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Higher order Hamiltonian fluid reduction of Vlasov equation

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Abstract

From the Hamiltonian structure of the Vlasov equation, we build a Hamiltonian model for the first three moments of the Vlasov distribution function, namely, the density, the momentum density and the specific internal energy. We derive the Poisson bracket of this model from the Poisson bracket of the Vlasov equation, and we discuss the associated Casimir invariants.

1. Introduction

Ideal fluid and plasma dynamical systems have Hamiltonian structure in terms of noncanonical Poisson brackets [1, 2, 3] that is inherited from the Hamiltonian structure of underlying microphysics. Indeed, a path can be traced, at least formally, from n-body dynamics represented as a kinetic theory via Liouville's equation, to mean field kinetic theories, such as that of the Vlasov equation, to fluid models, with interacting electromagnetic fields (e.g., [4]). Along this path one takes various kinds of moments in order to obtain closures, self-contained reduced models, and it has been shown that the Hamiltonian structure can be traced from the Liouville equation to the BBGKY hierarchy, to Vlasov theory [5]. Similarly, the Hamiltonian structures of some fluid systems have been obtained from that of Vlasov theory [6, 1, 7, 8] by a moment reduction using only the density and fluid velocity (e.g., [9, 10]). The main purpose of the present paper is to derive Hamiltonian fluid closures that allow for pressure or entropy dynamics, and thus provide a Poisson bracket derivation of the more complete Hamiltonian theory of Ref. [11].

The derivation proceeds by projecting the Vlasov Poisson bracket onto a complete (infinite) set of velocity (or momentum) moments. In general, brackets obtained by truncation of this bracket by dropping higher order moments do not satisfy the Jacobi identity [12]. Here we develop a procedure for recovering the Jacobi identity by introduction of a single scalar field that plays the role of a thermodynamic variable, e.g., entropy, scalar pressure, or energy. In order

to transmit the essential idea in an uncluttered fashion, we restrict here to the case of a single spatial dimension.

Moment reductions are of practical concern because integrating a kinetic description describing collisionless plasmas is a most challenging task, one that is desired, e.g., for designing realistic fusion devices and understanding naturally occurring plasmas. Consequently, beginning from the noncanonical Hamiltonian description of Vlasov theory, a variety of models have been obtained for reduced kinetic descriptions (e.g., [13, 14, 15, 16, 17]). In addition, reductions to ordinary differential equations, as well as to fluid equations, have been obtained and applied to specific physical problems. For example, such reductions have been used for describing vortex dynamics in a variety of configurations [18, 19, 20, 21], self-gravitating ellipsoids [22, 23], and laser plasma interaction physics [24, 25, 26]. Not all of these reductions are Hamiltonian, e.g., the quadratic moment reductions of [23, 19, 20, 21, 26] are Hamiltonian, while the higher degree reductions of [18] and those of [24, 25], although energy conserving, are not Hamiltonian (as was explicitly shown in [12]). As noted above, the goal here is to explore Hamiltonian fluid closures, which could then be further reduced for numerical computation.

The paper is organized as follows. In Sec. 2 we briefly give some background material. In Sec. 3 we first state our main result and then provide the derivation of our Hamiltonian model which is analysed in Sec. 4. Finally, in Sec. 5 we summarize and conclude. In addition the paper contains several appendices with explicit calculations pertaining to the direct proof of the Jacobi identity.

2. Background

We begin with the Hamiltonian formulation of the Vlasov equation, which describes the time evolution of a distribution function of non-colliding particles in phase space $f : \mathcal{D} \to \mathbb{R}_+$ (where $\mathcal{D} \subset \mathbb{R}^{2n}$ and n is the dimension of the configuration space of the particles). Usually the Vlasov equation is coupled to Maxwell's equations or to a single elliptic equation, e.g., Poisson equation for gravitational or electrostatic interaction. However, since the crux of the present moment closure problem lies in the Vlasov part of the Poisson bracket, we will find it sufficient for our purposes to consider the case of particles subjected to an external (but possibly time-dependent) potential V. Thus, the equation we consider is given by

$$\partial_t f = -\mathbf{v} \cdot \nabla f + \nabla V \cdot \partial_{\mathbf{v}} f = [\mathcal{E}, f], \tag{1}$$

where the second equality follows from the particle Poisson bracket,

$$[f, g] := \nabla f \cdot \partial_{\mathbf{v}} g - \partial_{\mathbf{v}} f \cdot \nabla g \,, \tag{2}$$

and the particle energy $\mathcal{E} := |\mathbf{v}|^2/2 + V(\mathbf{x}, t)$.

The Vlasov equation possesses a Hamiltonian structure as a field theory that reflects the Hamiltonian character of the equations of motion of individual

particles. On the field theory level, the particle Hamiltonian structure translates into the existence of a noncanonical Poisson bracket $\{\cdot,\cdot\}_V$ such that the Vlasov equation can be reformulated as $\dot{f} = \{f, \mathcal{H}\}_V$, where the dot denotes the time derivative and the Hamiltonian $\mathcal{H}$, the total energy of the system, is given by

$$\mathcal{H}(f) = \int_{\mathcal{D}} f(\mathbf{x}, \mathbf{v}) \left(\frac{|\mathbf{v}|^2}{2} + V(\mathbf{x}, t) \right) \, \mathrm{d}^n x \mathrm{d}^n v. \tag{3}$$

The noncanonical Poisson bracket that is our concern here is the same as that for the Vlasov-Poisson system [6, 1]. However, this same bracket is ubiquitous in fluid and plasma physics and is of the Lie-Poisson type (see, e.g., Refs. [3, 2]). For example, this bracket occurs in the coupling to the electromagnetic field in Vlasov-Maxwell theory [6, 1, 7, 8], in general relativity [27, 28], and for vortex type dynamics (e.g., [1, 19, 29]). It is given by

$$\{F, G\}_V = \int_{\mathcal{D}} f(\mathbf{x}, \mathbf{v}) \left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right] \, \mathrm{d}^n x \mathrm{d}^n v, \tag{4}$$

where $\delta F/\delta f$ is the functional derivative of F with respect to f and the bracket $[\cdot,\cdot]$ is given by Eq. (2). In what follows, we assume periodic boundary conditions with respect to the spatial coordinates, i.e., they are defined on the n-dimensional torus $\mathbb{T}^n$, whereas velocities are defined on $\mathbb{R}^n$ with vanishing boundary conditions at infinity. Thus, our integration domain is $\mathcal{D} = \mathbb{T}^n \times \mathbb{R}^n$.

It is commonplace to consider fluid reductions of kinetic theories like Vlasov equation by taking velocity moments. However very few works focus on the Hamiltonian structure of the resulting models (or at least of its ideal part). One exception is the well-known Hamiltonian fluid reduction obtained from the Poisson bracket (4) using the Poisson subalgebra given by the functionals

$$F(f) = \bar{F} \left(\int f \, \mathrm{d}^n v, \int f \mathbf{v} \, \mathrm{d}^n v \right), \tag{5}$$

where the dynamical variables are reduced to the density $\rho = \int f \, \mathrm{d}^n v$ and the momentum density $\mathbf{M} = \int f \mathbf{v} \, \mathrm{d}^n v$. The reduced Poisson bracket is derived from bracket (4) by restriction to such functionals of the form of (5) and is equal to that given in Ref. [11], viz.

$$\{F, G\} = \int \left[\rho \frac{\delta G}{\delta \mathbf{M}} \cdot \nabla \frac{\delta F}{\delta \rho} + \mathbf{M} \cdot \left(\frac{\delta G}{\delta \mathbf{M}} \cdot \nabla \right) \frac{\delta F}{\delta \mathbf{M}} \right] \mathrm{d}^n x - (F \leftrightarrow G),$$

where $(F \leftrightarrow G)$ denotes the same quantity as that shown but with F and G interchanged in order to have an antisymmetric bracket. This bracket reduction is exact, i.e., if two functionals that depend on f only through ρ and $\mathbf{M}$ are inserted into Eq. (4), then the resulting functional $\{F, G\}$ only depends on the variables ρ and $\mathbf{M}$. However, Hamiltonian (3) does not belong to this subalgebra of functionals, since it depends on moments of order two of the distribution function f. An approximation leading to a completed reduction consists in

replacing the kinetic energy by $\int |\mathbf{M}|^2/(2\rho)\ \mathrm{d}^n x$, a quantity that belongs to the subalgebra, together with kinetic fluctuations through a specific internal energy function U that depends on ρ. With these assumptions, the reduced Hamiltonian becomes

$$\mathcal{H}(\rho, \mathbf{M}) = \int \left(\frac{|\mathbf{M}|^2}{2\rho} + \rho U(\rho) + \rho V \right)\ \mathrm{d}^n x.$$

Following Ref. [1], a specific entropy variable can be added to the system by allowing the internal energy U to depend on this specific entropy or any alternative thermodynamic variable and finding an appropriate algebra. Of course, with this procedure a specific connection to the second moment is lost and the derivation has an ad hoc flavor.

Another strategy consists in keeping the second order moments $\int f \mathbf{v} \otimes \mathbf{v}\ \mathrm{d}^n v$ as a dynamical variable, in which case there would be no approximation to be performed on Hamiltonian (3), although some dynamical information on the kinetic fluctuations could also be kept. However the set of functionals

$$F(f) = \bar{F}\left(\int f\ \mathrm{d}^n v, \int f \mathbf{v}\ \mathrm{d}^n v, \int f \mathbf{v} \otimes \mathbf{v}\ \mathrm{d}^n v \right), \tag{6}$$

does not constitute a Poisson subalgebra associated with bracket (4) since the whole hierarchy of higher order moments enters into play. This is because brackets of elements of the set of functionals (6) generate functions of order higher than two. To our knowledge there is no Hamiltonian system which contains a finite number of moments with among them, the first three moments as dynamical field variables. Finding such a model is the main objective of the present work and, as noted in Sec. 1, this is achieved by introducing a closure in terms of a single scalar field.

3. A higher order closure

In this section we state our answer for the one-dimensional case, returning to its derivation and further discussion in later sections. The one-dimensional case is much simpler than the n-dimensional one, but contains the same essential ingredients: a Poisson subalgebra for the functionals of the first two moments, and the intricacy of higher order moments. The generalization to higher dimensions is more involved, since more dynamical field variables are needed to characterize the second and third moments of the distribution function f.

Here, we derive a three-field Hamiltonian model for the first three moments of the distribution function, or equivalently for the density $\rho(x)$, the momentum density $M(x)$ and the specific internal energy $U(x)$. In a nutshell, the result that we obtain is that the Hamiltonian is naturally

$$\mathcal{H}(\rho, M, U) = \int \left(\frac{M^2}{2\rho} + \rho U + \rho V \right)\ \mathrm{d}x,$$

and the Poisson bracket is the following:

$$\{F,G\} = \int \left[\rho \frac{\delta G}{\delta M} \partial_x \frac{\delta F}{\delta \rho} + M \frac{\delta G}{\delta M} \partial_x \frac{\delta F}{\delta M} + U \left(\frac{\delta G}{\delta M} \partial_x \frac{\delta F}{\delta U} + \frac{\delta G}{\delta U} \partial_x \frac{\delta F}{\delta M} \right) \right.$$
$$\left. + \rho^2 \mathcal{Q} \frac{\delta G}{\delta U} \partial_x \frac{\delta F}{\delta U} \right] \mathrm{d}x - (F \leftrightarrow G), \quad (7)$$

where $\mathcal{Q} = \mathcal{Q}(2U/\rho^2)$ is an arbitrary function. The resulting equations of motion given by $\dot{F} = \{F, H\}$ for the density ρ, the fluid velocity $\mathrm{v} = M/\rho$ and the specific internal energy U are given by

$$\dot{\rho} = -\partial_x(\rho \mathrm{v}), \quad (8)$$
$$\dot{\mathrm{v}} = -\mathrm{v}\partial_x \mathrm{v} - \partial_x V - \rho^{-1}\partial_x(2\rho U), \quad (9)$$
$$\dot{U} = -\mathrm{v}\partial_x U - 2U\partial_x \mathrm{v} - \rho^{-1}\partial_x \left(\rho^4 \mathcal{Q}\right). \quad (10)$$

The first two equations are the usual equations for the density and the fluid velocity where the pressure is defined by $P = 2\rho U$, which is consistent with $\rho U = P/(\gamma - 1)$, γ being the one-dimensional adiabatic index for an ideal gas. The third equation is also standard except for the last term which provides for the departure of the distribution function f from a Maxwellian distribution (cf. Sec. 4). One of the main benefits of the model (8)–(10) is that this departure from a Maxwellian distribution can now be investigated using a purely fluid model. It should be noted that the models given by Eqs. (8)–(10) are all conservative, but only the ones where $\mathcal{Q}$ is of the prescribed form, namely $\mathcal{Q} = \mathcal{Q}(2U/\rho^2)$, are Hamiltonian, i.e., the Jacobi identity for bracket (7) is satisfied. Below we provide a derivation of the Poisson bracket (7)

In the one-dimensional case $n = 1$, the Vlasov equation (1) reduces to

$$\partial_t f = -v\partial_x f + \partial_x V \partial_v f\,,$$

and the associated Poisson bracket (4) becomes

$$\{F,G\} = \int f(x,v)\, \partial_x \frac{\delta F}{\delta f} \partial_v \frac{\delta G}{\delta f}\, \mathrm{d}x\mathrm{d}v - (F \leftrightarrow G). \quad (11)$$

We perform the change of dynamical field variables

$$f(x,v) \mapsto (P_0(x), P_1(x), \ldots, P_\infty(x))\,,$$

defined by

$$P_i(x) = \int v^i f(x,v)\, \mathrm{d}v,$$

with the hypothesis on f that this change of variables is well defined and invertible. The functionals are modified according to

$$F(f) = \tilde{F}(P_0, P_1, ..., P_\infty).$$

From this expression, using the chain rule for functional derivatives, we deduce the expression of $\delta F/\delta f$ as a function of the $\delta F/\delta P_i$,

$$\frac{\delta F}{\delta f} = \sum_{i=0}^{\infty} \frac{\delta \tilde{F}}{\delta P_i} v^i.$$

Consequently, bracket (11) becomes

$$\{\tilde{F}, \tilde{G}\} = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \int j P_{i+j-1} \frac{\delta \tilde{G}}{\delta P_j} \partial_x \frac{\delta \tilde{F}}{\delta P_i} \, \mathrm{d}x - (\tilde{F} \leftrightarrow \tilde{G}). \tag{12}$$

From this expression, it is straightforward to see that the subset of functionals of P_0 and P_1 is invariant (closed) under the bracket (12), and hence constitutes a Poisson subalgebra, since $i + j - 1 \leq 1$ for $i, j \leq 1$. This is used to obtain a two-field Hamiltonian fluid model [2]. As noted in Sec. 2, this property does not extend to higher order moments since the subsets of functionals that depend on $(P_i)_{i \leq N}$ with $N \geq 2$, are not invariant under the bracket (12), given that $2N - 1 > N$ for $N \geq 2$. Our purpose is to construct a Poisson bracket for functionals of P_0, P_1 and P_2 that is obtained by an appropriate closure on higher order moments. We consider the associated bracket acting on functionals of P_0, P_1 and P_2 obtained from a truncation of bracket (12):

$$\{F, G\} = \{F, G\}_J + \{F, G\}^*, \tag{13}$$

where

$$\{F, G\}_J = \int \left\{ P_0 \frac{\delta G}{\delta P_1} \partial_x \frac{\delta F}{\delta P_0} + P_1 \frac{\delta G}{\delta P_1} \partial_x \frac{\delta F}{\delta P_1} + P_2 \frac{\delta G}{\delta P_1} \partial_x \frac{\delta F}{\delta P_2} \right\} \mathrm{d}x - (F \leftrightarrow G), \tag{14}$$

and

$$\{F, G\}^* = \int 2 \left\{ P_1 \frac{\delta G}{\delta P_2} \partial_x \frac{\delta F}{\delta P_0} + P_2 \frac{\delta G}{\delta P_2} \partial_x \frac{\delta F}{\delta P_1} + P_3 \frac{\delta G}{\delta P_2} \partial_x \frac{\delta F}{\delta P_2} \right\} \mathrm{d}x - (F \leftrightarrow G),$$

which introduces the dependence on P_3. Thus, in order to be able to truncate the resulting system of equations, we consider imposing a closure constraint of the kind

$$P_3 = \mathcal{P}(x, P_0, P_1, P_2, \partial_x P_0, \partial_x P_1, \partial_x P_2, \dots),$$

meaning that $\mathcal{P}$ may depend on x explicitly, on the variables P_i (for $i \in \{0, 1, 2\}$) and their derivatives to all orders. In this section, we illustrate the method by considering constraints of the reduced kind $P_3 = \mathcal{P}(x, P_0(x), P_1(x), P_2(x))$.

However, we show in AppendixD that in order to have a Hamiltonian system, $\mathcal{P}$ cannot depend on the derivatives of the dynamical variables.

To preserve the Hamiltonian structure of the bracket, $\mathcal{P}$ must be such that the Jacobi identity is satisfied. From AppendixA, we find the following conditions on $\mathcal{P}$:

$$P_0\frac{\partial\mathcal{P}}{\partial P_1}+2P_1\frac{\partial\mathcal{P}}{\partial P_2}-3P_2=0, \tag{15}$$

$$P_0\frac{\partial\mathcal{P}}{\partial P_0}+2P_1\frac{\partial\mathcal{P}}{\partial P_1}-4\mathcal{P}+3P_2\frac{\partial\mathcal{P}}{\partial P_2}=0, \tag{16}$$

$$\frac{\partial\mathcal{P}}{\partial x}=0. \tag{17}$$

The third equation implies that $\mathcal{P}$ does not have any explicit dependence on the spatial coordinate (but it might have an implicit dependence on x through its dependence on P_0, P_1 and P_2). The first two equations are first-order linear partial differential equations which are solved using the method of characteristics. The calculation, detailed in AppendixC, gives

$$P_3=3\frac{P_1P_2}{P_0}-2\frac{P_1^3}{P_0^2}+\frac{P_0^4}{2}\mathcal{Q}\left(\frac{P_2}{P_0^3}-\frac{P_1^2}{P_0^4}\right), \tag{18}$$

where $\mathcal{Q}$ is an arbitrary function. Equation (18) provides then a family of Hamiltonian closures for a three-moment model, that is expressions for the fourth-order moment P_3 in terms of P_0, P_1 and P_2 such that, when inserted into the antisymmetric bilinear form (13), they yield a Poisson bracket. The calculation also suggests more adapted field variables, for instance, the use of the specific internal energy instead of P_2. Thus, we can perform the change of variables from (P_0, P_1, P_2) to (ρ, M, U) defined by $\rho = P_0$, $M = P_1$ and

$$U=\frac{1}{2P_0}\left(P_2-\frac{P_1^2}{P_0}\right),$$

and it is straightforward to see that the bracket (13) together with the condition (18) becomes Eq. (7). Given that $\mathcal{Q}$ must be of the form $\mathcal{Q}=\mathcal{Q}(2U/\rho^2)$, another convenient variable is given by $S=2U/\rho^2$, in terms of which the Poisson bracket (7) becomes

$$\{F,G\}=\int\left\{\rho\frac{\delta G}{\delta M}\partial_x\frac{\delta F}{\delta\rho}+M\frac{\delta G}{\delta M}\partial_x\frac{\delta F}{\delta M}-\frac{\delta G}{\delta M}\frac{\delta F}{\delta S}\partial_x S\right.$$
$$\left.+\frac{\mathcal{Q}(S)}{\rho^2}\frac{\delta G}{\delta S}\partial_x\frac{\delta F}{\delta S}\right\}\,\mathrm{d}x-(F\leftrightarrow G). \tag{19}$$

4. Analysis of the reduced fluid model

The reduced fluid model we analyse below is given by the Poisson bracket (19) and the Hamiltonian

$$\mathcal{H}(\rho,M,S)=\int\left(\frac{M^2}{2\rho}+\frac{\rho^3}{2}S+\rho V\right)\,\mathrm{d}x. \tag{20}$$

which is obtained from Eq. (3). First we analyse the case $\mathcal{Q} = 0$ which corresponds to a symmetric Vlasov distribution function.

4.1. Case $\mathcal{Q} = 0$

When $\mathcal{Q} = 0$, condition (18) on P_3 becomes

$$P_3 - 6MU - \frac{M^3}{\rho^2} = 0, \tag{21}$$

or equivalently,

$$\int (v - M/\rho)^3 f(x, v) \, \mathrm{d}v = 0.$$

Thus, its skewness is zero. For instance, a Maxwellian distribution,

$$f(x, v) = \frac{\rho}{\sqrt{4\pi U}} \exp\left(-\frac{(v - M/\rho)^2}{4U}\right), \tag{22}$$

from which we get $P_0 = \rho$, $P_1 = M$ and $P_2 = 2\rho U + M^2/\rho$, has a third moment P_3 which satisfies Eq. (21).

The Poisson bracket (19) with $\mathcal{Q} = 0$ is equivalent to that of [11], which is easily seen by introducing the variable $\sigma = \rho S$ and effecting the chain rule on $F[\rho, S, M] = \bar{F}[\rho, \sigma, M]$, which gives the functional derivative relations

$$\frac{\delta F}{\delta \rho} = \frac{\delta \bar{F}}{\delta \rho} + \frac{\sigma}{\rho}\frac{\delta \bar{F}}{\delta \sigma} \qquad \text{and} \qquad \frac{\delta F}{\delta S} = \rho \frac{\delta \bar{F}}{\delta \sigma}.$$

Also note that the Poisson bracket (19) with $\mathcal{Q} = 0$ is invariant under the change of variable $S \to \tilde{S}$ given by $\tilde{S} = \phi(S)$ with arbitrary but invertible ϕ. This follows from the chain rule expression,

$$\frac{\delta F}{\delta S} = \phi'(\psi(\tilde{S})) \frac{\delta \bar{F}}{\delta \tilde{S}},$$

where $\psi = \phi^{-1}$. This symmetry of the Poisson bracket is also expressed in the form of a family of Casimir invariants:

$$C(\rho, S) = \int \rho \, \kappa(S) \, \mathrm{d}x,$$

where κ is any scalar function of one variable. When κ is a constant, this translates the conservation of the total mass whereas in all the other cases, it expresses the conservation of the total entropy. After changing to the variable $\tilde{S}$, Hamiltonian (20) becomes

$$\mathcal{H}(\rho, M, \tilde{S}) = \int \left(\frac{M^2}{2\rho} + \frac{\rho^3}{2}\psi(\tilde{S}) + \rho V\right) \mathrm{d}x,$$

with the following corresponding equations of motion:

$$\begin{aligned}
\dot{\rho} &= -\partial_x(\rho \mathrm{v}), \\
\dot{\mathrm{v}} &= -\mathrm{v}\partial_x \mathrm{v} - \partial_x V - \rho^{-1}\partial_x(\rho^3 \psi(\tilde{S})), \\
\dot{\tilde{S}} &= -\mathrm{v}\partial_x \tilde{S}.
\end{aligned}$$

Upon identifying $\tilde{S}$ with the specific entropy, the above equations are easily recognized to be the equations of an ideal fluid with the polytropic (adiabatic) equation of state with pressure $P = \psi(\tilde{S})\rho^\gamma$. As noted in Sec. 3, $\gamma = (N + 2)/N = 3$, as is the case for one dimension, and $\rho U = P/2$. Thus, using the expression for the specific internal energy $U(\rho, \tilde{S}) = \psi(\tilde{S})\rho^2/2$, which exhibits a peculiar dependence upon ρ and a separability feature with $\tilde{S}$, we obtain the thermodynamic relations $P = \rho^2 \partial U/\partial\rho = \psi(\tilde{S})\rho^3$ and $T = \partial U/\partial\tilde{S} = \psi'\rho^3$. If initially $\tilde{S}$ is constant, it will remain so. Consequently, $P\rho^{-3} = \text{const} \geq 0$, where the inequality is chosen to ensure thermodynamic stability. This will be the case if the distribution function f is a local Maxwellian of the form (22).

Finally, for this $\mathcal{Q} = 0$ case, using the above we can remove the variable S of the Poisson bracket (19) in lieu of the pressure P via the functional chain rule relations,

$$\frac{\delta F}{\delta \rho} = \frac{\delta \bar{F}}{\delta \rho} + \frac{\partial P}{\partial \rho}\frac{\delta \bar{F}}{\delta P} \qquad \text{and} \qquad \frac{\delta F}{\delta S} = \frac{\partial P}{\partial S}\frac{\delta \bar{F}}{\delta P},$$

which results in a bracket that gives the equations of motion in the form usually encountered in plasma physics,

$$\dot{P} = -\mathrm{v}\partial_x P - 3P\partial_x \mathrm{v}. \tag{23}$$

Since the derivation of the bracket is straightforward, we do not include it here.

4.2. Case $\mathcal{Q} \neq 0$

First, we recall that the quantity $\mathcal{Q}$ provides a term in the equations of motion which would vanish if the distribution function were symmetric in v. Thus, the equations of motion that allow for this skewness are

$$\begin{aligned}
\dot{\rho} &= -\partial_x(\rho \mathrm{v}), \\
\dot{\mathrm{v}} &= -\mathrm{v}\partial_x \mathrm{v} - \partial_x V - \rho^{-1}\partial_x(\rho^3 S), \\
\dot{S} &= -\mathrm{v}\partial_x S - \frac{1}{2}\rho^{-3}\partial_x(\rho^4 \mathcal{Q}(S)).
\end{aligned}$$

We remark that a constant and uniform S is now only a solution of the third equation if it corresponds to a zero of $\mathcal{Q}$. Therefore, adiabatic processes are restricted to cases with a function $\mathcal{Q}$ possessing a zero. Cold processes are obtained when $\mathcal{Q}(0) = 0$.

If we assume that the external potential V is even in x and if we perform the change of variables given by $\bar{\rho}(x) = \rho(-x)$, $\bar{M}(x) = -M(-x)$ and $\bar{S}(x) = S(-x)$, the equations of motion (or equivalently Hamiltonian (20) and

bracket (19)) are unchanged provided that $\mathcal{Q}$ changes sign. Therefore if we assume that $\mathcal{Q}$ does not vanish on its domain $\mathbb{R}_+$, there are only three Casimir invariants, namely $\int \rho \, \mathrm{d}x$, $\int [M/\rho - \rho\kappa_0^2(S)/4] \, \mathrm{d}x$ and $\int \rho \, \kappa_0(S) \, \mathrm{d}x$ where $\kappa_0' = 1/\sqrt{|\mathcal{Q}|}$. Given this set of invariants, one can see that the case $\mathcal{Q} = 0$ is structurally unstable. Indeed, any small perturbation $\mathcal{Q} = \epsilon$ leads to the generation of a third Casimir invariant, which is by definition a conserved quantity. Furthermore, we can restrict ourselves to a positive $\mathcal{Q}$. In this case it is possible to further simplify the model by considering the change of variable $\tilde{S} = \phi(S)$. With this change of variable the bracket (19) becomes

$$\{F, G\} = \int \left\{ \rho \frac{\delta G}{\delta M} \partial_x \frac{\delta F}{\delta \rho} + M \frac{\delta G}{\delta M} \partial_x \frac{\delta F}{\delta M} - \frac{\delta G}{\delta M} \frac{\delta F}{\delta \tilde{S}} \partial_x \tilde{S} \right.$$
$$\left. + \frac{\mathcal{Q}(\phi^{-1}(\tilde{S}))}{\rho^2} \left(\phi'(\phi^{-1}(\tilde{S}))\right)^2 \frac{\delta G}{\delta \tilde{S}} \partial_x \frac{\delta F}{\delta \tilde{S}} \right\} \, \mathrm{d}x - (F \leftrightarrow G).$$

Furthermore, upon choosing ϕ such that

$$\mathcal{Q}(S) \left(\phi'(S)\right)^2 = 1,$$

the bracket and Hamiltonian become, respectively,

$$\{F, G\} = \int \left\{ \rho \frac{\delta G}{\delta M} \partial_x \frac{\delta F}{\delta \rho} + M \frac{\delta G}{\delta M} \partial_x \frac{\delta F}{\delta M} - \frac{\delta G}{\delta M} \frac{\delta F}{\delta \tilde{S}} \partial_x \tilde{S} \right.$$
$$\left. + \frac{1}{\rho^2} \frac{\delta G}{\delta \tilde{S}} \partial_x \frac{\delta F}{\delta \tilde{S}} \right\} \, \mathrm{d}x - (F \leftrightarrow G), \quad (24)$$

and

$$\mathcal{H}(\rho, M, \tilde{S}) = \int \left(\frac{M^2}{2\rho} + \frac{\rho^3}{2} \psi(\tilde{S}) + \rho V \right) \, \mathrm{d}x.$$

As a consequence, the arbitrariness in the definition of the model, namely $\mathcal{Q}$ in the bracket, can be put into the Hamiltonian, and more precisely in a modification of the specific internal energy $U(\rho, \tilde{S}) = \rho^2 \psi(\tilde{S})/2$. Thus, the equations of motion become

$$\begin{aligned}
\dot{\rho} &= -\partial_x(\rho \mathrm{v}), \\
\dot{\mathrm{v}} &= -\mathrm{v}\partial_x \mathrm{v} - \partial_x V - \rho^{-1} \partial_x(\rho^3 \psi(\tilde{S})), \\
\dot{\tilde{S}} &= -\mathrm{v}\partial_x \tilde{S} - \rho^{-1} \partial_x \left(\rho^2 \psi'(\tilde{S})\right).
\end{aligned}$$

Using this set of variables, the three Casimir invariants are the total mass $\int \rho \, \mathrm{d}x$, the total "generalised" velocity $\int [M/\rho - \rho\tilde{S}^2/4] \, \mathrm{d}x$ and the total entropy $\int \rho\tilde{S} \, \mathrm{d}x$. Introducing the new variables $\sigma = \rho\tilde{S}$ and $m = M/\rho - \rho\tilde{S}^2/4$, suggested by the form of the Casimir invariants, bracket (24) takes the remarkably simpler form

$$\{F, G\} = \int \left\{ \frac{\delta G}{\delta m} \partial_x \frac{\delta F}{\delta \rho} + \frac{\delta G}{\delta \sigma} \partial_x \frac{\delta F}{\delta \sigma} \right\} \, \mathrm{d}x - (F \leftrightarrow G).$$

Finally we observe that in the case $\mathcal{Q} \neq 0$, the evolution of the pressure is modified such that the term $-2\partial_x(\rho^4\mathcal{Q})$ has to be added in the right hand side of Eq. (23).

5. Summary

In summary, we have derived a family of Hamiltonian models for the first three moments of the distribution function, starting from the Vlasov equation in one-dimension. The procedure with the verification of the Jacobi identity clearly identifies restrictions on the possible fluid models to be considered and highlights natural variables. Using the Poisson structure of these models, we have discussed the Casimir invariants.

As noted in Sec. 1, the purpose of the present paper was to build in as direct way as possible Hamiltonian closures for higher order fluid models. It would be interesting to discuss dynamical consequences of the closures, and extend this approach to higher dimensions and higher order models with richer physical content by reinstating the coupling to self-consistent fields.

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AppendixA. Conditions on P_3 for bracket (13) to satisfy the Jacobi identity

The aim of this appendix is to find the conditions on

$$P_3 = \mathcal{P}(x, P_0(x), P_1(x), P_2(x)) \,,$$

such that the bracket (13) satisfies the Jacobi identity. First we notice that the bracket (14) satisfies the Jacobi identity (see AppendixB). As a consequence, the Jacobi identity for the bracket (13) reduces to

$$\{F, \{G, H\}\} + \circlearrowleft = \{F, \{G, H\}^*\}_J + \{F, \{G, H\}_J\}^* + \{F, \{G, H\}^*\}^* + \circlearrowleft \,.$$

where $\circlearrowleft$ designates the summation of the expression over circular permutations of the functionals F, G and H. Below, we detail the computation of the first contribution, $\{F, \{G, H\}^*\}_J$, and provide the results for the other two contributions, $\{F, \{G, H\}_J\}^*$ and $\{F, \{G, H\}^*\}^*$. Furthermore, in what follows, we

shall denote F_{P_i} the functional derivative of F with respect to the dynamical field variable P_i such that $F_{P_i} = \delta F/\delta P_i$. In order to compute $\{F, \{G, H\}^*\}_J$, we calculate the functional derivatives of $\{G, H\}^*$ by differentiating only with respect to the explicit dependence on the dynamical variables (see Ref. [1]). It has been shown that the other contributions with second order functional derivatives cancel in a very general way. This leads to

$$\begin{aligned}
\frac{\delta\{G,H\}^*}{\delta P_0} &= 2\partial_{P_0}\mathcal{P}\left(H_{P_2}\partial_x G_{P_2} - G_{P_2}\partial_x H_{P_2}\right), \\
\frac{\delta\{G,H\}^*}{\delta P_1} &= 2\left[H_{P_2}\partial_x G_{P_0} - G_{P_2}\partial_x H_{P_0} + \partial_{P_1}\mathcal{P}\left(H_{P_2}\partial_x G_{P_2} - G_{P_2}\partial_x H_{P_2}\right)\right], \\
\frac{\delta\{G,H\}^*}{\delta P_2} &= 2\left[H_{P_2}\partial_x G_{P_1} - G_{P_2}\partial_x H_{P_1} + \partial_{P_2}\mathcal{P}\left(H_{P_2}\partial_x G_{P_2} - G_{P_2}\partial_x H_{P_2}\right)\right],
\end{aligned}$$

where the notation $\partial_{P_i}\mathcal{P}$ indicates the partial derivative of $\mathcal{P}$ with respect to P_i. We thus obtain

$$\begin{aligned}
\{F,\{G,H\}^*\}_J = \int 2\,\{&P_0\left(\left[H_{P_2}\partial_x G_{P_0} - G_{P_2}\partial_x H_{P_0} + \partial_{P_1}\mathcal{P}\left(H_{P_2}\partial_x G_{P_2}\right.\right.\right. \\
&\left.\left.\left. - G_{P_2}\partial_x H_{P_2}\right)\right]\partial_x F_{P_0} - F_{P_1}\partial_x\left[\partial_{P_0}\mathcal{P}\left(H_{P_2}\partial_x G_{P_2} - G_{P_2}\partial_x H_{P_2}\right)\right]\right) \\
&+ P_1\left(\left[H_{P_2}\partial_x G_{P_0} - G_{P_2}\partial_x H_{P_0} + \partial_{P_1}\mathcal{P}\left(H_{P_2}\partial_x G_{P_2} - G_{P_2}\partial_x H_{P_2}\right)\right]\partial_x F_{P_1}\right. \\
&\left. - F_{P_1}\partial_x\left[H_{P_2}\partial_x G_{P_0} - G_{P_2}\partial_x H_{P_0} + \partial_{P_1}\mathcal{P}\left(H_{P_2}\partial_x G_{P_2} - G_{P_2}\partial_x H_{P_2}\right)\right]\right) \\
&+ P_2\left(\left[H_{P_2}\partial_x G_{P_0} - G_{P_2}\partial_x H_{P_0} + \partial_{P_1}\mathcal{P}\left(H_{P_2}\partial_x G_{P_2} - G_{P_2}\partial_x H_{P_2}\right)\right]\partial_x F_{P_2}\right. \\
&\left. - F_{P_1}\partial_x\left[H_{P_2}\partial_x G_{P_1} - G_{P_2}\partial_x H_{P_1} + \partial_{P_2}\mathcal{P}\left(H_{P_2}\partial_x G_{P_2} - G_{P_2}\partial_x H_{P_2}\right)\right]\right)\}\ \mathrm{d}x.
\end{aligned}$$

By circular permutation on (F, G, H), the terms of the type $P_0 H_{P_2}\partial_x G_{P_0}\partial_x F_{P_0}$ and $P_2\partial_x F_{P_1}\partial_x G_{P_1} H_{P_2}$ cancel. Using an integration by parts, we have

$$\begin{aligned}
\{F,\{G,H\}^*\}_J + \circlearrowleft = \int 2\,\{&P_0\partial_{P_1}\mathcal{P}\left(H_{P_2}\partial_x G_{P_2} - G_{P_2}\partial_x H_{P_2}\right)\partial_x F_{P_0} \\
&+\partial_x\left(P_0 F_{P_1}\right)\partial_{P_0}\mathcal{P}\left(H_{P_2}\partial_x G_{P_2} - G_{P_2}\partial_x H_{P_2}\right) + 2P_1\left[H_{P_2}\partial_x G_{P_0} - G_{P_2}\partial_x H_{P_0}\right. \\
&\left.+\partial_{P_1}\mathcal{P}\left(H_{P_2}\partial_x G_{P_2} - G_{P_2}\partial_x H_{P_2}\right)\right]\partial_x F_{P_1} + F_{P_1}\partial_x P_1\left[H_{P_2}\partial_x G_{P_0} - G_{P_2}\partial_x H_{P_0}\right. \\
&\left.+\partial_{P_1}\mathcal{P}\left(H_{P_2}\partial_x G_{P_2} - G_{P_2}\partial_x H_{P_2}\right)\right] + P_2\left(H_{P_2}\partial_x G_{P_0} - G_{P_2}\partial_x H_{P_0}\right)\partial_x F_{P_2} \\
&+ \partial_x\left(P_2 F_{P_1}\right)\partial_{P_2}\mathcal{P}\left(H_{P_2}\partial_x G_{P_2} - G_{P_2}\partial_x H_{P_2}\right) + F_{P_1}\partial_x P_2\left(H_{P_2}\partial_x G_{P_1}\right. \\
&\left. - G_{P_2}\partial_x H_{P_1}\right)\}\ \mathrm{d}x + \circlearrowleft .
\end{aligned}$$

Similarly, we obtain the following expressions for $\{F, \{G, H\}_J\}^*$ and $\{F, \{G, H\}^*\}^*$:

$$\begin{aligned}
\{F,\{G,H\}_J\}^* + \circlearrowleft = \int 2\,\{&P_1\partial_x F_{P_0}\left(H_{P_1}\partial_x G_{P_2} - G_{P_1}\partial_x H_{P_2}\right) \\
&+ \partial_x\left(P_1 F_{P_2}\right)\left(H_{P_1}\partial_x G_{P_0} - G_{P_1}\partial_x H_{P_0}\right) + P_2\partial_x F_{P_1}\left(H_{P_1}\partial_x G_{P_2} - G_{P_1}\partial_x H_{P_2}\right) \\
&+ \partial_x\left(P_2 F_{P_2}\right)\left(H_{P_1}\partial_x G_{P_1} - G_{P_1}\partial_x H_{P_1}\right) + F_{P_2}\partial_x\mathcal{P}\left(H_{P_1}\partial_x G_{P_2}\right. \\
&\left. - G_{P_1}\partial_x H_{P_2}\right)\}\ \mathrm{d}x + \circlearrowleft,
\end{aligned}$$

and

$$\{F,\{G,H\}^*\}^*+\circlearrowleft=\int 4\,\{P_1\partial_x F_{P_0}\left(H_{P_2}\partial_x G_{P_1}-G_{P_2}\partial_x H_{P_1}\right)$$
$$+P_1\partial_x F_{P_0}\partial_{P_2}\mathcal{P}\left(H_{P_2}\partial_x G_{P_2}-G_{P_2}\partial_x H_{P_2}\right)+P_2\partial_x F_{P_1}\partial_{P_2}\mathcal{P}\left(H_{P_2}\partial_x G_{P_2}\right.$$
$$\left.-G_{P_2}\partial_x H_{P_2}\right)+P_2\left(H_{P_2}\partial_x G_{P_0}-G_{P_2}\partial_x H_{P_0}\right)\partial_x F_{P_2}$$
$$+2\mathcal{P}\partial_x F_{P_2}\left(H_{P_2}\partial_x G_{P_1}-G_{P_2}\partial_x H_{P_1}\right)\}\ \mathrm{d}x+\circlearrowleft\,.$$

As a consequence, we have

$$\{F,\{G,H\}\}+\circlearrowleft=\int 2\,\{P_0\partial_{P_1}\mathcal{P}\left(H_{P_2}\partial_x G_{P_2}-G_{P_2}\partial_x H_{P_2}\right)\partial_x F_{P_0}$$
$$+\partial_x\left(P_0F_{P_1}\right)\partial_{P_0}\mathcal{P}\left(H_{P_2}\partial_x G_{P_2}-G_{P_2}\partial_x H_{P_2}\right)+2P_1\partial_{P_1}\mathcal{P}\left(H_{P_2}\partial_x G_{P_2}\right.$$
$$\left.-G_{P_2}\partial_x H_{P_2}\right)\partial_x F_{P_1}+F_{P_1}\partial_{P_1}\mathcal{P}\partial_x P_1\left(H_{P_2}\partial_x G_{P_2}-G_{P_2}\partial_x H_{P_2}\right)+3P_2\left(H_{P_2}\partial_x G_{P_0}\right.$$
$$\left.-G_{P_2}\partial_x H_{P_0}\right)\partial_x F_{P_2}+\partial_x\left(P_2F_{P_1}\right)\partial_{P_2}\mathcal{P}\left(H_{P_2}\partial_x G_{P_2}-G_{P_2}\partial_x H_{P_2}\right)+\left(H_{P_1}\partial_x G_{P_2}\right.$$
$$\left.-G_{P_1}\partial_x H_{P_2}\right)F_{P_2}\partial_x\mathcal{P}+2P_1\partial_x F_{P_0}\partial_{P_2}\mathcal{P}\left(H_{P_2}\partial_x G_{P_2}-G_{P_2}\partial_x H_{P_2}\right)+\left(H_{P_2}\partial_x G_{P_2}\right.$$
$$\left.-G_{P_2}\partial_x H_{P_2}\right)2P_2\partial_x F_{P_1}\partial_{P_2}\mathcal{P}+4\mathcal{P}\partial_x F_{P_2}\left(H_{P_2}\partial_x G_{P_1}-G_{P_2}\partial_x H_{P_1}\right)\}\ \mathrm{d}x+\circlearrowleft\,.$$

Furthermore, by definition, we have

$$\partial_x\mathcal{P}=\frac{\partial\mathcal{P}}{\partial x}+\partial_{P_0}\mathcal{P}\partial_x P_0+\partial_{P_1}\mathcal{P}\partial_x P_1+\partial_{P_2}\mathcal{P}\partial_x P_2,$$

where ∂_x and $\partial/\partial x$ are two distinct operators, the later acting solely on the explicit dependence on the spatial coordinate x. Consequently, one obtains

$$\{F,\{G,H\}\}+\circlearrowleft=\int 2\left(H_{P_2}\partial_x G_{P_2}-G_{P_2}\partial_x H_{P_2}\right)\{\partial_x F_{P_0}\left[P_0\partial_{P_1}\mathcal{P}+2P_1\partial_{P_2}\mathcal{P}\right.$$
$$\left.-3P_2\right]+\partial_x F_{P_1}\left[P_0\partial_{P_0}\mathcal{P}+2P_1\partial_{P_1}\mathcal{P}-4\mathcal{P}+3P_2\partial_{P_2}\mathcal{P}\right]-F_{P_1}\frac{\partial\mathcal{P}}{\partial x}\Big\}\ \mathrm{d}x+\circlearrowleft\,.$$

Therefore, in order for the bracket (13) to satisfy the Jacobi identity, the function $\mathcal{P}$ has to satisfy Eqs. (15)–(17).

AppendixB. Jacobi identity for bracket (14)

In this Appendix, the Jacobi identity is proved for brackets of the type:

$$\{F,G\}_M=\int\rho_i\left(G_{\rho_M}\partial_x F_{\rho_i}-F_{\rho_M}\partial_x G_{\rho_i}\right)\ \mathrm{d}x, \tag{B.1}$$

with implicit summation of the repeated index $i=1,\ldots,N$, and for $N,M\in\mathbb{N}$ and $M\leq N$. For the computation of $\{F,\{G,H\}_M\}_M$, we again use Morrison's lemma which states that only the functional derivatives with respect to the

explicit dependence on the variables matter for the Jacobi identity [1], that is to say we consider that

$$\frac{\delta\{G,H\}_M}{\delta\rho_i} = H_{\rho_M}\partial_x G_{\rho_i} - G_{\rho_M}\partial_x H_{\rho_i}, \tag{B.2}$$

assuming that the other contributions compensate through summation over circular permutation. Using an integration by parts, $\{F,\{G,H\}_M\}_M$ is rewritten as

$$\{F,\{G,H\}_M\}_M = \int \left\{ \rho_i \left[\frac{\delta\{G,H\}_M}{\delta\rho_M}\partial_x F_{\rho_i} - \partial_x F_{\rho_M}\frac{\delta\{G,H\}_M}{\delta\rho_i} \right] + \partial_x\rho_i F_{\rho_M}\frac{\delta\{G,H\}_M}{\delta\rho_i} \right\} \mathrm{d}x.$$

Inserting Eq. (B.2) into the previous equation leads to the following expression

$$\begin{aligned}\{F,\{G,H\}_M\}_M = \int \{&\rho_i\left[H_{\rho_M}\partial_x G_{\rho_M}\partial_x F_{\rho_i} - \partial_x F_{\rho_M} G_{\rho_M}\partial_x H_{\rho_i}\right] \\ &+ \rho_i\left[\partial_x F_{\rho_M} H_{\rho_M}\partial_x G_{\rho_i} - G_{\rho_M}\partial_x H_{\rho_M}\partial_x F_{\rho_i}\right] \\ &+ \partial_x\rho_i\left[F_{\rho_M} H_{\rho_M}\partial_x G_{\rho_i} - F_{\rho_M} G_{\rho_M}\partial_x H_{\rho_i}\right]\}\ \mathrm{d}x.\end{aligned}$$

Using circular permutation of (F,G,H), each line of the previous equation cancels out, and, as a consequence, bracket (B.1) satisfies the Jacobi identity.

AppendixC. Method of characteristics and closure

We use the method of characteristics in order to solve Eq. (15) by introducing three spectral parameters A, B and C. We obtain

$$\frac{\partial P_0}{\partial A} = 0, \quad \frac{\partial P_1}{\partial A} = P_0, \quad \frac{\partial P_2}{\partial A} = 2P_1 \quad \text{and} \quad \frac{\partial \mathcal{P}}{\partial A} = 3P_2,$$

whose solution is

$$\begin{aligned}P_0 &= P_0(B,C),\\ P_1 &= AP_0(B,C) + \alpha(B,C),\\ P_2 &= A^2P_0(B,C) + 2A\alpha(B,C) + \beta(B,C),\\ \mathcal{P} &= A^3P_0(B,C) + 3A^2\alpha(B,C) + 3A\beta(B,C) + \gamma(B,C),\end{aligned}$$

where α, β and γ are sufficiently regular functions in both their arguments. We choose the spectral parameters B and C such that

$$\alpha = 0, \quad P_0 = B \quad \text{and} \quad \beta = C,$$

which leads to

$$
\begin{aligned}
A &= \frac{P_1}{P_0}, \\
C &= P_2 - \frac{P_1^2}{P_0}, \\
\mathcal{P} &= 3\frac{P_1 P_2}{P_0} - 2\frac{P_1^3}{P_0^2} + \gamma\left(P_0, P_2 - \frac{P_1^2}{P_0}\right).
\end{aligned}
$$

We insert the solution for $\mathcal{P}$ in Eq. (16) and we obtain the following condition :

$$B\frac{\partial\gamma}{\partial B} + 3C\frac{\partial\gamma}{\partial C} = 4\gamma.$$

As previously, this equation is solved by using the method of characteristics, which results in

$$\gamma(B, C) = B^4\mathcal{Q}\left(\frac{C}{B^3}\right),$$

where $\mathcal{Q}$ is some sufficiently regular function. Thus, we finally obtain

$$P_3 = 3\frac{P_1 P_2}{P_0} - 2\frac{P_1^3}{P_0^2} + P_0^4\mathcal{Q}\left(\frac{P_2}{P_0^3} - \frac{P_1^2}{P_0^4}\right).$$

AppendixD. Independence of $\mathcal{Q}$ in the derivatives of the field variables

As stated above, the closure procedure exhibits a natural set of variables (ρ, M, S) defined such that $\rho = P_0$, $M = P_1$ and $S = (P_2 - P_1^2/P_0)/P_0^3$. Denoting respectively F_ρ, F_M and F_S the functional derivatives of F with respect to ρ, M and S, one has

$$F_{P_0} = F_\rho + \frac{1}{\rho}\left(\frac{M^2}{\rho^4} - 3S\right)F_S, \quad F_{P_1} = F_M - 2\frac{M}{\rho^4}F_S \quad \text{and} \quad F_{P_2} = \frac{1}{\rho^3}F_S.$$

According to this change of variable, bracket (13) becomes

$$\{F, G\} = \{F, G\}_1 + \{F, G\}_2, \tag{D.1}$$

where

$$
\begin{aligned}
\{F, G\}_1 &= \int \{\rho G_M \partial_x F_\rho + M G_M \partial_x F_M - G_M F_S \partial_x S\}\ \mathrm{d}x - (F \leftrightarrow G) \\
\{F, G\}_2 &= \int \mathcal{R}(G_S \partial_x F_S - F_S \partial_x G_S)\ \mathrm{d}x,
\end{aligned}
$$

and

$$\mathcal{R} = \frac{2}{\rho^4}\left(\frac{P_3}{\rho^2} - 3SM - \frac{M^3}{\rho^4}\right).$$

The constraint on $\mathcal{R}$ we impose is that it depends on the dynamical field variables and their derivatives, i.e., $\mathcal{R} = \mathcal{R}(x, \{\partial_x^n \rho\}_{n\in\mathbb{N}}, \{\partial_x^n M\}_{n\in\mathbb{N}}, \{\partial_x^n S\}_{n\in\mathbb{N}})$. We assume that $\mathcal{R}$ depends only on the first N derivatives of the field variables. We are looking for conditions such that the resulting bracket (D.1) satisfies the Jacobi identity. First we notice that the bracket $\{F, G\}_1$ satisfies the Jacobi identity [6]. Following the procedure used in AppendixA, one has to compute functional derivatives such as $\delta\{F, G\}_2/\delta\rho$. Due to the explicit dependence of $\mathcal{R}$ on the derivatives of the variables, this computation generates a series of terms, for example,

$$\frac{\delta\{G, H\}_2}{\delta\rho} = \sum_{n=0}^{N} (-1)^n \partial_x^n \left[\frac{\partial\mathcal{R}}{\partial\partial_x^n \rho} (H_S \partial_x G_S - G_S \partial_x H_S) \right].$$

Using similar techniques as in AppendixA (integration by parts and cancellations of terms by circular permutations), we eventually end up with

$$\begin{aligned}
\{F, \{G, H\}\} + \circlearrowleft = \int \Bigg\{ \sum_{n=0}^{N} \Bigg[& \partial_x^n (\rho \partial_x F_\rho) \frac{\partial\mathcal{R}}{\partial\partial_x^n M} + \partial_x^{n+1} (\rho F_M) \frac{\partial\mathcal{R}}{\partial\partial_x^n \rho} \\
& + 2\partial_x^n (M \partial_x F_M) \frac{\partial\mathcal{R}}{\partial\partial_x^n M} + \partial_x^n (F_M \partial_x M) \frac{\partial\mathcal{R}}{\partial\partial_x^n M} - \partial_x^n (F_S \partial_x S) \frac{\partial\mathcal{R}}{\partial\partial_x^n M} \\
& + \partial_x^n (F_M \partial_x S) \frac{\partial\mathcal{R}}{\partial\partial_x^n S} + 2\partial_x^n (\mathcal{R} \partial_x F_S) \frac{\partial\mathcal{R}}{\partial\partial_x^n S} + \partial_x^n (F_S \partial_x \mathcal{R}) \frac{\partial\mathcal{R}}{\partial\partial_x^n S} \Bigg] \\
& + 2\mathcal{R}\partial_x F_M - F_M \partial_x \mathcal{R} \Bigg\} (H_S \partial_x G_S - G_S \partial_x H_S) \, \mathrm{d}x + \circlearrowleft .
\end{aligned}$$

We remark that the terms which depend on F_ρ can not cancel through circular permutation. Thus, in order to satisfy the Jacobi identity, we must have

$$\sum_{n=0}^{N} \frac{\partial\mathcal{R}}{\partial\partial_x^n M} \sum_{m=0}^{n} \binom{n}{m} \partial_x^m \rho \partial_x^{n+1-m} F_\rho (H_S \partial_x G_S - G_S \partial_x H_S) + \circlearrowleft = 0,$$

where we used the generalised Leibniz rule. As this condition has to be verified for any F, one can prove by induction that $\forall n \in \mathbb{N}$, $\partial\mathcal{R}/\partial\partial_x^n M = 0$. As a consequence, we obtain

$$\begin{aligned}
\{F, \{G, H\}\} + \circlearrowleft = \int \Bigg\{ \sum_{n=0}^{N} \Bigg[& \partial_x^{n+1} (\rho F_M) \frac{\partial\mathcal{R}}{\partial\partial_x^n \rho} + \partial_x^n (F_M \partial_x S) \frac{\partial\mathcal{R}}{\partial\partial_x^n S} \\
& + 2\partial_x^n (\mathcal{R} \partial_x F_S) \frac{\partial\mathcal{R}}{\partial\partial_x^n S} + \partial_x^n (F_S \partial_x \mathcal{R}) \frac{\partial\mathcal{R}}{\partial\partial_x^n S} \Bigg] \\
& + 2\mathcal{R}\partial_x F_M - F_M \partial_x \mathcal{R} \Bigg\} (H_S \partial_x G_S - G_S \partial_x H_S) \, \mathrm{d}x + \circlearrowleft .
\end{aligned}$$

Next we consider the terms which include F_S, G_S and H_S. We notice that the terms $n = 0$ (which are proportional to $\partial_x F_s H_S \partial_x G_S$) vanish by circular

permutation. In order to make the terms $n \geq 1$ vanish, we need to impose

$$\sum_{n=1}^{N} \frac{\partial \mathcal{R}}{\partial \partial_x^n S} \left[2\partial_x^n \left(\mathcal{R}\partial_x F_S\right) + \partial_x^n \left(F_S \partial_x \mathcal{R}\right)\right] \left(H_S \partial_x G_S - G_S \partial_x H_S\right) + \circlearrowleft = 0.$$

As previously, we show by induction that $\forall n \in \mathbb{N}^*$, $\partial \mathcal{R}/\partial \partial_x^n S = 0$. This leads to

$$\{F, \{G, H\}\} + \circlearrowleft = \int \left\{ \sum_{n=0}^{N} \partial_x^{n+1} \left(\rho F_M\right) \frac{\partial \mathcal{R}}{\partial \partial_x^n \rho} + F_M \partial_x S \frac{\partial \mathcal{R}}{\partial S} \right.$$
$$\left. + 2\mathcal{R}\partial_x F_M - F_M \partial_x \mathcal{R} \right\} \left(H_S \partial_x G_S - G_S \partial_x H_S\right) \, \mathrm{d}x + \circlearrowleft .$$

Besides, by definition, we have

$$\partial_x \mathcal{R} = \frac{\partial \mathcal{R}}{\partial x} + \frac{\partial \mathcal{R}}{\partial S} \partial_x S + \sum_{n=0}^{N} \frac{\partial \mathcal{R}}{\partial \partial_x^n \rho} \partial_x^{n+1} \rho,$$

where ∂_x and $\partial/\partial x$ are distinct operators, the later acting solely on the explicit dependence on the spatial coordinate x. Consequently, one obtains

$$\{F, \{G, H\}\} + \circlearrowleft = \int \left\{ \sum_{n=0}^{N} \frac{\partial \mathcal{R}}{\partial \partial_x^n \rho} \sum_{m=0}^{n} \binom{n+1}{m} \partial_x^m \rho \partial_x^{n+1-m} F_M \right.$$
$$\left. + 2\mathcal{R}\partial_x F_M - F_M \frac{\partial \mathcal{R}}{\partial x} \right\} \left(H_S \partial_x G_S - G_S \partial_x H_S\right) \, \mathrm{d}x + \circlearrowleft .$$

This expression has to vanish for any set of functionals (F, G, H), so it requires that

$$\sum_{n=0}^{N} \frac{\partial \mathcal{R}}{\partial \partial_x^n \rho} \sum_{m=0}^{n} \binom{n+1}{m} \partial_x^m \rho \partial_x^{n+1-m} F_M + 2\mathcal{R}\partial_x F_M - F_M \frac{\partial \mathcal{R}}{\partial x} = 0.$$

Choosing $F = \int M \, \mathrm{d}x$ leads to the condition $\partial \mathcal{R}/\partial x = 0$, so there is no explicit dependence on the spatial variable. Also by induction, we have $\forall n \in \mathbb{N}^*$,

$$\frac{\partial \mathcal{R}}{\partial \partial_x^n \rho} = 0 \qquad \text{and} \qquad \frac{\partial \mathcal{R}}{\partial \rho} \rho + 2\mathcal{R} = 0.$$

Thus, the solution is given by

$$\mathcal{R} = \frac{\mathcal{Q}(S)}{\rho^2},$$

where $\mathcal{Q}$ is an arbitrary function.

[1] P. J. Morrison. *AIP Conf. Proc.*, 88:13–46, 1982.

[2] P. J. Morrison. *Rev. Mod. Phys.*, 70:467, 1998.

[3] J. E. Marsden and T. S. Ratiu. *Introduction to Mechanics and Symmetry.* New York: Springer, 1999.

[4] C. D. Levermore. *J. Stat. Phys.*, 83:1021, 1996.

[5] J. E. Marsden, P. J. Morrison, and A. Weinstein. *Cont. Math.*, 28:115–124, 1984.

[6] P. J. Morrison. *Phys. Rev. Lett.*, 45:790, 1980.

[7] J. E. Marsden and A. Weinstein. *Physica D*, 4:394, 1982.

[8] P. J. Morrison. *Phys. Plasmas*, 20:012104, 2013.

[9] J. Gibbons. *Physica D*, 3:503–511, 1981.

[10] J. Gibbons, D. D. Holm, and C. Tronci. *Phys. Lett. A.*, 372:1024–1033, 2008.

[11] P. J. Morrison and J. M. Greene. *Phys. Rev. Lett.*, 45:790–793, 1980. E **48**, 569 (1982).

[12] L. de Guillebon and C. Chandre. *Phys. Lett. A*, 376:3172, 2012.

[13] A. J. Brizard and T. S. Hahm. *Rev. Mod. Phys.*, 79:421, 2007.

[14] A. J. Brizard. *Commun. Nonlinear Sci. Numer. Simulat.*, 13:24, 2008.

[15] V. E. Zakharov and E. A. Kuznetsov. *Phys. Usp.*, 40:1087, 1997.

[16] P. Goswami, T. Passot, and P. L. Sulem. *Phys. Plasmas*, 12:102109–1, 2005.

[17] P. J. Morrison, M. Vittot, and L. de Guillebon. *Phys. Plasmas*, 20:032109, 2013.

[18] M. V. Melander, N. J. Zabusky, and A. S. Styczek. *J. Fluid Mech.*, 167:95–115, 1986.

[19] S. P. Meacham, P. J. Morrison, and G. R. Flierl. *Phys. Fluids*, 9:2310–2328, 1997.

[20] K. Ngan, S. P. Meacham, and P. J. Morrison. *Phys. Fluids*, 4:893–916, 1996.

[21] A. Crosby, E. R. Johnson, and P. J. Morrison. *Phys. Fluids*, 25:023602, 2013.

[22] S. Chandresekhar. *Ellipsoidal Figures of Equilibrium.* New York: Dover, 1987.

[23] P. J. Morrison, N. R. Lebovitz, and J. Biello. *Ann. Phys.*, 324:1747–1762, 2009.

[24] B. A. Shadwick, G. M. Tarkenton, and E. H. Esarey. *Phys. Rev. Lett.*, 93:175002–1, 2004.

[25] B. A. Shadwick, G. M. Tarkenton, E. H. Esarey, and C. B. Schroeder. *Phys. Plasmas*, 12:056710, 2005.

[26] B. A. Shadwick, G. M. Tarkenton, E. H. Esarey, and F. M. Lee. *Commun. Nonlinear Sci. Numer. Simulat.*, 17:2153–2160, 2012.

[27] H. E. Kandrup and P. J. Morrison. *Ann. Phys.*, 225:114–166, 1993.

[28] J. E. Marsden, R. Montgomery, P. J. Morrison, and W. B. Thompson. *Ann. Phys.*, 169:29–47, 1986.

[29] C. Chandre, P. J. Morrison, and E. Tassi. *Phys. Lett. A*, 378:956–959, 2014.