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Spectral Theory near Thresholds for Weak Interactions with Massive Particles

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Abstract

We consider a Hamiltonian describing the weak decay of the massive vector boson Z^0 into electrons and positrons. We show that the spectrum of the Hamiltonian is composed of a unique isolated ground state and a semi-axis of essential spectrum. Using an infrared regularization and a suitable extension of Mourre's theory, we prove that the essential spectrum below the boson mass is purely absolutely continuous.

Keywords: Mathematical models, quantum field theory, decay of gauge bosons, spectral theory, thresholds.
AMS subject classification codes: 47B25, 81Q10, 81T10, 83C47.

1 Introduction

In this paper, we study a mathematical model for the weak decay of the vector boson Z^0 into electrons and positrons. The model we consider is an example of models of the weak interaction that can be patterned according to the Standard Model of Quantum Field Theory. Another example, describing the weak decay of the intermediate vector bosons $W^\pm$ into the full family of leptons, has been

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considered previously in [5, 3]. Comparable models describing quantum electrodynamics processes can be constructed in a similar manner, see [6]. We also mention [12, 15] where the spectral analysis of some related abstract quantum field theory models have been studied.

Unlike [3], the physical phenomenon considered in the present paper only involves *massive* particles. In some respects, e.g. as far as the existence of a ground state is concerned, this feature considerably simplifies the spectral analysis of the Hamiltonian associated with the physical system we study. The main drawback is that, due to the positive masses of the particles, an infinite number of *thresholds* occur in the spectrum of the free Hamiltonian (i.e. the full Hamiltonian where the interaction between the different particles has been turned off). Understanding the nature of the spectrum of the full Hamiltonian near the thresholds as the interaction is turned on then becomes a subtle question. Spectral analysis near thresholds, in particular by means of perturbation theory, is indeed well-known to be a delicate subject. This is the main concern of the present work.

Our main result will provide a complete description of the spectrum of the Hamiltonian below the boson mass. We will show that the spectrum is composed of a unique isolated eigenvalue E (the ground state energy), and the semi-axis of essential spectrum $[E + \mathrm{m_e}, \infty)$, $\mathrm{m_e}$ being the electron mass. Moreover, using a version of Mourre's theory allowing for a non self-adjoint conjugate operator and requiring only low regularity of the Hamiltonian with respect to this conjugate operator, we will prove that the essential spectrum below the boson mass is purely absolutely continuous.

In order to prove our main results we use a spectral representation of the self-adjoint Dirac operator generated by the sequence of spherical waves. See [17] and Section 2. If we have been using the plane waves, for example the four ones associated with the helicity (see [29]), the two kernels $G^{(\alpha)}(\cdot)$ of the interaction (see below) would have had to satisfy an infrared regularization with respect to the fermionic variables. By our choice of the sequence of the spherical waves, the kernels of the interaction have to satisfy an infrared regularization for *only two values* of the discrete parameters characterizing the sequence of spherical waves. For any other value of the discrete parameters, we do not need to introduce an infrared regularization. Thus we have reduced the problem of proving that the spectrum is absolutely continuous in a neighborhood of a threshold to a simpler one, which still remains to be solved.

Before precisely stating our main results in Section 3, we begin with introducing in details the physical model we consider.

2 Description of the model

2.1 The Fock space of electrons, positrons and Z^0 bosons

2.1.1 Free Dirac operator

The energy of a free relativistic electron of mass $\mathrm{m_e}$ is described by the Dirac Hamiltonian (see [25, 29] and references therein)

$$H_D := \boldsymbol{\alpha} \cdot \frac{1}{i}\nabla + \boldsymbol{\beta}\,\mathrm{m_e},$$

acting on the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}^3; \mathbb{C}^4)$, with domain $\mathfrak{D}(H_D) = H^1(\mathbb{R}^3; \mathbb{C}^4)$. We use a system of units such that $\hbar = c = 1$. Here $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and $\boldsymbol{\beta}$ are the Dirac matrices in the standard form:

$$\boldsymbol{\beta} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad i = 1, 2, 3,$$

where σ_i are the usual Pauli matrices. The operator H_D is self-adjoint, and $\mathrm{spec}(H_D) = (-\infty, -\mathrm{m_e}] \cup [\mathrm{m_e}, +\infty)$.

The generalized eigenfunctions associated with the continuous spectrum of the Dirac operator H_D are labeled by the total angular momentum quantum numbers

$$j \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\}, \qquad m_j \in \{-j, -j+1, \dots, j-1, j\}, \tag{1}$$

and by the quantum numbers

$$\kappa_j \in \{\pm(j + \frac{1}{2})\}. \tag{2}$$

In the sequel, we will drop the index j and set

$$\gamma = (j,\, m_j,\, \kappa_j)\,, \tag{3}$$

and a sum over γ will thus denote a sum over $j \in \mathbb{N} + \frac{1}{2}$, $m_j \in \{-j, -j+1, \dots, j-1, j\}$ and $\kappa_j \in \{\pm(j + \frac{1}{2})\}$. We denote by Γ the set $\{(j,\, m_j,\, \kappa_j),\, j \in \mathbb{N} + \frac{1}{2},\, m_j \in \{-j, -j+1, \dots, j-1, j\},\, \kappa_j \in \{\pm(j + \frac{1}{2})\}\}$.

For $\mathbf{p} \in \mathbb{R}^3$ being the momentum of the electron, and $p := |\mathbf{p}|$, the continuum energy levels are given by $\pm\,\omega(p)$, where

$$\omega(p) := ({\mathrm{m_e}}^2 + p^2)^{\frac{1}{2}}\,. \tag{4}$$

We set the notation

$$\xi = (p,\, \gamma)\, \in \mathbb{R}_+ \times \Gamma. \tag{5}$$

The continuum eigenstates of H_D are denoted by (see Appendix A for a detailed description)

$$\psi_\pm(\xi, x) = \psi_\pm((p, \gamma), x)\,.$$

We then have

$$H_D\ \psi_\pm((p, \gamma), x) = \pm\,\omega(p)\ \psi_\pm((p, \gamma), x).$$

The generalized eigenstates $\psi_\pm$ are here normalized in such a way that

$$\begin{aligned}\int_{\mathbb{R}^3} \psi_\pm^\dagger((p,\gamma),x)\,\psi_\pm((p',\gamma'),x)\,\mathrm{d}x &= \delta_{\gamma\gamma'}\delta(p-p'),\\ \int_{\mathbb{R}^3} \psi_\pm^\dagger((p,\gamma),x)\,\psi_\mp((p',\gamma'),x)\,\mathrm{d}x &= 0\,.\end{aligned}$$

Here $\psi_\pm^\dagger((p,\gamma),\,x)$ is the adjoint spinor of $\psi_\pm((p,\gamma),\,x)$.

According to the hole theory [20, 25, 26, 29, 31], the absence in the Dirac theory of an electron with energy $E<0$ and charge e is equivalent to the presence of a positron with energy $-E>0$ and charge $-e$.

Let us split the Hilbert space $\mathfrak{H}=L^2(\mathbb{R}^3;\mathbb{C}^4)$ into

$$\mathfrak{H}_{c^-}=P_{(-\infty,-\mathrm{m_e}]}(H_D)\mathfrak{H} \quad \text{and} \qquad \mathfrak{H}_{c^+}=P_{[\mathrm{m_e},+\infty)}(H_D)\mathfrak{H}.$$

Here $P_I(H_D)$ denotes the spectral projection of H_D corresponding to the interval I.

Let $\Sigma:=\mathbb{R}_+\times\Gamma$. From now on, we identify the Hilbert spaces $\mathfrak{H}_{c^\pm}$ with

$$\mathfrak{H}_c:=L^2(\Sigma;\mathbb{C})\simeq\oplus_\gamma L^2(\mathbb{R}_+;\mathbb{C})\,,$$

by using the unitary operators $U_{c^\pm}$ defined from $\mathfrak{H}_{c^\pm}$ to $\mathfrak{H}_c$ as

$$(U_{c^\pm}\phi)(p,\gamma)=\text{L.i.m}\int\psi_\pm^\dagger((p,\gamma)\,,x)\,\phi(x)\,\mathrm{d}x\,. \tag{6}$$

On $\mathfrak{H}_c$, we define the scalar products

$$(g,\,h)=\int\overline{g(\xi)}h(\xi)\mathrm{d}\xi=\sum_{\gamma\in\Gamma}\int_{\mathbb{R}^+}\overline{g(p,\gamma)}h(p,\gamma)\,\mathrm{d}p\,. \tag{7}$$

In the sequel, we shall denote the variable $(p,\,\gamma)$ by $\xi_1=(p_1,\gamma_1)$ in the case of electrons, and $\xi_2=(p_2,\gamma_2)$ in the case of positrons, respectively.

2.1.2 The Fock space for electrons and positrons

Let

$$\mathfrak{F}_a:=\mathfrak{F}_a(\mathfrak{H}_c)=\bigoplus_{n=0}^{\infty}\otimes_a^n\mathfrak{H}_c,$$

be the Fermi-Fock space over $\mathfrak{H}_c$, and let

$$\mathfrak{F}_D:=\mathfrak{F}_a\otimes\mathfrak{F}_a$$

be the Fermi-Fock space for electrons and positrons, with vacuum Ω_D (see Appendix C for details).

2.1.3 Creation and annihilation operators for electrons and positrons

We set, for every $g \in \mathfrak{H}_c$,

$$\begin{aligned} b_{\gamma,+}(g) &= b_+(P_\gamma g)\,, \\ b^*_{\gamma,+}(g) &= b^*_+(P_\gamma g)\,, \end{aligned}$$

where P_γ is the projection of $\mathfrak{H}_c$ onto the γ-th component, and $b_+(P_\gamma g)$ and $b^*_+(P_\gamma g)$ are respectively the annihilation and creation operator for an electron defined in Appendix C.

As above, we set, for every $h \in \mathfrak{H}_c$,

$$\begin{aligned} b_{\gamma,-}(h) &= b_-(P_\gamma h)\,, \\ b^*_{\gamma,-}(h) &= b^*_-(P_\gamma h)\,, \end{aligned}$$

where $b_-(P_\gamma g)$ and $b^*_-(P_\gamma g)$ are respectively the annihilation and creation operator for a positron defined in Appendix C, according to which $b^\sharp_{\gamma,+}(g)$ and $b^\sharp_{\gamma-}(g)$ anticommute (see (89)).

As in [24, Chapter X], we introduce operator-valued distributions $b_\pm(\xi)$ and $b^*_\pm(\xi)$ that fulfill for $g \in \mathfrak{H}_c$,

$$\begin{aligned} b_\pm(g) &= \int b_{\gamma,\pm}(p)\,\overline{(P_\gamma g)\,(p)}\,\mathrm{d}\xi \\ b^*_\pm(g) &= \int b^*_{\gamma,\pm}(p)\,(P_\gamma g)\,(p)\,\mathrm{d}\xi \end{aligned}$$

where we used the notation of (7).

2.1.4 Fock space for the Z^0 boson.

Let $\mathfrak{S}$ be any separable Hilbert space. Let $\otimes^n_s\mathfrak{S}$ denote the symmetric n-th tensor power of $\mathfrak{S}$. The symmetric Fock space over $\mathfrak{S}$, denoted by $\mathfrak{F}_s(\mathfrak{S})$, is the direct sum

$$\mathfrak{F}_s(\mathfrak{S}) = \bigoplus_{n=0}^{\infty} \otimes^n_s \mathfrak{S}\ , \tag{8}$$

where $\otimes^0_s\mathfrak{S} \equiv \mathbb{C}$. The state $\Omega_s = (1, 0, 0, \ldots, 0, \ldots)$ denotes the vacuum state in $\mathfrak{F}_s(\mathfrak{S})$.

Let

$$\Sigma_3 := \mathbb{R}^3 \times \{-1,\, 0,\, 1\}\,.$$

The one-particle Hilbert space for the particle Z^0 is $L^2(\Sigma_3)$ with scalar product

$$(f,\, g) = \int_{\Sigma_3} \overline{f(\xi_3)} g(\xi_3) \mathrm{d}\xi_3\,, \tag{9}$$

with the notations

$$\xi_3 = (k,\, \lambda) \quad \text{and} \quad \int_{\Sigma_3} \mathrm{d}\xi_3 = \sum_{\lambda=-1,0,1} \int_{\mathbb{R}^3} \mathrm{d}k\,, \tag{10}$$

where $\xi_3 = (k,\, \lambda) \in \Sigma_3$.

The bosonic Fock space for the vector boson Z^0, denoted by $\mathfrak{F}_{Z^0}$, is thus

$$\mathfrak{F}_{Z^0} = \mathfrak{F}_s(L^2(\Sigma_3))\,. \tag{11}$$

For $f \in L^2(\Sigma_3)$, we define the annihilation and creation operators, denoted by $a(f)$ and $a^*(f)$ by

$$a(f) = \int_{\Sigma_3} \overline{f(\xi_3)} a(\xi_3) \mathrm{d}\xi_3 \tag{12}$$

and

$$a^*(f) = \int_{\Sigma_3} f(\xi_3) a^*(\xi_3) \mathrm{d}\xi_3 \tag{13}$$

where the operators $a(\xi_3)$ (respectively $a^*(\xi_3)$) are the bosonic annihilation (respectively bosonic creation) operator for the boson Z^0 (see e.g [21, 4, 5]).

2.2 The Hamiltonian

2.2.1 The free Hamiltonian

The quantization of the Dirac Hamiltonian H_D, denoted by $\mathbb{H}_D$, and acting on $\mathfrak{F}_D$, is given by

$$\mathbb{H}_D = \int \omega(p_1)\, b_+^*(\xi_1)\, b_+(\xi_1) \mathrm{d}\xi_1 \;+\; \int \omega(p_2)\, b_-^*(\xi_2)\, b_-(\xi_2) \mathrm{d}\xi_2,$$

with $\omega(p)$ given in (4). The operator $\mathbb{H}_D$ is the Hamiltonian of the quantized Dirac field.

Let $\mathfrak{D}_D$ denote the set of vectors $\Phi \in \mathfrak{F}_D$ for which $\Phi^{(r,s)}$ is smooth and has a compact support and $\Phi^{(r,s)} = 0$ for all but finitely many (r, s). Then $\mathbb{H}_D$ is well-defined on the dense subset $\mathfrak{D}_D$ and it is essentially self-adjoint on $\mathfrak{D}_D$. The self-adjoint extension will be denoted by the same symbol $\mathbb{H}_D$, with domain $\mathfrak{D}(\mathbb{H}_D)$.

The operators number of electrons and number of positrons, denoted respectively by N_+ and N_-, are given by

$$N_+ = \int b_+^*(\xi_1)\, b_+(\xi_1) \mathrm{d}\xi_1 \quad \text{and} \quad N_- = \int b_-^*(\xi_2)\, b_-(\xi_2) \mathrm{d}\xi_2\,. \tag{14}$$

They are essentially self-adjoint on $\mathfrak{D}_D$. Their self-adjoint extensions will be also denoted by N_+ and N_-.

We have

$$\mathrm{spec}(\mathbb{H}_D) = \{0\} \cup [\mathrm{m_e}, \infty).$$

The set $[\mathrm{m_e}, \infty)$ is the absolutely continuous spectrum of $\mathbb{H}_D$.

The Hamiltonian of the bosonic field, denoted by $\mathbb{H}_{\mathrm{Z}^0}$, acting on $\mathfrak{F}_{Z^0}$, is

$$\mathbb{H}_{\mathrm{Z}^0} := \int \omega_3(k)\, a^*(\xi_3) a(\xi_3)\, \mathrm{d}\xi_3$$

where

$$\omega_3(k) = \sqrt{|k|^2 + \mathrm{m}_{\mathrm{Z}^0}{}^2}. \tag{15}$$

The operator $\mathbb{H}_{\mathrm{Z}^0}$ is essentially self-adjoint on the set of vectors $\Phi \in \mathfrak{F}_{Z^0}$ such that $\Phi^{(n)}$ is smooth and has compact support and $\Phi^{(n)} = 0$ for all but finitely many n. Its self-adjoint extension is denoted by the same symbol.

The spectrum of $\mathbb{H}_{\mathrm{Z}^0}$ consists of an absolutely continuous spectrum covering $[\mathrm{m}_{\mathrm{Z}^0}, \infty)$ and a simple eigenvalue, equal to zero, whose corresponding eigenvector is the vacuum state $\Omega_s \in \mathfrak{F}_{Z^0}$.

The free Hamiltonian is defined on $\mathcal{H} := \mathfrak{F}_D \otimes \mathfrak{F}_{Z^0}$ by

$$H_0 = \mathbb{H}_D \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{H}_{\mathrm{Z}^0}\,. \tag{16}$$

The operator H_0 is essentially self-adjoint on $\mathfrak{D}(\mathbb{H}_D) \otimes \mathfrak{D}(\mathbb{H}_{\mathrm{Z}^0})$. Since $\mathrm{m_e} < \mathrm{m}_{\mathrm{Z}^0}$, the spectrum of H_0 is given by

$$\mathrm{spec}(H_0) = \{0\} \cup [\mathrm{m_e},\, \infty)\,.$$

More precisely,

$$\mathrm{spec}_{\mathrm{pp}}(H_0) = \{0\}, \quad \mathrm{spec}_{\mathrm{sc}}(H_0) = \emptyset, \quad \mathrm{spec}_{\mathrm{ac}}(H_0) = [\mathrm{m_e},\, \infty), \tag{17}$$

where $\mathrm{spec}_{\mathrm{pp}}$, $\mathrm{spec}_{\mathrm{sc}}$, $\mathrm{spec}_{\mathrm{ac}}$ denote the pure point, singular continuous and absolutely continuous spectra, respectively. Furthermore, 0 is a non-degenerate eigenvalue associated to the vacuum $\Omega_D \otimes \Omega_s$.

2.2.2 The Interaction

The interaction between the electrons/positrons and the boson vectors Z^0, in the Schrödinger representation, is given, up to coupling contant, by (see [18, (4.139)] and [32, (21.3.20)])

$$I = \int \overline{\Psi_e}(x)\gamma^\alpha(g'_V - \gamma_5)\Psi_e(x)Z_\alpha(x)\,\mathrm{d}x \; + h.c., \tag{18}$$

where γ^α, $\alpha = 0, 1, 2, 3$, and γ_5 are the Dirac matrices, g'_V is a real parameter such that $g'_V \simeq 0,074$ (see e.g [18]), $\Psi_e(x)$ and $\overline{\Psi_e}(x)$ are the Dirac fields for the electron e_- and the positron e_+ of mass $\mathrm{m_e}$, and Z_α is the massive boson field for Z^0.

With the notations of Subsection 2.1.1, $\Psi_e(x)$ is formally defined by

$$\Psi_e(x) = \int \psi_+(\xi,\, x)b_+(\xi) + \widetilde{\psi}_-(\xi,\, x)b^*_-(\xi)\,\mathrm{d}\xi,$$

where

$$\widetilde{\psi}_-(\xi,\, x) = \widetilde{\psi}_-((p,\gamma),\, x) = \psi_-((p,(j,-m_j,-\kappa_j)),x)\,. \tag{19}$$

The boson field Z_α is formally defined by (see e.g. [31, Eq. (5.3.34)]),

$$\begin{aligned} &Z_\alpha(x) \\ &= (2\pi)^{-\frac{3}{2}} \int \frac{\mathrm{d}\xi_3}{(2(|k|^2+\mathrm{m}_{\mathrm{Z}^0}{}^2)^{\frac{1}{2}})^{\frac{1}{2}}} \Big(\epsilon_\alpha(k,\lambda)a(\xi_3)\mathrm{e}^{ik.x} + \epsilon^*_\alpha(k,\lambda)a^*(\xi_3)\mathrm{e}^{-ik.x}\Big)\,, \end{aligned}$$

with $\xi_3 = (k,\, \lambda)$ according to (10), and where the vectors $\epsilon_\alpha(k, \lambda)$ are the polarizations vectors of the massive spin 1 bosons (see [31, Section 5.3]).

If one considers the full interaction I in (18) describing the decay of the gauge boson Z^0 into massive leptons and if one formally expands this interaction with respect to products of creation and annihilation operators, we are left with a finite sum of terms with kernels yielding singular operators which cannot be defined as closed operators. Therefore, in order to obtain a well-defined Hamiltonian (see e.g [11, 6, 7, 5, 3]), we replace these kernels by square integrable functions $G^{(\alpha)}$.

This implies in particular to introduce cutoffs for high momenta of electrons, positrons and Z^0 bosons. Moreover, we confine in space the interaction between the electrons/positrons and the bosons by adding a localization function $f(|x|)$, with $f \in \mathrm{C}_0^\infty([0, \infty))$. The interaction Hamiltonian is thus defined on $\mathcal{H} = \mathfrak{F}_D \otimes \mathfrak{F}_{Z^0}$ by

$$H_I = H_I^{(1)} + H_I^{(1)^*} + H_I^{(2)} + H_I^{(2)^*}\,, \tag{20}$$

with

$$\begin{aligned} H_I^{(1)} = \int \left(\int_{\mathbb{R}^3} f(|x|) \overline{\psi_+(\xi_1,\, x)} \gamma^\mu (g_V' - \gamma_5) \widetilde{\psi}_-(\xi_2,\, x) \frac{\epsilon_\mu(\xi_3)}{\sqrt{2\omega_3(k)}}\, \mathrm{e}^{ik\cdot x}\, \mathrm{d}x \right) \\ \times G^{(1)}(\xi_1,\, \xi_2,\, \xi_3) b_+^*(\xi_1) b_-^*(\xi_2) a(\xi_3)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3\,, \end{aligned} \tag{21}$$

$$\begin{aligned} H_I^{(1)^*} = \int \left(\int_{\mathbb{R}^3} f(|x|) \overline{\widetilde{\psi}_-(\xi_2,\, x)} \gamma^\mu (g_V' - \gamma_5) \psi_+(\xi_1,\, x) \frac{\epsilon_\mu^*(\xi_3)}{\sqrt{2\omega_3(k)}}\, \mathrm{e}^{-ik\cdot x}\, \mathrm{d}x \right) \\ \times \overline{G^{(1)}(\xi_1,\, \xi_2,\, \xi_3)} a^*(\xi_3) b_-(\xi_2) b_+(\xi_1)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3\,, \end{aligned} \tag{22}$$

$$\begin{aligned} H_I^{(2)} = \int \left(\int_{\mathbb{R}^3} f(|x|) \overline{\psi_+(\xi_1,\, x)} \gamma^\mu (g_V' - \gamma_5) \widetilde{\psi}_-(\xi_2,\, x) \frac{\epsilon_\mu^*(\xi_3)}{\sqrt{2\omega_3(k)}}\, \mathrm{e}^{-ik\cdot x}\, \mathrm{d}x \right) \\ \times G^{(2)}(\xi_1,\, \xi_2,\, \xi_3) b_+^*(\xi_1) b_-^*(\xi_2) a^*(\xi_3)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3\,, \end{aligned} \tag{23}$$

and

$$\begin{aligned} H_I^{(2)^*} = \int \left(\int_{\mathbb{R}^3} f(|x|) \overline{\widetilde{\psi}_-(\xi_2,\, x)} \gamma^\mu (g_V' - \gamma_5) \psi_+(\xi_1,\, x) \frac{\epsilon_\mu(\xi_3)}{\sqrt{2\omega_3(k)}}\, \mathrm{e}^{ik\cdot x}\, \mathrm{d}x \right) \\ \times \overline{G^{(2)}(\xi_1,\, \xi_2,\, \xi_3)} a(\xi_3) b_-(\xi_2) b_+(\xi_1)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3\,. \end{aligned} \tag{24}$$

Performing the integration with respect to x in the expressions above, we see that $H_I^{(1)}$ and $H_I^{(2)}$ can be written under the form

$$H_I^{(1)} := H_I^{(1)}(F^{(1)}) := \int F^{(1)}(\xi_1,\, \xi_2,\, \xi_3) b_+^*(\xi_1) b_-^*(\xi_2) a(\xi_3)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3\,, \tag{25}$$

$$H_I^{(2)} := H_I^{(2)}(F^{(2)}) := \int F^{(2)}(\xi_1,\, \xi_2,\, \xi_3) b_+^*(\xi_1) b_-^*(\xi_2) a^*(\xi_3)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3\,, \tag{26}$$

where, for $\alpha = 1, 2$,

$$F^{(\alpha)}(\xi_1,\, \xi_2,\, \xi_3) := h^{(\alpha)}(\xi_1, \xi_2, \xi_3) G^{(\alpha)}(\xi_1, \xi_2, \xi_3), \tag{27}$$

and $h^{(1)}(\xi_1, \xi_2, \xi_3)$, $h^{(2)}(\xi_1, \xi_2, \xi_3)$ are given by the integral over x in (21) and (23), respectively.

Our main result, Theorem 3.9 below, requires the coupling functions $F^{(\alpha)}(\xi_1,\, \xi_2,\, \xi_3)$ to be sufficiently regular near $p_1 = 0$ and $p_2 = 0$ (where, recall, $\xi_l = (p_l, \gamma_l)$ for $l = 1, 2$). The behavior of the generalized eigenstates $\psi_+(\xi,\, x)$ and $\psi_-(\xi,\, x)$ near $\xi = 0$, and therefore the behavior of $h^{(\alpha)}(\xi_1, \xi_2, \xi_3)$ near $p_1 = 0$ and $p_2 = 0$, will be analyzed in Appendix A.

2.2.3 The total Hamiltonian

Definition 2.1. *The Hamiltonian of the decay of the boson Z^0 into an electron and a positron is*

$$H := H_0 + gH_I\,.$$

where g is a real coupling constant.

3 Main results

For $p \in \mathbb{R}_+$, $j \in \{\frac{1}{2},\, \frac{3}{2},\, \cdots\}$, $\gamma = (j,\, m_j,\, \kappa_j)$ and $\gamma_j = j + \frac{1}{2}$, we define

$$A(\xi) = A(p, \gamma) := \frac{(2p)^{\gamma_j}}{\Gamma(\gamma_j)} \left(\frac{\omega(p) + \mathrm{m_e}}{\omega(p)}\right)^{\frac{1}{2}} \left(\int_0^\infty |f(r)| r^{2\gamma_j}(1 + (pr)^2)\mathrm{d}r\right)^{\frac{1}{2}}, \quad (28)$$

where Γ denotes Euler's Gamma function, and $f \in \mathrm{C}_0^\infty([0, \infty))$ is the localization function appearing in (21)–(24). We make the following hypothesis on the kernels $G^{(\alpha)}$.

Hypothesis 3.1. *For $\alpha = 1, 2$,*

$$\int A(\xi_1)^2 A(\xi_2)^2 (|k|^2 + \mathrm{m_{Z^0}}^2)^{\frac{1}{2}} \left|G^{(\alpha)}(\xi_1,\, \xi_2,\, \xi_3)\right|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 < \infty. \quad (29)$$

Remark 3.2. *Note that up to universal constants, the functions $A(\xi)$ in (28) are upper bounds for the integrals with respect to x that occur in (21). These bounds are derived using the inequality (see [31, Eq.(5.3.23)-(5.3.25)])*

$$\left|\frac{\epsilon_\mu(\xi_3)}{\sqrt{2\omega_3(k)}}\right| \leq C_{\mathrm{m_{Z^0}}}(1 + |k|^2)^{\frac{1}{4}}\,. \quad (30)$$

For $C_{\mathrm{m_{Z^0}}}$ being the constant defined in (30), and $C_Z = 156\, C_{\mathrm{m_{Z^0}}}$, let us define

$$K_1(G^{(\alpha)})^2 := {C_Z}^2 \left(\int A(\xi_1)^2 A(\xi_2)^2\, |G^{(\alpha)}(\xi_1,\, \xi_2,\, \xi_3)|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3\right),$$
$$K_2(G^{(\alpha)})^2 := {C_Z}^2 \left(\int A(\xi_1)^2 A(\xi_2)^2\, |G^{(\alpha)}(\xi_1,\, \xi_2,\, \xi_3)|^2 (|k|^2 + 1)^{\frac{1}{2}} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3\right). \quad (31)$$

Theorem 3.3 (Self-adjointness)**.** *Assume that Hypothesis 3.1 holds. Let $g_0 > 0$ be such that*

$$ {g_0}^2 \left(\sum_{\alpha=1,2} K_1 (G^{(\alpha)})^2 \right) (\frac{1}{{\mathrm{m_e}}^2} + 1) < 1 \,. \tag{32} $$

Then for any real g such that $|g| \leq g_0$, the operator $H = H_0 + gH_I$ is self-adjoint with domain $\mathfrak{D}(H_0)$. Moreover, any core for H_0 is a core for H.

Remark 3.4. *1) Combining* (17)*, relative boundedness of H_I with respect to H_0 (see Section 4) and standard perturbation theory of isolated eigenvalues (see e.g. [22]), we deduce that, for $|g| \ll \mathrm{m_e}$, $\inf \mathrm{spec}(H)$ is a non-degenerate eigenvalue of H. In other words, H admits a unique ground state.*

2) Let Q be the total charge operator

$$ Q = N_+ - N_- , $$

where N_+ and N_- are respectively the operator number of electrons and the operator number of positrons given by (14)*.*

The total Hamiltonian H commutes with Q, and $\mathcal{H}$ is decomposed with respect to the spectrum of the total charge operator as

$$ \mathcal{H} \simeq \oplus_{z \in \mathbb{Z}} \mathcal{H}_z . $$

Each $\mathcal{H}_z$ reduces H and by mimicking the proof given in [28] one proves that the ground state of H belongs to $\mathcal{H}_0$.

Theorem 3.3 follows from the Kato-Rellich Theorem together with standard estimates of creation and annihilation operators in Fock space, showing that the interaction Hamiltonian H_I is relatively bounded with respect to H_0. For the convenience of the reader, a sketch of the proof of Theorem 3.3 is recalled in Subsection 4.1.

For a self-adjoint operator A, we denote by $\mathrm{spec_{ess}}(A)$ the essential spectrum of A.

Theorem 3.5 (Localization of the essential spectrum)**.** *Assume that Hypothesis 3.1 holds and let g_0 be as in* (32)*. Then, for all $|g| \leq g_0$,*

$$ \mathrm{spec_{ess}}(H) = [\inf \mathrm{spec}(H) + \mathrm{m_e}, \infty). $$

Theorem 3.5 is proven in Subsection 4.2. Our proof is based on a method due to Dereziński and Gérard [9] that we adapt to our context.

To establish our next theorems, we need to strengthen the conditions on the kernels $G^{(\alpha)}$. Given a function $f \in L^1([0, \infty))$, we make the convention that the Fourier transform of f is the Fourier transform of the function $\tilde{f} \in L^1(\mathbb{R})$ defined by $\tilde{f}(p) = f(p)$ if $p \geq 0$ and $\tilde{f}(p) = f(-p)$ otherwise.

Hypothesis 3.6. *For $\alpha = 1, 2$, the kernels $G^{(\alpha)} \in L^2(\Sigma \times \Sigma \times \Sigma_3)$ satisfy*

(i) There exists a compact set $K \subset \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^3$ such that $G^{(\alpha)}(p_1, \gamma_1, p_2, \gamma_2, k, \lambda) = 0$ if $(p_1, p_2, k) \notin K$.

(ii) There exists $\varepsilon \geq 0$ such that

$$\sum_{\gamma_1, \gamma_2, \lambda} \int (1 + x_1^2 + x_2^2)^{1+\varepsilon} \left| \hat{G}^{(\alpha)}(x_1, \gamma_1, x_2, \gamma_2, k, \lambda) \right|^2 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}k < \infty,$$

where $\hat{G}^{(\alpha)}$ denote the Fourier transform of $G^{(\alpha)}$ with respect to the variables (p_1, p_2), and x_j is the variable dual to p_j.

(iii) If $\gamma_{1j} = 1$ or $\gamma_{2j} = 1$, where for $l = 1, 2$, $\gamma_{lj} = |\kappa_{j_l}|$ (with $\gamma_l = (j_l, m_{j_l}, \kappa_{j_l})$), and if $p_1 = 0$ or $p_2 = 0$, then $G^{(\alpha)}(p_1, \gamma_1, p_2, \gamma_2, k, \lambda) = 0$.

Remark 3.7. *1) The assumption that $G^{(\alpha)}$ is compactly supported in the variables (p_1, p_2, k) is an "ultraviolet" constraint that is made for convenience. It could be replaced by the weaker assumption that $G^{(\alpha)}$ decays sufficiently fast at infinity.*

2) Hypothesis 3.6 (ii) comes from the fact that the coupling functions $G^{(\alpha)}$ must satisfy some "minimal" regularity for our method to be applied. In fact, Hypothesis (ii) could be slightly improved with a refined choice of interpolation spaces in our proof (see Section 5 for more details). In Hypothesis 3.6 (iii), we need in addition an "infrared" regularization. We remark in particular that Hypotheses (ii) and (iii) imply that, if $\gamma_{1j} = 1$ or $\gamma_{2j} = 1$,

$$\left| G^{(\alpha)}(p_1, \gamma_1, p_2, \gamma_2, k, \lambda) \right| \lesssim |p_l|^{\frac{1}{2}+\varepsilon}, \quad l = 1, 2,$$

for $0 \leq \varepsilon < 1/2$. We emphasize, however, that this infrared assumption is required only in the case $\gamma_{lj} = 1$, that is, for $j = 1/2$. For all other $j \in \mathbb{N} + \frac{1}{2}$, we do not *need to impose any infrared regularization on the generalized eigenstates $\psi_\pm(p, \gamma)$; They are already regular enough.*

3) One verifies that Hypotheses 3.6(i) and 3.6(ii) imply Hypothesis 3.1.

Theorem 3.8 (Localization of the spectrum)**.** *Assume that Hypothesis 3.1 holds. There exists $g_1 > 0$ such that, for all $|g| \leq g_1$,*

$$\mathrm{spec}(H) = \{\inf \mathrm{spec}(H)\} \cup [\inf \mathrm{spec}(H) + \mathrm{m_e}, \infty).$$

In particular, H has no eigenvalue below its essential spectrum except for the ground state energy, $\inf \mathrm{spec}(H)$*, which is a simple eigenvalue.*

Theorem 3.9 (Absolutely continuous spectrum)**.** *Assume that Hypothesis 3.6 holds with $\varepsilon > 0$ in Hypothesis 3.6(ii). For all $\delta > 0$, there exists $g_\delta > 0$ such that, for all $|g| \leq g_\delta$, the spectrum of H in the interval*

$$[\inf \mathrm{spec}(H) + \mathrm{m_e}, \inf \mathrm{spec}(H) + \mathrm{m_{Z^0}} - \delta]$$

is purely absolutely continuous.

Remark 3.10. *In Theorem 5.5 below, we prove a stronger result than Theorem 3.9, which is of independent interest, namely we show that a* limiting absorption principle *holds for H in the interval* $[\inf\operatorname{spec}(H) + \mathrm{m_e}, \inf\operatorname{spec}(H) + \mathrm{m}_{Z^0} - \delta]$. *Another consequence of the limiting absorption principle of Theorem 5.5 is the local decay property* (70).

Theorems 3.8 and 3.9 are proven in Section 5. Our proofs rely on Mourre's Theory with a non-self adjoint conjugate operator. Such extensions of the usual conjugate operator theory [23, 2] have been considered in [19], [27], and later extended in [13, 14].

We use in this paper a conjugate operator, A, similar to the ones of [19] and [13, 14], and prove regularity of the total Hamiltonian with respect to this conjugate operator. Combined with a Mourre estimate, this regularity property allows us to deduce a virial theorem and a limiting absorption principle, from which we obtain Theorems 3.8 and 3.9.

Our main achievement consists in proving that the regularized physical interaction Hamiltonian H_I is regular enough for the Mourre theory to be applied, except for the terms associated to the "first" generalized eigenstates ($j = 1/2$). For the latter, we need to impose a non-physical infrared condition. To establish the regularity of H_I with respect to A, we use in particular real interpolation theory, together with a version of the Mourre theory requiring only low regularity of the Hamiltonian with respect to the conjugate operator.

We remark that if we make the further assumption that the kernels $G^{(\alpha)}$ are sufficiently regular with respect to the Z^0 variable k, similarly to what is assumed in Hypothesis 3.6(ii) for the variables p_1, p_2, it is possible to extend the result of Theorem 3.9 to the interval $[\inf\operatorname{spec}(H) + \mathrm{m_e}, M)$, for any $M > \inf\operatorname{spec}(H) + \mathrm{m_e}$. To do that, one would have to add to the conjugate operator A a term acting on $\mathfrak{F}_{Z^0}$, similar to the ones acting on $\mathfrak{F}_D$ (see (40)), which would yield a Mourre estimate on any interval of the form $[\inf\operatorname{spec}(H) + \mathrm{m_e}, M)$, $M > \inf\operatorname{spec}(H) + \mathrm{m_e}$. The regularity of $G^{(\alpha)}$ in p_1, p_2 and k would insure that H is regular enough with respect to A. For simplicity of exposition, we do not present the details of such an extension of Theorem 3.9 here.

Our paper is organized as follows. As mentioned above, Section 4 is devoted to the proof of Theorems 3.3 and 3.5, and Section 5 is devoted to the proof of Theorems 3.8 and 3.9. In Appendix A, we give the estimates on the generalized eigenfunctions of the Dirac operator that are used in this paper. In Appendix B, we recall the abstract results from Mourre's theory that we need. Finally, for the convenience of the reader, standard definitions and properties of creation and annihilation operators in Fock space are recalled in Appendix C.

4 Self-adjointness and localization of the essential spectrum

In this section we prove Theorems 3.3 and 3.5.

4.1 Self-adjointness

We sketch the standard proof of Theorem 3.3 relying on the Kato-Rellich Theorem.

Proof of Theorem 3.3. We use the N_τ estimates of [11] and follow the proof of [7, Theorem 2.6] (see also [6]). For

$$K_i(G)^2 := \sum_{\alpha=1,2} K_i(G^{(\alpha)})^2, \quad i = 1, 2\,, \tag{33}$$

and

$$C_{1,\beta} := (\frac{1}{\mathrm{m_e}^2} + 1 + 2\beta)^{\frac{1}{2}}, \quad C_{2,\beta\eta} := (\frac{\eta}{\mathrm{m_e}^2}(1+2\beta))^{\frac{1}{2}},$$
$$B_{1,\beta} := (1 + \frac{1}{2\beta})^{\frac{1}{2}}, \quad B_{2,\beta\eta} := (\eta(1+\frac{1}{2\beta}) + \frac{1}{4\eta})^{\frac{1}{2}}\,,$$

we obtain, for any $\psi \in \mathfrak{D}(H)$,

$$\|H_I\psi\| \leq (K_1(G)\, C_{1,\beta} + K_2(G) C_{2,\beta})\, \|H_0\psi\| + (K_1(G) B_{1,\beta} + K_2(G) B_{2,\beta\eta})\, \|\psi\|\,. \tag{34}$$

Therefore, with (32) and for β and η small enough, using the Kato-Rellich Theorem concludes the proof. □

If we note that $K_2(G) \geq K_1(G)$, and set

$$K(G) := K_2(G)\,, \quad C_{\beta\eta} := C_{1,\beta} + C_{2,\beta\eta}\,, \quad B_{\beta\eta} := B_{1,\beta} + B_{2,\beta\eta}\,,$$

we obtain from (34) the following relative bound:

Corollary 4.1. *For any $\psi \in \mathfrak{D}(H)$,*

$$\|H_I\psi\| \leq K(G)\,(C_{\beta\eta}\|H_0\psi\| + B_{\beta\eta}\|\psi\|)\ .$$

In the sequel, for the sake of simplicity, we shall use this relative bound instead of the stronger result (34).

4.2 Localization of the essential spectrum

In this subsection, we prove Theorem 3.5. We use the Dereziński-Gérard partition of unity [9] in a version that accommodates the Fermi-Dirac statistics and the CAR (such a partition of unity was used previously in [1]). Let

$$U_a : \mathfrak{F}_a(\mathfrak{H}_c \oplus \mathfrak{H}_c) \to \mathfrak{F}_a(\mathfrak{H}_c) \otimes \mathfrak{F}_a(\mathfrak{H}_c) = \mathfrak{F}_a \otimes \mathfrak{F}_a,$$

be defined by

$$U_a\Omega_a = \Omega_a \otimes \Omega_a$$
$$U_a b^*(\varphi_1 \oplus \varphi_2) = (b^*(\varphi_1) \otimes \mathbb{1} + (-1)^N \otimes b^*(\varphi_2))U_a,$$

where $(-1)^N$ denotes the bounded operator on $\mathfrak{F}_a$ defined by its restriction to $\otimes_a^r \mathfrak{h}_c$ as $(-1)^N u = (-1)^r u$ for any $u \in \otimes_a^r \mathfrak{h}_c$. Clearly, using the anti-commutation relations, U_a extends by linearity to a unitary map on $\mathfrak{F}_a(\mathfrak{H}_c \oplus \mathfrak{H}_c)$. Let $j_0 \in \mathrm{C}^\infty([0,\infty);[0,1])$ be such that $j_0 \equiv 1$ on $[0,1/2]$ and $j_0 \equiv 0$ on $[1,\infty)$, and let j_∞ be defined by the relation $j_0^2 + j_\infty^2 \equiv 1$. Let $y := i\nabla_p$ account for the position variable of the fermions. Given $R > 0$, we introduce the bounded operators $j_0^R := j_0(|y|/R)$ and $j_\infty^R := j_\infty(|y|/R)$ on $\mathfrak{F}_a(\mathfrak{H}_c)$. Let

$$\begin{aligned} j_a^R : \mathfrak{H}_c &\to \mathfrak{H}_c \oplus \mathfrak{H}_c \\ \varphi &\mapsto (j_0^R \varphi, j_\infty^R \varphi). \end{aligned}$$

Lifting the operator j_a^R to the Fock space $\mathfrak{F}_a(\mathfrak{H}_c)$ allows one to define a map $\Gamma(j_a^R) : \mathfrak{F}_a(\mathfrak{H}_c) \to \mathfrak{F}_a(\mathfrak{H}_c \oplus \mathfrak{H}_c)$. The Dereziński-Gérard partition of unity is defined by

$$\check{\Gamma}_a(j_a^R) : \mathfrak{F}_a \to \mathfrak{F}_a \otimes \mathfrak{F}_a, \qquad \check{\Gamma}_a(j_a^R) = U_a \Gamma(j_a^R).$$

Using the relation $j_0^2 + j_\infty^2 \equiv 1$, one easily verifies that $\check{\Gamma}_a(j_a^R)$ is isometric.

We construct a similar partition of unity, $\check{\Gamma}_s(j_s^R)$, acting on the bosonic Fock space $\mathfrak{F}_{Z^0} = \mathfrak{F}_s(L^2(\Sigma_3))$. It is defined by

$$\check{\Gamma}_s(j_s^R) : \mathfrak{F}_{Z^0} \to \mathfrak{F}_{Z^0} \otimes \mathfrak{F}_{Z^0}, \qquad \check{\Gamma}_s(j_s^R) = U_s \Gamma(j_s^R),$$

where

$$U_s : \mathfrak{F}_s(L^2(\Sigma_3) \oplus L^2(\Sigma_3)) \to \mathfrak{F}_{Z^0} \otimes \mathfrak{F}_{Z^0},$$

is the unitary operator defined by

$$\begin{aligned} &U_s \Omega_s = \Omega_s \otimes \Omega_s \\ &U_s a^*(\varphi_1 \oplus \varphi_2) = (a^*(\varphi_1) \otimes \mathbb{1} + \mathbb{1} \otimes a^*(\varphi_2)) U_s, \end{aligned}$$

and j_s^R is the bounded operator defined by

$$\begin{aligned} j_s^R : L^2(\Sigma_3) &\to L^2(\Sigma_3) \oplus L^2(\Sigma_3) \\ \varphi &\mapsto (j_0^R \varphi, j_\infty^R \varphi). \end{aligned}$$

Here we have used similar notations as above, namely $j_0^R := j_0(|y_3|/R)$ and $j_\infty^R := j_\infty(|y_3|/R)$, where $y_3 := i\nabla_k$ accounts for the position variable of the bosons.

Let N denote the number operator, acting either on $\mathfrak{F}_a$ or on $\mathfrak{F}_{Z^0}$. To shorten notations, we define the operators

$$N_0 := N \otimes \mathbb{1}, \qquad N_\infty := \mathbb{1} \otimes N,$$

acting on $\mathfrak{F}_a \otimes \mathfrak{F}_a$ and on $\mathfrak{F}_{Z^0} \otimes \mathfrak{F}_{Z^0}$.

We recall the following properties that can be easily proven using the definitions of the operators involved (see [1, 9]).

Lemma 4.2. *With the previous notations, we have the following properties.*

(i) Let $\varphi_1, \ldots, \varphi_n \in \mathfrak{H}_c$. Then

$$\check{\Gamma}_a(j_a^R) \prod_{i=1}^n b^*(\varphi_i)\Omega_a = \prod_{i=1}^n \big(b^*(j_0^R\varphi_i) \otimes \mathbb{1} + (-1)^N \otimes b^*(j_\infty^R\varphi_i)\big)\Omega_a \otimes \Omega_a.$$

Let $\varphi_1, \ldots, \varphi_n \in L^2(\Sigma_3)$. Then

$$\check{\Gamma}_s(j_s^R) \prod_{i=1}^n a^*(\varphi_i)\Omega_s = \prod_{i=1}^n \big(a^*(j_0^R\varphi_i) \otimes \mathbb{1} + \mathbb{1} \otimes a^*(j_\infty^R\varphi_i)\big)\Omega_s \otimes \Omega_s.$$

(ii) Let ω be an operator on $\mathfrak{H}_c$ such that the commutators $[\omega, j_\#^R]$, defined as quadratic forms on $\mathfrak{D}(\omega)$, extend to bounded operators on $\mathfrak{H}_c$, where $j_\#$ stands for j_0 and j_∞. Then

$$\big\|(N_0 + N_\infty)^{-\frac{1}{2}}\big((\mathrm{d}\Gamma(\omega) \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega))\check{\Gamma}_a(j_a^R) - \check{\Gamma}_a(j_a^R)\mathrm{d}\Gamma(\omega)\big)N^{-\frac{1}{2}}P_{\Omega_a}^\perp\big\| \\ \leq \big\|\check{\mathrm{ad}}_\omega(j_a^R)\big\|,$$

where P_{Ω_a} denotes the orthogonal projection onto the vacuum sector in $\mathfrak{F}_a$, and $\check{\mathrm{ad}}_\omega(j_a^R) := ([\omega, j_0^R], [\omega, j_\infty^R])$.

The same estimate holds if $\mathfrak{F}_a$, $\mathfrak{H}_c$, j_a^R, $\check{\Gamma}_a$ and Ω_a are replaced respectively by $\mathfrak{F}_{Z^0}$, $L^2(\Sigma_3)$, j_s^R, $\check{\Gamma}_s$ and Ω_s.

Recall that the total Hilbert space can be written as $\mathcal{H} = \mathfrak{F}_a \otimes \mathfrak{F}_a \otimes \mathfrak{F}_{Z^0}$. As in [1, 9], it is convenient to introduce an "extended" Hamiltonian, H^{ext}, acting on the "extended" Hilbert space

$$\mathcal{H}^{\mathrm{ext}} := \bigotimes_{i=1}^4 \mathfrak{F}_a \otimes \bigotimes_{j=1}^2 \mathfrak{F}_{Z^0}.$$

In our setting, the extended Hamiltonian is given by the expression

$$H^{\mathrm{ext}} := H_0^{\mathrm{ext}} + gH_I^{\mathrm{ext}},$$

where

$$\begin{aligned} H_0^{\mathrm{ext}} :=& \mathrm{d}\Gamma(H_D) \otimes \mathbb{1}_{\otimes^2\mathfrak{F}_a} \otimes \mathbb{1}_{\otimes^2\mathfrak{F}_{Z^0}} + \mathbb{1}_{\otimes^2\mathfrak{F}_a} \otimes \mathrm{d}\Gamma(H_D) \otimes \mathbb{1}_{\otimes^2\mathfrak{F}_{Z^0}} \\ &+ \mathbb{1}_{\otimes^4\mathfrak{F}_a} \otimes \mathrm{d}\Gamma(H_{Z^0}) \otimes \mathbb{1}_{\mathfrak{F}_{Z^0}} + \mathbb{1}_{\otimes^4\mathfrak{F}_a} \otimes \mathbb{1}_{\mathfrak{F}_{Z^0}} \otimes \mathrm{d}\Gamma(H_{Z^0}), \end{aligned}$$

and H_I^{ext} is given by (20)–(24), except that the creation and annihilation operators for the electrons, $b_+^\# = b^\# \otimes \mathbb{1} \otimes \mathbb{1}$, acting on $\mathcal{H} = \mathfrak{F}_a \otimes \mathfrak{F}_a \otimes \mathfrak{F}_{Z^0}$, are replaced by

$$b_+^{\#,0} := b^\# \otimes \mathbb{1}_{\otimes^3\mathfrak{F}_a} \otimes \mathbb{1}_{\otimes^2\mathfrak{F}_{Z^0}}$$

(acting on $\mathcal{H}^{\mathrm{ext}}$), likewise, the creation and annihilation operators for the positrons, $b_-^\# = (-1)^{N_+} \otimes b^\# \otimes \mathbb{1}$, are replaced by

$$b_-^{\#,0} := (-1)^{N_{+,0}} \otimes (-1)^{N_{+,\infty}} \otimes b^\# \otimes \mathbb{1}_{\mathfrak{F}_a} \otimes \mathbb{1}_{\otimes^2\mathfrak{F}_{Z^0}},$$

and the creation and annihilation operators for the bosons, $a^{\#}$, are replaced by

$$a^{\#,0} := \mathbb{1}_{\otimes^4\mathfrak{F}_a} \otimes a^{\#} \otimes \mathbb{1}_{\mathfrak{F}_{Z^0}}.$$

Here we have set

$$N_{+,0} := (N \otimes \mathbb{1}_{\mathfrak{F}_a}) \otimes \mathbb{1}_{\otimes^2\mathfrak{F}_a} \otimes \mathbb{1}_{\otimes^2\mathfrak{F}_{Z^0}}, \qquad N_{+,\infty} := (\mathbb{1}_{\mathfrak{F}_a} \otimes N) \otimes \mathbb{1}_{\otimes^2\mathfrak{F}_a} \otimes \mathbb{1}_{\otimes^2\mathfrak{F}_{Z^0}},$$

on $\mathcal{H}^{\mathrm{ext}}$. We define similarly the number operators

$$N_{-,0} := \mathbb{1}_{\otimes^2\mathfrak{F}_a} \otimes (N \otimes \mathbb{1}_{\mathfrak{F}_a}) \otimes \mathbb{1}_{\otimes^2\mathfrak{F}_{Z^0}}, \qquad N_{-,\infty} := \mathbb{1}_{\otimes^2\mathfrak{F}_a} \otimes (\mathbb{1}_{\mathfrak{F}_a} \otimes N) \otimes \mathbb{1}_{\otimes^2\mathfrak{F}_{Z^0}},$$

and

$$N_{Z^0,0} := \mathbb{1}_{\otimes^4\mathfrak{F}_a} \otimes \left(N \otimes \mathbb{1}_{\mathfrak{F}_{Z^0}}\right), \qquad N_{Z^0,\infty} := \mathbb{1}_{\otimes^4\mathfrak{F}_a} \otimes \left(\mathbb{1}_{\mathfrak{F}_{Z^0}} \otimes N\right),$$

and the creation and annihilation operators

$$b_+^{\#,\infty} := \mathbb{1}_{\mathfrak{F}_a} \otimes b^{\#} \otimes \mathbb{1}_{\otimes^2\mathfrak{F}_a} \otimes \mathbb{1}_{\otimes^2\mathfrak{F}_{Z^0}},$$

$$b_-^{\#,\infty} := (-1)^{N_{+,0}} \otimes (-1)^{N_{+,\infty}} \otimes \mathbb{1}_{\mathfrak{F}_a} \otimes b^{\#} \otimes \mathbb{1}_{\otimes^2\mathfrak{F}_{Z^0}},$$

and

$$a^{\#,\infty} := \mathbb{1}_{\otimes^4\mathfrak{F}_a} \otimes \mathbb{1}_{\mathfrak{F}_{Z^0}} \otimes a^{\#}.$$

Now, we introduce an isometric map, $\check{\Gamma}_R : \mathcal{H} \to \mathcal{H}^{\mathrm{ext}}$, by setting

$$\check{\Gamma}_R := \check{\Gamma}_a(j_a^R) \otimes \check{\Gamma}_a(j_a^R) \otimes \check{\Gamma}_s(j_s^R).$$

Theorem 3.5 will be a consequence of the following lemma.

Lemma 4.3. *Assume that Hypothesis 1 holds and let g_0 be as in* (32)*. Let $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$. Then, for all $|g| \leq g_0$,*

$$\left\|\check{\Gamma}_R\chi(H) - \chi(H^{\mathrm{ext}})\check{\Gamma}_R\right\| \to 0, \quad \textit{as } R \to \infty.$$

Proof. Using the Helffer-Sjöstrand functional calculus, we represent $\chi(H)$ as the integral

$$\chi(H) = \frac{1}{\pi}\int \frac{\partial\tilde{\chi}}{\partial\bar{z}}(z)(H-z)^{-1}\mathrm{d}\,\mathrm{Re}z\,\mathrm{d}\,\mathrm{Im}z,$$

where $\tilde{\chi} \in \mathrm{C}_0^\infty(\mathbb{C})$ denotes an almost analytic extension of χ satisfying $\tilde{\chi}|_{\mathbb{R}} = \chi$ and $|\partial_{\bar{z}}\tilde{\chi}(z)| \leq C_n|\mathrm{Im}\,z|^n$ for any $n \in \mathbb{N}$. The same representation holds for $\chi(H^{\mathrm{ext}})$, from which we deduce that

$$\begin{aligned}&\check{\Gamma}_R\chi(H) - \chi(H^{\mathrm{ext}})\check{\Gamma}_R\\ &= \frac{1}{\pi}\int \frac{\partial\tilde{\chi}}{\partial\bar{z}}(z)(H^{\mathrm{ext}}-z)^{-1}(H^{\mathrm{ext}}\check{\Gamma}_R - \check{\Gamma}_R H)(H-z)^{-1}\mathrm{d}\,\mathrm{Re}z\,\mathrm{d}\,\mathrm{Im}z.\end{aligned}$$

By Lemma 4.2(*ii*), together with

$$\left\|N_\#^{\frac{1}{2}}(H-z)^{-1}\right\| \leq C|\mathrm{Im}\,z|^{-1}, \qquad \left\|(H^{\mathrm{ext}}-z)^{-1}(N_{\#,0}+N_{\#,\infty})^{\frac{1}{2}}\right\| \leq C|\mathrm{Im}\,z|^{-1},$$

where $N_\#$ stands for N_+, N_- or N_{Z^0} (and likewise for $N_{\#,0}$ and $N_{\#,\infty}$), we obtain

$$\begin{aligned}&\big\|(H^{\rm ext}-z)^{-1}(H_0^{\rm ext}\check\Gamma_R-\check\Gamma_R H_0)(H-z)^{-1}\big\|\\&\le C\big(\big\|\check{\rm ad}_\omega(j_a^R)\big\|+\big\|\check{\rm ad}_{\omega_3}(j_s^R)\big\|\big)|{\rm Im}\,z|^{-2}.\end{aligned}\tag{35}$$

Here, ω is given by (4) and ω_3 is given by (15). Using e.g. pseudo-differential calculus, one easily verifies that $\big\|\check{\rm ad}_\omega(j_a^R)\big\|=\mathcal{O}(R^{-1})$ and $\big\|\check{\rm ad}_{\omega_3}(j_s^R)\big\|=\mathcal{O}(R^{-1})$, as $R\to\infty$. Hence, (35) combined with the properties of the almost analytic extension $\tilde\chi$ show that

$$\Big\|\int\frac{\partial\tilde\chi}{\partial\bar z}(z)(H^{\rm ext}-z)^{-1}(H_0^{\rm ext}\check\Gamma_R-\check\Gamma_R H_0)(H-z)^{-1}{\rm d}\,{\rm Re}z\,{\rm d}\,{\rm Im}z\Big\|=\mathcal{O}(R^{-1}).$$

It remains to estimate

$$\int\frac{\partial\tilde\chi}{\partial\bar z}(z)(H^{\rm ext}-z)^{-1}(H_I^{\rm ext}\check\Gamma_R-\check\Gamma_R H_I)(H-z)^{-1}{\rm d}\,{\rm Re}z\,{\rm d}\,{\rm Im}z.$$

The different interaction terms appearing in the definition (20) of H_I are treated in the same way. Consider for instance the interaction Hamiltonian $H_I^{(1)}$ given by (21), written under the form given in (25),

$$H_I^{(1)}=\int F^{(1)}(\xi_1,\,\xi_2,\,\xi_3)b_+^*(\xi_1)b_-^*(\xi_2)a(\xi_3)\,{\rm d}\xi_1{\rm d}\xi_2{\rm d}\xi_3\,,$$

with $F^{(1)}\in L^2({\rm d}\xi_1{\rm d}\xi_2{\rm d}\xi_3)$. We let $H_I^{(1),{\rm ext}}$ be defined by the same expression, except that the creation and annihilation operators b_+^*, b_-^*, a are replaced by $b_+^{*,0}$, $b_-^{*,0}$, a^0 defined above. Using Lemma 4.2(i), a straightforward computation gives

$$\begin{aligned}&H_I^{(1),{\rm ext}}\check\Gamma_R-\check\Gamma_R H_I^{(1)}\\&=\int j_1(|i\nabla_{p_1}|,|i\nabla_{p_2}|,|i\nabla_k|)F^{(1)}(\xi_1,\,\xi_2,\,\xi_3)b_+^{*,0}(\xi_1)b_-^{*,0}(\xi_2)a^0(\xi_3)\check\Gamma_R\,{\rm d}\xi_1{\rm d}\xi_2{\rm d}\xi_3\,,\\&+\sum_{l>1}\int j_l(|i\nabla_{p_1}|,|i\nabla_{p_2}|,|i\nabla_k|)F^{(1)}(\xi_1,\,\xi_2,\,\xi_3)b_+^{*,\sharp}(\xi_1)b_-^{*,\sharp}(\xi_2)a^\sharp(\xi_3)\check\Gamma_R\,{\rm d}\xi_1{\rm d}\xi_2{\rm d}\xi_3\,,\end{aligned}$$

where we have set $j_1(|y_1|,|y_2|,|y_3|)=1-j_0(|y_1|/R)j_0(|y_2|/R)j_0(|y_3|/R)$ and, for $l\neq 1$, $j_l(|y_1|,|y_2|,|y_3|)$ is of the form $j_l(|y_1|,|y_2|,|y_3|)=j_{\#1}(|y_1|/R)j_{\#2}(|y_2|/R)j_{\#3}(|y_3|/R)$ with $j_{\#i}=j_0$ or $j_{\#i}=j_\infty$, and at least one of the $j_{\#i}$'s is equal to j_∞. Moreover, $b_+^{*,\sharp}$ stands for $b_+^{*,0}$ or $b_+^{*,\infty}$, and likewise for $b_-^{*,\sharp}$ and $a^\sharp$.

It follows from the N_τ estimates (see [11]) that

$$\begin{aligned}&\big\|(H^{\rm ext}-z)^{-1}(H_I^{(1),{\rm ext}}\check\Gamma_R-\check\Gamma_R H_{I,(1)})(H-z)^{-1}\big\|\\&\le C|{\rm Im}\,z|^{-2}\sum_l\big\|j_l(|i\nabla_{p_1}|,|i\nabla_{p_2}|,|i\nabla_k|)F^{(1)}\big\|.\end{aligned}$$

Therefore, using the fact that

$$\big\|j_l(|i\nabla_{p_1}|,|i\nabla_{p_2}|,|i\nabla_k|)F^{(1)}\big\|\to 0,$$

as $R \to \infty$ and the properties of $\tilde{\chi}$, we deduce that

$$\Big\| \int \frac{\partial \tilde{\chi}}{\partial \bar{z}}(z)(H^{\mathrm{ext}} - z)^{-1}(H_I^{(1),\mathrm{ext}}\check{\Gamma}_R - \check{\Gamma}_R H_I^{(1)})(H - z)^{-1} \mathrm{d}\,\mathrm{Re} z\, \mathrm{d}\,\mathrm{Im} z \Big\| \to 0,$$

as $R \to \infty$. Since the other interaction terms in (20) are treated in the same way, this concludes the proof. □

We are now ready to prove Theorem 3.5.

Proof of Theorem 3.5. We prove that

$$\mathrm{spec}_{\mathrm{ess}}(H) \subset [\inf \mathrm{spec}(H) + \mathrm{m_e}, \infty). \tag{36}$$

Let $\chi \in \mathrm{C}_0^\infty((-\infty, \inf \mathrm{spec}(H) + \mathrm{m_e}))$. Since $\check{\Gamma}_R$ is isometric, we can write

$$\chi(H) = \check{\Gamma}_R^* \check{\Gamma}_R \chi(H) = \check{\Gamma}_R^* \chi(H^{\mathrm{ext}}) \check{\Gamma}_R + o_R(1), \tag{37}$$

where $o_R(1)$ stands for a bounded operator vanishing as $R \to \infty$. The last equality above follows from Lemma 4.3. Observing that $N_{\mathrm{tot},\infty} := N_{+,\infty} + N_{-,\infty} + N_{Z^0,\infty}$ commutes with H^{ext} and that

$$H^{\mathrm{ext}} \mathbb{1}_{[1,\infty)}(N_{\mathrm{tot},\infty}) \geq (\inf \mathrm{spec}(H) + \mathrm{m_e}) \mathbb{1}_{[1,\infty)}(N_{\mathrm{tot},\infty}),$$

we deduce that

$$\chi\big(H^{\mathrm{ext}}\big) = \mathbb{1}_{\{0\}}(N_{\mathrm{tot},\infty}) \chi\big(H^{\mathrm{ext}}\big).$$

Hence (37) yields

$$\begin{aligned} \chi(H) &= \check{\Gamma}_R^* \mathbb{1}_{\{0\}}(N_{\mathrm{tot},\infty}) \chi\big(H^{\mathrm{ext}}\big) \check{\Gamma}_R + o_R(1) \\ &= \check{\Gamma}_R^* \mathbb{1}_{\{0\}}(N_{\mathrm{tot},\infty}) \check{\Gamma}_R \chi(H) + o_R(1), \end{aligned} \tag{38}$$

where we used again Lemma 4.3 in the last equality. Inspecting the definition of the operator $\check{\Gamma}_R$, it is easy to see that

$$\check{\Gamma}_R^* \mathbb{1}_{\{0\}}(N_{\mathrm{tot},\infty}) \check{\Gamma}_R = \Gamma\big((j_0^R)^2\big) \otimes \Gamma\big((j_0^R)^2\big) \otimes \Gamma\big((j_0^R)^2\big).$$

Since

$$\Gamma\big((j_0^R)^2\big) \otimes \Gamma\big((j_0^R)^2\big) \otimes \Gamma\big((j_0^R)^2\big)(H_0 + i)^{-1}$$

is compact, and since $(H_0 + i)\chi(H)$ is bounded, we conclude that

$$\check{\Gamma}_R^* \mathbb{1}_{\{0\}}(N_{\mathrm{tot},\infty}) \check{\Gamma}_R \chi(H)$$

is compact. Therefore, by (38), the operator $\chi(H)$ is also compact, which proves (36).

To prove the converse inclusion, it suffices to construct, for any $\lambda \in (\inf \mathrm{spec}(H) + \mathrm{m_e}, \infty)$, a Weyl sequence associated to λ. This can be done in the same way as in [9, Theorem 4.1] or [1, Theorem 4.3]. We do not give the details. □

5 Proofs of Theorems 3.8 and 3.9

In this section, we prove Theorems 3.8 and 3.9 by applying a suitable version of Mourre's theory. We begin with defining the conjugate operator A that we consider in Subsection 5.1; We show that the semi-group generated by A preserves the form domain of the total Hamiltonian H. In Subsection 5.2, we establish regularity of H with respect to A and in Subsection 5.3, we prove a Mourre estimate. Putting all together, we finally deduce in Subsection 5.4 that the statements of Theorems 3.8 and 3.9 hold.

5.1 The conjugate operator and its associated semigroup

Let a be the operator on $L^2(\mathbb{R}_+)$ defined by the expression

$$a = \frac{i}{2}\big(f(p)\partial_p + \partial_p f(p)\big) = if(p)\partial_p + \frac{i}{2}f'(p), \tag{39}$$

where $f(p) := p^{-1}\omega(p) = p^{-1}\sqrt{p^2 + \mathrm{m_e}^2}$ and f' stands for the derivative of f. The operator a with domain $\mathrm{C}_0^\infty((0,\infty))$ is symmetric; its closure is denoted by the same symbol.

We construct the C_0-semigroup, w_t, associated with a. Let

$$g(p) := \int_0^p \frac{1}{f(r)}\mathrm{d}r = \sqrt{p^2 + \mathrm{m_e}^2} - \mathrm{m_e}.$$

Note that the function g is bijective on $[0,\infty)$, with inverse

$$g^{-1}(p) = \sqrt{(p + \mathrm{m_e})^2 - \mathrm{m_e}^2}.$$

For all $t \geq 0$, let ψ_t be defined on $[0,\infty)$ by $\psi_t(p) := 0$ if $p < \sqrt{(t + \mathrm{m_e})^2 - \mathrm{m_e}^2} = g^{-1}(t)$ and $\psi_t(p) := g^{-1}(-t + g(p))$ otherwise. Setting

$$(w_t u)(p) := \big(\partial_p \psi_t(p)\big)^{\frac{1}{2}} u(\psi_t(p)),$$

one easily verifies that w_t is the C_0-semigroup of isometries associated with a, namely $w_{t+s} = w_t w_s$ for $t, s \geq 0$, and $(\partial_t w_t u)|_{t=0}(p) = i(au)(p)$. We observe that a is maximal symmetric with deficiency index $n_+ = \dim \mathrm{Ker}(a^* - i) = 0$.

On $\mathfrak{H}_c = \oplus_\gamma L^2(\mathbb{R}_+)$, the operator $\oplus_\gamma a$ is still denoted by the symbol a. Our conjugate operator, A, acting on the full Hilbert space $\mathcal{H} = \mathfrak{F}_a \otimes \mathfrak{F}_a \otimes \mathfrak{F}_{Z^0}$, is then given by

$$A := \mathrm{d}\Gamma(a) \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(a) \otimes \mathbb{1}. \tag{40}$$

From the properties of a, we deduce that A is maximal symmetric and generates the C_0-semigroup

$$W_t := \Gamma(w_t) \otimes \Gamma(w_t) \otimes \mathbb{1}.$$

The adjoint semigroup, W_t^*, with generator $-A^*$, is given as follows: For any $p, t \geq 0$, let $\phi_t(p) := g^{-1}(t + g(p))$. One can verify that the adjoint semigroup of w_t is the C_0-semigroup of contractions given by

$$(w_t^* u)(p) = \big(\partial_p \phi_t(p)\big)^{\frac{1}{2}} u(\psi_t(p)).$$

We deduce that

$$W_t^* = \Gamma(w_t^*) \otimes \Gamma(w_t^*) \otimes \mathbb{1},$$

and that W_t^* is a C_0-semigroup (of contractions) on $\mathcal{H}$.

The form domain of H is denoted by

$$\mathcal{G} := \mathfrak{D}(|H|^{\frac{1}{2}}) = \mathfrak{D}(H_0^{\frac{1}{2}}).$$

Proposition 5.1. *For all $t \geq 0$, we have that*

$$W_t\, \mathcal{G} \subset \mathcal{G}, \quad W_t^*\, \mathcal{G} \subset \mathcal{G},$$

and

$$\big\| H_0^{\frac{1}{2}} W_t (H_0^{\frac{1}{2}} + \mathbb{1})^{-1} \big\| \leq 1, \quad \big\| H_0^{\frac{1}{2}} W_t^* (H_0^{\frac{1}{2}} + \mathbb{1})^{-1} \big\| \leq 1.$$

In particular, Hypothesis B.1 of Appendix B is satisfied.

Proof. We prove the statement for W_t^*, the proof for W_t is similar. First, we show that $w_t^*\, \mathfrak{D}(\omega) \subset \mathfrak{D}(\omega)$ and that

$$\big\| \omega^{-\frac{1}{2}}\, w_t\, \omega\, w_t^*\, \omega^{-\frac{1}{2}} \big\| \leq 1, \tag{41}$$

where, recall, ω is the multiplication operator by $\omega(p) = \sqrt{p^2 + \mathrm{m_e}^2}$ on $L^2(\mathbb{R}_+)$. For any $u \in \mathrm{C}_0^\infty((0,\infty))$, we have that

$$\|\omega\, w_t^*\, u\|^2 = \int \omega(p)^2 \big|\partial_p \phi_t(p)\big| \big|u(\phi_t(p))\big|^2 \mathrm{d}p.$$

Using the definition of ϕ_t, one sees that $\phi_t(p) \geq p$ for all $t \geq 0$, and hence

$$\|\omega\, w_t^*\, u\|^2 \leq \int \omega(\phi_t(p))^2 \big|\partial_p \phi_t(p)\big| \big|u(\phi_t(p))\big|^2 \mathrm{d}p = \|\omega\, u\|^2.$$

Since $\mathrm{C}_0^\infty((0,\infty))$ is a core for ω, this implies that $w_t^*\, \mathfrak{D}(\omega) \subset \mathfrak{D}(\omega)$ and that

$$\big\| \omega\, w_t^*\, \omega^{-1} \big\| \leq 1.$$

Using the fact that w_t is isometric and an interpolation argument, we obtain (41).

Now, let $\varphi \in \mathfrak{F}_{a,\mathrm{fin}}(\mathfrak{D}(\omega)) \otimes \mathfrak{F}_{a,\mathrm{fin}}(\mathfrak{D}(\omega)) \otimes \mathfrak{F}_{Z^0}$, where $\mathfrak{F}_{a,\mathrm{fin}}(\mathfrak{D}(\omega))$ denotes the set of vectors $(\varphi_0, \varphi_1, \dots)$ in $\oplus_{n=0}^\infty \otimes_a^n \mathfrak{D}(\omega)$ (algebraic tensor product) such that $\varphi_n = 0$ for all but finitely many n's. We compute

$$\begin{aligned}
\big\| H_0^{\frac{1}{2}} W_t^* \varphi \big\|^2 &= \big\langle \varphi, W_t H_0 W_t^* \varphi \big\rangle \\
&= \big\langle \varphi, \big(\mathrm{d}\Gamma(w_t w_t^*, w_t \omega w_t^*) \otimes \Gamma(w_t w_t^*) \otimes \mathbb{1} \\
&\qquad + \Gamma(w_t w_t^*) \otimes \mathrm{d}\Gamma(w_t w_t^*, w_t \omega w_t^*) \otimes \mathbb{1} \\
&\qquad + \Gamma(w_t w_t^*) \otimes \Gamma(w_t w_t^*) \otimes \mathrm{d}\Gamma(\omega_3) \big) \varphi \big\rangle,
\end{aligned}$$

where, for c_1, c_2 operators on $\mathfrak{H}_c$, the operator $\mathrm{d}\Gamma(c_1, c_2)$ on $\mathfrak{F}_a$ is defined by (see [1, 9])

$$\begin{aligned}
&\mathrm{d}\Gamma(c_1, c_2)\Omega_a = 0,\\
&\mathrm{d}\Gamma(c_1, c_2)|_{\otimes_a^n \mathfrak{H}_c} = \sum_{j=1}^{n} \underbrace{c_1 \otimes \cdots \otimes c_1}_{j-1} \otimes c_2 \otimes \underbrace{c_1 \otimes \cdots \otimes c_1}_{n-j}.
\end{aligned}$$

Combining (41), the bound $\|w_t w_t^*\| \leq 1$, and [1, Lemma 2.3] (see also [9, Lemma 2.8]), we obtain

$$\begin{aligned}
\big\|H_0^{\frac{1}{2}} W_t^* \varphi\big\|^2 &\leq \big\|(\mathrm{d}\Gamma(\omega)^{\frac{1}{2}} \otimes \mathbb{1} \otimes \mathbb{1})\varphi\big\|^2 + \big\|(\mathbb{1} \otimes \mathrm{d}\Gamma(\omega)^{\frac{1}{2}} \otimes \mathbb{1})\varphi\big\|^2\\
&\quad + \big\|(\mathbb{1} \otimes \mathbb{1} \otimes \mathrm{d}\Gamma(\omega_3)^{\frac{1}{2}})\varphi\big\|^2\\
&= \big\|H_0^{\frac{1}{2}} \varphi\big\|^2.
\end{aligned}$$

This concludes the proof. □

5.2 Regularity of the Hamiltonian with respect to the conjugate operator

Recall that the conjugate operator A is defined by the expressions (39) and (40). In this subsection, we prove the following proposition.

Proposition 5.2. *Assume that Hypothesis 3.6 holds. Let $|g| \ll \mathrm{m_e}$. Then we have that*

$$H \in \mathrm{C}^{1,1}(A_{\mathcal{G}}; A_{\mathcal{G}^*}),$$

in the sense of Hypothesis B.5 of Appendix B.

To prove Proposition 5.2, we use real interpolation. We have that

$$[H_0, iA] = N_+ \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes N_- \otimes \mathbb{1},$$

in the sense of quadratic forms on $\mathfrak{D}(H_0) \cap \mathfrak{D}(A)$. Since $\mathfrak{D}(H_0) \cap \mathfrak{D}(A)$ is a core for H_0 and since $N_+ \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes N_- \otimes \mathbb{1}$ is relatively H_0-bounded, Proposition 5.1 together with Proposition B.3 imply that H_0 belongs to $\mathrm{C}^1(A_{\mathcal{G}}; A_{\mathcal{G}^*})$. Next, since $[H_0, iA]$ commutes with A, we easily deduce that $H_0 \in \mathrm{C}^2(A_{\mathcal{G}}; A_{\mathcal{G}^*})$, and hence in particular $H_0 \in \mathrm{C}^{1,1}(A_{\mathcal{G}}; A_{\mathcal{G}^*})$. Here we recall that, for all $0 \leq \theta \leq 1$ and $1 \leq q < \infty$,

$$\begin{aligned}
\mathrm{C}^{\theta,q}(A_{\mathcal{G}}; A_{\mathcal{G}^*}) := \Big\{ &T \in \mathcal{B}(\mathcal{G}; \mathcal{G}^*), W_t^* T W_t - T \in \mathcal{B}(\mathcal{G}; \mathcal{G}^*) \text{ for all } t \in (0,1),\\
&\int_0^1 t^{-\theta q - 1} \big\|W_t^* T W_t - T\big\|^q_{\mathcal{B}(\mathcal{G}; \mathcal{G}^*)} \mathrm{d}t < \infty \Big\}. \qquad (42)
\end{aligned}$$

In order to prove that $H \in \mathrm{C}^{1,1}(A_{\mathcal{G}}; A_{\mathcal{G}^*})$, it remains to show that the interaction Hamiltonian $H_I \in \mathrm{C}^{1,1}(A_{\mathcal{G}}; A_{\mathcal{G}^*})$. Using in particular Proposition B.3, we see that it suffices, in fact, to verify that the commutator $[H_I, iA]$ belongs to $\mathcal{B}(\mathcal{G}; \mathcal{G}^*)$

and that $[H_I, iA] \in \mathrm{C}^{0,1}(A_{\mathcal{G}}; A_{\mathcal{G}^*})$. This is the purpose of the remainder of this section.

We use the notation (27). Using Hypothesis 3.6 and the estimates of Appendix A (see (80)–(81) and (83)–(86)), we can rewrite

$$F^{(\alpha)}(\xi_1,\, \xi_2,\, \xi_3) := \tilde{h}^{(\alpha)}(\xi_1, \xi_2, \xi_3)\tilde{G}^{(\alpha)}(\xi_1, \xi_2, \xi_3), \tag{43}$$

where $\tilde{h}^{(\alpha)}(\xi_1, \xi_2, \xi_3)$ is of the form

$$\tilde{h}^{(\alpha)}(\xi_1, \xi_2, \xi_3) = p_1 p_2 s^{(\alpha)}(\xi_1, \xi_2, \xi_3), \tag{44}$$

with $s^{(\alpha)}$ satisfying, for all $n, m \in \{0, 1, 2\}$,

$$\left|\partial_{p_1}^n \partial_{p_2}^m s^{(\alpha)}(\xi_1, \xi_2, \xi_3)\right| \lesssim p_1^{-n} p_2^{-m}, \tag{45}$$

in a neighborhood of 0.

Moreover the kernels $\tilde{G}^{(\alpha)}$ satisfy

(a_0) There exists a compact set $K \subset \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^3$ such that $\tilde{G}^{(\alpha)}(p_1, \gamma_1, p_2, \gamma_2, k, \lambda) = 0$ if $(p_1, p_2, k) \notin K$.

(b_0) There exists $\varepsilon > 0$ such that

$$\sum_{\gamma_1, \gamma_2, \lambda} \int (1 + x_1^2 + x_2^2)^{1+\varepsilon} \left|\hat{\tilde{G}}^{(\alpha)}(x_1, \gamma_1, x_2, \gamma_2, k, \lambda)\right|^2 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}k < \infty,$$

where, recall, $\hat{\tilde{G}}^{(\alpha)}$ denote the Fourier transform of $\tilde{G}^{(\alpha)}$ with respect to the variables (p_1, p_2), and x_l, $l = 1, 2$, is the variable dual to p_l.

(c_0) If $p_1 = 0$ or $p_2 = 0$, then $\tilde{G}^{(\alpha)}(p_1, \gamma_1, p_2, \gamma_2, k, \lambda) = 0$.

Our strategy consists in working with interaction operators of the form (20) with $H_I^{(1)}$, $H_I^{(2)}$ given by (25)–(26) and $F^{(1)}$, $F^{(2)}$ satisfying (43)–(45). We then use an interpolation argument for the kernels $\tilde{G}^{(\alpha)}$.

Lemma 5.3. *Consider the operator H_I of the form* (20) *with $H_I^{(1)}$, $H_I^{(2)}$ given by* (25)–(26) *and $F^{(1)}$, $F^{(2)}$ satisfying* (43)–(45).

(i) Suppose that $\tilde{G}^{(\alpha)} \in L^2(\Sigma \times \Sigma \times \Sigma_3)$ satisfy the following conditions

(i)(a) There exists a compact set $K \subset \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^3$ such that $\tilde{G}^{(\alpha)}(p_1, \gamma_1, p_2, \gamma_2, k, \lambda) = 0$ if $(p_1, p_2, k) \notin K$.

(i)(b)

$$\sum_{\gamma_1, \gamma_2, \lambda} \int (1 + x_1^2 + x_2^2) \left|\hat{\tilde{G}}^{(\alpha)}(x_1, \gamma_1, x_2, \gamma_2, k, \lambda)\right|^2 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}k < \infty.$$

(i)(c) If $p_1 = 0$ or $p_2 = 0$, then $\tilde{G}^{(\alpha)}(p_1, \gamma_1, p_2, \gamma_2, k, \lambda) = 0$.

Then $H_I' = [H_I, iA] \in \mathrm{C}^0(A_{\mathcal{G}}; A_{\mathcal{G}^*}) \equiv \mathcal{B}(\mathcal{G}; \mathcal{G}^*)$.

(ii) Suppose that $\tilde{G}^{(\alpha)} \in L^2(\Sigma \times \Sigma \times \Sigma_3)$ *satisfy the following conditions*

(ii)(a) There exists a compact set $K \subset \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^3$ *such that* $\tilde{G}^{(\alpha)}(p_1, \gamma_1, p_2, \gamma_2, k, \lambda) = 0$ *if* $(p_1, p_2, k) \notin K$.

(ii)(b)

$$\sum_{\gamma_1, \gamma_2, \lambda} \int (1 + x_1^2 + x_2^2)^3 \left| \hat{\tilde{G}}^{(\alpha)}(x_1, \gamma_1, x_2, \gamma_2, k, \lambda) \right|^2 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}k < \infty.$$

(ii)(c) If $p_1 = 0$ *or* $p_2 = 0$, *then* $D^\beta \tilde{G}^{(\alpha)}(p_1, \gamma_1, p_2, \gamma_2, k, \lambda) = 0$ *for all multi-index* $\beta = (\beta_1, \beta_2)$, $|\beta| \leq 2$, *with* $D^\beta = \partial^{\beta_1 + \beta_2} / \partial_{x_1^{\beta_1}} \partial_{x_2^{\beta_2}}$.

Then $H_I' = [H_I, iA] \in \mathrm{C}^1(A_{\mathcal{G}}; A_{\mathcal{G}^*})$.

Proof. (i) Recall that the conjugate operator A is defined by Eq. (40), with

$$a = if(p)\partial_p + \frac{i}{2} f'(p),$$

and $f(p) = p^{-1}\sqrt{p^2 + \mathrm{m_e}^2}$. We use the notation $a_l = if(p_l)\partial_{p_l} + \frac{i}{2} f'(p_l)$, for $l = 1, 2$. We then have that

$$[H_I,\, iA] = H_I(-ia_1 F) + H_I(-ia_2 F)\,, \tag{46}$$

in the sense of quadratic forms on $\mathfrak{D}(H_0) \cap \mathfrak{D}(A)$.

Recalling the notations $\xi_l = (p_l, \gamma_l)$, we compute

$$\begin{aligned}(a_1 F^{(\alpha)})(\xi_1, \xi_2, \xi_3) = &\Big[\frac{i}{2} p_1 p_2 f'(p_1) s^{(\alpha)}(\xi_1, \xi_2, \xi_3) + ip_2 f(p_1) s^{(\alpha)}(\xi_1, \xi_2, \xi_3) \\ &+ ip_1 p_2 f(p_1)(\partial_{p_1} s^{(\alpha)})(\xi_1, \xi_2, \xi_3)\Big] \tilde{G}^{(\alpha)}(\xi_1, \xi_2, \xi_3) \\ &+ ip_1 p_2 f(p_1) s^{(\alpha)}(\xi_1, \xi_2, \xi_3)(\partial_{p_1} \tilde{G}^{(\alpha)})(\xi_1, \xi_2, \xi_3). \end{aligned} \tag{47}$$

Using (45) and the definition of f, we see that the term in brackets satisfy

$$\begin{aligned}&\Big| \frac{i}{2} p_1 p_2 f'(p_1) s^{(\alpha)}(\xi_1, \xi_2, \xi_3) + ip_2 f(p_1) s^{(\alpha)}(\xi_1, \xi_2, \xi_3) + ip_1 p_2 f(p_1)(\partial_{p_1} s^{(\alpha)})(\xi_1, \xi_2, \xi_3) \Big| \\ &\lesssim p_1^{-1} p_2,\end{aligned}$$

in any compact set. Now, since $p_1 \mapsto \tilde{G}^{(\alpha)}(p_1, \gamma_1, \xi_2, \xi_3) \in H_0^1(\mathbb{R}_+)$ by the conditions $(i)(b)$ and $(i)(c)$, and since $\tilde{G}^{(\alpha)}$ is compactly supported in the variables (p_1, p_2, k) by the condition $(i)(a)$, we deduce that

$$p_1^{-1} p_2 \tilde{G}^{(\alpha)}(\xi_1, \xi_2, \xi_3) \in L^2(\mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3).$$

Here we used that

$$\left\| p_1^{-1} p_2 \tilde{G}^{(\alpha)}(\xi_1, \xi_2, \xi_3) \right\|_{L^2(\mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3)} \lesssim \left\| p_2 \partial_{p_1} \tilde{G}^{(\alpha)}(\xi_1, \xi_2, \xi_3) \right\|_{L^2(\mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3)},$$

by Hardy's inequality at the origin in $H^1_0(\mathbb{R}_+)$. Likewise, we have that

$$\big|ip_1p_2f(p_1)s^{(\alpha)}(\xi_1,\xi_2,\xi_3)\big| \lesssim 1,$$

in any compact set, and hence, using again that $p_1 \mapsto \tilde{G}^{(\alpha)}(p_1,\gamma_1,\xi_2,\xi_3) \in H^1_0(\mathbb{R}_+)$ and that $\tilde{G}^{(\alpha)}$ is compactly supported in the variables (p_1,p_2,k), it follows that

$$ip_1p_2f(p_1)s^{(\alpha)}(\xi_1,\xi_2,\xi_3)(\partial_{p_1}\tilde{G}^{(\alpha)})(\xi_1,\xi_2,\xi_3) \in L^2(\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3).$$

The previous estimates show that

$$(a_1F^{(\alpha)})(\xi_1,\xi_2,\xi_3) \in L^2(\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3),$$

and proceeding in the same way, one verifies that $(a_2F^{(\alpha)})(\xi_1,\xi_2,\xi_3) \in L^2(\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3)$. Using the expression (46) of the commutator $[H_I, iA]$ and the N_τ estimates of [11], we immediately deduce that $[H_I, iA] \in \mathcal{B}(\mathcal{G};\mathcal{G}^*) = \mathrm{C}^0(A_{\mathcal{G}};A_{\mathcal{G}^*})$.

(ii) It suffices to proceed similarly. More precisely, we compute the second commutator

$$\big[\big[H_I, iA\big], iA\big] = -H_I(a_1^2F) - H_I(a_2^2F) - 2H_I(a_1a_2F). \tag{48}$$

Computing a_1^2F, a_2^2F and a_1a_2F yields to several terms that are estimated separately. Each term, however, can be treated in the same way, using Hardy's inequality together with the assumptions $(ii)(a)$, $(ii)(b)$, $(ii)(c)$. We give an example. Consider the first term inside the brackets of (47) and apply to it the operator $if(p_1)\partial_{p_1}$. This gives in particular a term of the form

$$-\frac{1}{2}p_2f(p_1)f'(p_1)s^{(\alpha)}(\xi_1,\xi_2,\xi_3)\tilde{G}^{(\alpha)}(\xi_1,\xi_2,\xi_3),$$

that will appear in the expression of a_1^2F. From (45) and the definition of f, it follows that

$$\big|p_2f(p_1)f'(p_1)s^{(\alpha)}(\xi_1,\xi_2,\xi_3)\tilde{G}^{(\alpha)}(\xi_1,\xi_2,\xi_3)\big| \lesssim p_1^{-3}p_2\big|\tilde{G}^{(\alpha)}(\xi_1,\xi_2,\xi_3)\big|,$$

in any compact set. Since $p_1 \mapsto \tilde{G}^{(\alpha)}(p_1,\gamma_1,\xi_2,\xi_3) \in H^3_0(\mathbb{R}_+)$ by the conditions $(ii)(b)$ and $(ii)(c)$, and since $\tilde{G}^{(\alpha)}$ is compactly supported in the variables (p_1,p_2,k) by the condition $(ii)(a)$, we obtain as above that

$$p_2f(p_1)f'(p_1)s^{(\alpha)}(\xi_1,\xi_2,\xi_3)\tilde{G}^{(\alpha)}(\xi_1,\xi_2,\xi_3) \in L^2(\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3).$$

Here we used that

$$\big\|p_1^{-3}p_2\tilde{G}^{(\alpha)}(\xi_1,\xi_2,\xi_3)\big\|_{L^2(\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3)} \lesssim \big\|p_2\partial_{p_1}^3\tilde{G}^{(\alpha)}(\xi_1,\xi_2,\xi_3)\big\|_{L^2(\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3)},$$

by Hardy's inequality at the origin in $H^3_0(\mathbb{R}_+)$. Treating all the other terms in a similar manner, we deduce that

$$a_1^2F + a_2^2F + 2a_1a_2F \in L^2(\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3),$$

and therefore that $[[H_I, iA], iA] \in \mathcal{B}(\mathcal{G};\mathcal{G}^*)$. Together with Proposition 5.1, this shows (ii). □

Proof of Proposition 5.2. By the observation after the statement of Proposition 5.2, we already know that $H_0 \in \mathrm{C}^{1,1}(A_{\mathcal{G}}; A_{\mathcal{G}^*})$. Hence, to conclude the proof of Proposition 5.2, it suffices to verify that $H_I \in \mathrm{C}^{1,1}(A_{\mathcal{G}}; A_{\mathcal{G}^*})$. Recall that H_I is the sum of 4 terms, see (20). We consider for instance the first one, $H_I^{(1)}$. The other terms can be treated in the same way.

Let $K_0 \subset \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^3$ be a compact set. Let $S_{(i)}$ denote the set of all $\tilde{G}^{(1)} \in L^2(\Sigma \times \Sigma \times \Sigma_3)$ satisfying the conditions $(i)(a)$ (with $K = K_0$), $(i)(b)$ and $(i)(c)$, equipped with the norm

$$\|\tilde{G}^{(1)}\|_{S_{(i)}} := \sum_{\gamma_1,\gamma_2,\lambda} \int (1 + x_1^2 + x_2^2) \left|\hat{\tilde{G}}^{(1)}(x_1, \gamma_1, x_2, \gamma_2, k, \lambda)\right|^2 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}k.$$

Likewise, we denote by $S_{(ii)}$ the set of all $\tilde{G}^{(1)} \in L^2(\Sigma \times \Sigma \times \Sigma_3)$ satisfying the conditions $(ii)(a)$ (with $K = K_0$), $(ii)(b)$ and $(ii)(c)$, equipped with the norm

$$\|\tilde{G}^{(1)}\|_{S_{(ii)}} := \sum_{\gamma_1,\gamma_2,\lambda} \int (1 + x_1^2 + x_2^2)^3 \left|\hat{\tilde{G}}^{(1)}(x_1, \gamma_1, x_2, \gamma_2, k, \lambda)\right|^2 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}k.$$

By Lemma 5.3 and its proof, the map

$$S_{(i)} \ni \tilde{G}^{(1)} \mapsto H_I^{(1)'}(\tilde{h}^{(1)}\tilde{G}^{(1)}) \in \mathrm{C}^0(A_{\mathcal{G}}; A_{\mathcal{G}^*}) \tag{49}$$

is linear and continuous, and, likewise, the map

$$S_{(ii)} \ni \tilde{G}^{(1)} \mapsto H_I^{(1)'}(\tilde{h}^{(1)}\tilde{G}^{(1)}) \in \mathrm{C}^1(A_{\mathcal{G}}; A_{\mathcal{G}^*}) \tag{50}$$

is linear and continuous. Here we have used the notation

$$H_I^{(1)'}(\tilde{h}^{(1)}\tilde{G}^{(1)}) := [H_I^{(1)}(\tilde{h}^{(1)}\tilde{G}^{(1)}), iA].$$

By real interpolation, we deduce that

$$\big(S_{(i)}, S_{(ii)}\big)_{\theta,2} \ni \tilde{G}^{(1)} \mapsto H_I^{(1)'}(\tilde{h}^{(1)}\tilde{G}^{(1)}) \in \big(\mathrm{C}^0(A_{\mathcal{G}}; A_{\mathcal{G}^*}), \mathrm{C}^1(A_{\mathcal{G}}; A_{\mathcal{G}^*})\big)_{\theta,2}, \tag{51}$$

for all $0 \leq \theta \leq 1$.

Now, by [2, Section 5], we have that

$$\big(\mathrm{C}^0(A_{\mathcal{G}}; A_{\mathcal{G}^*}), \mathrm{C}^1(A_{\mathcal{G}}; A_{\mathcal{G}^*})\big)_{\theta,2} = \mathrm{C}^{\theta,2}(A_{\mathcal{G}}; A_{\mathcal{G}^*}), \tag{52}$$

for all $0 < \theta < 1$, and using the definition (42), one easily verifies that

$$\mathrm{C}^{\theta,2}(A_{\mathcal{G}}; A_{\mathcal{G}^*}) \subset \mathrm{C}^{0,1}(A_{\mathcal{G}}; A_{\mathcal{G}^*}).$$

On the other hand, from the definition of the interpolated space $\big(S_{(i)}, S_{(ii)}\big)_{\theta,2}$ and mimicking the method allowing one to compute the interpolation of Sobolev spaces (see e.g. [30]), it is not difficult to verify that, for $0 < \varepsilon < 2\theta < 1$, the set of all kernels $\tilde{G}^{(1)} \in L^2(\Sigma \times \Sigma \times \Sigma_3)$ satisfying the conditions (a_0), (b_0) and (c_0) stated above is included in $\big(S_{(i)}, S_{(ii)}\big)_{\theta,2}$. This shows, in particular, that $H_I^{(1)'} \in \mathrm{C}^{0,1}(A_{\mathcal{G}}; A_{\mathcal{G}^*})$, and hence that $H_I^{(1)} \in \mathrm{C}^{1,1}(A_{\mathcal{G}}; A_{\mathcal{G}^*})$. Since the other terms, $H_I^{(1)^*}$, $H_I^{(2)}$ and $H_I^{(2)^*}$, can be treated in the same way, this concludes the proof. □

5.3 The Mourre estimate

Given $\tilde{F} = (\tilde{F}^{(1)}, \tilde{F}^{(2)}) \in (\mathfrak{H}_c \otimes \mathfrak{H}_c \otimes L^2(\Sigma_3))^2$, and for $H_I^{(i)}(\tilde{F}^{(i)})$ given by (25)-(26), we define

$$H_I(\tilde{F}) = H_I^{(1)}(\tilde{F}^{(1)}) + (H_I^{(1)}(\tilde{F}^{(1)}))^* + H_I^{(2)}(\tilde{F}^{(2)}) + (H_I^{(2)}(\tilde{F}^{(2)}))^* .$$

Proposition 5.4. *Assume that Hypothesis 3.6 hold and let $\delta \in (0, \mathrm{m_e})$. There exist $g_\delta > 0$, $\mathrm{c}_\delta > 0$ and $\mathrm{C} \in \mathbb{R}$ such that, for all $|g| \leq g_\delta$, and for*

$$\Delta := [\delta, \mathrm{m}_{\mathrm{Z}^0} - \delta],$$

we have, in the sense of quadratic forms on $\mathcal{D}(A) \cap \mathcal{D}(H_0)$,

$$H' \equiv [H, iA] \geq \mathrm{c}_\delta \mathbb{1} - \mathrm{C} \mathbb{1}_\Delta^\perp (H - E) \langle H \rangle, \tag{53}$$

where we have set $E := \inf \mathrm{spec}(H)$, $\mathbb{1}_\Delta^\perp (H - E) := \mathbb{1} - \mathbb{1}_\Delta (H - E)$ and $\langle H \rangle := (\mathbb{1} + H^2)^{1/2}$.

Proof of Proposition 5.4. As in Subsection 5.2, we have, in the sense of quadratic forms on $\mathcal{D}(A) \cap \mathcal{D}(H_0)$,

$$[H_0, iA] = N_+ \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes N_- \otimes \mathbb{1} , \tag{54}$$

where N_+ (respectively N_-) is the number operator for electrons (respectively positrons) as defined in (14). In the sequel, by abuse of notation, we shall omit the identity operators in $N_+ \otimes \mathbb{1} \otimes \mathbb{1}$ and $\mathbb{1} \otimes N_- \otimes \mathbb{1}$ and denote them respectively again by N_+ and N_-.

Let $a_1 = a \otimes \mathbb{1} \otimes \mathbb{1}$ be the conjugate operator for electron acting on the p_1 variable in $\mathfrak{H}_c \otimes \mathfrak{H}_c \otimes L^2(\Sigma_3)$ and $a_2 = \mathbb{1} \otimes a \otimes \mathbb{1}$ be the conjugate operator for positron acting on the p_2 variable. As in (46), we have that

$$[H_I, iA] = H_I(-ia_1 F) + H_I(-ia_2 F) , \tag{55}$$

in the sense of quadratic forms on $\mathcal{D}(A) \cap \mathcal{D}(H_0)$. Here we recall that $a_1 F$ and $a_2 F$ belong to $L^2(\mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3)$ as follows from the estimates of Appendix A and Hypothesis 3.6 (see more precisely the proof of Lemma 5.3 (i)).

For $P_{\Omega_a \times \Omega_a} := P_{\Omega_a} \otimes P_{\Omega_a} \otimes \mathbb{1}$ being the projection onto the electron/positron vacuum, we have that

$$N_+ + N_- + P_{\Omega_a \times \Omega_a} \geq \mathbb{1}. \tag{56}$$

Since $H = H_0 + gH_I$, and for $E = \inf \mathrm{spec}(H)$, we obtain from (54)-(55) that

$$\begin{aligned} [H, iA] &= (N_+ + N_- + P_{\Omega_a \times \Omega_a}) - P_{\Omega_a \times \Omega_a} + g\,(H_I(-ia_1 F) + H_I(-ia_2 F)) \\ &\geq \mathbb{1} - P_{\Omega_a \times \Omega_a} + g\,(H_I(-ia_1 F) + H_I(-ia_2 F)) , \end{aligned} \tag{57}$$

where we used the operator inequality (56) in the last inequality. We estimate separately the two remainder terms occuring in the right hand side of (57).

Let us define a function $f_\Delta \in \mathrm{C}_0^\infty(\mathbb{R})$ such that $0 \leq f_\Delta \leq 1$ and

$$f_\Delta(\lambda) = \begin{cases} 1 & \text{if } \lambda \in [\delta, \mathrm{m}_{\mathrm{Z}^0} - \delta], \\ 0 & \text{if } \lambda < \delta/2 \text{ or } \lambda > \mathrm{m}_{\mathrm{Z}^0} - \delta/2. \end{cases} \tag{58}$$

We observe that

$$P_{\Omega_a \times \Omega_a} f_\Delta(H_0) = 0\,. \tag{59}$$

The last identity holds because $P_{\Omega_a \times \Omega_a}$ is a projection commuting with H_0 and because $\mathrm{supp}(f_\Delta) \cap \mathrm{spec}(H_0 P_{\Omega_a \times \Omega_a}) = \emptyset$. As in the proof of Lemma 4.3, let $\tilde{f} \in \mathrm{C}_0^\infty(\mathbb{C})$ denote an almost analytic extension of f_Δ satisfying $\tilde{f}|_{\mathbb{R}} = f_\Delta$ and $|\partial_{\bar{z}} \tilde{f}(z)| \leq C_n |\mathrm{Im}\, z|^n$ for any $n \in \mathbb{N}$. Thus, for $\mathrm{d}\tilde{f}(z) := -\frac{1}{\pi} \frac{\partial \tilde{f}}{\partial \bar{z}}(z)\, \mathrm{dRe}\, z\, \mathrm{dIm}\, z$, using Helffer-Sjöstrand functional calculus and the second resolvent equation, we obtain

$$\begin{aligned} f_\Delta(H - E) - f_\Delta(H_0) &= \int (H - E - z)^{-1} (H - E - H_0)(H_0 - z)^{-1}\, \mathrm{d}\tilde{f}(z) \\ &= \int (H - E - z)^{-1} g H_I(F)(H_0 - z)^{-1}\, \mathrm{d}\tilde{f}(z) \\ &\quad - E \int (H - E - z)^{-1} (H_0 - z)^{-1}\, \mathrm{d}\tilde{f}(z)\,. \end{aligned} \tag{60}$$

From Corollary 4.1, since Hypothesis 3.1 holds, there exists a constant C such that

$$\|H_I(F)(H_0 + 1)^{-1}\| \leq C K(G)\,, \tag{61}$$

where $h^{(\alpha)} G^{(\alpha)} = F^{(\alpha)}$ (see (27)) and $K(G) = K_2(G)$ is given by (31) and (33).

Therefore, with the inequality

$$\left\|(H_0 + 1)(H_0 - z)^{-1}\right\| \leq 1 + \frac{1 + |z|}{|\mathrm{Im}\, z|}\,, \tag{62}$$

and the properties of $\tilde{f}$, we obtain that there exists a constant $C_1 > 0$ depending only on f_Δ and $K(G)$ such that

$$\begin{aligned} &\left\| \int (H - E - z)^{-1} g H_I(F)(H_0 - z)^{-1}\, \mathrm{d}\tilde{f}(z) \right\| \\ &\leq |g| \int (1 + \frac{1 + |z|}{|\mathrm{Im}\, z|})\, \|(H - E - z)^{-1}\|\, \|H_I(G)(H_0 + 1)^{-1}\|\, \mathrm{d}\tilde{f}(z) \leq C_1\, |g|\,. \end{aligned} \tag{63}$$

Moreover, using again (61), standard perturbation theory yields that there exists $g_1 > 0$ such that for all $|g| \leq g_1$, we have

$$|E| \leq |g|\, \frac{K(G) B_{\beta\eta}}{1 - g_1 K(G) C_{\beta\eta}}\,, \tag{64}$$

where $B_{\beta\eta}$ and $C_{\beta\eta}$ are the positive constants defined in Subsection 4.1. Thus, there exists a constant C_2 depending on f_Δ and $K(G)$ such that

$$\left\| E \int (H - E - z)^{-1} (H_0 - z)^{-1}\, \mathrm{d}\tilde{f}(z) \right\| \leq C_2 |g|\,. \tag{65}$$

Inequalities (60), (63) and (65) give

$$\|f_\Delta(H-E) - f_\Delta(H_0)\| \leq (C_1 + C_2)\,|g|. \tag{66}$$

For shortness, let $\mathbb{1}_\Delta \equiv \mathbb{1}_\Delta(H-E)$ and $\mathbb{1}_\Delta^\perp \equiv \mathbb{1}_\Delta^\perp(H-E)$. We have that

$$\begin{aligned} -P_{\Omega_a\times\Omega_a} &= -\mathbb{1}_\Delta P_{\Omega_a\times\Omega_a}\mathbb{1}_\Delta - \mathbb{1}_\Delta P_{\Omega_a\times\Omega_a}\mathbb{1}_\Delta^\perp - \mathbb{1}_\Delta^\perp P_{\Omega_a\times\Omega_a}\mathbb{1}_\Delta - \mathbb{1}_\Delta^\perp P_{\Omega_a\times\Omega_a}\mathbb{1}_\Delta^\perp \\ &\geq -\mathbb{1}_\Delta P_{\Omega_a\times\Omega_a}\mathbb{1}_\Delta - \mathbb{1}_\Delta P_{\Omega_a\times\Omega_a}\mathbb{1}_\Delta^\perp - \mathbb{1}_\Delta^\perp P_{\Omega_a\times\Omega_a}\mathbb{1}_\Delta - \mathbb{1}_\Delta^\perp. \end{aligned} \tag{67}$$

Using (59) and (66), we obtain that

$$\begin{aligned} \big\|\mathbb{1}_\Delta P_{\Omega_a\times\Omega_a}\big\| &\leq \big\|f_\Delta(H-E)P_{\Omega_a\times\Omega_a}\big\| \\ &= \big\|\big(f_\Delta(H-E) - f_\Delta(H_0)\big)P_{\Omega_a\times\Omega_a}\big\| \leq (C_1+C_2)\,|g|, \end{aligned}$$

from which we deduce that

$$-\mathbb{1}_\Delta P_{\Omega_a\times\Omega_a}\mathbb{1}_\Delta - \mathbb{1}_\Delta P_{\Omega_a\times\Omega_a}\mathbb{1}_\Delta^\perp - \mathbb{1}_\Delta^\perp P_{\Omega_a\times\Omega_a}\mathbb{1}_\Delta \geq -3(C_1+C_2)\,|g|\,\mathbb{1}.$$

Together with (67), this shows that

$$-P_{\Omega_a\times\Omega_a} \geq -3(C_1+C_2)|g|\mathbb{1} - \mathbb{1}_\Delta^\perp. \tag{68}$$

To bound the last term in the right hand side of (57), it suffices to use the relative bound in Corollary 4.1 and the fact that Hypothesis 3.6 holds (and hence also Hypothesis 3.1), to obtain that the operators $H_I(-ia_lF)$ $(l=1,2)$ are norm relatively bounded with respect to H_0 with relative bounds depending on $K(G)$ and $K(-ia_lG)$. Therefore, there exists C_3 depending on $K(G)$ and $K(-ia_lG)$ such that

$$\begin{aligned} g\big(H_I(-ia_1F) + H_I(-ia_2F)\big) &\geq -C_3|g|\langle H\rangle \\ &= -C_3|g|\langle H\rangle\mathbb{1}_\Delta(H-E) - C_3|g|\langle H\rangle\mathbb{1}_\Delta^\perp(H-E) \\ &\geq -C_4|g|\mathbb{1}_\Delta(H-E) - C_3|g|\langle H\rangle\mathbb{1}_\Delta^\perp(H-E) \\ &\geq -C_4|g|\mathbb{1} - C_5|g|\langle H\rangle\mathbb{1}_\Delta^\perp(H-E), \end{aligned} \tag{69}$$

for some constants $C_4, C_5 \in \mathbb{R}$.

The estimates (57), (68) and (69) yield (53), which concludes the proof. □

5.4 Proofs of the main theorems

Proof of Theorem 3.8. As above, we use the notation $E = \inf\operatorname{spec}(H)$. The proof of Theorem 3.8 is divided into two main steps.

Step 1. Let $0 < \delta < \mathrm{m_e}$. There exists $g_\delta > 0$ such that, for all $0 \leq |g| \leq g_\delta$,

$$\inf\big(\operatorname{spec}(H)\setminus\{E\}\big) \geq \delta.$$

To prove this, we use the min-max principle. Let μ_2 denote the second point above E in the spectrum of H. The min-max principle implies that

$$\mu_2 \geq \inf_{\substack{\psi\in\mathfrak{D}(H),\,\|\psi\|=1,\\ \psi\in[\Omega_D\otimes\Omega_s]^\perp}} \langle\psi, H\psi\rangle = \inf_{\substack{\psi\in\mathfrak{D}(H),\,\|\psi\|=1,\\ \psi\in[\Omega_D\otimes\Omega_s]^\perp}} \big(\langle\psi, H_0\psi\rangle + g\langle\psi, H_I\psi\rangle\big),$$

where $[\Omega_D \otimes \Omega_s]^\perp$ denotes the orthogonal complement of the subspace spanned by $\Omega_D \otimes \Omega_s$ in the total Hilbert space $\mathcal{H}$. Since H_I is relatively bounded with respect to H_0, there exists a positive constant C such that $\langle \psi, H_I \psi \rangle \geq -\mathrm{C}\langle \psi, H_0 \psi \rangle$, and therefore

$$\mu_2 \geq \inf_{\substack{\psi \in \mathfrak{D}(H), \|\psi\| = 1, \\ \psi \in [\Omega_D \otimes \Omega_s]^\perp}} (1 - \mathrm{C}|g|)\langle \psi, H_0 \psi \rangle \geq (1 - \mathrm{C}|g|)\mathrm{m_e},$$

the last inequality being a consequence of (17). This proves Step 1.

Step 2. Let $0 < \delta < \mathrm{m_e}$. There exists $g_\delta > 0$ such that, for all $0 \leq |g| \leq g_\delta$,

$$\mathrm{spec}(H) \cap [\delta, \mathrm{m_e} + E) = \emptyset.$$

Observe that $E < 0$ satisfies $E \geq -\mathrm{C}|g|$ with C a positive constant, as follows from standard perturbation theory (see (64)), and therefore, for g_δ small enough and $|g| \leq g_\delta$, we have that $\delta < \mathrm{m_e} + E$. By Theorem 3.5, we know that $\inf \mathrm{spec}_{\mathrm{ess}}(H) = \mathrm{m_e} + E$. Thus we only have to show that H do not have discrete eigenvalue in the interval $[\delta, \mathrm{m_e} + E)$: This is a simple, usual consequence of the virial theorem (see Theorem B.4) combined with the Mourre estimate of Proposition 5.4. □

We introduce the notation $\langle A \rangle = (1 + A^* A)^{1/2} = (1 + |A|^2)^{1/2}$ for any closed operator A. As mentioned before, Theorem 3.9 is a consequence of the following stronger result, which itself follows from Propositions 5.1, 5.2, 5.4, and the abstract results of Appendix B.

Theorem 5.5 (Limiting absorption principle)**.** *Assume that Hypothesis 3.6 holds with $\varepsilon > 0$ in Hypothesis 3.6(ii). For all $\delta > 0$, there exists $g_\delta > 0$ such that, for all $|g| \leq g_\delta$ and $1/2 < s \leq 1$,*

$$\sup_{z \in \tilde{\Delta}} \|\langle A \rangle^{-s} (H - z)^{-1} \langle A \rangle^{-s}\| < \infty,$$

with $\Delta := [\inf \mathrm{spec}(H) + \mathrm{m_e}, \inf \mathrm{spec}(H) + \mathrm{m}_{\mathrm{Z^o}} - \delta]$ and $\tilde{\Delta} := \{z \in \mathbb{C}, \mathrm{Re}\, z \in \Delta, 0 < |\mathrm{Im}\, z| \leq 1\}$,. Moreover, the map $z \mapsto \langle A \rangle^{-s} (H - z)^{-1} \langle A \rangle^{-s} \in \mathcal{B}(\mathcal{H})$ is uniformly Hölder continuous of order $s - 1/2$ on $\tilde{\Delta}$ and the limits

$$\langle A \rangle^{-s} (H - \lambda - i0^\pm)^{-1} \langle A \rangle^{-s} := \lim_{\varepsilon \to 0^\pm} \langle A \rangle^{-s} (H - \lambda - i\varepsilon)^{-1} \langle A \rangle^{-s},$$

exist in the norm topology of $\mathcal{B}(\mathcal{H})$, uniformly in $\lambda \in \Delta$. Finally, the map $\lambda \mapsto \langle A \rangle^{-s} (H - \lambda - i0^\pm)^{-1} \langle A \rangle^{-s} \in \mathcal{B}(\mathcal{H})$ is uniformly Hölder continuous of order $s - 1/2$ on Δ and, for any $1/2 < s \leq 1$, H satisfies the local decay property

$$\left\| \langle A \rangle^{-s} e^{-itH} \mathbb{1}_\Delta(H) \langle A \rangle^{-s} \right\| \lesssim \langle t \rangle^{-s + \frac{1}{2}}, \tag{70}$$

for all $t \in \mathbb{R}$.

Proof. By Propositions 5.1, 5.2 and 5.4, we see that Hypotheses B.1, B.5 and B.6 of Appendix B are satisfied, the open interval I of Hypothesis B.6 being chosen, for instance, as $I = (\inf \mathrm{spec}(H) + \mathrm{m_e} - \delta, \inf \mathrm{spec}(H) + \mathrm{m}_{\mathrm{Z^o}} - \delta/2)$. Therefore we can apply Theorem B.7 with $J = \Delta$, which proves Theorem 5.5. □

A Generalized eigenfunctions of the free Dirac operator

In this section we describe the properties of the generalized eigenfunctions of the Dirac operator H_D introduced in subsection 2.1.1. More details can be found in [17, section 9.9, (44), (45), (63)].

Recall that the generalized eigenfunctions of H_D are labeled by the angular momentum quantum numbers

$$j \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}, \qquad m_j \in \{-j, -j+1, \ldots, j-1, j\},$$

and by the quantum numbers

$$\kappa_j \in \{\pm(j+\frac{1}{2})\}\,.$$

We define, for $\gamma_j := |\kappa_j|$,

$$\begin{aligned} g_{\kappa_j,\pm}(p,r) &= \frac{C_1^{\pm}}{|\omega(p)|^{\frac{1}{2}}} \frac{(2pr)^{\gamma_j}}{r} \frac{1}{2\sqrt{\pi}} \frac{\Gamma(\gamma_j)}{\Gamma(2\gamma_j+1)} \\ &\times \left\{ \mathrm{e}^{-ipr} \mathrm{e}^{i\eta_j} \gamma_j F(\gamma_j+1,\, 2\gamma_j+1,\, 2ipr) + \mathrm{e}^{ipr} \mathrm{e}^{-i\eta_j} \gamma_j F(\gamma_j+1,\, 2\gamma_j+1,\, -2ipr) \right\} \end{aligned} \tag{71}$$

with $C_1^+ = \sqrt{\omega(p)+\mathrm{m_e}}$ when we consider a positive energy $\omega(p) > \mathrm{m_e}$ and $C_1^- = \sqrt{\omega(p)-\mathrm{m_e}}$ when we consider a negative energy $-\omega(p) < -\mathrm{m_e}$.

We also define

$$\begin{aligned} f_{\kappa_j,\pm}(p,r) &= \frac{iC_2^{\pm}}{|\omega(p)|^{\frac{1}{2}}} \frac{1}{2\sqrt{\pi}} \frac{(2pr)^{\gamma_j}}{r} \frac{\Gamma(\gamma_j)}{\Gamma(2\gamma_j+1)} \\ &\times \left\{ \mathrm{e}^{-ipr} \mathrm{e}^{i\eta_j} \gamma_j F(\gamma_j+1,\, 2\gamma_j+1,\, 2ipr) - \mathrm{e}^{ipr} \mathrm{e}^{-i\eta_j} \gamma_j F(\gamma_j+1,\, 2\gamma_j+1,\, -2ipr) \right\} \end{aligned} \tag{72}$$

with $C_2^+ = \sqrt{\omega(p)-\mathrm{m_e}}$, for energies $\omega(p) > \mathrm{m_e}$ and $C_2^- = -\sqrt{\omega(p)+\mathrm{m_e}}$ for energies $-\omega(p) < -\mathrm{m_e}$.

The functions F that occur in (71) and (72) are the confluent hypergeometric functions. Their integral representations for $\gamma_j > 1/2$ are

$$F(\gamma_j+1,\, 2\gamma_j+1,\, \pm 2ipr) = \frac{\Gamma(2\gamma_j+1)}{\Gamma(\gamma_j+1)\Gamma(\gamma_j)} \int_0^1 \mathrm{e}^{\pm 2ipru} u^{\gamma_j} (1-u)^{\gamma_j} \mathrm{d}u\,. \tag{73}$$

The generalized eigenfunctions

$$\psi_{\pm,(j,m_j,\kappa_j)}(p,x) = \psi_{\pm,\gamma}(p,x) = \psi_{\pm}(\xi,x)\,,$$

where $+$ refers to positive energies $\omega(p) > \mathrm{m_e}$ and $-$ refers to negative energies $-\omega(p) < -\mathrm{m_e}$, fulfill

$$H_D\ \psi_{\pm}((p,\gamma),x) = \pm\,\omega(p)\ \psi_{\pm}((p,\gamma),x)\,,$$

and are defined by

$$\psi_{\pm,(j,m_j,\kappa_j)}(p,x) := \begin{pmatrix} ig_{\kappa_j,\pm}(p,r)\Phi^{(1)}_{m_j,\kappa_j}(\theta,\varphi) \\ -f_{\kappa_j,\pm}(p,r)\Phi^{(2)}_{(m_j,\kappa_j)}(\theta,\varphi) \end{pmatrix} \tag{74}$$

where the spinors $\Phi^{(1)}_{m_j,\kappa_j}$ and $\Phi^{(2)}_{m_j,\kappa_j}$ are orthogonal and defined by

$$\Phi^{(1)}_{m_j,(j+\frac{1}{2})}(\theta,\varphi) := \begin{pmatrix} -\sqrt{\frac{j-m_j+1}{2j+2}}Y_{j+\frac{1}{2},m_j-\frac{1}{2}}(\theta,\varphi) \\ \sqrt{\frac{j+m_j+1}{2j+2}}Y_{j+\frac{1}{2},m_j-\frac{1}{2}}(\theta,\varphi) \end{pmatrix} \tag{75}$$

$$\Phi^{(2)}_{m_j,(j+\frac{1}{2})}(\theta,\varphi) := \begin{pmatrix} \sqrt{\frac{j+m_j}{2j}}Y_{j-\frac{1}{2},m_j-\frac{1}{2}}(\theta,\varphi) \\ \sqrt{\frac{j-m_j}{2j}}Y_{j-\frac{1}{2},m_j+\frac{1}{2}}(\theta,\varphi) \end{pmatrix} \tag{76}$$

and

$$\begin{aligned} \Phi^{(1)}_{m_j,-(j+\frac{1}{2})}(\theta,\varphi) =& \Phi^{(2)}_{m_j,(j+\frac{1}{2})}(\theta,\varphi) \\ \Phi^{(2)}_{m_j,-(j+\frac{1}{2})}(\theta,\varphi) =& \Phi^{(1)}_{m_j,(j+\frac{1}{2})}(\theta,\varphi). \end{aligned} \tag{77}$$

It follows from (19) that

$$\widetilde{\psi}_{-,(j,m_j,\kappa_j)}(p,x) := \begin{pmatrix} ig_{-\kappa_j,-}(p,r)\Phi^{(1)}_{-m_j,-\kappa_j}(\theta,\varphi) \\ -f_{-\kappa_j,-}(p,r)\Phi^{(2)}_{-m_j,-\kappa_j}(\theta,\varphi) \end{pmatrix}$$

For positive energies $\omega(p) > \mathrm{m_e}$, we have the following estimates for the functions $g_{\kappa_j,\pm}$ and $f_{\kappa_j,\pm}$,

$$\begin{aligned} |g_{j+\frac{1}{2},+}(p,r)| &\leq \left(\frac{\omega(p)+\mathrm{m_e}}{\omega(p)}\right)^{\frac{1}{2}} \frac{p}{\sqrt{\pi}}(2pr)^{\gamma_j}\frac{1}{\Gamma(\gamma_j)}\,, \\ |f_{j+\frac{1}{2},+}(p,r)| &\leq \left(\frac{\omega(p)-\mathrm{m_e}}{\omega(p)}\right)^{\frac{1}{2}} \frac{2p}{\sqrt{\pi}}(2pr)^{\gamma_j-1}\frac{1}{\Gamma(\gamma_j)}\,, \\ |g_{-(j+\frac{1}{2}),+}(p,r)| &\leq \left(\frac{\omega(p)+\mathrm{m_e}}{\omega(p)}\right)^{\frac{1}{2}} \frac{2p}{\sqrt{\pi}}(2pr)^{\gamma_j-1}\frac{1}{\Gamma(\gamma_j)}\,, \\ |f_{-(j+\frac{1}{2}),+}(p,r)| &\leq \left(\frac{\omega(p)-\mathrm{m_e}}{\omega(p)}\right)^{\frac{1}{2}} \frac{p}{\sqrt{\pi}}(2pr)^{\gamma_j}\frac{1}{\Gamma(\gamma_j)}\,, \end{aligned} \tag{78}$$

and for negative energies $-\omega(p) < -\mathrm{m_e}$, we have

$$
\begin{aligned}
|g_{j+\frac{1}{2},-}(p,r)| &\leq \left(\frac{\omega(p)-\mathrm{m_e}}{\omega(p)}\right)^{\frac{1}{2}} \frac{p}{\sqrt{\pi}}(2pr)^{\gamma_j}\frac{1}{\Gamma(\gamma_j)}\,,\\
|f_{j+\frac{1}{2},-}(p,r)| &\leq \left(\frac{\omega(p)+\mathrm{m_e}}{\omega(p)}\right)^{\frac{1}{2}} \frac{2p}{\sqrt{\pi}}(2pr)^{\gamma_j-1}\frac{1}{\Gamma(\gamma_j)}\,,\\
|g_{-(j+\frac{1}{2}),-}(p,r)| &\leq \left(\frac{\omega(p)-\mathrm{m_e}}{\omega(p)}\right)^{\frac{1}{2}} \frac{2p}{\sqrt{\pi}}(2pr)^{\gamma_j-1}\frac{1}{\Gamma(\gamma_j)}\,,\\
|f_{-(j+\frac{1}{2}),-}(p,r)| &\leq \left(\frac{\omega(p)+\mathrm{m_e}}{\omega(p)}\right)^{\frac{1}{2}} \frac{p}{\sqrt{\pi}}(2pr)^{\gamma_j}\frac{1}{\Gamma(\gamma_j)}\,.
\end{aligned}
\tag{79}
$$

We also can bound the first and second derivatives. Below, we give such bounds for $|p| \leq 1$. For p larger than one, the functions are locally in L^q for any value of q.

There exists a constant C such that for $|p| \leq 1$, and for positive energies $\omega(p) > \mathrm{m_e}$ we have

$$
\begin{aligned}
\left|\frac{\partial}{\partial p}\, g_{j+\frac{1}{2},+}(p,r)\right| &\leq \frac{C}{\Gamma(\gamma_j)}\left[(2pr)^{\gamma_j} + pr(\gamma_j-1)(2pr)^{\gamma_j-1} + pr(2pr)^{\gamma_j-1}\right]\,,\\
\left|\frac{\partial}{\partial p}\, f_{j+\frac{1}{2},+}(p,r)\right| &\leq \frac{C}{\Gamma(\gamma_j)}\left[p(2pr)^{\gamma_j-1} + p^2r(\gamma_j-1)(2pr)^{\gamma_j-2} + p^2r(2pr)^{\gamma_j}\right]\,,\\
\left|\frac{\partial}{\partial p}\, g_{-(j+\frac{1}{2}),+}(p,r)\right| &\leq \frac{C}{\Gamma(\gamma_j)}\left[(2pr)^{\gamma_j-1} + pr(\gamma_j-1)(2pr)^{\gamma_j-2} + pr(2pr)^{\gamma_j}\right]\,,\\
\left|\frac{\partial}{\partial p}\, f_{-(j+\frac{1}{2}),+}(p,r)\right| &\leq \frac{C}{\Gamma(\gamma_j)}\left[p(2pr)^{\gamma_j} + p^2r(\gamma_j-1)(2pr)^{\gamma_j-1} + p^2r(2pr)^{\gamma_j-1}\right]\,,
\end{aligned}
\tag{80}
$$

and for $|p| \leq 1$ and negative energies $-\omega(p) < -\mathrm{m_e}$, we have

$$
\begin{aligned}
\left|\frac{\partial}{\partial p}\, g_{j+\frac{1}{2},-}(p,r)\right| &\leq \frac{C}{\Gamma(\gamma_j)}\left[p(2pr)^{\gamma_j} + p^2r(\gamma_j-1)(2pr)^{\gamma_j-1} + p^2r(2pr)^{\gamma_j-1}\right]\,,\\
\left|\frac{\partial}{\partial p}\, f_{j+\frac{1}{2},-}(p,r)\right| &\leq \frac{C}{\Gamma(\gamma_j)}\left[(2pr)^{\gamma_j-1} + pr(\gamma_j-1)(2pr)^{\gamma_j-2} + pr(2pr)^{\gamma_j}\right]\,,\\
\left|\frac{\partial}{\partial p}\, g_{-(j+\frac{1}{2}),-}(p,r)\right| &\leq \frac{C}{\Gamma(\gamma_j)}\left[p(2pr)^{\gamma_j-1} + p^2r(\gamma_j-1)(2pr)^{\gamma_j-2} + p^2r(2pr)^{\gamma_j}\right]\,,\\
\left|\frac{\partial}{\partial p}\, f_{-(j+\frac{1}{2}),-}(p,r)\right| &\leq \frac{C}{\Gamma(\gamma_j)}\left[(2pr)^{\gamma_j} + pr(\gamma_j-1)(2pr)^{\gamma_j-1} + pr(2pr)^{\gamma_j-1}\right]\,.
\end{aligned}
\tag{81}
$$

The estimates (80) and (81) yield, for a being the operator defined by (39),

and for positive energies $\omega(p) > \mathrm{m_e}$,

$$
\begin{aligned}
|a\, g_{j+\frac{1}{2},+}(p,r)| &\leq \frac{C}{\Gamma(\gamma_j)}\Big[\frac{\omega(p)}{p}\left((2pr)^{\gamma_j} + pr(\gamma_j-1)(2pr)^{\gamma_j-1} + pr(2pr)^{\gamma_j-1}\right) \\
&\qquad + \omega(p)(1+\frac{1}{p^2})p(2pr)^{\gamma_j})\Big], \\
|a\, f_{j+\frac{1}{2},+}(p,r)| &\leq \frac{C}{\Gamma(\gamma_j)}\Big[\frac{\omega(p)}{p}\left(p(2pr)^{\gamma_j-1} + p^2r(\gamma_j-1)(2pr)^{\gamma_j-2} + p^2r(2pr)^{\gamma_j}\right) \\
&\qquad + \omega(p)(1+\frac{1}{p^2})p^2(2pr)^{\gamma_j})\Big], \\
|a\, g_{-(j+\frac{1}{2}),+}(p,r)| &\leq \frac{C}{\Gamma(\gamma_j)}\Big[\frac{\omega(p)}{p}\left((2pr)^{\gamma_j-1} + pr(\gamma_j-1)(2pr)^{\gamma_j-2} + pr(2pr)^{\gamma_j}\right) \\
&\qquad + \omega(p)(1+\frac{1}{p^2})p(2pr)^{\gamma_j-1})\Big], \\
|a\, f_{-(j+\frac{1}{2}),+}(p,r)| &\leq \frac{C}{\Gamma(\gamma_j)}\Big[\frac{\omega(p)}{p}\left(p(2pr)^{\gamma_j} + p^2(\gamma_j-1)(2pr)^{\gamma_j-1} + p^2r(2pr)^{\gamma_j-1}\right) \\
&\qquad + \omega(p)(1+\frac{1}{p^2})p^2(2pr)^{\gamma_j})\Big],
\end{aligned}
\tag{82}
$$

And for negatives energies $-\omega(p) < -\mathrm{m_e}$, we get the same estimates for $|a\, g_{j+\frac{1}{2},-}(p,r)|$, $|a\, f_{j+\frac{1}{2},-}(p,r)|$, $|a\, g_{-(j+\frac{1}{2}),-}(p,r)|$ and $|a\, f_{-(j+\frac{1}{2}),-}(p,r)|$, respectively for $|a\, f_{-(j+\frac{1}{2}),+}(p,r)|$, $|a\, g_{-(j+\frac{1}{2}),+}(p,r)|$, $|a\, f_{j+\frac{1}{2},+}(p,r)|$ and $|a\, g_{j+\frac{1}{2},+}(p,r)|$.

Estimates for the second derivatives are given for (p,r) near $(0,0)$ by

$$
\left|\frac{\partial^2}{\partial p^2}\, g_{j+\frac{1}{2},+}(p,r)\right| \leq \frac{C\gamma_j^2}{\Gamma(\gamma_j)} p^{\gamma_j-1} r^{\gamma_j}, \tag{83}
$$

$$
\left|\frac{\partial^2}{\partial p^2}\, f_{j+\frac{1}{2},+}(p,r)\right| \leq \frac{C\gamma_j^2}{\Gamma(\gamma_j)} p^{\gamma_j-1} r^{\gamma_j-2}, \tag{84}
$$

$$
\left|\frac{\partial^2}{\partial p^2}\, g_{-(j+\frac{1}{2}),+}(p,r)\right| \leq \frac{C(\gamma_j-1)\gamma_j}{\Gamma(\gamma_j)} p^{\gamma_j-2} r^{\gamma_j-1} + \frac{C\gamma_j}{\Gamma(\gamma_j)} p^{\gamma_j} r^{\gamma_j-1}, \tag{85}
$$

$$
\left|\frac{\partial^2}{\partial p^2}\, f_{-(j+\frac{1}{2}),+}(p,r)\right| \leq \frac{C\gamma_j^2}{\Gamma(\gamma_j)} p^{\gamma_j} r^{\gamma_j-1}, \tag{86}
$$

and the same estimates for negatives energies hold respectively for $\left|\frac{\partial^2}{\partial p^2}\, f_{-(j+\frac{1}{2}),-}(p,r)\right|$, $\left|\frac{\partial^2}{\partial p^2}\, g_{-(j+\frac{1}{2}),-}(p,r)\right|$, $\left|\frac{\partial^2}{\partial p^2}\, f_{j+\frac{1}{2},-}(p,r)\right|$ and $\left|\frac{\partial^2}{\partial p^2}\, g_{j+\frac{1}{2},-}(p,r)\right|$.

B Mourre theory: abstract framework

In this section, we recall some abstract results from Mourre's theory that were used in Section 5. We work with an extension of the original Mourre theory [23] that

allows, in particular, the so-called *conjugate operator* to be maximal symmetric (not necessarily self-adjoint). Such an extension was considered in [19] and further refined in [13, 14] (see also [10, 16]). Here we mainly follow the presentation of [10].

Let $\mathcal{H}$ be a complex separable Hilbert space. Consider a self-adjoint operator H on $\mathcal{H}$ and a symmetric operator H' on $\mathcal{H}$ such that $\mathfrak{D}(H) \subset \mathfrak{D}(H')$. Let

$$\mathcal{G} := \mathfrak{D}(|H|^{\frac{1}{2}}),$$

equipped with the norm

$$\|\varphi\|_{\mathcal{G}}^2 := \big\||H|^{\frac{1}{2}}\varphi\big\|^2 + \|\varphi\|^2.$$

We set

$$\|\varphi\|_{\mathcal{G}^*}^2 := \big\|(|H| + \mathbb{1})^{-\frac{1}{2}}\varphi\big\|^2.$$

The dual space $\mathcal{G}^*$ of $\mathcal{G}$ identifies with the completion of $\mathcal{H}$ with respect to the norm $\|\cdot\|_{\mathcal{G}^*}$, and the operators H, H' identify with elements of $\mathcal{B}(\mathcal{G};\mathcal{G}^*)$, the set of bounded operators from $\mathcal{G}$ to $\mathcal{G}^*$.

Let A be a closed and maximal symmetric operator on $\mathcal{H}$. In particular, the deficiency indices $n_{\mp} = \dim \operatorname{Ker}(A^* \pm i)$ of A obey either $n_+ = 0$ or $n_- = 0$. We suppose that $n_+ = 0$, so that A generates a C_0-semigroup of isometries $\{W_t\}_{t\geq 0}$ (see e.g. [8, Theorem 10.4.4]). Recall that a C_0-semigroup on $[0,\infty)$ is, by definition, a map $t \mapsto W_t \in \mathcal{B}(\mathcal{H})$ such that $W_0 = \mathbb{1}$, $W_tW_s = W_{t+s}$ for $t, s \geq 0$, and $\text{w-}\lim_{t\to 0^+} W_t = \mathbb{1}$, where $\mathcal{B}(\mathcal{H})$ denotes the set of bounded operators on $\mathcal{H}$ and w-lim stands for weak limit. The fact that A is the generator of the C_0-semigroup $\{W_t\}_{t\geq 0}$ means that

$$\begin{aligned}
&\mathfrak{D}(A) = \big\{u \in \mathcal{H}, \lim_{t\to 0^+} (it)^{-1}(W_t u - u) \text{ exists}\big\},\\
&iAu = \lim_{t\to 0^+} t^{-1}(W_t u - u).
\end{aligned}$$

We make the following hypotheses.

Hypothesis B.1. *For all $t > 0$, W_t and W_t^* preserve $\mathcal{G}$ and, for all $\varphi \in \mathcal{G}$,*

$$\sup_{0<t<1} \|W_t\varphi\|_{\mathcal{G}} < \infty, \quad \sup_{0<t<1} \|W_t^*\varphi\|_{\mathcal{G}} < \infty.$$

In particular, $t \mapsto W_t|_{\mathcal{G}} \in \mathcal{B}(\mathcal{G})$ is a C_0-semigroup, and the extension of W_t to $\mathcal{G}^*$ (which will be denoted by the same symbol) defines a C_0-semigroup on $\mathcal{B}(\mathcal{G}^*)$ (see [10, Remark 1.4.1)]. Their generators are denoted by $A_{\mathcal{G}}$ and $A_{\mathcal{G}^*}$, respectively.

Hypothesis B.2. *The operator $H \in \mathcal{B}(\mathcal{G};\mathcal{G}^*)$ is of class $\mathrm{C}^1(A_{\mathcal{G}}; A_{\mathcal{G}^*})$, meaning that there exists a positive constant C such that, for all $0 \leq t \leq 1$,*

$$\|W_tH - HW_t\|_{\mathcal{B}(\mathcal{G};\mathcal{G}^*)} \leq \mathrm{C}t.$$

Moreover, for all $\varphi \in \mathfrak{D}(H)$,

$$\lim_{t\to 0^+} \frac{1}{t}\big(\langle\varphi, W_tH\varphi\rangle - \langle H\varphi, W_t\varphi\rangle\big) = \langle\varphi, H'\varphi\rangle.$$

Proposition B.3. *Suppose that Hypothesis B.1 holds and that the sesquilinear form* $[H, iA]$ *defined on* $\mathfrak{D}(A) \cap \mathcal{G}$ *by*

$$\langle u, [H, iA]v\rangle := i\langle u, HAv\rangle - i\langle A^*u, Hv\rangle,$$

extends to a bounded quadratic form on $\mathcal{G}$*. Then* H *is of class* $\mathrm{C}^1(A_{\mathcal{G}}; A_{\mathcal{G}^*})$ *in the sense of Hypothesis B.2, and the operator* $H' \in \mathcal{B}(\mathcal{G}; \mathcal{G}^*)$ *is the operator associated with the quadratic form* $[H, iA]$*.*

Under Hypotheses B.1 and B.2, we have the following version of the virial theorem.

Theorem B.4 (Virial Theorem)**.** *Assume Hypotheses B.1 and B.2. For any eigenstate* φ *of* H*, we have that*

$$\langle \varphi, H'\varphi\rangle = 0.$$

The limiting absorption principle stated in Theorem B.7 below requires some more regularity of H with respect to A:

Hypothesis B.5. *The operator* $H \in \mathcal{B}(\mathcal{G}; \mathcal{G}^*)$ *is of class* $\mathrm{C}^{1,1}(A_{\mathcal{G}}; A_{\mathcal{G}^*})$*, i.e.*

$$\int_0^1 \left\|[W_t, [W_t, H]]\right\|_{\mathcal{B}(\mathcal{G};\mathcal{G}^*)} \frac{\mathrm{d}t}{t^2} < \infty.$$

We recall that $\langle A\rangle = (1 + A^*A)^{1/2} = (1 + |A|^2)^{1/2}$ for any closed operator A. Our last hypothesis is a version of a strict Mourre estimate.

Hypothesis B.6. *There exist an open interval* $I \subset \mathbb{R}$ *and constants* $\mathrm{c}_0 > 0$*,* $\mathrm{C} \in \mathbb{R}$*, such that, in the sense of quadratic forms on* $\mathcal{D}(H)$*,*

$$H' \geq \mathrm{c}_0 \mathbb{1} - \mathrm{C}\mathbb{1}_I^{\perp}(H)\langle H\rangle, \tag{87}$$

where $\mathbb{1}_I^{\perp}(H) := \mathbb{1} - \mathbb{1}_I(H)$*.*

The following theorem shows that a limiting absorption principle holds for H in any compact interval where a Mourre estimate is satisfied in the sense of Hypothesis B.6. The proof of Theorem B.7 can be found in [13] (see also [19] for a similar result under slightly stronger assumptions).

Theorem B.7 (Limiting absorption principle)**.** *Assume that Hypotheses B.1, B.5 and B.6 hold. Let* $J \subset I$ *be a compact interval, where* I *is given by Hypothesis B.6, and let*

$$\tilde{J} = \{z \in \mathbb{C}, \operatorname{Re} z \in J, 0 < |\operatorname{Im} z| \leq 1\}.$$

For any $1/2 < s \leq 1$*, we have that*

$$\sup_{z \in \tilde{J}} \|\langle A\rangle^{-s}(H - z)^{-1}\langle A\rangle^{-s}\| < \infty,$$

and the map $z \mapsto \langle A\rangle^{-s}(H-z)^{-1}\langle A\rangle^{-s} \in \mathcal{B}(\mathcal{H})$ *is uniformly Hölder continuous of order* $s-1/2$ *on* $\tilde{J}$*. In particular, the limits*

$$\langle A\rangle^{-s}(H-\lambda-i0^{\pm})^{-1}\langle A\rangle^{-s} := \lim_{\varepsilon\to 0^{\pm}} \langle A\rangle^{-s}(H-\lambda-i\varepsilon)^{-1}\langle A\rangle^{-s},$$

exist in the norm topology of $\mathcal{B}(\mathcal{H})$*, uniformly in* $\lambda \in J$*. This implies that the spectrum of* H *in* J *is purely absolutely continuous. Moreover, the map* $\lambda \mapsto \langle A\rangle^{-s}(H-\lambda-i0^{\pm})^{-1}\langle A\rangle^{-s} \in \mathcal{B}(\mathcal{H})$ *is uniformly Hölder continuous of order* $s-1/2$ *on* J*.*

Remark B.8. *1) Theorem B.7 is established in [13] in the more general context of* singular Mourre theory*. More precisely, as shown in [13], the assumption that the commutator* H' *is relatively bounded with respect to* H *can be relaxed. This is of fundamental importance for the application to massless quantized fields considered in [14], but is not needed for the model studied in the present paper. Therefore, we content ourselves with the simpler setting of* regular Mourre theory *(i.e. we suppose that* H' *is* H*-bounded).*

2) The results in [13] are formulated under a stronger assumption than Hypothesis B.5, namely that $H \in \mathrm{C}^2(A_{\mathcal{G}}; A_{\mathcal{G}^*})$*. Nevertheless, as mentioned in [13], one can verify that Hypothesis B.5 is sufficient for Theorem B.7 to hold.*

3) By Fourier transform, Theorem B.7 implies the local decay property

$$\left\|\langle A\rangle^{-s}e^{-itH}\chi(H)\langle A\rangle^{-s}\right\| = \mathcal{O}(\langle t\rangle^{-s+\frac{1}{2}}),$$

for any $\chi \in \mathrm{C}_0^{\infty}(I;\mathbb{R})$ *and* $1/2 < s \leq 1$*.*

C Creation and annihilation operators in Fermi-Fock space

Let $\mathfrak{G}$ be any separable Hilbert space. Let $\otimes_a^n \mathfrak{G}$ denotes the antisymmetric n-th tensor power of $\mathfrak{G}$, appropriate to Fermi-Dirac statistics. We define the Fermi-Fock space over $\mathfrak{G}$, denoted by $\mathfrak{F}_a(\mathfrak{G})$, to be the direct sum

$$\mathfrak{F}_a(\mathfrak{G}) = \bigoplus_{n=0}^{\infty} \otimes_a^n \mathfrak{G},$$

where, by definition, we have set $\otimes_a^0 \mathfrak{G} := \mathbb{C}$. We shall denote by Ω_a the vacuum vector in $\mathfrak{F}_a(\mathfrak{G})$, i.e., the vector $(1,0,0,\cdots)$.

Let $\mathfrak{F}_a$ be the Fermi-Fock space over $\mathfrak{H}_c$,

$$\mathfrak{F}_a := \mathfrak{F}_a(\mathfrak{H}_c)\,.$$

The Fermi-Fock space for electrons and positrons, denoted by $\mathfrak{F}_D$, is the following Hilbert space

$$\mathfrak{F}_D := \mathfrak{F}_a \otimes \mathfrak{F}_a\,. \tag{88}$$

We denote by $\Omega_D := \Omega_a \otimes \Omega_a$ the vacuum of electrons and positrons. One has

$$\mathfrak{F}_D = \bigoplus_{r,s=0}^{\infty} \mathfrak{F}_a^{(r,s)},$$

where $\mathfrak{F}_a^{(r,s)} := (\otimes_a^r \mathfrak{H}_c) \otimes (\otimes_a^s \mathfrak{H}_c)$.

For every $\varphi \in \mathfrak{H}$ we define in $\mathfrak{F}_a(\mathfrak{H})$ the annihilation operator, denoted by $b(\varphi)$ as:

$$b(\varphi)\Omega = 0,$$

and, for any $n \in \mathbb{N}$,

$$\begin{aligned} & b(\varphi)\left(A_{n+1}(\varphi_1 \otimes \ldots \otimes \varphi_{n+1})\right) \\ & \quad = \frac{\sqrt{n+1}}{(n+1)!} \sum_{\sigma} sgn(\sigma)\left(\varphi, \varphi_{\sigma(1)}\right) \varphi_{\sigma(2)} \otimes \ldots \otimes \varphi_{\sigma(n+1)} \end{aligned}$$

where $\varphi_i \in \mathfrak{H}$. Note that the operator $b(\varphi)$ maps $\otimes_a^{n+1}\mathfrak{H}$ to $\otimes_a^n \mathfrak{H}$. It extends by linearity to a bounded operator on $\mathfrak{F}_a(\mathfrak{H})$.

The creation operator, denoted by $b^*(\varphi)$, is the adjoint of $b(\varphi)$. The operators $b^*(\varphi)$ and $b(\varphi)$ satisfy $\|b(\varphi)\| = \|b^*(\varphi)\| = \|\varphi\|$.

We now define the annihilation and creation operators in the Fermi-Fock space $\mathfrak{F}_D$ for electrons and positrons.

We first define the creation and annihilation operators for the electrons. For any $g \in \mathfrak{H}_c$, we define in $\mathfrak{F}_D = \mathfrak{F}_a \otimes \mathfrak{F}_a$ the annihilation operator, denoted by $b_+(g)$, as

$$b_+(g) := b(g) \otimes \mathbb{1}.$$

Observe that $b_+(g)$ maps $\mathfrak{F}_a^{(r+1,s)}$ into $\mathfrak{F}_a^{(r,s)}$ as follows

$$\begin{aligned} & b_+(g)\left(A_{r+1}(g_1 \otimes \ldots \otimes g_{r+1}) \otimes A_s(h_1 \otimes \ldots \otimes h_s)\right) \\ & = \left[b(g) A_{r+1}(g_1 \otimes \ldots \otimes g_{r+1})\right] \otimes A_s(h_1 \otimes \ldots \otimes h_s) \end{aligned}$$

The creation operator $b_+^*(g) = b^*(g) \otimes \mathbb{1}$ is the adjoint of $b_+(g)$. The operators $b_+^*(g)$ and $b_+(g)$ are bounded operators in $\mathfrak{F}_D$.

We set, for every $g \in \mathfrak{H}_c$,

$$\begin{aligned} b_{\gamma,+}(g) &= b_+(P_\gamma g) \\ b_{\gamma,+}^*(g) &= b_+^*(P_\gamma g) \end{aligned}$$

where P_γ is the projection of $\mathfrak{H}_c$ onto the γ-th component.

We next define the creation and annihilation operators for the positrons. For every $h \in \mathfrak{H}_c$, we define in $\mathfrak{F}_D$ the annihilation operator, denoted by $b_-(h)$, as

$$b_-(h) := (-1)^{N_e} \otimes b(h),$$

where $(-1)^{N_e}$ denotes the bounded operator on $\mathfrak{F}_a$ defined by its restriction to $\otimes_a^r \mathfrak{h}_c$ as $(-1)^{N_e} u = (-1)^r u$ for any $u \in \otimes_a^r \mathfrak{h}_c$.

In other words, $b_-(h)$ maps $\mathfrak{F}_a^{(r,s+1)}$ into $\mathfrak{F}_a^{(r,s)}$ as follows:

$$\begin{aligned}&b_-(h)(A_r(g_1\otimes\ldots\otimes g_r)\otimes A_{s+1}(h_1\otimes\ldots\otimes h_{s+1}))\\&=A_r(g_1\otimes\ldots\otimes g_r)\otimes[(-1)^r b(h)A_{s+1}(h_1\otimes\ldots\otimes h_{s+1})]\end{aligned}$$

The creation operator $b_-^*(h)=(-1)^{N_{\mathrm{e}}}\otimes b^*(h)$ is the adjoint of $b_-(h)$; $b_-^*(h)$ and $b_-(h)$ are bounded operators in $\mathfrak{F}_D$.

As above, we set, for every $h\in\mathfrak{H}_c$,

$$\begin{aligned}b_{\gamma,-}(h)&=b_-(P_\gamma h)\\b_{\gamma,-}^*(h)&=b_-^*(P_\gamma h)\,.\end{aligned}$$

A simple computation shows that the following anti-commutation relations hold

$$\{b_{\gamma,\pm}(g_1),b_{\beta,\pm}^*(g_2)\}=\delta_{\gamma,\beta}(P_\gamma g_1,P_\gamma g_2)_{L^2(\mathbb{R}_+)}\,,$$

and

$$\{b_{\gamma,+}^{\sharp_1}(g_1),b_{\beta,-}^{\sharp_2}(g_2)\}=0\,,$$

where g_1, $g_2\in\mathfrak{H}_c$, and $\sharp_i$ $(i=1,2)$ stand either for $*$ or for no symbol.

As in [24, chapter X], we introduce operator-valued distributions $b_{\gamma,\pm}(p)$ and $b_{\gamma,\pm}^*(p)$ that fulfills

$$\begin{aligned}b_{\gamma,\pm}(g)&=\int_{\mathbb{R}^+}b_{\gamma,\pm}(p)\,\overline{(P_\gamma g)\,(p)}\,\mathrm{d}p\\b_{\gamma,\pm}^*(g)&=\int_{\mathbb{R}^+}b_{\gamma,\pm}^*(p)\,(P_\gamma g)\,(p)\,\mathrm{d}p\end{aligned}$$

where $g\in\mathfrak{H}_c$.

We also define for $\xi=(p,\gamma)$,

$$b_\pm^\sharp(\xi):=b_{\gamma,\pm}^\sharp(p)\,.$$

Note that with the notation of (7), we have

$$b_\pm^\sharp(g)=\int b_\pm^\sharp(\xi)\overline{g(\xi)}\,\mathrm{d}\xi\,.$$

We now give a representation of $b_{\gamma,\pm}(p)$ and $b_{\gamma,\pm}^*(p)$. Recall that $\mathfrak{D}_D$ denote the set of smooth vectors $\Phi\in\mathfrak{F}_D$ for which $\Phi^{(r,s)}$ has a compact support and $\Phi^{(r,s)}=0$ for all but finitely many (r,s).

For every $\xi_1=(p,\gamma)$, $b_+(\xi_1)$ maps $\mathfrak{F}_a^{(r+1,s)}\cap\mathfrak{D}_D$ into $\mathfrak{F}_a^{(r,s)}\cap\mathfrak{D}_D$ and we have

$$\begin{aligned}&(b_+(\xi_1)\Phi)^{(r,s)}(p_1,\gamma_1,\ldots,p_r,\gamma_r;p_1',\gamma_1',\ldots,p_s',\gamma_s')\\&=\sqrt{r+1}\Phi^{(r+1,s)}(p,\gamma,p_1,\gamma_1,\ldots,p_r,\gamma_r;p_1',\gamma_1',\ldots,p_s',\gamma_s')\end{aligned}$$

$b_+^*(\xi_1)$ is then given by:

$$(b_+^*(\xi_1)\Phi)^{(r+1,s)}(p_1,\gamma_1,\ldots,p_{r+1},\gamma_{r+1};p_1',\gamma_1',\ldots,p_s',\gamma_s') =$$
$$\frac{1}{\sqrt{r+1}}\sum_{i=1}^{r+1}(-1)^{i+1}\delta_{\gamma_i\gamma}\delta(p-p_i)$$
$$\Phi^{(r,s)}(p_1,\gamma_1,\ldots,\widehat{p_i,\gamma_i},\ldots,p_{r+1},\gamma_{r+1};p_1',\gamma_1',\ldots,p_s',\gamma_s')$$

where $\widehat{\cdot}$ denotes that the i-th variable has to be omitted.

Similarly, for $\xi_2=(p',\gamma')$, $b_-(\xi_2)$ maps $\mathfrak{F}_a^{(r,s+1)}\cap\mathfrak{D}_D$ into $\mathfrak{F}_a^{(r,s)}\cap\mathfrak{D}_D$ such that

$$(b_-(\xi_2)\Phi)^{(r,s)}(p_1,\gamma_1,\ldots,p_r,\gamma_r;p_1',\gamma_1',\ldots,p_s',\gamma_s') =$$
$$((-1)^{N_+}\otimes b(\xi_2)\Phi)^{(r,s)}(p_1,\gamma_1,\ldots,p_r,\gamma_r;p_1',\gamma_1',\ldots,p_s',\gamma_s') =$$
$$\sqrt{s+1}(-1)^r\Phi^{(r,s+1)}(p_1,\gamma_1,\ldots,p_r,\gamma_r;p',\gamma',p_1',\gamma_1',\ldots,p_s',\gamma_s')$$

$b_-^*(\xi_2)$ is then given by

$$(b_-^*(\xi_2)\Phi)^{(r,s+1)}(p_1,\gamma_1,\ldots,p_r,\gamma_r;p_1',\gamma_1',\ldots,p_{s+1}',\gamma_{s+1}') =$$
$$\frac{1}{\sqrt{s+1}}(-1)^r\sum_{i=1}^{s+1}(-1)^{i+1}\delta_{\gamma',\gamma_i'}\delta(p'-p_i')$$
$$\Phi^{(r,s)}(p_1,\gamma_1,\ldots,p_r,\gamma_r;p_1',\gamma_1',\ldots,\widehat{p_i',\gamma_i'},\ldots,p_{s+1}',\gamma_{s+1}')$$

Let us recall that $\Phi^{(r,s)}$ is antisymmetric in the electron and the positron variables separately. We have

$$\{b_{\gamma,+}(p),b_{\gamma',+}^*(p')\}=\{b_{\gamma,-}(p),b_{\gamma',-}^*(p')\}=\delta_{\gamma,\gamma'}\delta(p-p')\,. \tag{89}$$

Any other anti-commutators equal zero.

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