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# Several notions of rank-width for countable graphs

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**Abstract** : We define several notions of rank-width for countable graphs. We compare, for each of them the width of a countable graph with the least upper-bound of the widths of its finite induced subgraphs. A width has the *compactness property* if these two values are equal. Our notion of *rank-width* that uses *quasi-trees* (trees where paths may have the order type of rational numbers) has this property. So has *linear rank-width*, based on arbitrary linear orders. A more natural notion of rank-width based on countable cubic trees (we call it *discrete rank-width*) has a weaker type of compactness: the corresponding width is at most twice the least upper bound of the widths of the finite induced subgraphs. The notion of discrete linear rank-width, based on discrete linear orders has no compactness property.

**Keywords** : Rank-width; linear rank-width; compactness; Koenig's Lemma.

## 1 Introduction

We consider the class  $\mathcal{G}$  of finite or countable, loop-free, undirected graphs without parallel edges. A *width measure* is a mapping  $wd : \mathcal{G} \rightarrow \mathbb{N} \cup \{\omega\}$  such that  $wd(G) \in \mathbb{N}$  if  $G$  is finite and  $wd(H) \leq wd(G)$  if  $H$  is isomorphic to an induced subgraph  $H'$  of  $G$ , which we denote by  $H' \subseteq_i G$ . We say that  $wd$  has

the compactness property if for every  $G \in \mathcal{G}$ ,  $wd(G)$  is equal to  $\overline{wd}(G)$  defined as  $\text{Sup}\{wd(H) \mid H \subseteq_i G \text{ and } H \text{ is finite}\}$ . We say that it has the compactness property for a gap function  $f : \mathbb{N} \rightarrow \mathbb{N}$  if, for every  $G \in \mathcal{G}$ ,  $wd(G)$  is finite if and only if  $\overline{wd}(G)$  is finite and in that case,  $wd(G) \leq f(\overline{wd}(G))$ .

Tree-width, path-width and clique-width (their definitions for finite graphs are in [CouEng]) are width measures. The natural extension of tree-width to countable graphs has the compactness property [KriTho]. That of clique-width has the compactness property for some gap function ([Cou1], Section 7 of [Blu]). Path-width has not for any gap function (see the remark after Theorem 2).

*Rank-width* is a width measure on finite graphs investigated first in [Oum] and [OumSey]. Its variant called *linear rank-width* (similar to path-width and linear clique-width, see [CouEng, Heg+]) has been investigated in particular in [AdiKan], [Gan] and [JKO]. For countable graphs, we define two notions of linear rank-width (both equal to linear rank-width for finite graphs) and two notions of rank-width (both equal to rank-width for finite graphs). Roughly speaking, *discrete linear rank-width* is based on a linear order of the vertices isomorphic to  $(\mathbb{N}, \leq)$  whereas *linear rank-width* is based on an arbitrary linear order. *Discrete rank-width* is based on a ternary tree with vertices at leaves, whereas *rank-width* is based on a generalized tree such that the unique "path" between two nodes is isomorphic to a suborder of  $(\mathbb{Q}, \leq)$ . We compare these four notions and examine whether they satisfy some compactness property. Our results are summarized in the following table.

<i>measure</i>	<i>compactness</i>
rank-width	yes, by Theorem 22
discrete rank-width	yes with gap, by Theorem 8
linear rank-width	yes by Theorem 2
discrete linear rank-width	no by Theorem 2

This investigation is based on ideas used in [Cou1] for studying the clique-width of countable graphs and in [CouDel] for defining the modular decomposition of such graphs.

## 2 Definitions, notation and basic facts.

All ordered sets, graphs and trees are finite or countable, and "countable" means "countably infinite". We denote by  $\omega$  the first infinite ordinal and also the linear order  $(\mathbb{N}, \leq)$ . The finite or infinite cardinality of a set  $X$  is  $|X|$ .

Isomorphisms of graphs, trees and partial orders are denoted by  $\cong$ .

## 2.1 Ordered sets

If  $(V, \leq)$  is a partial order,  $<$  denotes the corresponding strict partial order, and for  $X, Y \subseteq V$ ,  $X < Y$  means that each element of  $X$  is strictly smaller than each element of  $Y$ . The least upper bound of  $x$  and  $y$  is denoted by  $x \sqcup y$  if it exists.

We compare partial orders by *inclusion* as follows:  $(V, \leq) \sqsubseteq (W, \leq')$  if  $V \subseteq W$  and  $\leq$  is  $\leq' \cap (W \times W)$ ; we say that  $(V, \leq)$  is a *suborder* of  $(W, \leq')$ .

The next definitions concern linear orders  $(V, \leq)$ . (The canonical reference is the book [Ros].) An *interval* is a subset  $Y$  that satisfies the following *convexity property*:

$$\text{if } x \leq y \leq z \text{ and } x, z \in Y, \text{ then } y \in Y.$$

Particular notations for intervals are  $[x, y]$  denoting  $\{z \mid x \leq z \leq y\}$ ,  $]x, y]$  denoting  $\{z \mid x < z \leq y\}$ ,  $] - \infty, x]$  denoting  $\{y \mid y \leq x\}$  (even if  $V$  is finite),  $]x, +\infty]$  denoting  $\{y \mid x < y\}$  etc. A linear order is *discrete* if  $[x, y]$  is finite for every  $x$  and  $y$ .

A *Dedekind cut* is a pair of nonempty intervals  $(X, X^c)$  such that  $X \subseteq V$  and  $X < X^c$  ( $X^c$ , the complement of  $X$ , is  $V - X$ : the reference set is clearly  $V$ ).

A set  $X \subseteq V$  is *dense on an interval*  $[u, v]$  such that  $v > u$ , if  $X \cap [u', v'] \neq \emptyset$  for every  $u', v'$  such that  $u \leq u' < v' \leq v$ . It is *dense* if it is dense on all intervals.

## 2.2 Matrices

A *matrix* (intended to be an adjacency matrix) is a mapping  $M : X \times Y \rightarrow \{0, 1\}$  where  $X$  and  $Y$  are finite or countable sets and  $0, 1$  are the two elements of the field  $GF(2)$ . If  $U \subseteq X$  and  $W \subseteq Y$ , we denote by  $M[U, W]$  the matrix that is the restriction of  $M$  to  $U \times W$ . We call it a *submatrix* of  $M$ . A submatrix  $M[\{x\}, Y]$  is a *row* of  $M$ , and  $M[X, \{y\}]$  is a *column*. The *transposed* of  $M$ , denoted by  ${}^tM$ , is the matrix  ${}^tM : Y \times X \rightarrow \{0, 1\}$  such that  ${}^tM(y, x) := M(x, y)$ . The *rank* of  $M$ , defined as the maximum cardinality of an independent set of rows (equivalently, of columns) is denoted by  $rk(M)$ ; it belongs to  $\mathbb{N} \cup \{\omega\}$ . The relevant field is  $GF(2)$ . It is convenient to take  $rk(M[\emptyset, W]) = rk(M[U, \emptyset]) = 0$ .

We will only use the following classical facts about ranks, where  $M : X \times Y \rightarrow \{0, 1\}$ :

- (1)  $rk({}^tM) = rk(M)$ ,
- (2)  $rk(M[U, W]) \leq rk(M)$ ,
- (3)  $rk(M[U \cup U', W]) \leq rk(M[U, W]) + rk(M[U', W])$ ,
- (4) and, if  $X \cup Y$  is infinite:  
 $rk(M) = \sup\{rk(M[U, W]) \mid U \subseteq X, W \subseteq Y, U \text{ and } W \text{ are finite}\}$ .
- (5) If  $X$  or  $Y$  is finite, then  $rk(M) \leq \min\{|X|, |Y|\}$ .

From (1)-(3), we get :

(6) If  $M$  is a matrix :  $X \times X \rightarrow \{0, 1\}$  and  $(A, B, C)$  is a partition of  $X$  into three sets, we have:

$$rk(M[B, A \cup C]) \leq rk(M[A, B \cup C]) + rk(M[A \cup B, C]).$$

Here is another convenient way to express this fact :

(6') If  $A \subset D \subseteq X$  then :

$$rk(M[D - A, X - (D - A)]) \leq rk(M[A, X - A]) + rk(M[D, X - D]).$$

## 2.3 Graphs and trees

*Graphs.*

They will be undirected, without loops and parallel edges. The adjacency matrix of a graph  $G$  with vertex set  $V_G$  is  $M_G : V_G \times V_G \rightarrow \{0, 1\}$  such that  $M_G[u, v] = 1$  if and only if  $u$  and  $v$  are adjacent, which we denote by  $u -_G v$ .

The notations  $G \subseteq H$  (resp.  $G \subseteq_i H$ ) indicate that  $G$  is a subgraph (resp. an induced subgraph) of  $H$ . The induced subgraph of  $G$  with vertex set  $X$  is denoted by  $G[X]$ . Its adjacency matrix is a submatrix of  $M_G$ . We let  $G - X := G[V_G - X]$ .

A *width measure* on a class  $\mathcal{C}$  of graphs closed under taking induced subgraphs is a mapping  $wd : \mathcal{C} \rightarrow \mathbb{N} \cup \{\omega\}$  that has same value on two isomorphic graphs and is such that  $wd(G) \in \mathbb{N}$  if  $G$  is finite and  $wd(H) \leq wd(G)$  if  $H \subseteq_i G$ . Two width measures on  $\mathcal{C}$ ,  $wd$  and  $wd'$ , are *equivalent* if, for every  $G \in \mathcal{C}$ ,  $wd(G)$  and  $wd'(G)$  are both infinite or both finite, and in the latter case, we have  $wd(G) \leq f(wd'(G))$  and  $wd'(G) \leq g(wd(G))$  for some functions  $f$  and  $g : \mathbb{N} \rightarrow \mathbb{N}$ .

*Trees*

We will use undirected trees (without root). They are graphs, but we will call *nodes* their vertices for clarity because we will consider simultaneously graphs and trees representing them. The set of nodes of  $T$  is  $N_T$ . A *leaf* is a node of degree 1:  $L_T$  denotes the set of leaves of a tree  $T$ . Any two nodes  $x, y$  are linked by a unique finite path  $P_{T,x,y}$  whose set of nodes is denoted by  $[x, y]_T$ .

Let  $x$  and  $y$  be adjacent nodes. Then  $N_{T,x,y}$  is the set of nodes of the connected component of  $T - \{x\}$  that contains  $y$ , and  $L_{T,x,y} := N_{T,x,y} \cap L_T$ . (We write  $N_{x,y}$  and  $L_{x,y}$  if  $T$  is clear from the context). A tree is *leafy* if each set  $L_{T,x,y}$  is not empty.  $\mathcal{L}(T)$  denotes the set of sets  $L_{T,x,y}$ . A *cut in  $T$*  is an (unordered) pair  $\{N_{T,x,y}, N_{T,y,x}\}$  or  $\{\emptyset, N_T\}$ .

Let  $T$  and  $T'$  be two trees. We say that  $T'$  *is included in  $T$* , which we denote by  $T' \sqsubseteq T$ , if  $N_{T'} \subseteq N_T$  and, for every two edges  $x - y$  and  $u - v$  of  $T'$ , if  $w$

is a node of  $T$  common to  $P_{T,x,y}$  and  $P_{T,u,v}$ , then  $w \in \{x,y\} \cap \{u,v\}$ . In other words,  $T'$  is a *topological minor* of  $T$  (see [Die]) hence is obtained, for some  $X$ , from  $T - X$  by smoothing all its degree 2 nodes. Smoothing a degree 2 node  $x$  means contracting one of its two incident edges so that  $x$  disappears.

A tree is *cubic* (resp. *subcubic*) if its nodes have degree 1 or 3 (resp. degree at most 3).

From a finite subcubic tree  $T$ , we obtain a finite cubic tree  $Red(T)$  such that  $\mathcal{L}(Red(T)) = \mathcal{L}(T)$  by smoothing its degree 2 nodes: this means that we replace by an edge between  $x$  and  $y$  an induced path linking two nodes  $x$  and  $y$  that do not have degree 2.

From a countable subcubic tree  $T$ , we obtain a leafy cubic tree  $Red(T)$  such that  $\mathcal{L}(Red(T)) = \mathcal{L}(T)$  as follows:

- (1) we delete the nodes of  $N_{T,x,y}$  whenever  $L_{T,x,y} = \emptyset$  which produces a leafy subcubic tree  $T'$  such that  $\mathcal{L}(T') = \mathcal{L}(T) - \{\emptyset\}$ ;
- (2) then, we smooth the degree 2 nodes as for finite trees, and we obtain the desired leafy cubic tree  $Red(T)$ .

Note that  $Red(T)$  is finite if  $T$  is countable but has finitely many leaves. It can be empty (e.g., if  $T$  is a biinfinite path).

#### *Rooted trees.*

We will also use rooted trees, directed in such a way that every node is accessible from the root by a directed path. If  $T$  is such a tree,  $\leq_T$  is the partial order on  $N_T$  such that  $x \leq_T y$  if and only if  $y$  is on the path from the root to  $x$ .

Let  $A$  be a finite alphabet of the form  $\{0, \dots, d\}$  for some nonnegative integer  $d$ . A language  $L \subseteq A^*$  is *prefix-free* if there are no two words  $u$  and  $uv$  in  $L$  with  $v \neq \varepsilon$  ( $\varepsilon$  is the empty word). Then  $Pref(L)$  denotes the set of all prefixes of  $L$  (we have  $L \subseteq Pref(L)$ ). The directed graph  $T(L) := (Pref(L), \rightarrow)$  such that  $u \rightarrow ua$  if  $u \in A^*$ ,  $a \in A$  and  $ua \in Pref(L)$  is a directed tree with root  $\varepsilon$ . Its set of leaves is  $L$ . We call *degree* of  $T(L)$  the maximal outdegree of a node; it is at most  $d + 1$ . Furthermore,  $Pref(L)$  is linearly ordered by  $\leq_{lex}$ . The partial order  $\leq_{T(L)}$  on  $Pref(L)$  is the reverse of  $\leq_{pref}$ , the prefix order. The linearly ordered set  $(L, \leq_{lex})$  is called the *frontier* of  $T(L)$ . If  $w \in A^*$  the set  $wA^* \cap L$  is an interval of  $(L, \leq_{lex})$ . Every nonempty linear order  $(V, \leq)$  is isomorphic to  $(L, \leq_{lex})$  for some *maximal prefix-free* language on  $\{0, 1\}$ . Note that  $L$  is maximal (for inclusion) if and only if, for all words  $u, v$ ,  $u0v \in L$  implies  $u1w \in L$  for some  $w$ , and vice-versa; in this case, every node of  $T(L)$  that is not a leaf has two sons.

## 2.4 Rank-width

In this section, we only consider finite graphs. We first define *linear rank-width*. Let  $G$  be a finite graph and  $\leq$  a linear order on  $V_G$ . Its cuts are of the form

$(] - \infty, x], [y, +\infty[)$  where  $y$  is the successor of  $x$  in  $\leq$ . For every cut  $(X, X^c)$  of  $\leq$ , we define  $rw d(G, \leq, X) := rk(M_G[X, X^c])$ . We define  $rw d(G, \leq)$  as the least upper bound of the values  $rw d(G, \leq, X)$  over all cuts  $(X, X^c)$ . The linear rank-width of  $G$ , denoted by  $lrwd(G)$  is the smallest value of  $rw d(G, \leq)$  over all linear orders  $\leq$  on  $V_G$ .

We now define *rank-width*. A *layout* of a finite graph  $G$  is a finite subcubic tree  $T$  such that  $L_T = V_G$ . For adjacent nodes  $x, y$  of  $T$ , we have  $L_{y,x} = V_G - L_{x,y}$ . We define the *rank-width of  $G$  relative to  $T$*  as:

$$rw d(G, T) := \max\{rk(M_G[L_{x,y}, L_{y,x}]) \mid x, y \text{ are adjacent nodes}\}.$$

The *rank-width* of  $G$  is  $rw d(G) := \min\{rw d(G, T) \mid T \text{ is a layout of } G\}$ . As  $rw d(G, Red(T)) = rw d(G, T)$ , the same value is obtained if we take the minimum over cubic layouts. Its linear rank-width could be defined equivalently as the smallest relative rank-width  $rw d(G, T)$  such that  $T$  is a layout of  $G$  that is a *comb*, i.e., the syntactic tree of a term of the form  $x_1 * (x_2 * (x_3 * (\dots * x_n) \dots))$  where  $*$  is a binary function symbol and  $x_1, x_2, x_3, \dots, x_n$  is an enumeration of  $V_G$ .

Cographs and trees have rank-width 1 and unbounded linear rank-width [AdlKan].

### 3 Linear rank-width of countable graphs

**Definition 1:** *Three notions of linear rank-width.*

Let  $G$  be a graph and  $\leq$  a linear order on  $V_G$ . For every cut  $(X, X^c)$  of  $\leq$ , we define  $lrwd(G, \leq, X)$  as  $rk(M_G[X, X^c])$ . We define  $lrwd(G, \leq)$  as the least upper bound of the ranks  $lrwd(G, \leq, X)$  over all cuts  $(X, X^c)$ .

As we defined  $rk(M[\emptyset, Y]) = 0$ , we can allow cuts  $(X, X^c)$  with one empty component. This does not change the notion of linear rank-width but simplifies writings. Note that if  $(V_G, \leq)$  is isomorphic to  $\omega$ , then  $lrwd(G, \leq, X)$  is finite for each  $X$ , and so, if  $lrwd(G, \leq) = \omega$ , the least upperbound of the numbers  $lrwd(G, \leq, X)$  is not reached for any cut  $X$ .

We define  $lrwd(G)$ , the *linear rank-width* of  $G$ , as the smallest value of  $lrwd(G, \leq)$  over all linear orders  $\leq$  on  $V_G$ , and  $dldrwd(G)$ , the *discrete linear rank-width* of  $G$  as the smallest value of  $lrwd(G, \leq)$  over all linear orders  $\leq$  on  $V_G$  that are finite or isomorphic to  $\omega$ . A third notion is  $dldrwd_{\mathbb{Z}}(G)$ , the smallest value of  $lrwd(G, \leq)$  for all *discrete* linear orders  $\leq$  on  $V_G$ , i.e., those that are isomorphic to a suborder of  $(\mathbb{Z}, \leq)$ . It is clear that

$$lrwd(G) \leq dldrwd_{\mathbb{Z}}(G) \leq dldrwd(G).$$

The verification that these three values are width measures is routine. They are equal on finite graphs. We will prove below (Proposition 3) that  $dlrwd$  and  $dlrwd_{\mathbb{Z}}$  are equivalent and we will not consider  $dlrwd_{\mathbb{Z}}$  any more.

**Theorem 2 :** (1) Linear rank-width has the compactness property.

(2) There is a graph  $G$  such that  $dlrwd(G) = \omega$  and  $\overline{lrwd}(G) = 1$ . Hence, discrete linear rank-width has not the compactness property, even with respect to a gap function.

**Proof:** (1) Let  $G$  be countable. Clearly,  $\overline{lrwd}(G) \leq lrwd(G)$  since  $lrwd$  is a width measure.

For proving the other inequality, we assume that  $\overline{lrwd}(G) = k < \omega$ . Let  $G_1 \subseteq_i G_2 \subseteq_i \dots \subseteq_i G_n \subseteq_i \dots$  be an increasing sequence of finite induced subgraphs of  $G$  whose union is  $G$ . For each  $i$ , the finite set  $\mathcal{L}_i$  of linear orders on  $V_{G_i}$  witnessing that  $lrwd(G_i) \leq k$  is not empty. Furthermore, the restriction to  $V_{G_i}$  of an order in  $\mathcal{L}_{i+1}$  is in  $\mathcal{L}_i$ . By Koenig's Lemma, there is an increasing sequence  $(\leq_i)_{i \geq 1}$  of orders in  $\mathcal{L}_i$  whose union is a linear order  $\leq$  on  $V_G$ . (This order need not be discrete.)

Assume for getting a contradiction that  $lrwd(G, \leq) > k$ . There exists a cut  $(X, X^c)$  such that  $rk(M_G[X, X^c]) \geq k + 1$  hence (cf. Section 2.2) finite subsets  $Y \subseteq X, Z \subseteq X^c$  such that  $rk(M_G[Y, Z]) \geq k + 1$ . There exists  $i$  such that  $V_{G_i}$  contains  $Y \cup Z$ . Hence, the cut  $(X', V_{G_i} - X')$  of  $G_i$  such that  $X' := \{x \in V_{G_i} \mid x \leq_i y \text{ for some } y \in Y\}$  shows that  $lrwd(G_i, \leq_i) \geq k + 1$ . Hence,  $\leq_i$  does not witness that  $lrwd(G_i) \leq k$ . Contradiction. It follows that  $lrwd(G, \leq) \leq k$ , hence  $lrwd(G) \leq k = \overline{lrwd}(G)$ .

(2) We let  $P$  be a countable path with one vertex of degree 1 and all others of degree 2. It is isomorphic to  $(\mathbb{N}, suc)$  where  $suc$  is the successor function. Let  $G$  be the union of  $\omega$  pairwise disjoint copies of  $P$ , each denoted by  $P_n$ , for  $n \in \mathbb{N}$ , i.e.,  $G = P_0 \oplus P_1 \oplus \dots \oplus P_n \oplus \dots$

We have  $lrwd(P_n) = 1$  for each  $n$ , with corresponding linear orders isomorphic to  $\omega$ . It follows easily that  $G$  has linear rank-width 1 with linear order  $\omega + \omega + \dots + \omega + \dots$ . We now prove that if  $\leq$  is a linear order of  $V_G$  isomorphic to  $\omega$ , then  $lrwd(G, \leq) = \omega$ . Let  $\leq$  be such an order. The vertices of  $P_n$  are  $v_{n,i}$  for  $i \geq 0$ , with  $v_{n,i}$  adjacent to  $v_{n,i+1}$ . Let  $r_{n,i}$  be the rank of  $v_{n,i}$  with respect to  $\leq$  (ranks start at 1, not at 0). We have  $r_{n,i} \neq r_{n',i'}$  if  $(n, i) \neq (n', i')$ .

Let  $k$  be any positive integer and  $p := \max\{r_{0,0}, \dots, r_{k,0}\}$ . For each  $i = 0, \dots, k$ , let  $m_i$  be such that  $r_{i,m_i} \geq p + 1$ . We let  $(X, X^c)$  be the cut of  $\leq$  such that  $X$  is the set of vertices of rank at most  $p$ . Hence,  $X$  contains  $v_{0,0}, \dots, v_{k,0}$  and  $X^c$  contains  $v_{0,m_0}, \dots, v_{k,m_k}$ . As there are in  $G$  paths between  $v_{i,0}$  and  $v_{i,m_i}$  for each  $i = 0, \dots, k$  that are vertex disjoint, there are  $k+1$  pairwise disjoint edges between  $X$  and its complement. Hence,  $lrwd(G, \leq, X) \geq k$  and so,  $lrwd(G, \leq) \geq \omega$ . It follows that  $dlrwd(G) = \omega$ .  $\square$

*Remark :* For defining the path-width of a countable graph one can take as path of a path-decomposition either  $\mathbb{N}$  or  $\mathbb{Z}$  with successor relation as adjacency.



(The other conditions are naturally as in the case of finite graphs). The same graphs have finite path-width, as for rank-width (cf. the next proposition). The previous proof can be adapted to show that  $P_0 \oplus P_1 \oplus \dots \oplus P_n \oplus \dots$  has infinite path-width whereas its finite subgraphs have path-width 1. Hence, path-width has not the compactness property for any gap function.

**Proposition 3** : For every countable graph  $G$  such that  $dlrwd_{\mathbb{Z}}(G)$  or  $dlrwd(G)$  is finite, we have:

$$dlrwd_{\mathbb{Z}}(G) \leq dlrwd(G) \leq 2 \cdot dlrwd_{\mathbb{Z}}(G).$$

**Proof** : The first inequality is clear. For proving the other, we assume that  $dlrwd_{\mathbb{Z}}(G) = k$ , with linear order  $\leq$  showing that  $dlrwd_{\mathbb{Z}}(G) \leq k$ . If  $(V_G, \leq)$  is isomorphic to  $\omega$  or  $\omega^{-1}$ , then this order shows that  $dlrwd(G) = k$ . Assume now that  $(V_G, \leq)$  is isomorphic to  $\mathbb{Z}$ , so that the vertices of  $G$  can be denoted by  $v_i$  for  $i \in \mathbb{Z}$ . Let us now consider the linear order  $\leq'$  such that  $v_0 <' v_{-1} <' v_1 <' v_{-2} <' v_2 <' \dots <' v_{-n} <' v_n <' \dots$  so that  $(V_G, \leq')$  is isomorphic to  $\omega$ . We claim that  $lrwd(G, \leq') \leq 2k$ .

Consider a cut  $(X, X^c)$  of  $\leq'$  such that  $X := \{v_0, v_{-1}, v_1, v_{-2}, v_2, \dots, v_{-n}, v_n\}$ . Let  $Y := \{v_{n+1}, v_{n+2}, \dots\}$  and  $Z := \{v_{-(n+1)}, v_{-(n+2)}, \dots\} = X^c - Y$ . We have

$$M_G[X, X^c] = M_G[X, Y \cup Z]$$

and  $rk(M_G[X, Y \cup Z]) \leq rk(M_G[X, Y]) + rk(M_G[X, Z])$  (cf. Section 2.2).

The pair  $(X \cup Z, Y)$  is a cut for  $\leq$ . Hence,  $rk(M_G[X \cup Z, Y]) \leq k$ . We have  $rk(M_G[X, Y]) \leq rk(M_G[X \cup Z, Y]) \leq k$  and similarly,  $(Z, X \cup Y)$  is a cut for  $\leq$  and  $rk(M_G[Z, X]) = rk(M_G[X, Z]) \leq k$ . Hence  $rk(M_G[X, X^c]) \leq 2k$ . The proof is similar if  $X = \{v_0, v_{-1}, v_1, v_{-2}, v_2, \dots, v_{n-1}, v_{-n}\}$ . Hence,  $dlrwd(G) \leq 2k$ .  $\square$

**Definition 4** : *Sufficient set of cuts.*

Let  $(V, \leq)$  be a linear order. A set of cuts  $\mathcal{C}$  of this order is *sufficient* if for every two finite subsets  $Y, Z$  of  $V$  such that  $Y < Z$  (this notation means that  $y < z$  for every  $y$  in  $Y$  and  $z$  in  $Z$ ) there is in  $\mathcal{C}$  a cut  $(X, X^c)$  such that  $Y \subseteq X$  and  $Z \subseteq X^c$ .

If  $x$  has successor  $y$  (i.e., if  $[x, y] = \{x, y\}$ ), then every sufficient set of cuts must contain  $(] - \infty, x], [y, +\infty[)$ . Hence, if  $\leq$  is discrete, the set of all cuts is the only sufficient set.

If  $V$  is countable, then the set of all cuts may be uncountable but the countable set  $\{ (] - \infty, x], [x, +\infty[) \mid x \in V \}$  is sufficient. There may exist other sufficient sets. As a second example, consider  $(\mathbb{Q}, \leq)$  and a dense subset  $D$  of  $\mathbb{Q}$ , e.g., the set of numbers of the form  $\pm m/10^{p^2}$  for  $m, p \in \mathbb{N}$ . Then, the set  $\{ (] - \infty, x], [x, +\infty[) \mid x \in D \}$  is sufficient and countable.

Finally, consider  $(L, \leq_{lex})$  where  $L$  is a prefix-free language on  $\{0, 1\}$ . The set of pairs  $(X, Y)$  such that, for some proper prefix  $u$  of a word of  $L$  we take  $X = L \cap u0\{0, 1\}^*$  and  $Y = L \cap u1\{0, 1\}^*$  is a countable and sufficient set of cuts.

**Proposition 5:** Let  $G$  be a graph,  $\leq$  a linear order on  $V_G$  and  $\mathcal{C}$  a sufficient set of cuts. We have

$$lrwd(G, \leq) = \max\{rk(M_G[X, X^c]) \mid (X, X^c) \in \mathcal{C}\}.$$

The proof is straightforward by inspection of the proof of the first part of Theorem 2. This proposition shows that the linear rank-width of a graph can be defined from countable sufficient sets of cuts.

## 4 Rank-width of countable graphs

Discrete rank-width is the natural generalization of rank-width to countable graphs.

### 4.1 Discrete rank-width

**Definition 6 :** *Discrete rank-width*

A *discrete layout* of a countable graph  $G$  is a subcubic tree  $T$  whose set of leaves contains  $V_G$ . The sets  $L_{T,x,y}$  are defined as for finite graphs, but they can be empty. The *discrete rank-width* of  $G$  is defined as for finite graphs: if  $T$  is a discrete layout, we define :

$$rwd(G, T) := \max\{rk(M_G[L_{T,x,y}, L_{T,y,x}]) \mid x, y \text{ are adjacent nodes}\}.$$

The *discrete rank-width* of  $G$  is  $drwd(G) := \min\{rwd(G, T) \mid T \text{ is a discrete layout of } G\}$ .

As technical variant, we can define a discrete layout of  $G$  as a pair  $(T, \varphi)$  where  $\varphi$  is an injection :  $V_G \rightarrow L_T$  and  $T$  is subcubic. The associated width is then defined by replacing  $L_{T,u,v}$  by  $\varphi^{-1}(L_{T,u,v})$  for all  $u, v$ . The associated notion of discrete rank-width is clearly the same. This variant will make some proofs easier to write.

**Proposition 7 :** In the definition of discrete rank-width, the minimum can be taken over all leafy cubic layouts such that  $V_G = L_T$ .

**Proof sketch :** Let  $T$  be a discrete layout such that  $V_G \subseteq L_T$ . Similarly as in Section 2.3, we remove the nodes of the sets  $N_{T,x,y}$  such that  $N_{T,x,y} \cap V_G = \emptyset$  that we call the *useless nodes*. We obtain a leafy subcubic tree  $T'$  such that  $V_G = L_{T'}$ . By smoothing its degree 2 nodes, we get a leafy cubic tree  $T''$  that is a layout of  $G$  such that  $rwd(G, T'') = rwd(G, T)$ , which proves the result.  $\square$

**Theorem 8** : Discrete rank-width does not have the compactness property. It has this property with gap function  $\lambda n.2n$ .

**Proof:** We let  $\{A, A^c\}$  be a partition of  $\mathbb{Q}$  into two dense subsets. We define  $G$  to have vertex set  $\mathbb{Q}$  and an edge between  $p$  and  $q > p$  if and only if  $q \in A$ . Each finite induced subgraph of  $G$  is a threshold graph and has linear rank-width 1 (witnessed by the usual order on  $\mathbb{Q}$ ).

Let  $T$  be a discrete and cubic layout of  $G$  such that  $V_G = L_T$ . We claim that  $rdw(G, T) \geq 2$ .

To prove the claim we need only find two adjacent nodes  $x, y$  of  $T$  and vertices  $a, b, c, d$  of  $G$  such that there is a path  $a - b - c - d$ ,  $a$  is not adjacent to  $d$ ,  $\{a, c\} \subseteq L_{T,x,y}$  and  $\{b, d\} \subseteq L_{T,y,x}$  because then,  $rk(M_G[L_{T,x,y}, L_{T,y,x}]) \geq rk(M_G[\{a, c\}, \{b, d\}]) = 2$ .

**Claim** : If  $X_1, \dots, X_p$  are pairwise disjoint sets such that  $X_1 \cup \dots \cup X_p$  is dense on  $[u, v]$ , then some set  $X_i$  is dense on  $[u', v']$  for some  $u', v'$  such that  $u \leq u' < v' \leq v$ .

**Proof sketch:** We consider the case  $p = 2$ . If  $X_1$  is not dense on  $[u, v]$ , (cf. Section 2.1) then  $X_1 \cap [u', v'] = \emptyset$  for some  $u', v'$  such that  $u \leq u' < v' \leq v$ . Then  $X_1 \cup X_2$  is dense on  $[u', v']$ , hence  $X_2$  is dense on this interval. The general case follows by induction on  $p$ .  $\square$

We now look for a path  $a - b - c - d$  showing that  $rdw(G, T) \geq 2$ . We choose a leaf  $c$  of  $T$  belonging to  $A$  and another leaf  $a$  such that  $a < c$ . There is a path  $a - x_1 - \dots - x_p - c$  in  $T$ . Let  $y_1, \dots, y_p$  be the neighbours of  $x_1, \dots, x_p$  respectively that are not on this path.

*Case 1:* For some  $i$ ,  $A \cap L_{x_i, y_i}$  and  $A^c \cap L_{x_i, y_i}$  are dense on  $[a, c]$ . There exist  $b \in A \cap L_{x_i, y_i}$  and  $d \in A^c \cap L_{x_i, y_i}$  such that  $a < b < d < c$ . Between  $a, b, c, d$ , we have the edges  $a - b, b - c, d - c$  and  $a - c$ . Hence, we have a path  $a - b - c - d$  with no edge between  $a$  and  $d$ ,  $\{a, c\} \subseteq L_{y_i, x_i}$  and  $\{b, d\} \subseteq L_{x_i, y_i}$ . We are done.

*Case 2:* Case 1 does not hold. The set  $(A \cap L_{x_1, y_1}) \cup \dots \cup (A \cap L_{x_p, y_p})$  is dense on  $[a, c]$ . Hence, by the claim,  $A \cap L_{x_i, y_i}$  is dense on  $[a', c'] \subseteq [a, c]$  for some  $i$ . Now,  $(A^c \cap L_{x_1, y_1}) \cup \dots \cup (A^c \cap L_{x_p, y_p})$  is dense on  $[a', c']$ . Hence, again by the claim,  $A^c \cap L_{x_j, y_j}$  is dense on  $[a'', c''] \subseteq [a', c']$  for some  $j$ . If  $i = j$  we argue as in Case 1. Otherwise, assume  $i < j$  and then  $L_{x_j, y_j} \subset L_{y_i, x_i}$ . Hence  $A \cap L_{x_i, y_i}$  and  $A^c \cap L_{y_i, x_i}$  are dense on  $[a'', c'']$ . Hence, we can find  $u$  and  $w$  in  $A \cap L_{x_i, y_i} \cap [a'', c'']$  and  $v$  and  $z$  in  $A^c \cap L_{y_i, x_i} \cap [a'', c'']$  such that  $a'' < v < u < z < w < c''$ . This gives a path  $u - v - w - z$  showing that  $rk(M_G[L_{x_i, y_i}, L_{y_i, x_i}]) \geq rk(M_G[\{u, w\}, \{v, z\}]) = 2$ .

The discrete rank-width of  $G$  is thus at least 2.

Its linear rank-width is 1 and its discrete rank-width is 2, as a consequence of the second assertion that will be proved in Theorem 26, Section 4.3.  $\square$

As for discrete orders, a countable layout has countably many cuts, and all of them are necessary. The notion of a sufficient set of cuts is pointless.

## 4.2 Quasi-trees

The good notion of rank-width, where "good" means that it has the compactness property will be based on certain generalized trees, called *quasi-trees*. Roughly speaking, a quasi-tree is to a tree what a countable linear order is to a discrete one (a linear order isomorphic to a suborder of  $(\mathbb{Z}, \leq)$ ).

**Definition 9 :** We associate with each tree  $T$  its ternary *betweenness relation*,  $B_T \subseteq N_T \times N_T \times N_T$  defined as follows:

$B_T(x, y, z)$  holds if and only if  $x, y, z$  are pairwise different and  $y \in [x, z]_T$  (where  $[x, z]_T$  is the set of nodes of the unique path in  $T$  between  $x$  and  $z$ , cf. Section 2.3)

We let  $Q(T) := (N_T, B_T)$ .

**Proposition 10:** The betweenness relation  $B = B_T$  of a finite or countable tree  $T$  satisfies the following properties for all  $u, x, y, z$  in  $N_T$ :

A1 :  $B(x, y, z) \Rightarrow x \neq y \neq z \neq x$ .

A2 :  $B(x, y, z) \Rightarrow B(z, y, x)$ .

A3 :  $B(x, y, z) \Rightarrow \neg B(x, z, y)$ .

A4 :  $B(x, y, z) \wedge B(y, z, u) \Rightarrow B(x, y, u) \wedge B(x, z, u)$ .

A5 :  $B(x, y, z) \wedge B(x, u, y) \Rightarrow B(x, u, z) \wedge B(u, y, z)$ .

A6 :  $B(x, y, z) \wedge B(x, u, z) \Rightarrow$

$y = u \vee (B(x, u, y) \wedge B(u, y, z)) \vee (B(x, y, u) \wedge B(y, u, z))$ .

A7 :  $x \neq y \neq z \neq x \Rightarrow$

$B(x, y, z) \vee B(x, z, y) \vee B(y, x, z) \vee (\exists u. B(x, u, y) \wedge B(y, u, z) \wedge B(x, u, z))$ .

**Proof :** Straightforward verification from definitions.  $\square$

**Definitions 11 :** *Quasi-trees and related notions*

(a) A *quasi-tree* is a structure  $T = (N_T, B_T)$  such that  $N_T$ , called the set of *nodes*, has at least 3 elements, and  $B_T$  is a ternary relation that satisfies conditions A1-A7.

In A1, one could replace  $x \neq y \neq z \neq x$  by  $x \neq z$  because A2 and A3 imply  $x \neq y \neq z$ .

A quasi-tree satisfies the following properties :

(1) The four cases of the conclusion of A7 are exclusive. This is clear for the first three by A3. Let us now assume that we have  $B(x, y, z) \wedge [B(x, u, y) \wedge B(y, u, z) \wedge B(x, u, z)]$ . Then  $B(u, y, z)$  holds by A5 and we get a contradiction by A3 and A2<sup>1</sup>. Two other similar verifications yield the result.

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<sup>1</sup>In order to simplify the writing of similar proofs, we will not mention the use of A2.

(2) In the fourth case of the conclusion of A7, there is at most one  $u$  satisfying  $B(x, u, y) \wedge B(y, u, z) \wedge B(x, u, z)$ . To prove this, we assume that we also have  $B(x, v, y) \wedge B(y, v, z) \wedge B(x, v, z)$  with  $v \neq u$ . By A6, we have  $B(x, v, u) \wedge B(v, u, y)$  or  $B(x, u, v) \wedge B(u, v, y)$ . Assume the first. As  $B(x, u, y)$  holds, A5 yields  $B(v, u, y)$  and, similarly,  $B(v, u, z)$ . We have  $B(y, u, z)$  and  $B(y, v, z)$ , hence, by A6, we have  $B(y, u, v) \wedge B(u, v, z)$  or  $B(y, v, u) \wedge B(v, u, z)$ . A3 excludes each of these two cases because of  $B(v, u, y)$  and  $B(v, u, z)$ . Hence, we must have  $u = v$ .

The following definitions concern a quasi-tree  $T = (N, B)$ .

(b) If  $x, y, z$  are pairwise distinct, we denote by  $M(x, y, z)$  ( $M$  stands for "middle"; we write  $M_T$  if we need to specify  $T$ ), the node  $x$  (resp.  $y, z$ ) if  $B(y, x, z)$  (resp.  $B(x, y, z), B(x, z, y)$ ) holds, and, otherwise, the unique node  $u$  satisfying  $B(x, u, y) \wedge B(y, u, z) \wedge B(x, u, z)$ .

(c) A quasi-tree  $T' = (N', B')$  is a *sub-quasi-tree* of  $T$ , which we denote by  $T' \sqsubseteq T$ , if  $N' \subseteq N$  and  $B' = B \cap (N' \times N' \times N')$ . Then,  $M_{T'}$  is the restriction of  $M_T$  to  $N_{T'}$ .

Let  $X \subseteq N$ . Then  $(X, B \cap (X \times X \times X))$  is a sub-quasi-tree of  $T$  if and only if  $X$  is closed under  $M_T$ . In all cases, there is a least set  $\widehat{X}$  that contains  $X$  and is such that  $(\widehat{X}, B \cap (\widehat{X} \times \widehat{X} \times \widehat{X}))$  is a sub-quasi-tree of  $T$ : it is the least set  $Y$  such that  $Y \supseteq X \cup M_T(Y, Y, Y)$ .

If  $X$  is a *line*, i.e. a set  $X \subseteq N$  satisfying the following property that is stronger than A7:

A7' : If  $x, y, z$  are three elements of  $X$ , then we have :

$$B(x, y, z) \vee B(x, z, y) \vee B(y, x, z),$$

then  $\widehat{X} = X$ .

(d) A *leaf* of  $T$  is a node  $z$  such that  $B(x, z, y)$  holds for no  $x, y$ . A quasi-tree is *leafy* if every node  $z$  is a leaf or *is between* two leaves, i.e., satisfies  $B(x, z, y)$  for some leaves  $x$  and  $y$ . If  $x$  is a leaf of  $T$ , then  $T - \{x\}$  defined as  $(N', B \cap (N' \times N' \times N'))$  where  $N' := N - \{x\}$  is a quasi-tree.

(e) If  $S_1, \dots, S_n, \dots$  are quasi-trees forming an increasing sequence for inclusion, i.e., such that  $S_1 \sqsubseteq \dots \sqsubseteq S_n \sqsubseteq \dots$  then, their *union* is the quasi-tree  $T = (N, B)$  such that  $N := \bigcup_{n \geq 1} N_{S_n}$  and  $B := \bigcup_{n \geq 1} B_{S_n}$ . We have  $S_n \sqsubseteq T$  for each  $n$ . We denote  $T$  by  $\bigcup_{n \geq 1} S_n$ .

(f) We let  $G(T)$  be the graph with vertex set  $N$  and an undirected edge between  $x$  and  $y$  if  $B(x, y, z)$  holds. If  $T$  is a quasi-tree, this graph is connected

(with diameter 1 or 2) by A7. Any two leaves are at distance 2 but a quasi-tree may have no leaf.  $\square$

**Definitions 12 :** *Convexity and intervals.*

Let  $T = (N, B)$  be a quasi-tree.

(a) A subset  $X$  of  $N$  is *convex* (with respect to  $T$ ) if:

$$B(x, y, z) \wedge x \in X \wedge z \in X \implies y \in X.$$

(b) If  $x, y \in N$ , we denote by  $]x, y[_B$  the convex set  $\{z \in N \mid B(x, z, y) \text{ holds}\}$  and by  $[x, y]_B$ ,  $]x, y]_B$  and  $[x, y[_B$  the sets, respectively,  $]x, y[_B \cup \{x, y\}$ ,  $]x, y[_B \cup \{y\}$  and  $]x, y[_B \cup \{x\}$ . We call them *intervals*, which will be justified by Lemma 13(1) below.

(c) We say that  $T$  is *discrete* if each interval  $]x, y[_B$  is finite. If  $T = Q(T')$  for some tree  $T'$ , then the intervals  $]x, y[_B$  of  $T$  are the sets  $]x, y[_{T'}$ , and so,  $T$  is a discrete quasi-tree.

(d) The *convex hull* of a set  $X \subseteq N$ , denoted by  $\overline{X}$ , is the union of the intervals  $]x, y[_B$  for  $x, y \in X$ . We will prove that it is actually convex and is thus the least convex set containing  $X$ .  $\square$

**Lemma 13 :** Let  $T = (N, B)$  be a quasi-tree and  $r$  one of its nodes.

(1) The following binary relation is a partial order:

$$x \leq y :\Leftrightarrow x = y \vee y = r \vee B(x, y, r).$$

Its minimal elements are the leaves of  $T$  except  $r$  if it is a leaf, and  $r$  is its maximal element. Each interval  $]x, r[_B$  is linearly ordered.

(2) Every two nodes  $x$  and  $y$  have a least upperbound denoted by  $x \sqcup y$ . It is equal to  $M(x, y, r)$  if  $x, y, r$  are pairwise distinct. If  $x$  and  $y$  are incomparable, then  $B(x, x \sqcup y, y)$  holds. Hence, a convex set is closed under  $\sqcup$ .

(3) For any three distinct nodes  $x, y, w$ , we have:

$$B(x, w, y) \iff (x < w \leq x \sqcup y) \vee (y < w \leq x \sqcup y),$$

$$\text{which is equivalent to } ]x, y[_B = ]x, x \sqcup y[_B \cup ]x \sqcup y, y[_B.$$

(4) If  $T$  is discrete, then  $]x, r[_B$  is of the form  $\{z_0, \dots, z_n\}$  with  $x = z_0 < z_1 < \dots < z_n = r$ . If  $i < j$  and  $1 \leq k \leq n - 1$ , we have  $B(z_i, z_k, z_j)$  if and only if  $i < k < j$ .

**Proof:** The verification is easy for (1). That of (2) is not hard either, but we give it. Let  $x, y$  be incomparable.

*Case 1 :*  $B(x, r, y)$  holds. We have  $x < r$  and  $y < r$ . Let  $u$  be such that  $x \leq u$  and  $y \leq u$ . We want to prove that  $u = r$ . We can have neither  $x = u$  nor  $y = u$

because  $x, y$  are incomparable. Consider the case where  $B(x, u, r) \wedge B(y, u, r)$  holds. By A5, we have  $B(u, r, y)$ , hence  $B(u, r, y) \wedge B(y, u, r)$  and a contradiction with A3. Hence,  $u = r$ .

*Case 2* :  $B(x, r, y)$  does not hold and  $z = M(x, y, r)$  is the unique node (cf. A7 and the unicity property of Definition 11(a)) such that  $B(x, z, r) \wedge B(y, z, r) \wedge B(x, z, y)$ . If  $x \leq u$  and  $y \leq u$ , we want to prove that  $z \leq u$ . We have neither  $x = u$  nor  $y = u$ . If  $u = r$ , then  $z \leq u$ . Otherwise, we have  $B(x, u, r) \wedge B(y, u, r)$ . If  $z = u$  or  $B(z, u, r)$ , we are done. Otherwise, by A6, we have  $B(u, z, r) \wedge B(x, u, z) \wedge B(y, u, z)$ . But with  $B(x, z, y)$  and by A5, we get  $B(u, z, y)$  and a contradiction with A3. This case cannot hold and so, we get  $z \leq u$ . Hence,  $z$  is the least upper bound of  $x$  and  $y$ .

(3)  $\Leftarrow$ : Assume  $x < w \leq x \sqcup y$ . If  $w = x \sqcup y$ , then  $B(x, w, y)$  holds by (2). Otherwise,  $x < w < x \sqcup y$ . Hence, we have  $B(x, w, r)$  and  $B(w, x \sqcup y, r)$ . We have thus  $B(x, w, x \sqcup y)$  by A5. Since we also have  $B(x, x \sqcup y, y)$ , we have  $B(x, w, y)$  again by A5. If  $y < w \leq x \sqcup y$  the proof is similar.

$\Rightarrow$ : Assume we have  $B(x, w, y)$ . Since we also have  $B(x, x \sqcup y, y)$ , then, by A6, we have  $w = x \sqcup y$  or  $B(x, w, x \sqcup y)$  or  $B(x \sqcup y, w, y)$ . Consider the case  $B(x, w, x \sqcup y)$ . We also have  $B(x, x \sqcup y, r)$ , hence,  $B(x, w, r)$  and  $B(w, x \sqcup y, r)$  by A5 and so,  $x < w < x \sqcup y$ . If  $B(x \sqcup y, w, y)$  holds, the proof is similar and the case  $w = x \sqcup y$  also gives the desired result.

(4) We prove the property for all pairs  $(x, r)$  by induction on the cardinality  $c(x, r)$  of  $[x, r]_B$  that is finite since  $T$  is discrete. Assuming  $x \neq r$ , we have  $c(x, r) \geq 2$ . If  $c(x, r) = 2$  then the result holds with  $n = 1$ .

Otherwise, there is  $u$  such that  $B(x, u, r)$  holds. By A1 and A5, we have  $[x, u]_B \subset [x, r]_B$  and  $[u, r]_B \subset [x, r]_B$ . Hence, by the induction hypothesis we have :  $x = z_0 < z_1 < \dots < z_m = u$ ,  $[x, u]_B = \{z_0, \dots, z_m\}$ ,  $u = z_m < z_{m+1} < \dots < z_n = r$  and  $[u, r]_B = \{z_m, \dots, z_n\}$  which gives the result.  $\square$

It follows from Assertions (1) and (2) that  $(N_T, \leq)$  is a *join-tree* in the sense of [CouDel] with maximal element  $r$  (the least upper bound of  $x$  and  $y$  is also called their *join*), and that for all  $x, y, z$ , the least upper bound  $(x \sqcup y) \sqcup z$  belongs to  $\{x \sqcup y, x \sqcup z, y \sqcup z\}$  and this set consists of at most 2 elements that are comparable.

**Corollary 14** : Let  $T = (N, B)$  be a quasi-tree and  $X \subseteq N$ . Let  $r \in X$ , from which we get  $\leq$  and  $\sqcup$ .

- (1) The set  $\overline{X}$  is convex.
- (2) The set  $X$  is closed under  $\sqcup$  if and only if it is closed under  $M$ .
- (3) If  $X$  is finite, the set  $\widehat{X}$  is finite.

The mapping  $M$  is defined in Definition 11(a) and the sets  $\widehat{X}$  and  $\overline{X}$  are in Definitions 11(d) and 12(d).

**Proof:** (1) Let us choose a node  $r \in X$ . Let  $Y$  be the union of the intervals  $[x, r]_B$  for all  $x \in X$ . By (3) of Lemma 13, we have:

$$[x, y]_B = [x, x \sqcup y]_B \cup [x \sqcup y, y]_B \subseteq [x, r]_B \cup [y, r]_B.$$

Hence,  $[x, y]_B \subseteq Y$  if  $x, y \in X$ , and so,  $Y = \overline{X}$ . We get also, by the same observation, that  $Y$  is convex. As every convex set containing  $X$  must contain  $\overline{X}$ , the latter set is the least one containing  $X$  and can be called its convex hull, as we anticipated in Definition 12.

(2) If  $X$  is closed under  $M$ , then, it is closed under  $\sqcup$  by Lemma 13(2). Assume conversely that  $X$  is closed under  $\sqcup$ . Let  $x, y, z \in X$  be pairwise distinct. The only case to consider is when  $u := M(x, y, z)$  satisfies  $B(x, u, y) \wedge B(y, u, z) \wedge B(x, u, z)$ , and we aim to prove that  $u \in A := \{x \sqcup y, x \sqcup z, y \sqcup z\}$ . The set  $A$  consists of at most two comparable elements. Assume that  $x \sqcup y \leq x \sqcup z$ . By the first three assertions of Lemma 13, we get  $u = x \sqcup y$ . We omit the routine proof. The proofs for the two other cases are similar.

(3) For every set  $Y \subseteq N$ , we let  $m(Y) := \{x \sqcup y \mid x, y \in Y\}$ . This set contains  $Y$  as we can have  $x = y$ . We have observed that for all  $x, y, z$ , we have  $(x \sqcup y) \sqcup z \in A := \{x \sqcup y, x \sqcup z, y \sqcup z\}$ , hence  $m(m(Y)) = m(Y)$ . It follows that  $m(X)$  is the least set containing  $X$  that is closed under  $\sqcup$ . By (2),  $\widehat{X} = m(X)$ . If  $y \notin m(X)$ , then  $m(m(X) \cup \{y\})$  consists of  $m(X) \cup \{y\}$  and at most one additional element (a consequence of Lemma 13(1)). Hence, if  $X$  is finite with at least 3 nodes,  $|m(X)| \leq 2 \cdot |X| - 2$ .  $\square$

**Lemma 15** : Let  $S = (N, B)$  be a quasi-tree. For pairwise distinct  $x, y, z$ , we have :

$$[x, y]_B \cap [y, z]_B = \{y\} \text{ if and only if } [x, z]_B = [x, y]_B \cup [y, z]_B.$$

**Proof:** Let  $x, y, z$  be distinct and  $\leq$  be the partial order of Lemma 13 with maximal element  $y$ . Hence  $x \sqcup z \leq y$ . Then by this lemma we have:

$$\begin{aligned} [x, z]_B &= [x, x \sqcup z]_B \cup [x \sqcup z, z]_B, \\ [x, y]_B &= [x, x \sqcup z]_B \cup [x \sqcup z, y]_B, \\ [z, y]_B &= [z, x \sqcup z]_B \cup [x \sqcup z, y]_B \text{ and} \\ \{x \sqcup z\} &= [x, x \sqcup z]_B \cap [x \sqcup z, z]_B = \\ &= [x, x \sqcup z]_B \cap [x \sqcup z, y]_B = [z, x \sqcup z]_B \cap [x \sqcup z, y]_B. \end{aligned}$$

It follows that

$$\begin{aligned} [x, y]_B \cap [y, z]_B &= \{y\} \text{ if and only if } y = x \sqcup z \\ &\text{if and only if } [x, z]_B = [x, y]_B \cup [y, z]_B. \square \end{aligned}$$



**Proposition 16** : (1) Every discrete quasi-tree  $S = (N, B)$  is  $Q(T)$  for the tree  $T$  with set of nodes  $N$  and adjacency relation  $x -_T y$  defined by  $[x, y]_B = \{x, y\}$ . The leaves of  $T$  are those of  $S$ .

(2) Let  $T$  be a tree and  $S = (N, B)$  a quasi-tree such that  $N_T \subseteq N$ . Then  $Q(T) \sqsubseteq S$  if and only if for every edge of  $T$  with ends  $x$  and  $y$ , then  $[x, y]_B \cap N_T = \{x, y\}$ . It follows that for trees  $T$  and  $T'$  we have  $T \sqsubseteq T'$  if and only if  $Q(T) \sqsubseteq Q(T')$ .

**Proof:** (1) Let  $S$  be a discrete quasi-tree. By Lemma 13(4), each interval  $[x, y]_B$  is of the form  $\{x, z_1, \dots, z_{n-1}, y\}$  with  $x = z_0 < z_1 < \dots < z_{n-1} < y = z_n$  ( $<$  is as in Lemma 13(1) with maximal element  $y$ ) and  $[z_i, z_{i+1}]_B = \{z_i, z_{i+1}\}$  for each  $i$ , so that we have  $x -_T z_1 -_T \dots -_T z_{n-1} -_T y$ . Hence, the graph  $T$  with vertex set  $N$  and adjacency relation  $-_T$  as defined in the statement is connected.

Consider two adjacent edges :  $x -_T y -_T z, z \neq x$ . We have  $[x, y]_B \cap [y, z]_B = \{y\}$ . By Lemma 15, we have  $[x, z]_B = \{x, y, z\}$  hence  $B(x, y, z)$  holds.

Consider now a path  $z_1 -_T z_2 -_T \dots -_T z_n$  with  $n \geq 3$ . By the previous remark, we have  $B(z_1, z_2, z_3), B(z_2, z_3, z_4)$ , hence  $B(z_1, z_3, z_4)$  holds. Then we have  $B(z_3, z_4, z_5)$  and  $B(z_1, z_4, z_5)$ , and finally, by repeating the argument,  $B(z_1, z_{n-1}, z_n)$ . Hence,  $T$  has no edge between  $z_1$  and  $z_n$ . It has no cycle, hence it is a tree.

From this proof, one obtains that the leaves of  $T$  are those of  $S$ .

(2) It is clear that if  $Q(T) \sqsubseteq S$ , then  $[x, y]_B \cap N_T = \{x, y\}$  for every edge of  $T$  with ends  $x$  and  $y$ . If  $T \sqsubseteq T'$  for trees  $T$  and  $T'$  then  $Q(T) \sqsubseteq Q(T')$ .

The other proofs are routine.  $\square$

**Definition 17** : *Directions*

Let  $S = (N, B)$  be a quasi-tree and  $x$  a node of  $S$ .

(a) We say that  $y, z \in N - \{x\}$  are the *same direction relative to  $x$*  if, either  $y = z$  or  $B(y, z, x)$  or  $B(z, y, x)$  or  $B(y, u, x) \wedge B(z, u, x)$  holds for some node  $u$ . Equivalently,  $y \sqcup z < x$  where  $<$  is the strict partial order with maximal element  $x$  (cf. Lemma 13(1)). Hence, if  $B(y, x, z)$  holds,  $y$  and  $z$  are not in the same direction relative to  $x$  (by Fact (1) of Definition 11). This relation is an equivalence, denoted by  $y \sim_x z$ , and its classes are the *directions relative to  $x$* .

(b) The *degree* of  $x$  is the number of classes of  $\sim_x$ . A node has degree 1 if and only if it is a leaf. We say that  $S$  is *cubic* (resp. *subcubic*) if its nodes have degree 1 or 3 (resp. degree at most 3). If  $S = Q(T)$  for a tree  $T$ , then a direction around  $x$  is associated with a neighbour  $y$  of  $x$  and is  $N_{x,y}$ , the set of nodes of the connected component of  $T - \{x\}$  that contains  $y$ .

If  $S' \sqsubseteq S$ , then the degree of a node of  $S'$  is at most its degree in  $S$ . If  $S_1, \dots, S_n, \dots$  is an increasing sequence of quasi-trees and  $S = \bigcup_{n \geq 1} S_n$ , then the degree of a node of  $S$  is the least upperbound of its degrees in the quasi-trees  $S_n$  to which it belongs.

**Definition 18** : *Cuts in quasi-trees.*

A *cut* of a quasi-tree  $S = (N, B)$  is a partition  $\{X, X^c\}$  of  $N$  into two convex sets. We consider an empty set as convex and so  $\{\emptyset, N\}$  is a cut (cf. Definition 1 for a similar fact). The following facts are clear:

- (1) If  $S = Q(T)$  for a tree  $T$ , the cuts of  $S$  are those of  $T$ .
- (2) If  $\{X, X^c\}$  is a cut of  $S := \bigcup_{n \geq 1} S_n$ , then each pair  $\{X \cap N_{S_n}, N_{S_n} - X\}$  is a cut of  $S_n$ .
- (3) If  $S_1, \dots, S_n, \dots$  is an increasing sequence of quasi-trees and for each  $n$ ,  $\{X_n, Y_n\}$  is a cut of  $S_n$ ,  $X_n = X_{n+1} \cap N_{S_n}$  and  $Y_n = Y_{n+1} \cap N_{S_n}$ , then  $\{\bigcup_{n \geq 1} X_n, \bigcup_{n \geq 1} Y_n\}$  is a cut of  $\bigcup_{n \geq 1} S_n$ .

**Lemma 19 :** Let  $S$  and  $S'$  be quasi-trees such that  $S \sqsubseteq S'$ . The cuts of  $S$  are the pairs  $\{X' \cap N_S, Y' \cap N_S\}$  such that  $\{X', Y'\}$  is a cut of  $S'$ .

**Proof:** It is clear that  $\{X' \cap N_S, Y' \cap N_S\}$  is a cut of  $S$  if  $\{X', Y'\}$  is one of  $S'$ . Let conversely  $\{X, Y\}$  be a cut of  $S$ . Choose an element  $r$  of  $Y$  (if  $Y$  is empty, we take  $\{X', Y'\} = \{\emptyset, N_{S'}\}$ ). We will use the associated partial order  $\leq$  on  $N_{S'}$  with maximal element  $r$ . We first prove that  $X$  is closed under  $\sqcup$  (this least upperbound function is relative to  $\leq$  on  $N_{S'}$ ). Let  $x, y \in X$ ; then  $x \sqcup y = M_{S'}(x, y, r)$  by Lemma 13(2), hence is equal to  $M_S(x, y, r)$  and belongs to  $X$  as  $X$  is convex in  $S$  and  $B(x, x \sqcup y, y)$  holds.

We define  $X' := \{u \in N_{S'} \mid u \leq x \text{ for some } x \in X\}$  and  $Y' := N_{S'} - X'$ . We prove that  $X = X' \cap N_S$  so that  $Y = Y' \cap N_S$ .

Clearly,  $X \subseteq X' \cap N_S$ . For the other inclusion, let  $y \in X' \cap N_S$ . Then  $y \leq x$  for some  $x \in X$ . If  $y \notin X$ , then  $y \in Y$ , we have  $B(y, x, r)$  and  $Y$  is not convex. We now prove that  $X'$  is convex. Let  $x', y' \in X'$ . Then,  $x' \leq x \in X, y' \leq y \in X$ , hence  $x' \sqcup y' \leq x \sqcup y$ , but  $x \sqcup y \in X$  as observed above, hence, by Lemma 13(3), every element of  $N_{S'}$  between  $x'$  and  $y'$  is below  $x \sqcup y$  or equal to it, hence is in  $X'$ . To prove that  $Y = Y' \cap N_S$  is convex, we consider  $x, y \in Y' \cap N_S$ . If some  $z \in X'$  is between them, then we have  $x \leq z \leq x \sqcup y$  or  $y \leq z \leq x \sqcup y$  and we have respectively  $x$  or  $y$  in  $X'$ , contradicting the initial choice. Hence,  $\{X', Y'\}$  is a cut of  $S'$ .  $\square$

### 4.3 Rank-width of countable graphs

We now define "the good notion" of rank-width for countable graphs.

**Definition 20:** *Rank-width.*

A *layout* of a graph  $G$  is a subcubic quasi-tree  $S$  whose set of leaves  $L_S$  contains  $V_G$ , or contains  $\varphi(V_G)$  for some injective mapping  $\varphi : V_G \rightarrow L_S$ . In the latter case, the layout is  $(S, \varphi)$ . Its *width*  $rwd(G, S)$  is the least upperbound of the ranks of the matrices

$M_G[X \cap V_G, X^c \cap V_G]$  (or  $M_G[\varphi^{-1}(X), \varphi^{-1}(X^c)]$ ) over all cuts  $\{X, X^c\}$  of  $S$ . The rank-width of  $G$ , denoted by  $rw(G)$ , is the minimal width of all its layouts. It is a width measure because if  $H \subseteq_i G$ , every layout of  $G$  is one of  $H$  (with useless nodes) and, by Fact (2) of Section 2.2, the associated rank-width is no larger. It is clear that:

$$\overline{rw}(G) = \overline{drwd}(G) \leq rw(G) \leq drwd(G).$$

The following proposition shows that the rank-width of a finite graph  $G$  defined as above is the same as the classical one recalled in Section 2.4.

**Proposition 21:** Rank-width can be defined with respect to cubic and leafy layouts whose leaves are the vertices of the considered graph.

**Proof:** Let  $S = (N, B)$  be a layout of a graph  $G$  such that  $V_G \subseteq L_S$ . We first delete some useless nodes of  $S$  as follows.

We choose a vertex  $r$  of  $G$ . It is a leaf of  $S$ . We let  $N'$  be the union of the intervals  $[x, r]_B$  for all vertices  $x$  of  $G$ . It follows from Lemma 13 and Corollary 14 that  $N'$  is closed under  $\sqcup$  and thus also under  $M$ . Hence  $S' := (N', B')$  where  $B' := B \cap (N' \times N' \times N')$  is a quasi-tree that, furthermore, is leafy and subcubic. It is a layout of  $G$  whose leaves are the vertices of  $G$ .

Next we remove from  $N'$  the nodes of degree 2 in  $S'$  and we get  $N'' \subseteq N'$ . Then  $S'' := (N'', B'')$  where  $B'' := B \cap (N'' \times N'' \times N'')$  is a quasi-tree because the deleted nodes cannot be the nodes  $u$  that are necessary to satisfy A7. Hence,  $S''$  is a leafy quasi-tree and a cubic layout of  $G$  as desired. We have  $S'' \sqsubseteq S$ . By Lemma 19, every cut  $\{X, Y\}$  of  $S''$  is  $\{U \cap N'', W \cap N''\}$  for some cut  $\{U, W\}$  of  $S'$ . Hence,

$$rk(M_G[X \cap V_G, Y \cap V_G]) \leq rk(M_G[U \cap V_G, W \cap V_G]) \leq k.$$

It follows that rank-width defined with respect to cubic and leafy layouts whose leaves are the vertices of the considered graph  $G$  is no larger than  $rw(G)$ .  $\square$

**Theorem 22 :** Rank-width has the compactness property.

**Proof:** It is similar to that of Theorem 2(1) with layouts instead of linear orders.

We first make some easy observations. If  $T$  is a finite cubic tree with leaves  $1, \dots, n$ ,  $n \geq 3$ , then its internal nodes (those that are not leaves) can be denoted in a unique way by the integers  $\{-n, \dots, -3\}$ , in such a way that  $i$  is adjacent to  $-i$  for each  $i = 3, \dots, n$ . We call such a tree a *standard tree*. If we delete the "last leaf"  $n$  and we smooth the degree 2 node  $-n$ , we get the standard tree  $Red(T - n)$ .

Let  $G$  be a countable graph such that  $\overline{rw}(G) = k < \omega$ . Without loss of generality, we can assume that  $V_G = \mathbb{N} - \{0\}$ . We let  $G_3 \subseteq_i G_4 \subseteq_i \dots \subseteq_i$

$G_n \subseteq_i \dots$  be the increasing sequence of finite induced subgraphs of  $G$  such that  $V_{G_n} = [n]$ . Their union is  $G$ . For each  $n \geq 3$ , we let  $\mathcal{L}_n$  be the set of layouts of  $V_{G_n}$  that witness that  $lrwd(G_n) \leq k$  and that are standard trees.

Clearly,  $Red(T-n) \in \mathcal{L}_{n-1}$  if  $T \in \mathcal{L}_n$ . The sets  $\mathcal{L}_n$  are finite and not empty. By Koenig's Lemma, there is an increasing sequence  $(T_n)_{n \geq 3}$  of layouts in  $\mathcal{L}_n$  such that  $Red(T_n - n) = T_{n-1}$  (for  $n > 3$ ). It is increasing with respect to the notion of inclusion defined in Section 2.3, hence  $Q(T_n) \subseteq Q(T_{n+1})$  for all  $n$ . The union of the finite quasi-trees  $Q(T_n)$  is a quasi-tree  $S$  with set of leaves equal to  $V_G$ . It is cubic by the final remark in Definition 17(b), hence is a layout of  $G$ .

We now prove that  $rwid(G, S) \leq k$ . Assume for getting a contradiction that  $rwid(G, S) > k$ . By Fact (4) of Section 2.2, there exists a cut  $\{X, X^c\}$  of  $S$  and finite subsets  $Y \subseteq X \cap V_G, Z \subseteq X^c \cap V_G$  such that the  $rk(M_G[Y, Z]) \geq k + 1$ . There exists  $i$  such that  $V_{G_i} \supseteq Y \cup Z$ . Hence, the cut  $\{X \cap N_{T_i}, N_{T_i} - X\}$  of  $Q(T_i)$  yields  $rk(M_G[X \cap V_{G_i}, V_{G_i} - X]) \geq k + 1$  and  $T_i$  does not witness that  $rwid(G_i) \leq k$ . Contradiction. It follows that  $rwid(G, \leq) \leq k$ , hence  $rwid(G) \leq k = rwid(G)$ .  $\square$

**Definition 23 :** Let  $S$  be a quasi-tree. A set of cuts  $\mathcal{C}$  of  $S$  is *sufficient* if, for every two finite sets of leaves  $Y, Z$  such that  $\overline{Y} \cap \overline{Z} = \emptyset$ , there is a cut  $\{X, X^c\}$  in  $\mathcal{C}$  such that  $Y \subseteq X$  and  $Z \subseteq X^c$ .

**Example 24 :** Let  $S$  be a quasi-tree with countably many leaves. Let  $\mathcal{C}$  be the set of cuts of the form  $\{D, D^c\}$  such that  $D$  is a direction relative to some node  $x$ . Let us prove that it is sufficient. Let  $Y$  and  $Z$  be nonempty and satisfy the condition of Definition 23. Let  $r \in Z$  and let  $\leq$  be the associated partial order with maximal element  $r$ . Let  $y$  be the least upper bound of  $Y$ . It belongs to  $\overline{Y}$  by Lemma 13 and  $y < r$  (otherwise,  $r \in \overline{Y}$ ). Let  $E := \{u \mid u \leq y\}$ . This set contains  $Y$  and is convex, hence it contains  $\overline{Y}$ . Its complement  $E^c$  contains  $Z$  and is convex. To prove this last point, assume that  $w, w' \in E^c$ . Any node  $v$  between  $w$  and  $w'$  satisfies  $w \leq v$  or  $w' \leq v$  (by Lemma 13(3)). If  $v \in E$ , then  $w$  or  $w'$  belongs also to  $E$ , contradicting the assumption. It remains to prove that  $E^c$  is  $D$ , the direction of  $r$  relative to  $y$ . Consider any node  $w \in E^c$ . Then  $y < y \sqcup w$ . If  $y \sqcup w = r$ , then  $B(y, r, w)$  holds. If  $y \sqcup w < r$ , then  $B(y, y \sqcup w, r)$  and  $B(y, y \sqcup w, w)$  hold. In both cases,  $w$  and  $r$  are in the same direction relative to  $y$  and so  $w \in D$ . If  $w \in E - \{y\}$  then  $B(w, y, r)$  holds and  $w \notin D$ . Hence,  $D = E^c$  is the direction of  $r$  relative to  $y$ . The countable set  $\mathcal{C}$  is sufficient.  $\square$

The following proposition shows that rank-width can be defined with respect to sufficient sets of cuts.

**Proposition 25 :** Let  $G$  be a graph with layout  $S$ , and  $\mathcal{C}$  be a sufficient set of cuts of  $S$ . Then  $rwid(G, S)$  is the least upper bound of the ranks  $rk(M_G[X \cap V_G, X^c \cap V_G])$  for  $\{X, X^c\} \in \mathcal{C}$ .

**Proof:** Let  $rwid'(G, S)$  be the least upper bound of the ranks  $rk(M_G[X \cap V_G, X^c \cap V_G])$  for  $\{X, X^c\} \in \mathcal{C}$ . Clearly,  $rwid'(G, S) \leq rwid(G, S)$ . Assume that  $rwid'(G, S) = k$ , and for getting a contradiction, that  $rwid(G, S) \geq k + 1$ . There

is a cut  $\{X, X^c\}$  of  $S$  and finite sets of vertices of  $G$  (hence sets of leaves)  $Y, Z$  such that  $Y \subseteq X$ ,  $Z \subseteq X^c$  and  $rk(M_G[Y, Z]) = k + 1$ . As  $X$  and  $X^c$  are convex, we have  $\overline{Y} \subseteq X$  and  $\overline{Z} \subseteq X^c$ , hence  $\overline{Y} \cap \overline{Z} = \emptyset$ . There is a cut  $\{U, U^c\} \in \mathcal{C}$  such that  $Y \subseteq U$ ,  $Z \subseteq U^c$  and so  $rk(M_G[U \cap V_G, U^c \cap V_G]) \geq k + 1$ , which contradicts the hypothesis that  $rw d'(G, S) = k$ . Hence,  $rw d(G, S) = k$ .  $\square$

**Theorem 26** : For every graph  $G$ , we have  $drwd(G) \leq 2rw d(G)$ .

**Proof:** We will show how to transform a layout of a countable graph  $G$  of width  $k$  into a discrete one of width at most  $2k$ . We will use definitions and facts from Section 2.3 without recalling them. To make the proof more clear, we first consider a special case. We show how to transform a linear order of  $V_G$  of width  $k$  into a discrete layout (not a linear order) of width at most  $2k$ .

Let  $\leq$  be such a linear order. There is a maximal prefix-free language  $L \subseteq \{0, 1\}^*$  and a bijection  $\varphi : V_G \rightarrow L$  such that  $(V_G, \leq)$  is isomorphic to  $(L, \leq_{lex})$ . Then  $T := T(L)$  is a rooted and directed binary tree, and we let  $T' := Und(T)$  be obtained from  $T$  by omitting edge directions. It is a subcubic layout of  $G$ . We will prove that  $rk(G, T') \leq 2k$ .

Consider adjacent nodes  $x, y$  such that  $y$  is a son of  $x$  in  $T$ . Then  $L_{x,y}$  is an interval of  $(L, \leq_{lex})$ . Clearly,  $L_{y,x} = Y \cup Z$  where  $Y$  and  $Z$  are two (possibly empty) intervals such that  $Y < L_{x,y} < Z$ . Since  $(Y \cup L_{x,y}, Z)$  and  $(Y, L_{x,y} \cup Z)$  are cuts we have:

$$\begin{aligned} rk(M_G[\varphi^{-1}(L_{x,y} \cup Y), \varphi^{-1}(Z)]) &\leq k \text{ and} \\ rk(M_G[\varphi^{-1}(Y), \varphi^{-1}(L_{x,y} \cup Z)]) &\leq k. \end{aligned}$$

By Fact (6) of Section 2.4, we get:

$$\begin{aligned} rk(M_G[\varphi^{-1}(L_{x,y}), \varphi^{-1}(L_{y,x})]) &= \\ rk(M_G[\varphi^{-1}(L_{x,y}), \varphi^{-1}(Y) \cup \varphi^{-1}(Z)]) &\leq 2k. \end{aligned}$$

We now consider a cubic and leafy layout  $S = (N, B)$  of  $G$  (assumed countable) of width  $k$  such that  $V_G = L_S$ . Its nodes are enumerated into an infinite sequence. We define an infinite sequence of lines (cf. Definition 11(c)) whose union is  $S$  :

- (1) we choose a leaf  $r$ , from which we get a partial order  $\leq$  on  $N$  (cf. Lemma 13(1); the notation does not indicate the dependence on  $r$ ),
- (2) we let  $N_0$  be a maximal line containing  $r$  (maximality is for inclusion);
- (3) we let  $N_{i+1}$  be a maximal line that contains  $r$  and the first element of  $N - (N_0 \cup \dots \cup N_i)$  (according to the chosen enumeration of  $N$ ).

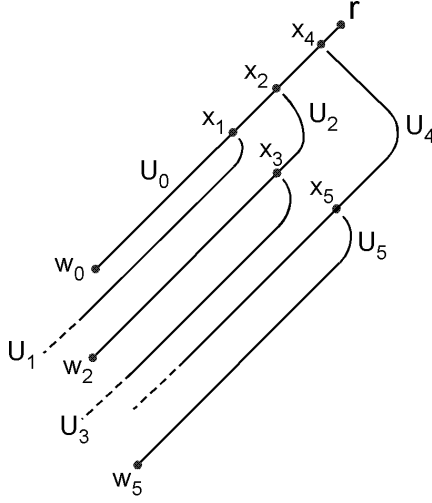


Figure 1: A structured quasi-tree.

As each line  $N_i$  contains  $r$  and at most one other leaf, the sets  $N - (N_0 \cup \dots \cup N_i)$  are not empty and the sequence  $(N_i)_{i \geq 0}$  is infinite. By Lemma 13(1), each line  $N_i$  is linearly ordered by  $\leq$ . We do not have  $N_i \subseteq N_j$  if  $i \neq j$ . We define:

$$U_0 := N_0 \text{ and } U_{i+1} := N_{i+1} - (N_0 \cup \dots \cup N_i).$$

Hence  $U_{i+1} \neq \emptyset$  and there is a node  $x_{i+1} \in N_0 \cup \dots \cup N_i$  such that

$$]-\infty, x_{i+1}[ = U_{i+1} \text{ and } [x_{i+1}, r] = N_{i+1} \cap (N_0 \cup \dots \cup N_i).$$

Clearly,  $N$  is the union of the pairwise disjoint lines  $U_i$ . We call  $(U_i)_{i \geq 0}$  a *structuring* of  $S$  (this notion is also used in [CouDel]).

The nodes  $x_i$  are pairwise distinct because if  $x_i = x_j$  and  $i < j$ , then  $x_i$  would have degree at least 4, whereas  $S$  is cubic.

We illustrate this definition with the example of Figure 1. Lines  $U_0, U_2, U_5$  have minimal elements respectively  $w_0, w_2, w_5$  that are leaves of  $S$ . This figure shows lines  $U_0, \dots, U_5$  but other lines,  $U_6, \dots$  etc. may branch from the lower parts of  $U_1, U_3, U_4$ . Hence, the represented graph has other vertices than  $r, w_0, w_2, w_5$ .

**Claim 1:** Let  $(X, Y)$  be a cut of  $U_i$  and  $\tilde{X} := \{x \mid x \leq x' \text{ for some } x' \in X\}$ . Then  $\{\tilde{X}, N - \tilde{X}\}$  is a cut of  $S$ .

**Proof:** It is clear that if  $x, y \in \tilde{X}$ , then  $x \sqcup y \in \tilde{X}$  because  $X$  is convex and by Lemma 13(2), hence  $\tilde{X}$  is convex by Lemma 13(3). The set  $N - \tilde{X}$  is convex because if  $B(x, y, z)$  holds we have  $x < y \vee z < y$  by Lemma 13(3), so we cannot have  $x \notin \tilde{X} \wedge y \in \tilde{X} \wedge z \notin \tilde{X}$ .  $\square$

Our next objective is to build an injective mapping  $\varphi : N \rightarrow \{0, 1, 2\}^*$  that encodes the nodes of  $S$  by words, hence, by the nodes of a rooted directed tree  $T$ . Section 2.3 contains the relevant definitions and basic facts.

For each  $i$ , we let  $\psi_i$  be an isomorphism  $:(U_i, \leq) \rightarrow (L_i, \leq_{lex})$  where  $L_i \subseteq \{0, 1\}^*$  is maximal prefix-free. We define a mapping  $\varphi_i : U_i \rightarrow \{0, 1, 2\}^*2$  as follows:

$$\varphi_0(u) := \psi_0(u)2 \text{ for } u \in U_0 = N_0.$$

$$\varphi_{i+1}(u) := \varphi_j(x_{i+1})\psi_{i+1}(u)2 \text{ if } u \in U_{i+1} \text{ and } x_{i+1} \in U_j, j \leq i.$$

and finally:

$$\varphi(u) := \varphi_i(u) \text{ if } u \in U_i.$$

**Claim 2 :** The mapping  $\varphi$  is injective.

**Proof:** Assume for a contradiction that  $\varphi_i(u) = \varphi_{i'}(u')$ ,  $u \neq u'$  with  $i + i'$  minimal. We cannot have  $i = i' = 0$ .

*Case 1:*  $i = i' > 1$ , then :

$$\varphi_i(u) = \varphi_j(x_i)\psi_i(u)2 \text{ and } \varphi_{i'}(u') = \varphi_j(x_i)\psi_i(u')2.$$

But  $\psi_i(u), \psi_i(u') \in \{0, 1\}^*$  and since the word  $\varphi_j(x_i)$  finishes with 2, we must have  $\psi_i(u) = \psi_i(u')$  hence,  $u = u'$ , contradicting the initial assumption.

*Case 2:*  $i < i'$ . We cannot have  $i = 0$ . Hence, we have:

$$\varphi_i(u) = \varphi_j(x_i)\psi_i(u)2 \text{ and } \varphi_{i'}(u') = \varphi_{j'}(x_{i'})\psi_{i'}(u')2.$$

But  $\psi_i(u), \psi_{i'}(u') \in \{0, 1\}^*$  and since the words  $\varphi_j(x_i)$  and  $\varphi_{j'}(x_{i'})$  finish with letter 2, we must have  $\varphi_j(x_i) = \varphi_{j'}(x_{i'})$  and  $\psi_i(u) = \psi_{i'}(u')$ . As  $j + j' < i + i'$ , we have  $j = j'$  and thus  $x_i = x_{i'}$  by minimality of  $i + i'$ . Hence  $i = i'$  and  $u = u'$  since  $\psi_i$  is a bijection. We get again a contradiction.

Hence,  $\varphi$  is injective.  $\square$

A similar proof yields:

**Claim 3 :** If  $\varphi(u)$  is a proper prefix of  $\varphi(u')$  then  $u$  is not a leaf of  $S$ .

We define  $W := \varphi(N) \subseteq \{0, 1, 2\}^*$ . Then  $Pref(W)$  is the set of nodes of a rooted and directed tree  $T$ , furthermore ordered by  $\leq_{lex}$ . Although  $T$  is defined with 3 letters, it is binary because if  $w \in \{0, 1, 2\}^*$  and  $w2 \in Pref(W)$  then  $w0, w1 \notin Pref(W)$ .

**Claim 4 :** The maximal elements of  $W$  for the prefix order correspond by  $\varphi$  to the leaves of  $S$  hence, to the vertices of  $G$  except  $r$ .

**Proof:** Every vertex of  $G$  except  $r$  is a leaf of  $S$ , hence the minimal element  $w_i$  of some line  $U_i$ . Hence,  $w_i$  is not  $x_j$  for any  $j$  and  $\varphi(w_i)$  is a maximal word in  $W$  by the definition of  $\varphi$ .

Conversely, let  $u \in U_i$  be not the minimal. Its degree is 3 as  $S$  is cubic. Two directions relative to  $u$  are those of  $r$  and any element of  $U_i$  below  $u$  ( $U_i$

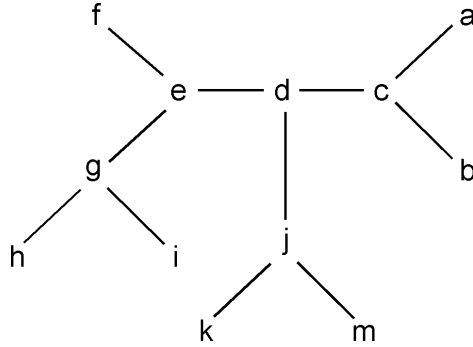


Figure 2: A tree  $A$ .

is contained in the union of these two directions). There is a third direction. It must contain a leaf. Hence  $u = x_j$  for some  $j > i$ , and thus  $\varphi(u)$  is not maximal in  $W$ .  $\square$

For an example, Figure 2 shows a finite tree  $A$  with nodes  $a, b, \dots, k, m$ . We let  $S := Q(A)$ . The construction of  $T$  is better illustrated with finite trees and quasi-trees. We choose  $a$  as root  $r$ . The lines, their orders and their associated nodes  $x_i$  are:

$$\begin{aligned}
 N_0 &: f < e < d < c < a, \\
 N_1 &: b < c < a, & x_1 = c, \\
 N_2 &: h < g < e < d < c < a, & x_2 = e, \\
 N_3 &: i < g < e < d < c < a, & x_3 = g, \\
 N_4 &: k < j < d < c < a, & x_4 = d, \\
 N_5 &: m < j < d < c < a, & x_5 = j.
 \end{aligned}$$

An associated graph has vertices  $a, b, f, h, i, k, m$ .

The tree  $T$  that encodes  $S$  is shown on Figure 3. With each node labelled by 2, we indicate the corresponding node of  $S$ . The nodes  $f, e, d, c, a$  forming  $U_0$  are encoded by the words, respectively, 0002, 0012, 012, 102, 112a. The nodes  $h, g$  forming  $U_2$  are encoded by 001202 and 001212. The node  $i$  forming  $U_3$  is encoded by 0012122.

We now continue the proof. By adding nodes to  $S$ , we made it into a discrete quasi-tree, hence, a tree. The vertices of  $G$  are the leaves of  $S$ , and these leaves correspond by  $\varphi$  to the leaves of  $T$  and its root.

The tree  $T' := \text{Und}(T)$  is subcubic and leafy. Its leaves ( $r$  is a leaf of  $T$ ) correspond by  $\varphi$  to the vertices of  $G$ . Hence,  $(T', \varphi')$  is a discrete layout of  $G$ , where  $\varphi'$  is the restriction of  $\varphi$  to  $V_G$ . We prove that  $\text{rwd}(G, T') \leq 2k$ .

Consider an edge of  $T$  between  $x$  and one of its sons  $y$ . We must prove that:



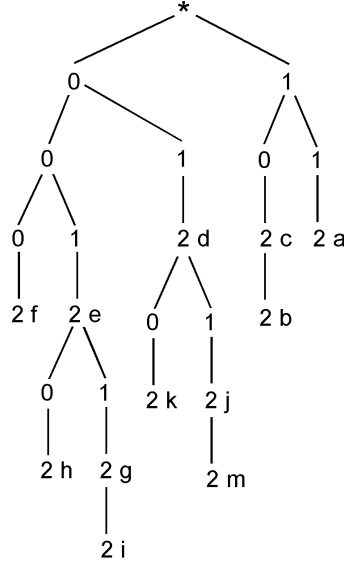


Figure 3: A directed tree that encodes  $S$ .

$$rk(M_G[\varphi^{-1}(N_{x,y}) \cap V_G, \varphi^{-1}(N_{T',y,x}) \cap V_G] \leq 2k.$$

There are three cases:  $y = x0$ ,  $y = x1$  and  $y = x2$ .

*Case  $y = x0$ .*

We have  $x0 = \varphi_j(x_i)2v$  for some  $i, j \in \mathbb{N}$  and  $v \in \{0, 1\}^*$ . Hence  $x0$  is one of the nodes not in  $W$ , introduced for the linear order of  $U_i$ . Hence,  $\varphi_i^{-1}(N_{T',x,y})$  is an interval of  $(U_i, \leq)$ , say  $Y$ . There are two (possibly empty) intervals  $X$  and  $Z$  such that  $X < Y < Z$ . The pairs  $(X, Y \cup Z)$  and  $(X \cup Y, Z)$  are cuts of  $(U_i, \leq)$ .

By Claim 1,  $\{\widetilde{X}, N_S - \widetilde{X}\}$  and  $\{\widetilde{X \cup Y}, N_S - \widetilde{X \cup Y}\}$  are cuts of  $S$ , hence:

$$rk(M_G[\widetilde{X} \cap V_G, (N_S - \widetilde{X}) \cap V_G]) \leq k, \text{ and}$$

$$rk(M_G[\widetilde{X \cup Y} \cap V_G, (N_S - \widetilde{X \cup Y}) \cap V_G]) \leq k.$$

Now observe that :

$$\varphi^{-1}(N_{T',x,y}) = \widetilde{X \cup Y} - \widetilde{X}, \text{ hence:}$$

$$\varphi^{-1}(N_{T',x,y}) \cap V_G = (\widetilde{X \cup Y} \cap V_G) - (\widetilde{X} \cap V_G).$$

The result follows then from fact (6') of Section 2.4.

*Case  $y = x1$ .* Same proof.

*Case  $y = x2$ .*

Here,  $x$  cannot be the root (otherwise, by the construction,  $N_0$  would consist of a unique vertex). Consider its father  $z$ .

Since  $S$  is cubic (and not just subcubic) we cannot have  $x = z2$ . It is easy to see that :

$$\begin{aligned} rk(M_G[\varphi^{-1}(N_{T',x,y}) \cap V_G, \varphi^{-1}(N_{T',y,x}) \cap V_G]) &= \\ rk(M_G[\varphi^{-1}(N_{T',x,y}) \cap V_G, \varphi^{-1}(N_{T',x,z}) \cap V_G]), \end{aligned}$$

because  $x$  has degree 2 in  $T'$  and so, does not encode a vertex, as it is not a leaf of  $T'$ . Hence, the upperbounds used in the previous two cases apply and yield the desired conclusion.

This concludes the proof.  $\square$

**End of the proof of Theorem 8:** Discrete rank-width has the compactness property with gap function  $\lambda n.2n$ : let  $G$  be countable. We have  $\overline{drwd}(G) = \overline{rwd}(G)$ . Assume this value is  $k < \omega$ . Then  $rwd(G) = k$  and  $drwd(G) \leq 2k$  by Theorem 26.  $\square$

## 5 Conclusion

We have defined several notions of rank-width for countable undirected graphs and studied their compactness properties. For finite directed graphs, several notions of rank-width have been defined in [KanRao]. We think that the definitions and results of this article can be extended to countable directed graphs in a straightforward manner.

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