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Note on diffeomorphism extension for observer design

Vincent Andrieu*, Laurent Praly†

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Abstract

An often encountered way of designing an observer is to use coordinates different from the given ones (or the ones of interest). This is done via the construction of a diffeomorphism (maybe obtained following the extension of an injective immersion) and an extended vector field on which is designed the observer. The estimated state is then obtained by employing the left inverse of the diffeomorphism (which may be difficult to obtain). A possible solution to overcome this difficulty is to use a diffeomorphism extension. A first preliminary result is given in this note.

1 Introduction

Consider the system:

$$\dot{x} = f(x) \quad , \quad y = h(x) \quad , \quad (1)$$

with x in $\mathcal{A} \subset \mathbb{R}^n$ is an open forward invariant set and y in \mathbb{R} .

An often encountered way of designing an observer is to use coordinates different from the given ones (or the ones of interest). This is done via the construction of a diffeomorphism (maybe obtained following the extension of an injective immersion as in [1]) $\tau^* : \mathcal{A} \rightarrow \mathbb{R}^n$ and a (extended) vector field $\varphi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ such that

$$\dot{\xi} = \varphi(\xi, y) \quad , \quad y = h(\tau(\xi)) \quad , \quad \xi \in \tau^*(\mathcal{A}) \quad (2)$$

is the image of the system (1) with the diffeomorphism τ^* and such that the system

$$\dot{\hat{\xi}} = \varphi(\hat{\xi}, y) \quad ,$$

defines an observer for the system (2). In other words, we have,

$$\lim_{t \rightarrow +\infty} \left| \hat{\Xi}(\hat{\xi}, \xi, t) - \Xi(\xi, t) \right| = 0 \quad , \quad \forall (\xi, \hat{\xi}) \in \tau^*(\mathcal{A}) \times \mathbb{R}^m$$

An observer for the system (1) is then defined as

$$\hat{x} = \tau(\hat{\xi}) \quad , \quad \dot{\hat{\xi}} = \varphi(\hat{\xi}, y) \quad , \quad (3)$$

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where τ is an inverse of τ^* . A possible way to implement this observer is as follows (see for instance [2] and [1])

$$\dot{\hat{x}} = f(\hat{x}) + \left(\frac{\partial \tau^*}{\partial x}(\hat{x}) \right)^{-1} [\varphi(\tau^*(\hat{x}), y) - \varphi(\tau^*(\hat{x}), h(\hat{x}))]$$

But this expression makes sense only in the set in which $\left(\frac{\partial \tau^*}{\partial x}(\hat{x}) \right)^{-1}$ is well defined, i.e. as long as its solutions \hat{x} is in \mathcal{A} . A possible solution to overcome this difficulty is to extend the diffeomorphism outside \mathcal{A} . Indeed, if we construct a diffeomorphism $\tau_e^* : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that for all x in \mathcal{A} , we have $\tau^*(x) = \tau_e^*(x)$ then a possible implementation of the observer would simply be:

$$\dot{\hat{x}} = f(\hat{x}) + \left(\frac{\partial \tau_e^*}{\partial x}(\hat{x}) \right)^{-1} [\varphi(\tau_e^*(\hat{x}), y) - \varphi(\tau_e^*(\hat{x}), h(\hat{x}))]$$

In this note, based on [3] we investigate the possibility of extended a diffeomorphism.

2 Diffeomorphism extension

In the following, we denote by $\mathcal{B}_R(0)$ the open ball in \mathbb{R}^n which is centered at 0 and has radius R .

Proposition 1 (Extension of the diffeomorphism on a ball) *Assume there exists a C^2 function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\psi(0) = 0$ and $\psi : \mathcal{B}_R(0) \rightarrow \psi(\mathcal{B}_R(0)) \subset \mathbb{R}^n$ is a diffeomorphism. Then for any δ in $(0, R)$, there exists a diffeomorphism $\psi_e : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\psi_e(x) = \psi(x)$ for all x in the closure $\text{cl}(\mathcal{B}_{R-\delta}(0))$.*

Proof : Let δ in $(0, R)$, we denote

$$T = \sqrt{\frac{R}{R-\delta}} > 1 \quad , \quad R_2 = R \sqrt{\frac{R-\delta}{R}} \quad , \quad R_1 = R - \delta .$$

Note that we have

$$\mathcal{B}_{TR_1}(0) = \mathcal{B}_{R_2}(0) \quad , \quad \mathcal{B}_{TR_2}(0) = \mathcal{B}_R(0) .$$

Consider the function $\varphi : \mathcal{B}_{R_2}(0) \times (-T, T) \rightarrow \mathbb{R}^n$ defined as

$$\begin{aligned} \varphi(x, t) &= \left(\frac{\partial \psi}{\partial x}(0) \right)^{-1} \frac{\psi(xt)}{t} \quad \text{for } t \neq 0 \quad , \\ \varphi(x, 0) &= x . \end{aligned}$$

We study the properties of the function $x \mapsto \varphi_t(x) = \varphi(x, t)$ with t fixed in $(-T, T)$.

- Let x_a and x_b be in $\mathcal{B}_{R_2}(0)$ satisfying

$$\varphi(x_a, t) = \varphi(x_b, t) .$$

If $t \neq 0$, this implies readily

$$\psi(x_a t) = \psi(x_b t) .$$

Since the pair $(x_a t, x_b t)$ is in $\mathcal{B}_R(0)$ for all t in $(-T, T)$ and since the function ψ is injective on this set, we obtain

$$x_a = x_b .$$

If $t = 0$, we get this last inequality directly. So we have established the injectivity of φ_t on $\mathcal{B}_{R_2}(0)$ for all t in $(-T, T)$.

- we have

$$\begin{aligned} \frac{\partial \varphi_t}{\partial x}(x) &= \left(\frac{\partial \psi}{\partial x}(0) \right)^{-1} \frac{\partial \psi}{\partial x}(xt) \quad \text{for } t \neq 0 \\ \frac{\partial \varphi_0}{\partial x}(x) &= \text{Id} \end{aligned}$$

and

$$\lim_{t \rightarrow 0} \left(\frac{\partial \psi}{\partial x}(0) \right)^{-1} \frac{\partial \psi}{\partial x}(xt) = \text{Id}$$

Hence φ_t is C^1 on $\mathcal{B}_{R_2}(0)$. Moreover, since by assumption the matrix $\frac{\partial \psi}{\partial x}(xt)$ is invertible for all t in $(-T, T)$ and x in $\mathcal{B}_{R_2}(0)$. φ_t is full rank on $\mathcal{B}_{R_2}(0)$.

We conclude that, for all t in $(-T, T)$, the function $x \mapsto \varphi_t(x) = \varphi(x, t)$ is a diffeomorphism from $\mathcal{B}_{R_2}(0)$ onto $\varphi_t(\mathcal{B}_{R_2}(0))$. Let φ_t^{-1} denote its inverse map.

Similarly, let us study the properties of the function ρ defined as :

$$\begin{aligned} \rho(x, t) &= \frac{1}{t^2} \left[\frac{\partial \psi}{\partial x}(xt) xt - \psi(xt) \right] \quad \text{for } t \neq 0 , \\ \rho(x, 0) &= \frac{1}{2} x' \left(\frac{\partial^2 \psi}{\partial x \partial x}(0) \right) x \end{aligned}$$

- The function ψ being C^2 , and satisfying $\psi(0) = 0$, we have

$$\lim_{t \rightarrow 0} \frac{\psi(xt) - \frac{\partial \psi}{\partial x}(0) xt}{t^2} = \frac{1}{2} \lim_{t \rightarrow 0} \frac{\frac{\partial \psi}{\partial x}(xt) x - \frac{\partial \psi}{\partial x}(0) x}{t} = \frac{1}{2} x' \left(\frac{\partial^2 \psi}{\partial x \partial x}(0) \right) x$$

This implies that the function ρ is well defined and the function $t \mapsto \rho(x, t)$ is continuous.

- We get

$$\frac{\partial \rho}{\partial x}(x, t) = \frac{\partial^2 \psi}{\partial x \partial x}(xt) x \quad \forall t \in (-T, T) .$$

Since xt is in $\mathcal{B}_R(0)$ for all t in $(-T, T)$ and x in $\mathcal{B}_{R_2}(0)$ and ψ is C^2 on \mathbb{R}^n , this implies that the function $x \mapsto \rho(x, t)$ is Lipschitz on the closure $\mathcal{B}_{R_2}(0)$ uniformly in t in $(-T, T)$.

Now we observe that we have

$$\begin{aligned} \overline{\varphi(x, t)} &= \frac{\partial \varphi}{\partial t}(x, t) = \left(\frac{\partial \psi}{\partial x}(0) \right)^{-1} \rho(x, t) \\ &= \left(\frac{\partial \psi}{\partial x}(0) \right)^{-1} \rho(\varphi_t^{-1}(\varphi(x, t)), t) \end{aligned}$$

This motivates us for considering the time varying system

$$\dot{z} = \zeta(z, t) = \left(\frac{\partial \psi}{\partial x}(0) \right)^{-1} \rho(\varphi_t^{-1}(z), t) \quad (4)$$

is well defined on the open set

$$\mathcal{O} = \left\{ (z, t) : t \in (-T, T), z \in \varphi_t(\mathcal{B}_{R_2}(0)) \right\}$$

where it is continuous and Lipschitz in z . It follows that $t \mapsto \varphi(x, t)$ is the unique solution on $(-T, T)$ of this system which goes through x at $t = 0$. With $Z(z, t)$ denoting a solution of (4), we have

$$Z(x, t) = \varphi(x, t) \quad \forall (x, t) \in \mathcal{B}_{R_2}(0) \times (-T, T) .$$

The above time varying system can be extended to $\mathbb{R}^n \times (-T, T)$ as

$$\dot{z} = \zeta_e(z, t) = \begin{cases} 0 & \text{if } z \notin \varphi_t(\mathcal{B}_{R_2}(0)) , \\ \chi(\varphi_t^{-1}(z)) \left(\frac{\partial \psi}{\partial x}(0) \right)^{-1} \rho(\varphi_t^{-1}(z), t) & \text{if } z \in \varphi_t(\mathcal{B}_{R_2}(0)) , \end{cases} \quad (5)$$

where $\chi : \mathbb{R}^n \rightarrow \mathbb{R}^+$ is a C^1 function satisfying

$$\chi(x) = \begin{cases} 0 & \text{if } x \notin \mathcal{B}_{R_2}(0) , \\ 1 & \text{if } x \in \mathcal{B}_{R_1}(0) . \end{cases}$$

This extended system is continuous on $\mathbb{R}^n \times (-T, T)$ and Lipschitz in z . So it has well defined and unique solutions on $(-T, T)$ for any initial condition z in \mathbb{R}^n . We denote by $Z_e(z, t)$ such a solution going through z at time $t = 0$. From this definition and since 1 is in $(0, T)$, the function $z \mapsto Z_e(z, 1)$ is a diffeomorphism. Also if, for some $\tau \geq 0$, a solution $Z(z, t)$ of (4) is in $\varphi_t(\mathcal{B}_{R_1}(0))$, for all t in $[0, \tau]$, then $t \mapsto Z_e(z, t)$ is solution of (5) at least on $[0, \tau]$. But by definition, when x is in \mathcal{B}_{R_1} , $\varphi(x, t)$ is in $\varphi_t(\mathcal{B}_{R_1}(0))$ for all t in $(-T, T)$. With uniqueness of solutions, this implies

$$Z(x, 1) = \varphi(x, 1) \quad \forall x \in \mathcal{B}_{R_1}(0) .$$

We are now ready to define the extension ψ_e as

$$\psi_e(x) = \frac{\partial \psi}{\partial x}(0) Z(x, 1) .$$

Since $\frac{\partial \psi}{\partial x}(0)$ is an invertible matrix, it is a diffeomorphism on \mathbb{R}^n . Moreover we have

$$\psi_e(x) = \frac{\partial \psi}{\partial x}(0) \varphi(x, 1) = \psi(x) \quad \forall x \in \mathcal{B}_{R_1}(0) .$$

□

Note that to realize the extended diffeomorphism we have to integrate the model (5) which depends on the inverse of the function φ_t . Hence, we come back to our preliminary problem.

Note however that for all x in \mathbb{R}^n and t in $[0, 1]$ such that $Z_e(x, t)$ is in $\varphi_t(\mathcal{B}_{R_2}(0))$, if we note $v = \varphi_t^{-1}(Z_e(x, t))$ we have,

$$\frac{\partial \varphi_t}{\partial t}(v) + \frac{\partial \varphi_t}{\partial x}(v) = \chi(v) \left(\frac{\partial \psi}{\partial x}(0) \right)^{-1} \rho(v, t)$$

where by definition

$$\frac{\partial \varphi_t}{\partial t}(v) = \frac{\partial \varphi}{\partial t}(v, t) = \left(\frac{\partial \psi}{\partial x}(0) \right)^{-1} \rho(v, t) .$$

Consequently, for all x in \mathbb{R}^n and t in $[0, 1]$ such that $Z_e(x, t)$ is in $\varphi_t(\mathcal{B}_{R_2}(0))$, if we note $v = \varphi_t^{-1}(Z_e(x, t))$ we have,

$$\dot{v} = (\chi(v) - 1) \left(\frac{\partial \varphi_t}{\partial x}(v) \right)^{-1} \frac{\partial \varphi_t}{\partial t}(v)$$

Note however that this is still difficult to implement. Indeed, in this case, $Z_e(x, t)$ is solution to this time dynamical system:

$$\begin{cases} \dot{z} = 0, & z \notin \varphi_t(\mathcal{B}_{R_2}) \\ z = \varphi_t(v), & \dot{v} = (\chi(v) - 1) \left(\frac{\partial \varphi_t}{\partial x}(v) \right)^{-1} \frac{\partial \varphi_t}{\partial t}(v), & z \in \varphi_t(\mathcal{B}_{R_2}) \end{cases}$$

However, if we have the property that, if $t \mapsto \varphi_t(\mathcal{B}_{R_2})$ is a strictly decreasing set valued map. Then this system can be implemented without inverting the function.

References

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