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ON CLOSED IDEALS IN THE BIG LIPSCHITZ ALGEBRAS OF ANALYTIC FUNCTIONS

BRAHIM BOUYA AND MOHAMED ZARRABI

Abstract. In this paper we study the closed ideals in the big Lipschitz algebras of analytic functions on the unit disk. More precisely we give the smallest closed ideal with given hull and inner factor.

1. Introduction and statement of main result

Let $\mathbb{D}$ be the open unit disk of the complex plane and $\mathbb{T}$ its boundary. By $\mathcal{H}^\infty$ we denote the space of all bounded analytic functions on $\mathbb{D}$. The big Lipschitz algebra is defined by the following

$$Lip_\alpha := \Big\{ f \in \mathcal{H}^\infty : \sup_{\substack{z,w\in\mathbb{D}\\ z\neq w}} \frac{|f(z)-f(w)|}{|z-w|^\alpha} < +\infty \Big\},$$

where $0 < \alpha \leq 1$ is a real number. It is clear that Lip_α is included in $\mathcal{A}(\mathbb{D})$, the usual disk algebra of all analytic functions f on $\mathbb{D}$ that are continuous on $\overline{\mathbb{D}}$. It is well known that Lip_α is a non separable commutative Banach algebra when equipped with the norm

$$\|f\|_\alpha := \|f\|_\infty + \sup_{\substack{z,w\in\mathbb{D}\\ z\neq w}} \frac{|f(z)-f(w)|}{|z-w|^\alpha},$$

where $\|f\|_\infty := \sup_{z\in\mathbb{D}} |f(z)|$ is the supremum norm. We note that

$$\|f\|'_\alpha := \|f\|_\infty + \sup_{z\in\mathbb{D}} (1-|z|)^{1-\alpha}|f'(z)|,$$

defines an equivalent norm on Lip_α, see for example [5, Theorem 5.1]. From now on, we denote by $U \in \mathcal{H}^\infty$ an inner function and by $\mathbb{E} \subseteq \mathbb{T}$ a closed set such that $\mathbb{E} \supseteq \sigma(U) \cap \mathbb{T}$, where

$$\sigma(U) := \{\lambda \in \overline{\mathbb{D}} \ : \ \liminf_{\substack{z\to\lambda\\ z\in\mathbb{D}}} |U(z)| = 0\},$$

is called the spectrum of U, see [15, Pages. 62-63]. It is known that $\sigma(U) = \overline{\mathbb{Z}_U} \cup \mathrm{supp}(\mu_U)$, where $\mathbb{Z}_U$ is the zero set in $\mathbb{D}$ of U and $\mathrm{supp}(\mu_U)$ is the closed support of the singular measure μ_U associated to the singular part of U. We set

$$\mathcal{I}_{\mathcal{A}(\mathbb{D})}(\mathbb{E}, U) := \{f \in \mathcal{A}(\mathbb{D}) : \ f_{|\mathbb{E}} \equiv 0 \text{ and } f/U \in \mathcal{H}^\infty\}.$$

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The structure of closed ideals in the disk algebra was given independently by Beurling and Rudin, see [8, Page. 85] and [17]. They proved that if $\mathcal{I}$ is a closed ideal of $\mathcal{A}(\mathbb{D})$, then $\mathcal{I} = \mathcal{I}_{\mathcal{A}(\mathbb{D})}(\mathbb{E}_{\mathcal{I}}, U_{\mathcal{I}})$, where

$$\mathbb{E}_{\mathcal{I}} := \{\xi \in \mathbb{T} : f(\xi) = 0, \ \forall f \in \mathcal{I}\}$$

is known as the hull of $\mathcal{I}$ and $U_{\mathcal{I}}$ is the greatest inner common divisor of the inner parts of the non-zero functions in $\mathcal{I}$. Later in [2, 3, 11, 13, 18, 20] the authors described the complete structure of the closed ideals in some separable Banach algebras of analytic functions. They proved that they are standard in the sense of the above Beurling and Rudin characterization. However, the structure of the closed ideals of non separable Banach algebras of analytic functions seems to be much more difficult, see [6, 7, 9] and references therein for the analytic case and [19] for the non analytic case.

We set

$$\mathcal{I}_\alpha(\mathbb{E}, U) := \mathcal{I}_{\mathcal{A}(\mathbb{D})}(\mathbb{E}, U) \cap Lip_\alpha$$

and

$$\mathcal{J}_\alpha(\mathbb{E}, U) := \Big\{ f \in \mathcal{I}_\alpha(\mathbb{E}, U) : \lim_{\delta \to 0} \sup_{z \in \mathbb{E}(\delta)} (1 - |z|)^{1-\alpha} |f'(z)| = 0 \Big\},$$

where

$$\mathbb{E}(\delta) := \{z \in \mathbb{D} \ : \ d(z, \mathbb{E}) \leq \delta\}, \qquad 0 < \delta < 1,$$

and $d(z, \mathbb{E})$ notes the Euclidian distance from the point $z \in \mathbb{D}$ to $\mathbb{E}$. The spaces $\mathcal{I}_\alpha(\mathbb{E}, U)$ and $\mathcal{J}_\alpha(\mathbb{E}, U)$ are clearly closed ideals of the algebra Lip_α and $\mathcal{J}_\alpha(\mathbb{E}, U) \subseteq \mathcal{I}_\alpha(\mathbb{E}, U)$. It is known that there exists a non zero function $f \in Lip_\alpha$ with boundary zero set $\mathbb{E}$ and inner factor U if and only if the following condition holds

$$\int_0^{2\pi} \log d(e^{i\theta}, \mathbb{E} \cup \mathbb{Z}_U) d\theta > -\infty, \tag{1.1}$$

see Theorem 4 below. So under condition (1.1), we have $\mathbb{E}_{\mathcal{I}} = \mathbb{E}$ and $U_{\mathcal{I}} = U$ when $\mathcal{I}$ equals $\mathcal{I}_\alpha(\mathbb{E}, U)$ or $\mathcal{J}_\alpha(\mathbb{E}, U)$. For every closed ideal $\mathcal{I} \subseteq Lip_\alpha$ we obviously have $\mathcal{I} \subseteq \mathcal{I}_\alpha(\mathbb{E}_{\mathcal{I}}, U_{\mathcal{I}})$. On the other hand, T. V. Pederson proved in [16, Theorem 4.1] that $\mathcal{J}_\alpha(\mathbb{E}_{\mathcal{I}}, U_{\mathcal{I}}) \subseteq \mathcal{I}$, for every $\mathcal{I}$ such that $\mathbb{E}_{\mathcal{I}}$ is a countable set. A result of this type was stated first in [7] by Hedenmalm in the algebras $\mathcal{H}^\infty$ and Lip_1, for closed ideals $\mathcal{I}$ such that $\mathbb{E}_{\mathcal{I}}$ is a single point. We also note that the closed ideals with countable hull in many different separable Banach algebras were characterized in [1]. In [7, 16] the authors use the classical resolvent method (also called the Carleman transform) which seems to be difficult to apply when $\mathbb{E}_{\mathcal{I}}$ is uncountable. In this paper we show that the above inclusion always holds. To do this we give an adaptation in the space $\mathcal{J}_\alpha(\mathbb{E}, U)$ of Korenblum's functional approximation method [11], see also [2, 14]. Our main result is the following Theorem.

Theorem 1. *Let $\mathcal{I} \subseteq Lip_\alpha$ be a closed ideal, where $0 < \alpha \leq 1$. Then $\mathcal{J}_\alpha(\mathbb{E}_{\mathcal{I}}, U_{\mathcal{I}}) \subseteq \mathcal{I}$.*

It follows that for every closed ideal $\mathcal{I}$ of Lip_α, $\mathcal{J}_\alpha(\mathbb{E}_{\mathcal{I}}, U_{\mathcal{I}}) \subseteq \mathcal{I} \subseteq \mathcal{I}_\alpha(\mathbb{E}_{\mathcal{I}}, U_{\mathcal{I}})$. We note that for $s \geq 0$, it is shown in [16, Corollary 4.7] that the set of closed ideals lying between $\mathcal{J}_\alpha(\{1\}, \psi_s)$ and $\mathcal{I}_\alpha(\{1\}, \psi_s)$ is uncountable, where ψ_s is the following singular function

$$\psi_s(z) := e^{s\frac{z+1}{z-1}}, \qquad z \in \mathbb{D}.$$

We also obtain the following Corollary.

Corollary 1. *The closed ideal $\mathcal{J}_\alpha(\mathbb{E}, U)$ is principal and is generated by any function $g \in \mathcal{J}_\alpha(\mathbb{E}, U)$ with inner factor U and boundary zero set $\mathbb{E}$.*

To prove Theorem 1 we extend some approximation results obtained in [16] by using the factorization property (also called the F-property) of the space $\mathcal{J}_\alpha(\mathbb{E}) := \mathcal{J}_\alpha(\mathbb{E}, 1)$, which we state in the following Theorem.

Theorem 2. *Let $g \in Lip_\alpha$ be a function and $V \in \mathcal{H}^\infty$ be an inner function dividing g, that is $g/V \in \mathcal{H}^\infty$. If $g \in \mathcal{J}_\alpha(\mathbb{E})$, then $g/V \in \mathcal{J}_\alpha(\mathbb{E})$.*

We note that Lip_α possesses the F-property; If $f \in Lip_\alpha$ and $V \in \mathcal{H}^\infty$ is an inner function such that $f/V \in \mathcal{H}^\infty$ then $f/V \in Lip_\alpha$ and $\|f/V\|_\alpha \leq c_\alpha \|f\|_\alpha$, where c_α is a positive constant independent of the functions f and V, see [21].
The remaining of this paper is organized as follows: In section 2, we use Theorem 2 to give the proof of Theorem 1. Section 3 contains the proof of Theorem 2. The last section is devoted to presenting an elementary proof of Theorem 2 in the case $0 < \alpha < 1$.

2. Proof of Theorem 1

2.1. **Some technical results.** For $f \in \mathcal{H}^\infty$ we denote by U_f and O_f the inner and the outer factor of f. By B_f the Blaschke product with zeros

$$\mathbb{Z}_f := \{z \in \mathbb{D} \ : \ f(z) = 0\},$$

counting the multiplicities. For a closed ideal $\mathcal{I}$ of Lip_α, we set $\mathbb{Z}_{\mathcal{I}} := \bigcap_{f \in \mathcal{I}} \mathbb{Z}_f$ and we denote by $B_{\mathcal{I}}$ the Blaschke product with zeros $\mathbb{Z}_{\mathcal{I}}$, counting the multiplicities. In fact $B_{\mathcal{I}}$ is the Blaschke product factor of $U_{\mathcal{I}}$. We need the following result to show the next one.

Lemma 1. *Let $p \in \mathbb{N}$ be a number. The set $\mathcal{J}_\alpha(\mathbb{E}, U) \cap \mathcal{I}_\alpha^p(\mathbb{E})$ is dense in $\mathcal{J}_\alpha(\mathbb{E}, U)$, where*

$$\mathcal{I}_\alpha^p(\mathbb{E}) := \{f \in Lip_\alpha : \ \exists C > 0, \ |f(\xi)| \leq C d^p(\xi, \mathbb{E}) \ \ \textit{for all } \xi \in \mathbb{T}\}.$$

Proof. Here, we will just point out the steps in the proof of [16, Proposition 5.3] that prove the present lemma. For a real number $\delta \in (0, 1)$, we let $\mathbb{E}_{1,\delta}$ and $\mathbb{E}_{2,\delta}$ be two closed disjoint subsets of $\mathbb{T}$ such that $\mathbb{E} \subseteq \mathbb{E}_{1,\delta} \subseteq \overline{\mathbb{E}(\delta)}$ and $\mathbb{E}_f = \mathbb{E}_{1,\delta} \cup \mathbb{E}_{2,\delta}$. By using [16, Proposition 5.4], we have $O_f = O_{1,\delta} \times O_{2,\delta}$, where $O_{i,\delta} \in Lip_\alpha$ are outer functions such that $\mathbb{E}_{O_{i,\delta}} = \mathbb{E}_{i,\delta}$, $(i = 1, 2)$. We have $\mathbb{T} \setminus \mathbb{E}_{1,\delta} = \bigcup_{n=1}^{\infty} (a_n, b_n)$, where $(a_n, b_n) \subseteq \mathbb{T} \setminus \mathbb{E}_{1,\delta}$ is an open arc joining the points $a_n, b_n \in \mathbb{E}_{1,\delta}$. For $N \in \mathbb{N}$, we define F_N to be the outer function with boundary modulus defined as follows

$$|F_N(\xi)| := \begin{cases} |O_{1,\delta}(\xi)|, & \text{if } \ \xi \in \Omega_N, \\ 1, & \text{if } \ \xi \in \mathbb{T} \setminus \Omega_N, \end{cases}$$

where $\Omega_N := \bigcup_{n=N+1}^{\infty} (a_n, b_n)$. Since the set $\mathbb{E} \setminus \partial\Omega_N$ is finite we can set $\mathbb{E} \setminus \partial\Omega_N := \{c_1, c_2, ..., c_{m_N}\}$. Also, we define

$$K_{i,\mu}(z) := \frac{z - c_i}{z - c_i(1+\mu)}, \qquad z \in \mathbb{D}.$$

In [16, Pages 52-53] it is shown that for every $\varepsilon > 0$ there exist parameters δ, t, N, q, μ and p such that the function

$$h := \Big(\prod_{i=1}^{m_N} K_{i,\mu}\Big)^p F_N^q O_{1,\delta}^t$$

belongs to $\mathcal{A}(\mathbb{D})$ and $\|fh - f\|_\alpha \leq \varepsilon$. Hence every function $f \in \mathcal{J}_\alpha(\mathbb{E}, U)$ can be approximated by functions in $\mathcal{J}_\alpha(\mathbb{E}, U) \cap \mathcal{I}_\alpha^p(\mathbb{E})$, using the simple fact that $\mathcal{J}_\alpha(\mathbb{E}, U) \subseteq \mathcal{J}_\alpha(\mathbb{E})$ and that U divides fh. So $\mathcal{J}_\alpha(\mathbb{E}, U) \cap \mathcal{I}_\alpha^p(\mathbb{E})$ is dense in $\mathcal{J}_\alpha(\mathbb{E}, U)$. This finishes the proof of Lemma 1. □

To prove our main Theorem we need the following Proposition which in particular gives an answer to the question (2) in [16, Page 47].

Proposition 3. *Let $f \in Lip_\alpha$ be a function such that $f \in \mathcal{J}_\alpha(\mathbb{E}_f)$. Then*

$$\overline{Lip_\alpha\ f} = \mathcal{J}_\alpha(\mathbb{E}_f, U_f).$$

Proof. Let $f \in \mathcal{J}_\alpha(\mathbb{E}_f)$ be a function. It is clear that $\overline{Lip_\alpha\ f} \subseteq \mathcal{J}_\alpha(\mathbb{E}_f, U_f)$. We have to show that $\mathcal{J}_\alpha(\mathbb{E}_f, U_f) \subseteq \overline{Lip_\alpha\ f}$. Using Lemma 1 it is sufficient to show that $\mathcal{J}_\alpha(\mathbb{E}_f, U_f) \cap \mathcal{I}_\alpha^p(\mathbb{E}_f) \subseteq \overline{Lip_\alpha\ f}$ for some $p \in \mathbb{N}$. Let $g \in \mathcal{J}_\alpha(\mathbb{E}_f, U_f) \cap \mathcal{I}_\alpha^p(\mathbb{E}_f)$ be a function and suppose that $p > 2\alpha$. We note that $O_f \in \mathcal{J}_\alpha(\mathbb{E}_f)$, by Theorem 2. According to the proof of [16, Proposition 5.2] the function g can be approximated by functions of the form ghO_f, where $h \in Lip_\alpha$, (see assertions $(i) - (ii)$ in [16, p. 48] and assertions $(11) - (a) - (b)$ in [16, p. 50]). By using the F–property $g/U_f \in Lip_\alpha$. Then $hgO_f = h\,(g/U_f)\,f \in Lip_\alpha\ f$. It follows that $g \in \overline{Lip_\alpha\ f}$. Hence $\mathcal{J}_\alpha(\mathbb{E}_f, U_f) \cap \mathcal{I}_\alpha^p(\mathbb{E}_f) \subseteq \overline{Lip_\alpha\ f}$. The proof of Proposition 3 is finished. □

The following space

$$\mathcal{A}^1(\mathbb{D}) := \{f \in \mathcal{A}(\mathbb{D}) \ : \ f' \in \mathcal{A}(\mathbb{D})\},$$

endowed with norm

$$\|f\|_{\mathcal{A}^1} := \|f\|_\infty + \|f'\|_\infty, \qquad f \in \mathcal{A}^1(\mathbb{D}),$$

is a Banach algebra. Clearly $\mathcal{A}^1(\mathbb{D})$ is continuously embedded in Lip_α. The following theorem is proved in [10, Theorems 2 and 4], see also [22, Theorem].

Theorem 4. *Let f be a nonzero function in Lip_α. Then the closed set $\mathbb{E}_f \cup \mathbb{Z}_f$ satisfies the condition* (1.1)*. Conversely if $\mathbb{E} \cup \mathbb{Z}_U$ satisfies the condition* (1.1)*, then there exists a function $f \in \mathcal{A}^1(\mathbb{D})$ such that $U_f = U$, $\mathbb{E}_f = \mathbb{E}$ and $\mathbb{E}_{f'} \supseteq \mathbb{E}$.*

Now we can give the proof of our main Theorem by using Proposition 3 and Theorem 4.

2.2. **Proof of Theorem 1.** Let $\mathcal{I} \subseteq Lip_\alpha$ be a closed ideal. Since $\mathcal{A}^1(\mathbb{D})$ is continuously embedded in Lip_α then $\mathcal{I}_1 := \mathcal{A}^1(\mathbb{D}) \cap \mathcal{I}$ is a closed ideal of $\mathcal{A}^1(\mathbb{D})$. It is clear that $\mathbb{E}_{\mathcal{I}} \subseteq \mathbb{E}_{\mathcal{I}_1}$ and $U_{\mathcal{I}}$ divides $U_{\mathcal{I}_1}$. Now, let $f \in \mathcal{I} \setminus \{0\}$ be a function. It is easily seen that $f_1 := fO_f$ belongs to $\mathcal{J}_\alpha(\mathbb{E}_{f_1})$. Then $\overline{Lip_\alpha\ f_1} = \mathcal{J}_\alpha(\mathbb{E}_{f_1}, U_{f_1})$, by using Proposition 3. Since $\mathbb{E}_{f_1} = \mathbb{E}_f$, $U_{f_1} = U_f$ and $f_1 \in \mathcal{I}$ then $\mathcal{J}_\alpha(\mathbb{E}_f, U_f) \subseteq \mathcal{I}$.
By Theorem 4 there exists a function $g \in \mathcal{A}^1(\mathbb{D})$ such that $U_g = U_f$, $\mathbb{E}_g = \mathbb{E}_f$ and $\mathbb{E}_{g'} \supseteq \mathbb{E}_f$. It is clear that $g \in \mathcal{J}_\alpha(\mathbb{E}_f, U_f)$. Then $g \in \mathcal{I}$ and by consequence $g \in \mathcal{I}_1$. We conclude that $U_{\mathcal{I}_1}$ divides U_f and $\mathbb{E}_{\mathcal{I}_1} \subseteq \mathbb{E}_f$ for every function $f \in \mathcal{I} \setminus \{0\}$. So $\mathbb{E}_{\mathcal{I}_1} = \mathbb{E}_{\mathcal{I}}$ and $U_{\mathcal{I}_1} = U_{\mathcal{I}}$. According to the structure of closed ideals in $\mathcal{A}^1(\mathbb{D})$ given in [12]

$$\{f \in \mathcal{A}^1(\mathbb{D}) \ : \ f/U_{\mathcal{I}_1} \in \mathcal{H}^\infty \text{ and } f = f' = 0 \text{ on } \mathbb{E}_{\mathcal{I}_1}\} \subseteq \mathcal{I}_1. \tag{2.1}$$

It follows that

$$\{f \in \mathcal{A}^1(\mathbb{D}) \ : \ f/U_{\mathcal{I}} \in \mathcal{H}^\infty \text{ and } f = f' = 0 \text{ on } \mathbb{E}_{\mathcal{I}}\} \subseteq \mathcal{I}. \tag{2.2}$$

By using Theorem 4 there exists a function $f_0 \in \mathcal{A}^1(\mathbb{D})$ such that $U_{f_0} = U_{\mathcal{I}}$, $\mathbb{E}_{f_0} = \mathbb{E}_{\mathcal{I}}$ and $\mathbb{E}_{f_0'} \supseteq \mathbb{E}_{\mathcal{I}}$. Then $f_0 \in \mathcal{I}$ by (2.2). It follows that $\overline{Lip_\alpha\ f_0} \subseteq \mathcal{I}$. Since $f_0 \in \mathcal{J}_\alpha(\mathbb{E}_{f_0})$ then $\overline{Lip_\alpha\ f_0} = \mathcal{J}_\alpha(\mathbb{E}_{\mathcal{I}}, U_{\mathcal{I}})$, by using Proposition 3. Hence $\mathcal{J}_\alpha(\mathbb{E}_{\mathcal{I}}, U_{\mathcal{I}}) \subseteq \mathcal{I}$. The proof of Theorem 1 is completed.

2.3. **Proof of Corollary 1.** It follows clearly from Theorem 1 that $\mathcal{J}_\alpha(\mathbb{E}, U)$ is generated by any function $g \in \mathcal{J}_\alpha(\mathbb{E}, U)$ such that $U_g = U$ and $\mathbb{E}_g = \mathbb{E}$. So we have just to check that such functions exist. If $\mathcal{J}_\alpha(\mathbb{E}, U) \neq \{0\}$ then $\mathbb{E} \cup \mathbb{Z}_U$ satisfies the Condition (1.1) by Theorem 4. Now the existence of such functions follows again from Theorem 4, which finishes the proof of Corollary 1.

3. Proof of Theorem 2

Let g be a nonzero function in $\mathcal{J}_\alpha(\mathbb{E})$ such that V divides U_g. We set $k := g/V$. For a real number $\delta \in (0,1)$, we let $\mathbb{E}_{1,\delta}$ and $\mathbb{E}_{2,\delta}$ be two closed disjoint subsets of $\mathbb{T}$ such that $\mathbb{E} \subseteq \mathbb{E}_{1,\delta} \subseteq \overline{\mathbb{E}(\delta)}$ and $\mathbb{E}_g = \mathbb{E}_{1,\delta} \cup \mathbb{E}_{2,\delta}$. By using [16, Proposition 5.4], we have $O_g = O_{1,\delta} \times O_{2,\delta}$, where $O_{i,\delta} \in Lip_\alpha$ are outer functions such that $\mathbb{E}_{O_{i,\delta}} = \mathbb{E}_{i,\delta}$, $(i = 1, 2)$. The function $O_{i,\delta}$ is constructed such that $\log|O_{i,\delta}| = \chi_i \log|O_g| = \chi_i \log|g|$ on $\mathbb{T}$, where χ_i is a function such that $0 \leq \chi_i \leq 1$. This implies in particular that $|O_{i,\delta}| \leq |g| + 1$. We have

$$(gO_{1,\delta}^t - g)' = tO_{1,\delta}^t O_{1,\delta}' O_{2,\delta} U_g + (O_{1,\delta}^t - 1)g'.$$

Then

$$\|gO_{1,\delta}^t - g\|_\alpha' \leq \nu(t,\delta) + (\|g\|_\infty + 2) \sup_{z \in \mathbb{E}(2\delta)} (1 - |z|)^{1-\alpha} |g'(z)|, \tag{3.1}$$

where $t \in (0,1)$ is a real number and

$$\nu(t,\delta) := \|gO_{1,\delta}^t - g\|_\infty + t\|O_{1,\delta}\|_\infty^t \|O_{2,\delta}\|_\infty \|O_{1,\delta}\|_\alpha' + \|g\|_\alpha' \sup_{z \in \mathbb{D} \setminus \mathbb{E}(2\delta)} |O_{1,\delta}^t - 1|.$$

It is plain to see that, for every real number $\delta' > 0$,

$$\lim_{t\to 0^+} \sup_{z\in\mathbb{D}\setminus\mathbb{E}_{1,\delta}(\delta')} |O^t_{1,\delta}(z) - 1| = 0. \tag{3.2}$$

It follows

$$\lim_{t\to 0^+} \sup_{z\in\mathbb{D}\setminus\mathbb{E}(2\delta)} |O^t_{1,\delta}(z) - 1| = 0, \tag{3.3}$$

by using the fact that $\mathbb{D}\setminus\mathbb{E}(2\delta) \subseteq \mathbb{D}\setminus\mathbb{E}_{1,\delta}(\delta)$. From (3.2) and the fact that g is continuous on $\overline{\mathbb{D}}$ and vanishes on $\mathbb{E}_{1,\delta}$, we get that

$$\lim_{t\to 0^+} \|gO^t_{1,\delta} - g\|_\infty = 0. \tag{3.4}$$

Thus, for a fixed $\delta \in (0,1)$, we have

$$\lim_{t\to 0^+} \nu(t,\delta) = 0, \tag{3.5}$$

by using (3.3) and (3.4). Now, since $g \in \mathcal{J}_\alpha(\mathbb{E}, U)$ then

$$\lim_{\delta\to 0^+} \sup_{z\in\mathbb{E}(2\delta)} (1-|z|)^{1-\alpha}|g'(z)| = 0. \tag{3.6}$$

We deduce from (3.1), (3.5) and (3.6) that for every $\delta \in (0,1)$ there exists a number $t(\delta) > 0$ such that

$$\lim_{\delta\to 0^+} \|gO^{t(\delta)}_{1,\delta} - g\|_\alpha = 0. \tag{3.7}$$

By using the F-property of Lip_α

$$\|k(O^{t(\delta)}_{1,\delta} - 1)\|_\alpha \le c_\alpha \|g(O^{t(\delta)}_{1,\delta} - 1)\|_\alpha, \qquad \text{for all } 0<\delta<1, \tag{3.8}$$

where $c_\alpha > 0$ is a constant independent of δ. Hence

$$\lim_{\delta\to 0^+} \|k(O^{t(\delta)}_{1,\delta} - 1)\|_\alpha = 0. \tag{3.9}$$

By computing the derivative we see easily that $kO^{t(\delta)}_{1,\delta} \in \mathcal{J}_\alpha(\mathbb{E})$, for all $0<\delta<1$. Hence $k \in \mathcal{J}_\alpha(\mathbb{E})$, as consequence of the fact that $\mathcal{J}_\alpha(\mathbb{E})$ is closed and (3.9). This finishes the proof of Theorem 2.

Remark 5. *In the appendix below we give an elementary proof of Theorem 2 for* $0<\alpha<1$ *based on an estimation of some classical Toeplitz operators. However we do not know how to extend this proof to the limit case* $\alpha = 1$.

4. Appendix: A Toeplitz method for the F-property of $\mathcal{J}_\alpha(\mathbb{E})$

In this section we consider the spaces Lip_α such that $0<\alpha<1$. The proof in the following section is inspired from [21, Page. 8]. Let $\mathbb{E} \subseteq \mathbb{T}$ be a closed set. We define in Lip_α the following Toeplitz operator

$$T_V(g)(z) := \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{g(\zeta)\overline{V(\zeta)}}{\zeta - z} d\zeta, \qquad z \in \mathbb{D},$$

where $V \in \mathcal{H}^\infty$ is a function. We start with the following Proposition.

Proposition 6. *Let $g \in Lip_\alpha$ where $0 < \alpha < 1$ is a real number. For every function $V \in \mathcal{H}^\infty$, we have*

$$\sup_{z\in\mathbb{D}}(1-|z|)^{1-\alpha}|\big(T_V(g)\big)'(z)| < +\infty. \tag{4.1}$$

If moreover $g \in \mathcal{J}_\alpha(\mathbb{E})$, then we have

$$\lim_{\delta\to 0}\sup_{z\in\mathbb{E}(\delta)}(1-|z|)^{1-\alpha}|\big(T_V(g)\big)'(z)| = 0, \tag{4.2}$$

uniformly with respect to all functions V such that $\|V\|_\infty \leq 1$.

Proof. Let $g \in Lip_\alpha$ where $0 < \alpha < 1$ is a real number. We have

$$\begin{aligned}\big(T_V(g)\big)'(z) &= \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{g(\zeta)\overline{V(\zeta)}}{(\zeta-z)^2}d\zeta \\ &= \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{(g(\zeta)-g(z/|z|))\overline{V(\zeta)}}{(\zeta-z)^2}d\zeta, \qquad z\in\mathbb{D}. \end{aligned}\tag{4.3}$$

It follows

$$\begin{aligned} & (1-|z|)^{1-\alpha}|\big(T_V(g)\big)'(z)| \\ \leq\ & \frac{\|V\|_\infty(1-|z|)^{1-\alpha}}{2\pi}\int_{\mathbb{T}}\frac{|g(\zeta)-g(z/|z|)|}{|\zeta-z|^2}|d\zeta| \qquad (4.4)\\ \leq\ & \|V\|_\infty\|g\|_\alpha(1-|z|)^{1-\alpha}\int_{\mathbb{T}}\frac{|\zeta-z/|z||^\alpha}{|\zeta-z|^2}|d\zeta|, \qquad z\in\mathbb{D}. \qquad (4.5)\end{aligned}$$

The following classical equality

$$|e^{it}-|z|e^{i\theta}|^2 = (1-|z|)^2 + 4|z|\sin^2(\frac{1}{2}(\theta-t)),$$

gives the following one

$$\frac{|\zeta-z/|z||^\alpha}{|\zeta-z|^2} = \frac{2^\alpha|\sin(\frac{1}{2}(\theta-t))|^\alpha}{(1-|z|)^2+4|z|\sin^2(\frac{1}{2}(\theta-t))}, \tag{4.6}$$

where $z := |z|e^{i\theta} \in \mathbb{D}$ and $\zeta := e^{it} \in \mathbb{T}$. Therefore

$$\int_{\mathbb{T}}\frac{|\zeta-z/|z||^\alpha}{|\zeta-z|^2}|d\zeta| \leq c\int_0^\pi\frac{s^\alpha}{(1-|z|)^2+s^2}ds \leq c_\alpha(1-|z|)^{\alpha-1}, \qquad z\in\mathbb{D}, \tag{4.7}$$

where c and c_α are constants. By combining (4.5) and (4.7),

$$(1-|z|)^{1-\alpha}|\big(T_V(g)\big)'(z)| \leq c_\alpha\|V\|_\infty\|g\|_\alpha, \qquad z\in\mathbb{D}, \tag{4.8}$$

which proves (4.1).
Now we suppose that $g \in \mathcal{J}_\alpha(\mathbb{E})$ and that $\|V\|_\infty \leq 1$. Let $\varepsilon > 0$ be a positive number. It follows from [16, Proposition 3.1], there exists a real number $0 < \delta < 1$ such that

$$|g(\zeta)-g(\xi)| \leq \varepsilon|\zeta-\xi|^\alpha, \qquad \zeta,\xi\in\overline{\mathbb{E}(\delta)}\cap\mathbb{T}.$$

For a point $z \in \mathbb{E}(\delta)$,

$$\begin{aligned}
& \int_{\mathbb{T}} \frac{|g(\zeta)-g(z/|z|)||V(\zeta)|}{|\zeta-z|^2}|d\zeta| \\
\leq & \int_{\mathbb{T}} \frac{|g(\zeta)-g(z/|z|)|}{|\zeta-z|^2}|d\zeta| \\
= & \int_{\overline{\mathbb{E}(\delta)}\cap\mathbb{T}} \frac{|g(\zeta)-g(z/|z|)|}{|\zeta-z|^2}|d\zeta| + \int_{\mathbb{T}\setminus\overline{\mathbb{E}(\delta)}} \frac{|g(\zeta)-g(z/|z|)|}{|\zeta-z|^2}|d\zeta| \\
\leq & \ \varepsilon \int_{\mathbb{T}} \frac{|\zeta-z/|z||^\alpha}{|\zeta-z|^2}|d\zeta| + \|g\|_\alpha \int_{\mathbb{T}\setminus\overline{\mathbb{E}(\delta)}} \frac{|\zeta-z/|z||^\alpha}{|\zeta-z|^2}|d\zeta|. \qquad (4.9)
\end{aligned}$$

Let $0<\delta'<\delta/2$ and $z \in \mathbb{E}(\delta')$. By using (4.6),

$$\begin{aligned}
\int_{\mathbb{T}\setminus\overline{\mathbb{E}(\delta)}} \frac{|\zeta-z/|z||^\alpha}{|\zeta-z|^2}|d\zeta| & \leq c\int_{s\geq\frac{\delta-\delta'}{2}} \frac{s^\alpha}{(1-|z|)^2+s^2}ds \\
& \leq c_\alpha(1-|z|)^{\alpha-1}\int_{u\geq\frac{\delta-\delta'}{2\delta'}} \frac{u^\alpha}{1+u^2}du, \quad z\in\mathbb{E}(\delta'), \qquad (4.10)
\end{aligned}$$

where c and c_α are constants not depending on δ and δ'. Hence, for sufficiently small δ',

$$\int_{\mathbb{T}} \frac{|g(\zeta)-g(z/|z|)||V(\zeta)|}{|\zeta-z|^2}|d\zeta| \leq \varepsilon c'_\alpha(1-|z|)^{\alpha-1}, \qquad z\in\mathbb{E}(\delta'), \qquad (4.11)$$

by combining (4.7), (4.9) and (4.10). We deduce that (4.2) holds as consequence of (4.4) and (4.11). This finishes the proof of Lemma 6. □

The following Corollary gives the F-property of the spaces Lip_α and $\mathcal{J}_\alpha(\mathbb{E})$ directly from Proposition 6.

Corollary 2. *Let $g \in Lip_\alpha$ where $0<\alpha<1$ is a real number. Let $V \in \mathcal{H}^\infty$ be an inner function dividing U_g. We have the following assertions*

1. *The function g/V belongs to Lip_α.*
2. *If $g \in \mathcal{J}_\alpha(\mathbb{E})$, then $g/V \in \mathcal{J}_\alpha(\mathbb{E})$.*

Proof. Since $V \in \mathcal{H}^\infty$ is an inner function dividing U_g, then $T_V(g) = g/V$. The proof of the assertions 1 and 2 are deduced by applying Proposition 6. □

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