section $s \in \Gamma(Y, \mathcal{O}_Y(D))$, that is, an effective Weil divisor D' linearly equivalent to the round-down $[D]$ of D , we denote by Y_s the open subset $Y \setminus \text{Supp}(D')$ of Y .

Definition 1.2. ([1, Definition 2.5 and 2.7]) A *proper-polyhedral divisor*, noted pp-divisor, is a polyhedral divisor $\mathcal{D} = \sum \Delta_i \otimes D_i$ on Y which satisfies the following properties:

- 1) Each D_i is an effective divisor and $\mathcal{D}(u)$ is a \mathbb{Q} -Cartier divisor on Y for every $u \in \sigma^{\vee} \cap M$.
- 2) $\mathcal{D}(u)$ is semi-ample for each $u \in \sigma^{\vee} \cap M$, that is, for some $n \in \mathbb{Z}_{>0}$ the open subsets Y_s , where $s \in \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(nu)))$, cover Y .
- 3) $\mathcal{D}(u)$ is big for each $u \in \text{relint}(\sigma^{\vee}) \cap M$, that is, for some $n \in \mathbb{Z}_{>0}$ there exist a section $s \in \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(nu)))$ such that Y_s is affine.

Recall [1, Definition 2.1] that a variety Y is said to be semi-projective if $\Gamma(Y, \mathcal{O}_Y)$ is finitely generated and Y is projective over $Y_0 = \text{Spec}(\Gamma(Y, \mathcal{O}_Y))$. Given a pp-divisor \mathcal{D} on Y , the graded algebra

$$A = \bigoplus_{u \in \sigma^{\vee} \cap M} A_u = \bigoplus_{u \in \sigma^{\vee} \cap M} \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(u))).$$

is finitely generated, and $\text{Spec}(A)$ is a \mathbb{T} -variety for $\mathbb{T} = \text{Spec}(\mathbb{C}[M]) \simeq (\mathbb{C}^*)^k$. More precisely Altmann and Hausen, showed the following:

Theorem 1.1. [1] *For any pp-divisor \mathcal{D} on a normal semi-projective variety Y , the scheme*

$$\mathbb{S}(Y, \mathcal{D}) = \text{Spec}\left(\bigoplus_{u \in \sigma^{\vee} \cap M} \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(u)))\right)$$

is a normal affine \mathbb{T} -variety of dimension $\dim(Y) + \dim(\mathbb{T})$. Conversely any normal affine \mathbb{T} -variety is isomorphic to an $\mathbb{S}(Y, \mathcal{D})$ for suitable Y and \mathcal{D} .

1.2. Determining the semi-projective variety. The semi-projective variety Y is not unique, however there exists a natural construction, which we will use in the remainder of the article. It can be summarized as follows ([1, section 6]).

Let $X = \text{Spec}(\bigoplus_{u \in M} A_u)$ be an affine variety endowed with an effective action of the torus $\mathbb{T} = \text{Spec}(\mathbb{C}[M])$. For each $u \in M$ the set of semistable points

$$X^{ss}(u) := \{x \in X / \exists n \in \mathbb{Z}_{\geq 0} \text{ and } f \in A_{nu} \text{ such that } f(x) \neq 0\}$$

is an open \mathbb{T} -invariant subset of X which admits a good \mathbb{T} -quotient

$$Y_u = X^{ss}(u) // \mathbb{T} = \text{Proj}_{A_0}(\bigoplus_{n \in \mathbb{Z}_{\geq 0}} A_{nu}).$$

Following [1, section 6], there exists a fan $\Lambda \in M_{\mathbb{Q}}$ generated by a finite collection of cones λ such that the following holds:

1) For any u and u' in the relative interior of λ , $X^{ss}(u) = X^{ss}(u')$. We denote $W_\lambda = X^{ss}(u)$ for any $u \in \text{relint}(\lambda)$

2) If γ is a face of λ , W_λ is an open subset of W_γ . Let $W = \bigcap_{\lambda \in \Lambda} W_\lambda = \lim_{\leftarrow} W_\lambda$.

The quotient maps $q_\lambda : W_\lambda \rightarrow W_\lambda // \mathbb{T}$ form an inverse system indexed by the cones in Λ , whose inverse limit exist as a morphism $q : W \rightarrow Z = \lim_{\leftarrow} Y_\lambda$. The desired semi-projective variety Y is the normalization of the closure of the image of W by q .

$$\begin{array}{ccccccc} W & \longrightarrow & W_\lambda & \longrightarrow & W_\gamma & \longrightarrow & X \\ \downarrow q & & \downarrow q_\lambda & & \downarrow q_\gamma & & \downarrow q_0 \\ Z & \longrightarrow & Y_\lambda & \longrightarrow & Y_\gamma & & \\ & & & & & \searrow & \\ & & & & & & Y_0 = \text{Spec}(A_0) \end{array}$$

1.3. Maps of proper polyhedral divisor. Let Y and Y' be normal semi-projective varieties, N and N' be lattices and $\sigma \subset N_{\mathbb{Q}}$, $\sigma' \subset N'_{\mathbb{Q}}$ be pointed cones. Let $\mathcal{D} = \sum \Delta_i \otimes D_i$ and $\mathcal{D}' = \sum \Delta'_i \otimes D'_i$ be pp-divisors on Y and Y' respectively with corresponding tail cones σ and σ' .

Definition 1.3. [1, Definition 8.3]1) For a morphism $\varphi : Y \rightarrow Y'$ such that $\varphi(Y)$ is not contained in $\text{Supp}(D'_i)$ for any i , the polyhedral pull-back of \mathcal{D}' is defined by :

$$\varphi^*(\mathcal{D}') := \sum \Delta'_i \otimes \varphi^*(D'_i)$$

Where $\varphi^*(D'_i)$ is the usual pull-back of D'_i . It is a polyhedral divisor on Y with tail cone σ' .

2) For a linear map $F : N \rightarrow N'$ such that $F(\sigma) \subset \sigma'$, the polyhedral push forward is defined as :

$$F_*(\mathcal{D}) := \sum (F(\Delta_i) + \sigma') \otimes D_i$$

It is also a polyhedral divisor on Y with tail cone σ' .

An equivariant morphism from $\mathbb{S}(Y, \mathcal{D})$ to $\mathbb{S}(Y', \mathcal{D}')$ is given by a homomorphism of algebraic groups $\psi : \mathbb{T} \rightarrow \mathbb{T}'$ and a morphism $\phi : \mathbb{S}(Y, \mathcal{D}) \rightarrow \mathbb{S}(Y', \mathcal{D}')$ satisfying $\phi(\lambda.x) = \psi(\lambda).\phi(x)$. Every such morphism is uniquely determined by a triple (φ, F, f) defined as above consisting of a dominant morphism $\varphi : Y \rightarrow Y'$, a linear map $F : N \rightarrow N'$ as above and a *plurifunction* $f \in N' \otimes_{\mathbb{Z}} \mathbb{C}(Y)^*$ such that :

$$\varphi^*(\mathcal{D}') \leq F_*(\mathcal{D}) + \text{div}(f).$$

The identity map of a pp-divisor is the triple $(\text{id}, \text{id}, 1)$ and the composition of two maps (φ, F, f) and (φ', F', f') is $(\varphi' \circ \varphi, F' \circ F, F'_*(f) \cdot \varphi^*(f'))$.

1.4. Determining proper polyhedral divisors. A method to determine a possible pp-divisor \mathcal{D} ([1, section 11]) associated to a \mathbb{T} -variety X with $\mathbb{T} = (\mathbb{C}^*)^k$ is to embed X as a \mathbb{T} -stable subvariety of a toric variety. The calculation is then reduced to the toric case by considering an embedding in \mathbb{A}^m with linear action for m sufficiently large. In other words, X is realized as a $(\mathbb{C}^*)^k$ -stable subvariety of a $(\mathbb{C}^*)^m$ -toric variety. The inclusion of $(\mathbb{C}^*)^k$ corresponds to an inclusion of the lattice of characters \mathbb{Z}^k of \mathbb{T} into \mathbb{Z}^m . We obtain the exact sequence:

$$0 \longrightarrow \mathbb{Z}^k \xrightarrow[F]{} \mathbb{Z}^m \xrightarrow[P]{} \mathbb{Z}^m / \mathbb{Z}^k \longrightarrow 0,$$

\xleftarrow{s}

where F is given by the action of $(\mathbb{C}^*)^k$ on \mathbb{A}^m and s is a section of F . The $(\mathbb{C}^*)^m$ -toric variety is determined by the first integral vectors v_i of the unidimensional cone generated by the i -th column vector of P as rays in a \mathbb{Z}^m lattice, and each v_i correspond to a divisor. The support of D_i is the intersection between X and the divisor corresponding to v_i . The tail cone is $\sigma := s(\mathbb{Q}_{\geq 0}^m \cap F(\mathbb{Q}))$, and the polytopes are $\Pi_i = s(\mathbb{R}_{\geq 0}^m \cap P^{-1}(v_i))$.

2. ACTIONS OF FINITE ABELIAN GROUPS

Let $X = \text{Spec}(A)$ be a normal affine variety with an effective action of a torus \mathbb{T} and let G be a finite abelian group of order $d \geq 2$ whose action on X commutes with that of \mathbb{T} . The goal of this section is to determine the relationship between the Altmann-Hausen representations of X and those of $X//G = \text{Spec}(A^G)$.

Let Y be a semi-projective variety equipped with an action $\psi : G \times Y \rightarrow Y$ of an algebraic group G . If \mathcal{D}_G is a G -invariant pp-divisor, i.e a pp-divisor such that $\psi(g, \cdot)^* \mathcal{D}_G = \mathcal{D}_G$ for every $g \in G$, then for every $u \in \sigma^\vee \cap M$ the space $A_u = \Gamma(Y, \mathcal{O}(\mathcal{D}_G(u)))$ of global sections $\mathcal{O}(\mathcal{D}_G(u))$ is endowed with a G -action. It follows that $\mathbb{S}(Y, \mathcal{D}_G) = \text{Spec}(\bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(u))))$ admits an action of G commuting with that of \mathbb{T} .

Theorem 2.1. *Let X be a \mathbb{T} -variety and let G be a finite abelian group acting on X such that the two actions commute. Then the following hold:*

- 1) *There exist a semi-projective variety Y endowed with an action of G and a G -invariant pp-divisor \mathcal{D}_G on Y such that X is $\mathbb{T} \times G$ equivariantly isomorphic to $\mathbb{S}(Y, \mathcal{D}_G)$.*
- 2) *Moreover $X//G$ is equivariantly isomorphic to the \mathbb{T}' -variety $\mathbb{S}(Y//G, \mathcal{D}')$ where \mathcal{D}' can be chosen such that $F_*(\mathcal{D}_G) = \varphi_G^*(\mathcal{D}')$, where $\varphi_G : Y \rightarrow Y//G$ denotes the quotient morphism and $F : M^\vee \rightarrow M'^\vee$ is a linear map induced by the inclusion between the character lattices M' of \mathbb{T}' and M of \mathbb{T} (see 2.3).*

We will divide the proof in several steps. First we will prove that the action of G on X induces an action of G on Y . Secondly we will consider the case where the orbits of the G -action are included in the orbits of the \mathbb{T} -action and finally we consider the case where the action of $G \times \mathbb{T}$ is effective on X .

Lemma 2.1. *Let Y a quasi-projective variety endowed with an action of a finite group G and let $\widehat{Y} \rightarrow Y$ be the normalization of Y . Then the action of G lifts to an action on \widehat{Y} and the induced morphism $\widehat{Y}//G \rightarrow Y//G$ is the normalization of $Y//G$.*

Proof. Since Y is quasi-projective and G is finite, every $x \in X$ admits a G -invariant affine open neighborhood. The normalization being a local operation, we may assume that Y is affine. Using the universal properties of the normalization and of the quotient we obtain the following commutative diagram:

$$\begin{array}{ccc} Y & \rightarrow & Y//G \\ \uparrow & & \uparrow \\ \widehat{Y} & \rightarrow & \widehat{Y//G} \\ & \searrow & \uparrow \\ & & \widehat{Y//G} \end{array}$$

Thus $\mathbb{C}[\widehat{Y//G}] \subset \mathbb{C}[\widehat{Y}]^G$. Conversely, let $f \in \mathbb{C}[\widehat{Y}]^G$. Then $g.f = f$ for all g in G and there exists a monic polynomial P with coefficients in $\mathbb{C}[Y]$ such that $P(f) = 0$. Since G is finite, $Q = \prod_{g \in G} g.P$ is a monic polynomial with G -invariant coefficients and $G(f) = 0$. So $f \in \mathbb{C}[\widehat{Y//G}]$. \square

Corollary 2.1. *Let X be a \mathbb{T} -variety and suppose that a finite abelian group G acts on X such the two actions commute. Then there exists a semi-projective variety Y and a pp-divisor \mathcal{D} on Y such that X is $G \times \mathbb{T}$ equivariantly isomorphic to $\mathbb{S}(Y, \mathcal{D})$ and the action of G on $\mathbb{S}(Y, \mathcal{D})$ induces an action of G on Y .*

Proof. We consider the construction of Y given in section 1.2. Since the action of G and \mathbb{T} commute, for every $\lambda \in \Lambda$ the subset $X^{ss}(u)$ with $u \in \text{relint}(\lambda)$ is G -stable. Thus $W := \bigcap_{\lambda \in \Lambda} W_\lambda$ is also G -stable. Since $q' : W \rightarrow Z$ is the quotient by \mathbb{T} , the action of G on W induces one on $q'(W)$. The closure $\overline{q'(W)}$ is again G -stable, and since $\overline{q'(W)}$ is quasi-projective it follows from lemma 2.1 that the action of G lifts to an action on Y . \square

Lemma 2.2. *Let $X = \text{Spec}(A)$ be a \mathbb{T} -variety and let G a finite abelian group acting on X such that the two actions commute. Then there exists a G -invariant pp-divisor \mathcal{D}_G defined on Y such that X is equivariantly isomorphic to $\mathbb{S}(Y, \mathcal{D}_G)$.*

Proof. By lemma 2.1 the action of G on X induces an action of G on Y . By the proof of Theorem 3.4 in [1], a pp-divisor on Y corresponding to X is determined by the choice of a homomorphism h from M into the fraction field of A with the property that for every $u \in M$, $h(u)$ is semi-invariant of weight u . Namely, if $u \in \sigma^\vee \cap M$ is any saturated element, that is, $u \in \sigma^\vee \cap M$ such that $\bigoplus_{n \in \mathbb{N}} A_{nu}$ is generated in degree 1, then there exist a unique Cartier divisor $\mathcal{D}(u)$ such that $A_u = h(u) \cdot \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(u)))$: its local equations on open subsets Y_s with $s \in A_u$ are $h(u)/s$. By definition $h(u) = \frac{f}{g}$ where f and g are both non zero and $f \in A_{u_1}$, $g \in A_{u_2}$ such that $u_1 - u_2 = u$. Since A_u is G -stable for all $u \in M$, we can choose $f \in A_{u_1}$, $g \in A_{u_2}$ semi-invariant for the action of G with $u_1 - u_2 = u$ so that $h(u) = f/g$ is also semi-invariant for G . The corresponding divisor $\mathcal{D}(u)$ is then G -invariant. In the case of a general $u \in \sigma^\vee \cap M$, we can choose a saturated multiple nu and define $\mathcal{D}(u) = \mathcal{D}(nu)/n$. \square

To complete the proof of Theorem 2.1, we divide the argument into two cases. First we consider the situation where G is a subgroup of \mathbb{T} and secondly where the action of $G \times \mathbb{T}$ is effective.

Lemma 2.3. *Let X be the \mathbb{T} -variety $\mathbb{S}(Y, \mathcal{D})$ and let G be a finite abelian subgroup of $\mathbb{T} = \text{Spec}(\mathbb{C}[M])$. Then $X' = X//G$ is a \mathbb{T}' -variety where $\mathbb{T}' \simeq \mathbb{T}/G$ and is equivariantly isomorphic to the $\mathbb{S}(Y, F_*(\mathcal{D}))$ where $F : N = M^\vee \rightarrow N' = (M')^\vee$ is the linear map induced by the inclusion between the character lattices M' of \mathbb{T}' and M of \mathbb{T} .*

Proof. Let Y be as in 1.2. Since by hypothesis the G -orbits are contained in \mathbb{T} -orbits, the induced G -action on Y is trivial. In this case, for each $u \in \sigma^\vee \cap M$, A_u^G is either A_u or $\{0\}$. Letting M' be the sublattice M generated by the elements $u \in \sigma^\vee \cap M$ such that $A_u^G \neq 0$,

$$X' = X//G = \text{Spec}\left(\bigoplus_{u \in \sigma^\vee \cap M'} A_u^G\right)$$

is a \mathbb{T}' -variety where $\mathbb{T}' = \text{Spec}(\mathbb{C}[M'])$ is a torus of the same dimension as \mathbb{T} . The inclusion $M' \hookrightarrow M$ gives rise the desired linear map $F : N = M^\vee \rightarrow N' = M'^\vee$. \square

Remark 2.1. This case corresponds to the map of pp-divisors $(\text{id}, F, 1)$ defined in as 1.3. Indeed the quotient morphism $\varphi : Y \rightarrow Y//G$ is the identity.

Lemma 2.4. *Let X be a normal affine variety with an effective action of $G \times \mathbb{T}$ where G is a finite abelian group. Then there exists a semi-projective variety Y on which G acts and a G -invariant pp-divisor \mathcal{D}_G on Y such that X is $G \times \mathbb{T}$ -equivariantly isomorphic to $\mathbb{S}(Y, \mathcal{D}_G)$.*

Moreover $X//G$ is \mathbb{T} -equivariantly isomorphic to $\mathbb{S}(Y//G, \mathcal{D}')$ where $\mathcal{D}_G = \varphi_G^*(\mathcal{D}')$.

Proof. By lemmas 2.1 and 2.2, Y is endowed with an action of G , and we can assume that X is equivariantly isomorphic to $\mathbb{S}(Y, \mathcal{D}_G)$. Since \mathcal{D}_G is G -stable, for each $u \in \sigma^\vee \cap M$, $\Gamma(Y, \mathcal{O}(\mathcal{D}_G(u)))$ is a G -invariant submodule of $\Gamma(X, \mathcal{O}_X)$ and moreover there exists \mathcal{D}' satisfies $\varphi_G^*(\mathcal{D}') = \mathcal{D}_G$. Therefore, $\Gamma(X//G, \mathcal{O}_{X//G}) = \left(\bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Y, \mathcal{O}(\mathcal{D}_G(u)))\right)^G = \bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Y, \mathcal{O}(\mathcal{D}_G(u)))^G$.

By assumption, $\varphi : Y \rightarrow Y//G$ is the quotient morphism, and \mathcal{D}' satisfying $\varphi_G^*(\mathcal{D}') = \mathcal{D}_G$. Thus

$$\begin{aligned} \Gamma(Y, \mathcal{O}(\mathcal{D}_G(u)))^G &= \{f \in \mathbb{C}(Y)^G, \text{div}(f) + \mathcal{D}_G(u) \geq 0\} \cup \{0\} \\ &= \{h \in \mathbb{C}(Y//G), \varphi^*(\text{div}(h) + \mathcal{D}'(u)) \geq 0\} \cup \{0\} \\ &= \{h \in \mathbb{C}(Y//G), \text{div}(h) + \mathcal{D}'(u) \geq 0\} \cup \{0\}. \end{aligned}$$

We conclude that $X//G \simeq \text{Spec}\left(\bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Y//G, \mathcal{O}(\mathcal{D}'(u)))\right)$. \square

Remark 2.2. This lemma is the analogue of 4.1 in [2], in which Demazure established a similar result for algebras constructed from \mathbb{Q} -divisors. This case corresponds to the map of proper polyhedral divisors $(\varphi_G, \text{id}, 1)$ defined as in 1.3.

Proof. (of Theorem 2.1) Consider a finite abelian group G acting on $X = \mathbb{S}(Y, \mathcal{D})$ whose action commutes with that of \mathbb{T} . By virtue of lemmas 2.1 and 2.2, we may assume that G acts on Y and that \mathcal{D} is G -invariant. Then we let H be the subgroup of $G \times \mathbb{T}$ consisting of elements which act trivially on X . We let $G_0 \subset G$ and $\mathbb{T}_0 \subset \mathbb{T}$ be the images of H by the two projections and we let $G' = G/G_0$ and $\mathbb{T}' = \mathbb{T}/\mathbb{T}_0$. Applying lemma 2.3 to X equipped with the action of G_0 , we obtain a variety $X//G_0$ endowed with an effective action of $G' \times \mathbb{T}'$ to which the lemma 2.4 can be applied. Any map $(\varphi_G, F, 1)$ is obtained by composing maps of the two types above. \square

3. APPLICATIONS IN THE CASE $\mathbb{T} = \mathbb{C}^*$

3.1. Basic examples of \mathbb{C}^* -actions. The coordinate ring of a normal affine variety $X = \text{Spec}(A)$ equipped with an effective \mathbb{C}^* -action is \mathbb{Z} -graded in a natural way via $A = \bigoplus_{n \in \mathbb{Z}} A_n$ where $A_n := \{f \in A / f(\lambda \cdot x) = \lambda^n f(x)\}$. The semi-projective variety associated to the Altmann-Hausen representation of X is the irreducible component which correspond to the normalization of the closure of the image of W by q' (see 1.2) in the fiber product :

$$Y(X) := Y_-(X) \times_{Y_0(X)} Y_+(X)$$

where $Y_0(X) = X//\mathbb{C}^* = \text{Spec}(A_0)$, $Y_\pm(X) = \text{Proj}_{A_0}\left(\bigoplus_{n \in \mathbb{Z}_{\geq 0}} A_{\pm n}\right)$.

A \mathbb{C}^* -action said to be *hyperbolic* if there is at least one $n_1 < 0$ and one $n_2 > 0$ such that A_{n_1} and A_{n_2} are nonzero. In this case, the tail cone σ is equal to $\{0\}$ (see 1.4). If in addition X is smooth, then $Y(X)$ is in fact equal to the fiber product which is itself isomorphic to the blow-up of $Y_0(X)$ with center at the closed subscheme defined by the ideal $\mathcal{I} = \langle A_d \cdot A_{-d} \rangle$ where $d > 0$ is chosen so that $\bigoplus_{n \in \mathbb{Z}} A_{dn}$ is generated by A_0 and $A_{\pm d}$ ([8] Theorem 1.9 and proposition 1.4).

In what follows, we denote by $\pi : \hat{\mathbb{A}}^n_{(I)} \rightarrow \mathbb{A}^n$ the blow-up of the ideal (I) in $\mathbb{A}^n_{(x_1, \dots, x_n)} = \text{Spec}(\mathbb{C}[x_1, \dots, x_n])$.

Given an irreducible and reduced hypersurface $H = \{f(x_1, \dots, x_n) = 0\} \subset \mathbb{A}^n$ containing the origin, the hypersurface $X_{n,p,f}$ of $\mathbb{A}^{n+2} = \text{Spec}(\mathbb{C}[x_1, \dots, x_n][y, t])$ defined by the equation

$$\frac{f(x_1 y, \dots, x_n y)}{y} + t^p = 0$$

comes equipped with an effective \mathbb{C}^* -action induced by the linear one $\lambda \cdot (x_1, \dots, x_n, y, t) = (\lambda^p x_1, \dots, \lambda^p x_n, \lambda^{-p} y, \lambda t)$ on \mathbb{A}^{n+2} . We have $\mathbb{A}^{n+2} // \mathbb{C}^* \simeq \mathbb{A}^{n+1} = \text{Spec}(\mathbb{C}[u_1, \dots, u_{n+1}])$ via $u_i = x_i y$ for $i = 1, \dots, n$ and $u_{n+1} = y t^p$.

Proposition 3.1. *The variety $X_{n,p,f}$ is equivariantly isomorphic to $\mathbb{S}(\tilde{\mathbb{A}}^n_{(u_1, \dots, u_n)}, \mathcal{D})$ for with $\mathcal{D} = \left\{ \frac{1}{p} \right\} D + [0, \frac{1}{p}] E$, where E is the exceptional divisor of the blow up and D is the strict transform of the hypersurface $H \subset \mathbb{A}^n$.*

Proof. We determine $Y(X_{n,p,f})$ and the pp-divisor \mathcal{D} using the method described in sections 1.2 and 1.3. We consider the exact sequence :

$$0 \longrightarrow \mathbb{Z} \xrightarrow{F} \mathbb{Z}^{n+2} \xrightarrow{P} \mathbb{Z}^{n+1} \longrightarrow 0$$

where $F = {}^t(p, \dots, p, -p, 1)$, $P = \begin{pmatrix} & & 1 & 0 \\ & \text{I}_n & \vdots & \vdots \\ & & 1 & 0 \\ 0 & \dots & 0 & 1 & p \end{pmatrix}$ I_n being the identity matrix of rank $n \times n$ and

$s = (0, \dots, 0, 1)$.

The fan in \mathbb{Z}^{n+2} is generated by the rays $\{v_i\}_{i=1,\dots,n+2}$ where v_i is the first integral vector of the unidimensional cone generated by the i -th column vector of P . It corresponds to the blow up of the origin in \mathbb{A}^{n+1} , as a toric variety.

The variety Y is equal to the strict transform by $\pi : \tilde{\mathbb{A}}_{(u_1, \dots, u_n)}^{n+1} \rightarrow \mathbb{A}^{n+1} \simeq \mathbb{A}^{n+2} // \mathbb{C}^*$ of $\{f(u_1, \dots, u_n) + u_{n+1} = 0\} \subset \mathbb{A}^{n+1}$, thus $Y \simeq \tilde{\mathbb{A}}^n$.

Since $\sigma := s(\mathbb{Q}_{\geq 0}^m \cap F(\mathbb{Q}))$ is $\{0\}$, applying the formula $\Pi_i = s(\mathbb{R}_{\geq 0}^m \cap P^{-1}(v_i))$, we deduce that \mathcal{D} has the form $\left\{\frac{1}{p}\right\} D + [0, \frac{1}{p}] E$, where D corresponds to the restriction to Y of the toric divisor given by the ray v_{n+2} . It is the restriction of $\{u_{n+1} = yt^p = 0\}$ to Y thus D is the strict transforms of the hypersurface $H \subset \mathbb{A}^n$. The divisor E corresponds to the restriction to Y of the toric divisor given by v_{n+1} , that is, the exceptional divisor. \square

Example 3.1. Specializing the above construction we obtain examples of linear hyperbolic \mathbb{C}^* -actions on \mathbb{A}^3 which will be building blocks for further applications :

a) Choosing $n = 2$ and $f(x_1, x_2) = x_1$, we obtain that $X_{2, x_1, p}$ is isomorphic to $\mathbb{S}(\tilde{\mathbb{A}}_{(u, v)}^2, \mathcal{D})$ with $\mathcal{D} = \left\{\frac{1}{p}\right\} D + [0, \frac{1}{p}] E$, where E is the exceptional divisor of the blow up and D is the strict transform of the line $\{u = 0\} \subset \mathbb{A}^2$. Thus $X_{2, x_1, p} \subset \mathbb{A}^4$ is isomorphic to \mathbb{A}^3 equipped with the \mathbb{C}^* -action : $\lambda \cdot (x_2, y, t) = (\lambda^p x_2, \lambda^{-p} y, \lambda t)$.

b) In particular, if $p = 1$ then $X_{2, x_1, 1}$ is isomorphic to $\mathbb{S}(\tilde{\mathbb{A}}_{(u, v)}^2, \mathcal{D})$ with $\mathcal{D} = \{1\} D + [0, 1] E$. Since $\mathcal{D} = \{1\} D + [0, 1] E$ is equivalent to $\mathcal{D}' = [-1, 0] E$, we have that $X_{2, x_1, 1}$ is equivariantly isomorphic to $\mathbb{S}(\tilde{\mathbb{A}}_{(u, v)}^2, \mathcal{D}')$.

Example 3.2. Choosing $n = 2$ and $f(x_1, x_2) = x_1 + (x_1^d + x_2^d)^l$ yields that

$$X_{2, p, f} = \{x_1 + y^{dl-1}(x_1^d + x_2^d)^l + t^p = 0\} \subset \mathbb{A}^4$$

is isomorphic to $\mathbb{S}(\tilde{\mathbb{A}}_{(u, v)}^2, \mathcal{D} = \left\{\frac{1}{p}\right\} D + [0, \frac{1}{p}] E)$, where E is the exceptional divisor of the blow up and D is the strict transform of the curve $\{v + (v^d + u^d)^l = 0\} \subset \mathbb{A}^2$. Note that in contrast with the previous example, $X_{2, p, f}$ is not isomorphic to \mathbb{A}^3 . Indeed, if it were, then by the result of Koras-Russell [7], the \mathbb{C}^* action on $X_{2, p, f}$ would be linearizable. By considering the linear action induced on the tangent space of the fixed point, we find that $X_{2, p, f}$ would have to be equivariantly isomorphic to $X_{2, x_1, p}$ for some p . On the other hand it follows from [1, corollary 8.12] that two pp-divisors \mathcal{D}_i , defined on Y_i respectively with the same tail cone, define equivariantly isomorphic varieties $\mathbb{S}(Y_i, \mathcal{D}_i)$ if and only if there exist projective birational morphisms $\psi_i : Y_i \rightarrow Y$ and a pp-divisor \mathcal{D} on Y such that $\mathcal{D}_i \simeq \psi_i^*(\mathcal{D})$ $i = 1, 2$. This would induce an automorphism ϕ of $\tilde{\mathbb{A}}^2$, such that $\phi^*(f) = x_1$, which is not possible, since a general fiber of f is singular.

3.2. Koras-Russell threefolds. Smooth affine, contractible threefolds with a hyperbolic \mathbb{C}^* -action whose quotient is isomorphic to \mathbb{A}^2/G where G is a finite cyclic group have been classified by Koras and Russell [5], in the context of the linearization problem for \mathbb{C}^* -actions on \mathbb{A}^3 [7]. These threefolds, which we call Koras-Russell threefolds, provide examples of \mathbb{T} -varieties of complexity two. According to [6] they admit the following description:

Let a', b' and c' be pairwise prime natural numbers with $b' \geq c'$ and let $\mu_{a'}$, the group of a' -th roots of unity, act on $\mathbb{A}^2 = \text{Spec}(\mathbb{C}[u, v])$ by $(u, v) \rightarrow (\lambda^{c'} u, \lambda^{b'} v)$ where $\lambda \in \mu_{a'}$. Consider a semi-invariant polynomial f of weight congruent to b' modulo a' and with the property that $L = \{f = 0\}$ is isomorphic to a line and meets the axis $u = 0$ transversely at the origin and at $r - 1 \geq 1$ other points. With these assumptions the polynomial $s^{-c'} f(s^{c'} u, s^{b'} v)$ can be rewritten in the form $F(w, u, v)$ with $w = s^{a'}$ where F is semi-invariant of weight b' for the \mathbb{C}^* -action $(w, u, v) \mapsto (\lambda^{-a'} w, \lambda^{c'} u, \lambda^{b'} v)$. Then for any choice of pairwise prime integers $(\alpha_1, \alpha_2, \alpha_3)$ such that $\gcd(\alpha_1, a') = \gcd(\alpha_2, b') = \gcd(\alpha_2, c') = 1$, the hypersurface $X = \{(x, y, z, t) \in \mathbb{A}^4/t^{\alpha_3} + F(y^{\alpha_1}, z^{\alpha_2}, x) = 0\}$ is a Koras-Russell threefold.

Here we mainly consider two families of such threefolds:

1) The first kind is defined by equations of the form:

$$\{x + x^d y + z^{\alpha_2} + t^{\alpha_3} = 0\},$$

where $2 \leq d, 2 \leq \alpha_2 < \alpha_3$ with $\gcd(\alpha_2, \alpha_3) = 1$ and equipped with the \mathbb{C}^* -action induced by the linear one on \mathbb{A}^4 with weights $(\alpha_2 \alpha_3, -(d-1)\alpha_2 \alpha_3, \alpha_3, \alpha_2)$. These correspond to the choice of $f = u + v + v^d$.

2) The second type is defined by

$$\{x + y(x^d + z^{\alpha_2})^l + t^{\alpha_3} = 0\},$$

where $2 \leq d$, $1 \leq l$, $2 \leq \alpha_2 < \alpha_3$ with $\gcd(\alpha_2, d) = \gcd(\alpha_2, \alpha_3) = 1$ and equipped with the \mathbb{C}^* -action induced by the linear one on \mathbb{A}^4 with weights $(\alpha_2\alpha_3, -(dl-1)\alpha_2\alpha_3, d\alpha_3, \alpha_2)$. These correspond to the choice of $f = v + (u + v^d)^l$.

To obtain the Altmann-Hausen representation for these threefolds, we will exploit the fact that they arise as \mathbb{C}^* -equivariant bi-cyclic covers of \mathbb{A}^3 . We will see that the polyhedral coefficients are related with the choice of $(\alpha_1, \alpha_2, \alpha_3)$ and the divisors are related with the choice of the fiber $L = \{f = 0\}$ in the construction above.

3.3. The Russell Cubic. We begin with the Russell cubic $X = \{x + x^2y + z^2 + t^3 = 0\}$ in $\mathbb{A}^4 = \text{Spec}(\mathbb{C}[x, y, z, t])$ which corresponds to the choice $a' = b' = c' = 1$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$ and $f(u, v) = u + v + v^2$ in the construction above. By construction X is equipped with the \mathbb{C}^* -action induced by the linear one on \mathbb{A}^4 with weights $(6, -6, 3, 2)$. The algebraic quotient $X//\mathbb{C}^*$ is isomorphic to $\mathbb{A}_{(u,v)}^2 = \text{Spec}(\mathbb{C}[u, v])$ where $u = yz^2$ and $v = yx$.

Proposition 3.2. (see also [4]) *The Russell Cubic X is isomorphic to $\mathbb{S}(\tilde{\mathbb{A}}_{(u,v)}^2, \mathcal{D})$ for*

$$\mathcal{D} = \left\{ \frac{1}{2} \right\} D_3 + \left\{ -\frac{1}{3} \right\} D_2 + \left[0, \frac{1}{6} \right] E,$$

where E is the exceptional divisor of $\pi : \tilde{\mathbb{A}}_{(u,v)}^2 \rightarrow \mathbb{A}^2$, and where D_2 and D_3 are the strict transforms of the curves $\{u = 0\}$ and $\{u + v + v^2 = 0\}$ in \mathbb{A}^2 respectively.

Proof. The two projections $\Phi_2 = \text{pr}_{x,y,t} : X \rightarrow X_2 = \mathbb{A}^3$ and $\Phi_3 = \text{pr}_{x,y,z} : X \rightarrow X_3 = \mathbb{A}^3$ express X as cyclic Galois covers of \mathbb{A}^3 of degrees 2 and 3 respectively, whose Galois groups μ_2 and μ_3 act on X by $\xi \cdot (x, y, z, t) = (x, y, \xi z, t)$ and $\zeta \cdot (x, y, z, t) = (x, y, z, \zeta t)$ respectively. Furthermore these two actions commute and the quotient $X_6 = X/(\mu_2 \times \mu_3)$ is isomorphic to $\mathbb{A}^3 = \text{Spec}(\mathbb{C}[x, y, z^2])$. Letting $A = \bigoplus_{n \in \mathbb{Z}} A_n$ be the coordinate ring of X equipped with the grading corresponding to the given \mathbb{C}^* -action, we have in fact $X_\ell = \text{Spec}(\bigoplus_{n \in \mathbb{Z}} A_{\ell n})$, $\ell = 2, 3, 6$. This yields a \mathbb{C}^* -equivariant commutative diagram

$$\begin{array}{ccc} & X & \\ \Phi_2 \swarrow & & \searrow \Phi_3 \\ X_2 = X//\mu_2 & & X_3 = X//\mu_3 \\ & \searrow \Phi_6 & \swarrow \\ & X_6 = X//(\mu_2 \times \mu_3) & \end{array}$$

where \mathbb{C}^* acts linearly on X_2 , X_3 and X_6 with weights $(3, -3, 1)$, $(2, -2, 1)$ and $(1, -1, 1)$ respectively.

Furthermore since the action of $\mu_2 \times \mu_3$ on X factors through that of \mathbb{C}^* we deduce from Theorem 2.1 that Φ_2 corresponds to the map of proper polyhedral divisors $(\text{id}, F_2, 1)$ and Φ_3 corresponds to the map of proper polyhedral divisors $(\text{id}, F_3, 1)$ where $F_\ell^*(\mathcal{D}) = \ell\mathcal{D}$, $\ell = 2, 3, 6$. The semi-projective varieties $Y(X)$ and $Y(X_\ell)$, $\ell = 2, 3, 6$ are all isomorphic. As observed earlier, $A_0 = \mathbb{C}[u, v]$ with $u = yz$ and $v = yx$ so that $Y_0(X_6) \simeq Y_0(X) = \mathbb{A}_{(u,v)}^2$. We further observe that $A_{-6n} = A_0 \cdot y^n \subset A$ because all semi-invariant polynomials of negative weights divisible by 6 are divisible by y . This implies that $Y_-(X_6) \simeq \text{Proj}(\bigoplus_{n \in \mathbb{Z}_{\geq 0}} A_0 \cdot y^n) \simeq Y_0(X)$.

Finally, $\bigoplus_{n \in \mathbb{Z}_{\geq 0}} A_{6n} \simeq \text{Sym}_{A_0} A_6$ where A_6 is the free A_0 -submodule of A generated by x and z . Therefore

$$Y(X) \simeq Y(X_6) = Y_-(X_6) \times_{Y_0(X_6)} Y_+(X_6) \simeq Y_+(X_6)$$

is isomorphic to the blow-up $\tilde{\mathbb{A}}_{(u,v)}^2$ of $Y_0(X) = \mathbb{A}^2$ at the origin. It remains to determine the pp-divisor \mathcal{D} . We will construct it from those \mathcal{D}_2 and \mathcal{D}_3 corresponding to X_2 and X_3 respectively.

By Proposition 3.1, $X_2 = \mathbb{S}(\tilde{\mathbb{A}}_{(u,v)}^2, \mathcal{D}_2 = \{\frac{1}{3}\}D_2 + [0, \frac{1}{3}]E)$ where D_2 is the strict transform of the curve $\{u = 0\}$ and E is the exceptional divisor and $X_3 = \mathbb{S}(\tilde{\mathbb{A}}_{(u',v)}^2, \mathcal{D}_3 = \{\frac{1}{2}\}D_3 + [0, \frac{1}{2}]E)$ where D_3 is the strict transform of the curve $\{u' = 0\}$ and E is the exceptional divisor. Theorem 2.1 implies in turn that $2\mathcal{D} \sim \mathcal{D}_2 = \{\frac{1}{3}\}D_2 + [0, \frac{1}{3}]E$ and $3\mathcal{D} \sim \mathcal{D}_3 = \{\frac{1}{2}\}D_3 + [0, \frac{1}{2}]E$. Thus $\mathcal{D}_2 + \mathcal{D} = \mathcal{D}_3$ and we conclude that $\mathcal{D} = \{\frac{1}{2}\}D_3 + \{-\frac{1}{3}\}D_2 + [0, \frac{1}{6}]E$. \square

Remark. The choice of the coefficients is not unique since $\mathcal{D}' \sim \mathcal{D} + \text{div}(f)$ for any rational function f on Y . This corresponds for example to $\mathcal{D}' \sim \mathcal{D} + D_3 + E$ and more generally for any pair $(a, b) \in \mathbb{Z}^2$ such that $3a + 2b = 1$ we have that $\mathcal{D} \sim \{\frac{a}{2}\}D_3 + \{\frac{b}{3}\}D_2 + [0, \frac{1}{6}]E$.

3.4. Koras Russell threefolds of the first kind. Now we will show that a similar method can be used to present all Koras-Russell threefolds of the form $X = \{x + x^d y + z^{\alpha_2} + t^{\alpha_3} = 0\}$ in $\mathbb{A}^4 = \text{Spec}(\mathbb{C}[x, y, z, t])$. Namely, we consider a cyclic cover V of X with algebraic quotient $V//\mathbb{C}^*$ isomorphic to $\mathbb{A}^2 = \text{Spec}(\mathbb{C}[u, v])$ where $u = yz^{\alpha_2}$ and $v = yx$. A representation of V is obtained by the same method as in the previous case and the representation of X is deduced by applying again Theorem 2.1.

The categorical quotient $X//\mathbb{C}^*$ is isomorphic to $\mathbb{A}_{(u,v)}^2//\mu_{d-1}$ where μ_{d-1} acts by $\xi \cdot (u, v) = (\xi u, \xi v)$. So we consider V a finite cyclic cover of X given by the equation $X = \{x + x^d y^{d-1} + z^{\alpha_2} + t^{\alpha_3} = 0\}$ in $\mathbb{A}^4 = \text{Spec}(\mathbb{C}[x, y, z, t])$, equipped with the \mathbb{C}^* -action induced by the linear one on \mathbb{A}^4 with weights $(\alpha_2 \alpha_3, -\alpha_2 \alpha_3, \alpha_3, \alpha_2)$. Furthermore $\mu_{\alpha_2} \times \mu_{\alpha_3} \times \mu_{d-1}$ acts on V by $(\zeta, \epsilon, \xi) \cdot (x, y, z, t) \rightarrow (x, \xi y, \zeta z, \epsilon t)$. Observe that the action of $\mu_{\alpha_2} \times \mu_{\alpha_3}$ factors through that of \mathbb{C}^* . This yields the following diagram of quotient morphisms:

$$\begin{array}{ccc} & V & \\ \Phi_{\alpha_2} \swarrow & \downarrow \Phi_{\mu_{d-1}} & \searrow \Phi_{\alpha_3} \\ \mathbb{A}^3 \simeq V//\mu_{\alpha_2} & X = V//\mu_{d-1} & \mathbb{A}^3 \simeq V//\mu_{\alpha_3} \end{array}$$

By Theorem 2.1, Φ_{α_2} corresponds to the map of proper polyhedral divisors $(\text{id}, F_{\alpha_2}, 1)$ and Φ_{α_3} corresponds to the map of proper polyhedral divisor $(\text{id}, F_{\alpha_3}, 1)$ where $F_{\ell}^*(\mathcal{D}) = \ell\mathcal{D}$, $\ell = 2, 3, 6$. In addition we obtain that $Y(V)$ is isomorphic to the blow-up $\tilde{\mathbb{A}}_{(u,v)}^2$ of \mathbb{A}^2 at the origin on which $\mu_{\alpha_2} \times \mu_{\alpha_3} \times \mu_{d-1}$ acts by $(\zeta, \epsilon, \xi) \cdot (u, v) = (\xi u, \xi v)$. This leads to the following diagram:

$$\begin{array}{ccc} & Y(V) & \\ \simeq \swarrow & \downarrow \varphi_{\mu_{d-1}} & \searrow \simeq \\ Y(V_{\alpha_2}) & Y(X) \simeq Y(V)//\mu_{d-1} & Y(V_{\alpha_3}) \end{array}$$

Using example 3.1 we obtain Altman-Hausen representations of $V//\mu_{\alpha_2}$ and $V//\mu_{\alpha_3}$ in the form $\mathbb{S}(\tilde{\mathbb{A}}_{(u,v)}^2, \mathcal{D}_{\alpha_2} = \{\frac{1}{\alpha_3}\}D_{\alpha_2} + [0, \frac{1}{\alpha_3}]E)$ where D_{α_2} is the strict transform of the curve $\{u = 0\}$, E is the exceptional divisor and $\mathbb{S}(\tilde{\mathbb{A}}_{(u',v)}^2, \mathcal{D}_{\alpha_3} = \{\frac{1}{\alpha_2}\}D_{\alpha_3} + [0, \frac{1}{\alpha_2}]E)$ where D_{α_3} is the strict transform of the curve $\{u' = 0\}$, E is the exceptional divisor. This implies that V is isomorphic to $\mathbb{S}(\tilde{\mathbb{A}}_{(u,v)}^2, \mathcal{D})$ for

$$\mathcal{D} = \left\{ \frac{a}{\alpha_2} \right\} D_{\alpha_3} + \left\{ \frac{b}{\alpha_3} \right\} D_{\alpha_2} + \left[0, \frac{1}{\alpha_2 \alpha_3} \right] E (*),$$

where E is the exceptional divisor of $\pi : \tilde{\mathbb{A}}_{(u,v)}^2 \rightarrow \mathbb{A}^2$, D_{α_2} and D_{α_3} are the strict transforms of the curves $\{u = 0\}$ and $\{u + v + v^d = 0\}$ in $\mathbb{A}_{(u,v)}^2$ respectively, and $(a, b) \in \mathbb{Z}^2$ are chosen such that $a\alpha_3 + b\alpha_2 = 1$. Applying Theorem 2.1 we obtain

Proposition 3.3. *The Koras-Russell threefold $X = \{x + x^d y + z^{\alpha_2} + t^{\alpha_3} = 0\}$ in $\mathbb{A}^4 = \text{Spec}(\mathbb{C}[x, y, z, t])$ is isomorphic to $\mathbb{S}(\tilde{\mathbb{A}}_{(u,v)}^2//\mu_{d-1}, \mathcal{D}')$ for*

$$\mathcal{D}' = \left\{ \frac{a}{\alpha_2} \right\} D'_{\alpha_3} + \left\{ \frac{b}{\alpha_3} \right\} D'_{\alpha_2} + \left[0, \frac{1}{(d-1)\alpha_2 \alpha_3} \right] E'$$

where $\mathcal{D} = \varphi_{\mu_{d-1}}^*(\mathcal{D}')$, \mathcal{D} is defined in the relation $(*)$ and $D'_{\alpha_3}, D'_{\alpha_2}$ are prime divisors and E' is the exceptional divisor of the blow-up of the singularity in $\mathbb{A}^2//\mu_{d-1}$.

3.5. Koras Russell threefolds of the second kind. For Koras-Russell threefolds of the second kind $X = \{x + y(x^d + z^{\alpha_2})^l + t^{\alpha_3} = 0\}$ in $\mathbb{A}^4 = \text{Spec}(\mathbb{C}[x, y, z, t])$ the construction will be slightly different due to the fact that the variables z and t do no longer play symmetric roles. We will consider again a cyclic cover V of X , but in this case $V//\mu_{\alpha_2}$ will not be isomorphic to \mathbb{A}^3 . Recall that by definition, α_2 and d are coprime. We consider a bi-cyclic cover $V = \{x + y^{dl-1}(x^d + z^{d\alpha_2})^l + t^{\alpha_3} = 0\}$ of X of order $d \times (dl - 1)$, which we decompose as a cyclic cover $\phi_d : V \rightarrow V_d = \{x + y^{dl-1}(x^d + z^{\alpha_2})^l + t^{\alpha_3} = 0\}$ of degree d , followed by a cyclic cover $\phi_{dl-1} : V_d \rightarrow X$ of degree $dl - 1$. The hypersurface V is equipped with the \mathbb{C}^* -action induced by the linear one on \mathbb{A}^4 with weights $(\alpha_2\alpha_3, -\alpha_2\alpha_3, \alpha_3, \alpha_2)$ and with the action of $\mu_{\alpha_2} \times \mu_{\alpha_3} \times \mu_{dl-1} \times \mu_d$ defined by $(\zeta, \epsilon, \xi, \delta) \cdot (x, y, z, t) = (x, \xi y, \zeta \delta z, \epsilon t)$. The action of $\mu_{\alpha_2} \times \mu_{\alpha_3}$ on V factors through that of \mathbb{C}^* and we obtain the following diagram:

$$\begin{array}{ccccc}
& & V & & \\
& \swarrow \Phi_{\alpha_2} & \downarrow \Phi_{\mu_d} & \searrow \Phi_{\alpha_3} & \\
V_{\alpha_2} = V//\mu_{\alpha_2} & & V_d = V//\mu_d & & \mathbb{A}^3 \simeq V_{\alpha_3} = V//\mu_{\alpha_3} \\
& & \downarrow \Phi_{\mu_{dl-1}} & & \\
& & X = V//(\mu_d \times \mu_{dl-1}) & & .
\end{array}$$

By Theorem 2.1, considering Φ_{α_3} , we obtain that $Y(V)$ is isomorphic to the blow-up $\tilde{\mathbb{A}}_{(u,v)}^2$ of \mathbb{A}^2 where $u = yz^{\alpha_2}$ and $v = yx$ on which $\mu_{\alpha_2} \times \mu_{\alpha_3} \times \mu_{dl-1} \times \mu_d$ acts by $(\zeta, \epsilon, \xi, \delta) \cdot (u, v) = (\xi \delta^{\alpha_2} u, \xi v)$. We obtain the following quotient diagram:

$$\begin{array}{ccccc}
& & Y(V) & & \\
& \swarrow \simeq & \downarrow \varphi_{\mu_d} & \searrow \simeq & \\
Y(V_{\alpha_2}) & & Y(V_d) & & Y(V_{\alpha_3}) \\
& & \downarrow \varphi_{\mu_{dl-1}} & & \\
& & Y(X) & & .
\end{array}$$

Now by Proposition 3.1 $V_{\alpha_2} = \mathbb{S}(\tilde{\mathbb{A}}_{(u,v)}^2, \mathcal{D}_{\alpha_2})$, $\mathcal{D}_{\alpha_2} = \{\frac{1}{\alpha_3}\}D_{\alpha_2} + [0, \frac{1}{\alpha_3}]E$ where D_{α_2} is the strict transform of the curve $\{v + (v^d + u^d)^l = 0\}$ and E is the exceptional divisor, and $V_{\alpha_3} = \mathbb{S}(\tilde{\mathbb{A}}_{(u,v)}^2, \mathcal{D}_{\alpha_3})$, $\mathcal{D}_{\alpha_3} = \{\frac{1}{\alpha_2}\}D_{\alpha_3} + [0, \frac{1}{\alpha_2}]E$ where D_{α_3} is the strict transform of the curve $\{u = 0\}$ and E is the exceptional divisor. Thus $V = \mathbb{S}(\tilde{\mathbb{A}}_{(u,v)}^2, \mathcal{D})$ for

$$\mathcal{D} = \left\{ \frac{a}{\alpha_2} \right\} D_{\alpha_3} + \left\{ \frac{b}{\alpha_3} \right\} D_{\alpha_2} + \left[0, \frac{1}{\alpha_2 \alpha_3} \right] E,$$

where E is the exceptional divisor of $\pi : \tilde{\mathbb{A}}_{(u,v)}^2 \rightarrow \mathbb{A}^2$, and where D_{α_2} and D_{α_3} are the respective strict transforms of the curves $\{v + (v^d + u^d)^l = 0\}$ and $\{u = 0\}$ in \mathbb{A}^2 and $(a, b) \in \mathbb{Z}^2$ $a\alpha_3 + b\alpha_2 = 1$. Note that the choice of \mathcal{D} up to linear equivalence does not depend of the choice on $(a, b) \in \mathbb{Z}^2$.

Now we deduce from Theorem 2.1 that $V_d = \mathbb{S}(\tilde{\mathbb{A}}_{(u',v'^d)}^2, \mathcal{D}_d)$ for

$$\mathcal{D}_d = \left\{ \frac{a'}{\alpha_2} \right\} D_{d,\alpha_3} + \left\{ \frac{b'}{\alpha_3} \right\} D_{d,\alpha_2} + \left[0, \frac{1}{\alpha_2 \alpha_3} \right] E_d (**),$$

where $a' = a/d$, $b' = b$, E_d is the exceptional divisor of $\pi : \tilde{\mathbb{A}}_{(u',v'^d)}^2 \rightarrow \mathbb{A}^2$ due to the fact that $\tilde{\mathbb{A}}_{(u',v')}/\mu_d \simeq \tilde{\mathbb{A}}_{(u',v'^d)}$ for the action of μ_d as above, and where D_{d,α_2} and D_{d,α_3} are the strict transforms of the curves $\{v' + (u' + v'^d)^l = 0\}$ and $\{u' = 0\}$ ($u' = \varphi_d(u^d)$) in $\mathbb{A}^2 = \text{Spec}(\mathbb{C}[u', v'])$ respectively. Applying again Theorem 2.1 we obtain :

Proposition 3.4. *A Koras-Russell threefold $X = \{x + y(x^d + z^{\alpha_2})^l + t^{\alpha_3} = 0\}$ in $\mathbb{A}^4 = \text{Spec}(\mathbb{C}[x, y, z, t])$ is isomorphic to $\mathbb{S}(\tilde{\mathbb{A}}_{(u',v'^d)}^2//\mu_{dl-1}, \mathcal{D}_{d(dl-1)})$ for*

$$\mathcal{D}_{d(d-1)} = \left\{ \frac{a'}{\alpha_2} \right\} D_{d(d-1), \alpha_3} + \left\{ \frac{b'}{\alpha_3} \right\} D_{d(d-1), \alpha_2} + \left[0, \frac{1}{(d-1)\alpha_2\alpha_3} \right] E_{d(d-1)},$$

where $\mathcal{D}_d = \varphi_{\mu_{d-1}}^*(\mathcal{D}_{d(d-1)})$, \mathcal{D}_d is defined in the relation (**) and $E_{d(d-1)}$ is the exceptional divisor of the blow-up of the singularity in $\mathbb{A}^2 // \mu_{d-1}$.

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