On quaternionic functions: I. Local theory

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ON QUATERNIONIC FUNCTIONS: I. LOCAL THEORY

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Abstract. Several sets of quaternionic functions are described and studied with respect to hyperholomorphy, addition and (non commutative) multiplication, on open sets of $\mathbb{H}$. The aim is to get a local function theory.

Contents

1. Introduction

We first recall the definition of the non commutative field $\mathbb{H}$ of quaternions using pairs of complex numbers and a modified Cauchy-Fueter operator (section 2) that have been introduced in [2]. We will only use right multiplication; the (right) inverse of a nonzero quaternion being defined. We will consider C^∞ $\mathbb{H}$-valued quaternionic functions defined on an open set U of $\mathbb{H}$, whose behavior mimics the behavior of holomorphic functions near their zeroes on an open set of $\mathbb{C}$. If such a function does not identically vanish over U, it has an (algebraic) inverse which is defined almost everywhere on U. Finally, we describe properties of hyperholomorphic functions with respect to addition and multiplication.

In section 3, we characterize the quaternionic functions which are, almost everywhere, hyperholomorphic and whose inverses are hyperholomorphic almost everywhere, on U, as the solutions of a system of two non linear PDE. We find non trivial examples of a solution, showing that the considered space of functions is significant: we will call these functions hypermeromorphic. There is a preliminary announcement in [4]

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In section 4, we describe a subspace $\mathcal{H}_U$ of hyperholomorphic and hypermeromorphic functions defined almost everywhere on U, having "good properties for addition and multiplication"; we obtain again systems of non linear PDE, and we give first results on spaces of functions of strictly positive dimension as vector spaces.

In the next paper II. Global theory, we will consider globalization of the introduced notions, define Hamilton 4-manifolds analogous to Riemann manifolds for $\mathbb{H}$ instead of $\mathbb{C}$, and give examples of such manifolds; our ultimate aim is to describe a class of four dimensional manifolds.

2. Quaternions. $\mathbb{H}$-valued functions. Hyperholomorphic functions

Quaternions, $\mathbb{H}$-valued functions have been defined in [2].

2.1. **Quaternions.** If $q \in \mathbb{H}$, then $q = z_1 + z_2\mathbf{j}$ where $z_1, z_2 \in \mathbb{C}$, hence $\mathbb{H} \cong \mathbb{C}^2 \cong \mathbb{R}^4$ as complex or real vector space. We have: $z_1\mathbf{j} = \mathbf{j}\overline{z}_1$ (by computation in real coordinates); by definition, the *modulus* of q is $\parallel q \parallel = (|z_1|^2 + |z_2|^2)^{\frac{1}{2}}$.

Let $*$ denote the *(right) multiplication* in $\mathbb{H}$:

Recall the noncommutativity of the multiplication:

$qq' = (z_1 + z_2\mathbf{j}) * (z'_1 + z'_2\mathbf{j}) = (z_1z'_1 - z_2\overline{z}'_2) + (z_1z'_2 + z_2\overline{z}'_1)\mathbf{j} = a + b\mathbf{j}$

$q'q = (z_1z'_1 - z_2\overline{z}'_2) + (z'_1z_2 + z'_2\overline{z}_1)\mathbf{j} = a + b'\mathbf{j}$,

where $b = z_1z'_2 + z_2\overline{z}'_1)$ and $b' = z'_1z_2 + z'_2\overline{z}_1$.

Commutativity when q and $q' \in \mathbb{R}$.

The *conjugate* of q is $\overline{q} = \overline{z}_1 - z_2\mathbf{j}$. $q * \overline{q} = (z_1 + z_2\mathbf{j}) * (\overline{z}_1 - z_2\mathbf{j}) = |z_1|^2 - z_1z_2\mathbf{j} + z_2\mathbf{j}\overline{z}_1 - z_2\mathbf{j}z_2\mathbf{j} = |z_1|^2 + |z_2|^2 = |q|$, then: the *(right) inverse* of $q = z_1 + z_2\mathbf{j}$ is: $q^{-1} = (|z_1|^2 + |z_2|^2)^{-1}\overline{q} = (|z_1|^2 + |z_2|^2)^{-1}(\overline{z}_1 - z_2\mathbf{j}) = |q|^{-1}\overline{q}$. Moreover: $(|z_1|^2 + |z_2|^2)^{-1}(\overline{z}_1 - z_2\mathbf{j}) * (z_1 + z_2\mathbf{j}) = 1$, so the right inverse of q^{-1} is q.

2.2. **Quaternionic functions.** Let U be an open set of $\mathbb{H} \cong \mathbb{C}^2$ and $f \in C^\infty(U, \mathbb{H})$, then $f = f_1 + f_2\mathbf{j}$, where $f_1, f_2 \in C^\infty(U, \mathbb{C})$. The complex valued functions f_1, f_2 will be called the *components* of f.

Remark that $\mathbb{H}$ is a real vector space in which real analysis is valid, in particular differential forms, distributions and currents are defined in $\mathbb{H}$. Remark that $\dfrac{\partial f_1}{\partial \overline{z}_1}\mathbf{j} = \mathbf{j}\dfrac{\partial \overline{f}_1}{\partial z_1}$ and analogous relations.

2.3. **Definitions.** Let U be an open neighborhood of 0 in $\mathbb{H} \cong \mathbb{C}^2$.
(a) *From now on, we will consider the quaternionic functions* $f = f_1 + f_2\mathbf{j}$ *having the following properties*:

(i) when f_1 and f_2 are not holomorphic, the set $Z(f_1) \cap Z(f_2)$ is *discrete* on U;

(ii) for every $q \in Z(f_1) \cap Z(f_2)$, $J_q^{\alpha}(.)$ denoting the *jet of order* α *at* q [6], let $m_i = \sup_{\alpha_i} J_q^{\alpha_i}(f_i) = 0$; m_i, $i = 1, 2$, is finite.

Define: $m_q = \inf_i m_i$ as the *order of the zeroe* q *of* f.

(b) We will also consider *the quaternionic functions defined almost everywhere on* U (i.e. outside of a subset of U of Lebesgue measure 0, more precisely outside a finite set of C^∞ hypersurfaces).

2.4. **Modified Cauchy-Fueter operator $\mathcal{D}$. Hyperholomorphic functions.** The modified Cauchy-Fueter operator $\mathcal{D}$ and hyperholomorphic functions have been defined in [2, 5].

For $f \in C^\infty(U, \mathbb{H})$, with $f = f_1 + f_2\mathbf{j}$, where $f_1, f_2 \in C^\infty(U, \mathbb{C})$,

$$\mathcal{D}f(q) = \frac{1}{2}\Big(\frac{\partial}{\partial \overline{z}_1} + \mathbf{j}\frac{\partial}{\partial \overline{z}_2}\Big)f(q) = \frac{1}{2}\Big(\frac{\partial f_1}{\partial \overline{z}_1} - \frac{\partial \overline{f}_2}{\partial z_2}\Big)(q) + \mathbf{j}\frac{1}{2}\Big(\frac{\partial f_1}{\partial \overline{z}_2} + \frac{\partial \overline{f}_2}{\partial z_1}\Big)(q).$$

A function $f \in C^\infty(U, \mathbb{H})$ is said to be *hyperholomorphic* if $\mathcal{D}f = 0$.

Characterization of the hyperholomorphic function f on U:

$$\frac{\partial f_1}{\partial \overline{z}_1} - \frac{\partial \overline{f}_2}{\partial z_2} = 0; \ \frac{\partial f_1}{\partial \overline{z}_2} + \frac{\partial \overline{f}_2}{\partial z_1} = 0, \text{ on } U. \tag{1}$$

The conditions: f_1 is holomorphic and f_2 is holomorphic are equivalent. So *holomorphic functions with values in* $\mathbb{C}$ *will be identified with hyperholomorphic functions* f *such that* $f_2 = 0$.

Proposition 2.1. *The set* $\mathcal{H}$ *of hyperholomorphic functions such that the sum of two of them satisfies the above conditions* $(i), (ii)$ *is an* $\mathbb{H}$*-right vector space.*

Proof. Let $f' = f'_1 + f'_2\mathbf{j}$, $f'' = f''_1 + f''_2\mathbf{j}$ be two hyperholomorphic functions satisfying the above conditions $(i), (ii)$, then, for $\lambda', \lambda'' \in \mathbb{H}$, $\lambda' f' + \lambda'' f''$ has the same properties. □

Proposition 2.2. *The set* $\mathcal{H}'$ *of almost everywhere defined hyperholomorphic functions is an* $\mathbb{H}$*-right vector space.*

Proposition 2.3. *Let* f', f'' *be two almost everywhere defined hyperholomorphic functions. Then, their product* $f' * f''$ *satisfies:*

$$\mathcal{D}(f' * f'') = \mathcal{D}f' * \mathbf{j}f'' + \Big(f'\big(\frac{\partial}{\partial \overline{z}_1}\big) + \overline{f}'\mathbf{j}\frac{\partial}{\partial \overline{z}_2}\Big)f''$$

Proof. $f' = f'_1 + f'_2\mathbf{j}$, $f'' = f''_1 + f''_2\mathbf{j}$ be two hyperholomorphic functions.

We have: $f' * f'' = (f'_1 + f'_2\mathbf{j})(f''_1 + f''_2\mathbf{j}) = f'_1 f''_1 - f'_2\overline{f}''_2 + (f'_1 f''_2 + f'_2\overline{f}''_1)\mathbf{j}$

Compute

$$\frac{1}{2}\big(\frac{\partial}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial}{\partial\overline{z}_2}\big)\big(f'_1 f''_1 - f'_2\overline{f}''_2 + (f'_1 f''_2 + f'_2\overline{f}''_1)\mathbf{j}\big)$$

By derivation of the first factors of the sum $f' * f''$, we get the first term:

$$\frac{1}{2}\big(\frac{\partial f'_1}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial f'_1}{\partial\overline{z}_2}\big)(f''_1 + f''_2\mathbf{j}) + \frac{1}{2}\big(\frac{\partial f'_2}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial f'_2}{\partial\overline{z}_2}\big)\mathbf{j}\mathbf{j}(\overline{f}''_2 - \overline{f}''_1\mathbf{j})$$

$$= \frac{1}{2}\big(\frac{\partial f'_1}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial f'_1}{\partial\overline{z}_2}\big)(f''_1 + f''_2\mathbf{j}) + \frac{1}{2}\big(\frac{\partial f'_2\mathbf{j}}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial f'_2\mathbf{j}}{\partial\overline{z}_2}\big)\mathbf{j}(f''_2\mathbf{j} + f''_1) = \mathcal{D}f' * \mathbf{j}f''$$

By derivation in

$$\frac{1}{2}\big(\frac{\partial}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial}{\partial\overline{z}_2}\big)\big(f'_1 f''_1 + f'_2\mathbf{j}f''_2\mathbf{j} + (f'_1 f''_2\mathbf{j} + f'_2\mathbf{j}f''_1)\big)$$

of the second factors of the sum $f' * f''$, we get the second term (up to factor $\frac{1}{2}$):

$$\begin{aligned}
f'_1\frac{\partial f''_1}{\partial\overline{z}_1} + \overline{f}'_1\mathbf{j}\frac{\partial f''_1}{\partial\overline{z}_2} + f'_1\frac{\partial f''_2}{\partial\overline{z}_1}\mathbf{j} + \overline{f}'_1\mathbf{j}\frac{\partial f''_2}{\partial\overline{z}_2}\mathbf{j} \\
+ f'_2\mathbf{j}\frac{\partial f''_2}{\partial\overline{z}_1}\mathbf{j} + \overline{f}'_2\mathbf{j}\frac{\partial f''_2}{\partial\overline{z}_2} + f'_2\mathbf{j}\frac{\partial f''_1}{\partial\overline{z}_1} + \overline{f}'_2\mathbf{j}\mathbf{j}\frac{\partial f''_1}{\partial\overline{z}_2} \\
= (f'_1 + f'_2\mathbf{j})(\frac{\partial}{\partial\overline{z}_1})(f''_1 + f''_2\mathbf{j}) + (\overline{f}'_1 + \overline{f}'_2\mathbf{j})\mathbf{j}\frac{\partial}{\partial\overline{z}_2}(f''_1 + f''_2\mathbf{j}) \\
= \big((f'_1 + f'_2\mathbf{j})(\frac{\partial}{\partial\overline{z}_1}) + (\overline{f}'_1 + \overline{f}'_2\mathbf{j})\mathbf{j}\frac{\partial}{\partial\overline{z}_2}\big)(f''_1 + f''_2\mathbf{j}) \\
= \big(f'(\frac{\partial}{\partial\overline{z}_1}) + \overline{f}'\mathbf{j}\frac{\partial}{\partial\overline{z}_2}\big)f''.
\end{aligned}$$

□

If the components of f' and f'' are real, the second term is:

$$\frac{1}{2}(f'_1 + f'_2\mathbf{j})(\frac{\partial}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial}{\partial\overline{z}_2})(f''_1 + f''_2\mathbf{j}) = f' * \mathcal{D}f''$$

i.e.

Corollary 2.4. *The set $\mathcal{H}_{\mathbb{R}}$ of almost everywhere defined hyperholomorphic functions whose components are real is an $\mathbb{R}$-right algebra.*

2.5. **Remark.** The hyperholomorphic functions $f = f_1 + f_2\mathbf{j}$ such that $f_2 = 0$ satisfy:

$\mathcal{D}f(q) = \frac{1}{2}\big(\frac{\partial}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial}{\partial\overline{z}_2}\big)f(q) = \frac{1}{2}\frac{\partial f_1}{\partial\overline{z}_1}(q) + \mathbf{j}\frac{1}{2}\frac{\partial f_1}{\partial\overline{z}_2}(q)$ i.e. are holomorphic in (z_1, z_2) with values in $\mathbb{C}$. More generally, we will consider *their product with a quaternion. Their set is the $\mathbb{H}$-right algebra of holomorphic functions.*

3. Almost everywhere hyperholomorphic functions whose inverses are almost everywhere hyperholomorphic

3.1. **Null set and inverse of a quaternionic function.** Let $f = f_1 + f_2\mathbf{j}$ be a quaternionic function on U. The null set $Z(f)$ satisfies: $f_1 = 0$; $f_2 = 0$, then $Z(f)$ is of measure 0 in U.

Ex.: $f_1 = \overline{z}_1$; $f_2 = \overline{z}_2$, then $Z(f) = \{0\}$.

Note that if f is holomorphic, then, $f_2 \equiv 0$ and $Z(f)$ is a complex hypersurface in $\mathbb{C}^2$.

We call *inverse* of a function $f : q \mapsto f(q)$, the function, defined almost everywhere on U: $f^{-1} : q \mapsto f(q)^{-1}$; then: $f^{-1} = |f|^{-1}\overline{f}$, where $\overline{f}$ is the (quaternionic) conjugate of f, and $f^{-1} = (|f_1|^2 + |f_2|^2)^{-1}(\overline{f}_1 - f_2\mathbf{j})$.

3.2. **Inversion and hyperholomorphy.** *Assume f to be hyperholomorphic and $Z(f) = \{0\}$, then $\dfrac{1}{f}$ is not necessarily hyperholomorphic outside $\{0\}$.*

Ex.: $f = \overline{z}_1 + \overline{z}_2\mathbf{j}$, then

$$\frac{1}{f} = (z_1\overline{z}_1 + z_2\overline{z}_2)^{-1}(z_1 - \overline{z}_2\mathbf{j}); \;\; \mathcal{D}(\frac{1}{f}) \neq 0,$$

3.2.1. *Definition.* Behavior of f^{-1} at $q \in Z(f) = Z(f_1) \cap Z(f_2)$. Denoting $J^\alpha_q(.)$ the *jet of order α at q* [6], let $n_1 = \sup J^\alpha_q(|f|\overline{f}_1^{-1})$; $n_2 = \sup J^\alpha_q(|f|f_2^{-1})$

Define: $n_q = \sup\limits_i n_i$, $i = 1, 2$ as the *order of the pole q of f^{-1}*.

3.2.2. *Inverse of a holomorphic function.* Let $f = f_1 + 0\mathbf{j}$ be a hyperholomorphic function. Then $f^{-1} = f_1^{-1} + 0\mathbf{j}$ and f^{-1} is hyperholomorphic outside of the complex hypersurface $Z(f)$. Remark that $Z(f)$ is a subvariety of complex dimension 1, then of measure zero, in U.

We will consider *almost everywhere defined* hyperholomorphic functions on U. Ex.: holomorphic, meromorphic functions.

Let $f = f_1 + f_2\mathbf{j}$ be a hyperholomorphic function and $g = g_1 + g_2\mathbf{j} = |f|^{-1}(\overline{f}_1 - f_2\mathbf{j})$ its inverse; so $g_1 = |f|^{-1}\overline{f}_1$; $g_2 = -|f|^{-1}f_2$, where $|f| = (f_1\overline{f}_1 + f_2\overline{f}_2)$.

3.2.3. *A chacracterisation.*

Proposition 3.1. *The following conditions are equivalent*

(*i*) *the function $f = f_1 + f_2\mathbf{j}$ and its right inverse are hyperholomorphic, when they are defined:*

(*ii*) *we have the equations:*

$$(\overline{f}_1 - f_1)\frac{\partial\overline{f}_1}{\partial z_1} - \overline{f}_2\frac{\partial f_2}{\partial z_1} - f_2\frac{\partial\overline{f}_1}{\partial\overline{z}_2} = 0,$$

$$\overline{f}_2\frac{\partial f_1}{\partial z_1}+\frac{\partial\overline{f}_2}{\partial z_1}(\overline{f}_1-f_1)-f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_2}=0.$$

Proof. Let $f=f_1+f_2\mathbf{j}$ be a hyperholomorphic function and $g=g_1+g_2\mathbf{j}=|f|^{-1}(\overline{f}_1-f_2\mathbf{j})$ its inverse; so $g_1=|f|^{-1}\overline{f}_1$; $g_2=-|f|^{-1}f_2$, where $|f|=(f_1\overline{f}_1+f_2\overline{f}_2)$.

$$\begin{aligned}
\mathcal{D}g(q) &= \frac12\big(\frac{\partial}{\partial\overline{z}_1}+\mathbf{j}\frac{\partial}{\partial\overline{z}_2}\big)g(q)=\frac12\big(\frac{\partial g_1}{\partial\overline{z}_1}-\frac{\partial\overline{g}_2}{\partial z_2}\big)(q)+\mathbf{j}\frac12\big(\frac{\partial g_1}{\partial\overline{z}_2}+\frac{\partial\overline{g}_2}{\partial z_1}\big)(q)\\
\frac{\partial g_1}{\partial\overline{z}_1} &= |f|^{-1}\frac{\partial\overline{f}_1}{\partial\overline{z}_1}-|f|^{-2}\overline{f}_1\big(\frac{\partial f_1}{\partial\overline{z}_1}\overline{f}_1+f_1\frac{\partial\overline{f}_1}{\partial\overline{z}_1}+\frac{\partial f_2}{\partial\overline{z}_1}\overline{f}_2+f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_1}\big)\\
-\frac{\partial\overline{g}_2}{\partial z_2} &= |f|^{-1}\frac{\partial\overline{f}_2}{\partial z_2}-|f|^{-2}\overline{f}_2\big(\frac{\partial f_1}{\partial z_2}\overline{f}_1+f_1\frac{\partial\overline{f}_1}{\partial z_2}+\frac{\partial f_2}{\partial z_2}\overline{f}_2+f_2\frac{\partial\overline{f}_2}{\partial z_2}\big)\\
\frac{\partial g_1}{\partial\overline{z}_2} &= |f|^{-1}\frac{\partial\overline{f}_1}{\partial\overline{z}_2}-|f|^{-2}\overline{f}_1\big(\frac{\partial f_1}{\partial\overline{z}_2}\overline{f}_1+f_1\frac{\partial\overline{f}_1}{\partial\overline{z}_2}+\frac{\partial f_2}{\partial\overline{z}_2}\overline{f}_2+f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_2}\big)\\
\frac{\partial\overline{g}_2}{\partial z_1} &= -|f|^{-1}\frac{\partial\overline{f}_2}{\partial z_1}+|f|^{-2}\overline{f}_2\big(\frac{\partial f_1}{\partial z_1}\overline{f}_1+f_1\frac{\partial\overline{f}_1}{\partial z_1}+\frac{\partial f_2}{\partial z_1}\overline{f}_2+f_2\frac{\partial\overline{f}_2}{\partial z_1}\big)
\end{aligned}$$

$$\begin{aligned}
&2|f|^2\mathcal{D}g\\
&= (f_1\overline{f}_1+f_2\overline{f}_2)\big(\frac{\partial\overline{f}_1}{\partial\overline{z}_1}+\frac{\partial\overline{f}_2}{\partial z_2}\big)-\overline{f}_1f_1\frac{\partial\overline{f}_1}{\partial\overline{z}_1}-\overline{f}_1\overline{f}_1\frac{\partial f_1}{\partial\overline{z}_1}-\overline{f}_1f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_1}-\overline{f}_1\overline{f}_2\frac{\partial f_2}{\partial\overline{z}_1}\\
&\quad-\overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_2}-f_1\overline{f}_2\frac{\partial\overline{f}_1}{\partial z_2}-\overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_2}-f_2\overline{f}_2\frac{\partial\overline{f}_2}{\partial z_2}\\
&\quad+\mathbf{j}\Big((f_1\overline{f}_1+f_2\overline{f}_2)\big(\frac{\partial\overline{f}_1}{\partial\overline{z}_2}-\frac{\partial\overline{f}_2}{\partial z_1}\big)-\overline{f}_1\overline{f}_1\frac{\partial f_1}{\partial\overline{z}_2}-\overline{f}_1f_1\frac{\partial\overline{f}_1}{\partial\overline{z}_2}-\overline{f}_1\overline{f}_2\frac{\partial f_2}{\partial\overline{z}_2}-\overline{f}_1f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_2}\\
&\quad+\overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_1}+f_1\overline{f}_2\frac{\partial\overline{f}_1}{\partial z_1}+\overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_1}+f_2\overline{f}_2\frac{\partial\overline{f}_2}{\partial z_1}\Big)
\end{aligned}$$

Use the fact: f is hyperholomorphic:

$$\frac{\partial f_1}{\partial\overline{z}_1}-\frac{\partial\overline{f}_2}{\partial z_2}=0;\ \frac{\partial f_1}{\partial\overline{z}_2}+\frac{\partial\overline{f}_2}{\partial z_1}=0 \tag{1}$$

$$2|f|^2\mathcal{D}g=f_1\overline{f}_1\frac{\partial\overline{f}_2}{\partial z_2}+f_2\overline{f}_2\frac{\partial\overline{f}_1}{\partial\overline{z}_1}-\overline{f}_1\overline{f}_1\frac{\partial f_1}{\partial\overline{z}_1}-\overline{f}_1f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_1}-\overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_2}-\overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_2}+\overline{f}_2\frac{\partial f_2}{\partial\overline{z}_1}(f_1-\overline{f}_1)+$$
$$+\mathbf{j}\Big(+f_2\overline{f}_2\frac{\partial\overline{f}_1}{\partial\overline{z}_2}-\overline{f}_1\overline{f}_2\frac{\partial f_2}{\partial\overline{z}_2}-\overline{f}_1f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_2}+\overline{f}_1\frac{\partial f_1}{\partial\overline{z}_2}(f_1-\overline{f}_1)+\overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_1}+f_1\overline{f}_2\frac{\partial\overline{f}_1}{\partial z_1}+\overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_1}\Big)$$

f being hyperholomorphic, g hyperholomorphic is equivalent to the system of two equations:

$$+f_1\overline{f}_1\frac{\partial\overline{f}_2}{\partial z_2}+f_2\overline{f}_2\frac{\partial\overline{f}_1}{\partial\overline{z}_1}-\overline{f}_1\overline{f}_1\frac{\partial f_1}{\partial\overline{z}_1}-\overline{f}_1f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_1}-\overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_2}-\overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_2}+\overline{f}_2\frac{\partial f_2}{\partial\overline{z}_1}(f_1-\overline{f}_1)=0$$
$$+f_2\overline{f}_2\frac{\partial\overline{f}_1}{\partial\overline{z}_2}-\overline{f}_1\overline{f}_2\frac{\partial f_2}{\partial\overline{z}_2}-\overline{f}_1f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_2}+\overline{f}_1\frac{\partial f_1}{\partial\overline{z}_2}(f_1-\overline{f}_1)+\overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_1}+f_1\overline{f}_2\frac{\partial\overline{f}_1}{\partial z_1}+\overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_1}=0$$

f_1 and f_2 satisfy, by conjugation of the second equation:

$$+f_2\overline{f}_2\frac{\partial f_1}{\partial z_1}-f_1f_1\frac{\partial\overline{f}_1}{\partial z_1}-f_1\overline{f}_2\frac{\partial f_2}{\partial z_1}+f_2\frac{\partial\overline{f}_2}{\partial z_1}(\overline{f}_1-f_1)+f_1\overline{f}_1\frac{\partial f_2}{\partial\overline{z}_2}-f_1f_2\frac{\partial\overline{f}_1}{\partial\overline{z}_2}-f_2f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_2}=0$$

$$+\overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_1}+f_1\overline{f}_2\frac{\partial\overline{f}_1}{\partial z_1}.+\overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_1}+f_2\overline{f}_2\frac{\partial\overline{f}_1}{\partial\overline{z}_2}-\overline{f}_1\overline{f}_2\frac{\partial f_2}{\partial\overline{z}_2}-\overline{f}_1f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_2}+\overline{f}_1\frac{\partial f_1}{\partial\overline{z}_2}(f_1-\overline{f}_1)=0$$

Using (1), we get:

$$+f_2\overline{f}_2\frac{\partial f_1}{\partial z_1}+f_1(\overline{f}_1-f_1)\frac{\partial\overline{f}_1}{\partial z_1}-f_1\overline{f}_2\frac{\partial f_2}{\partial z_1}+f_2\frac{\partial\overline{f}_2}{\partial z_1}(\overline{f}_1-f_1)-f_1f_2\frac{\partial\overline{f}_1}{\partial\overline{z}_2}-f_2f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_2}=0$$

$$+\overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_1}+(f_1-\overline{f}_1)\overline{f}_2\frac{\partial\overline{f}_1}{\partial z_1}+\overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_1}+\overline{f}_1\frac{\partial f_1}{\partial\overline{z}_2}(f_1-\overline{f}_1)+f_2\overline{f}_2\frac{\partial\overline{f}_1}{\partial\overline{z}_2}.-\overline{f}_1f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_2}=0$$

Assume $f_1\neq 0$, $f_2\neq 0$

$$\overline{f}_1\big(f_2\overline{f}_2\frac{\partial f_1}{\partial z_1}+f_1(\overline{f}_1-f_1)\frac{\partial\overline{f}_1}{\partial z_1}-f_1\overline{f}_2\frac{\partial f_2}{\partial z_1}+f_2\frac{\partial\overline{f}_2}{\partial z_1}(\overline{f}_1-f_1)-f_1f_2\frac{\partial\overline{f}_1}{\partial\overline{z}_2}-f_2f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_2}\big)=0$$

$$-f_2\big(+\overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_1}+(f_1-\overline{f}_1)\overline{f}_2\frac{\partial\overline{f}_1}{\partial z_1}+\overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_1}-\overline{f}_1\frac{\partial\overline{f}_2}{\partial z_1}(f_1-\overline{f}_1)+f_2\overline{f}_2\frac{\partial\overline{f}_1}{\partial\overline{z}_2}-\overline{f}_1f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_2}\big)=0$$

By sum:

$$\overline{f}_1\big(f_1(\overline{f}_1-f_1)\frac{\partial\overline{f}_1}{\partial z_1}-f_1\overline{f}_2\frac{\partial f_2}{\partial z_1}+f_2\frac{\partial\overline{f}_2}{\partial z_1}(\overline{f}_1-f_1)-f_1f_2\frac{\partial\overline{f}_1}{\partial\overline{z}_2}\big)$$

$$-f_2\big((f_1-\overline{f}_1)\overline{f}_2\frac{\partial\overline{f}_1}{\partial z_1}+\overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_1}-\overline{f}_1\frac{\partial\overline{f}_2}{\partial z_1}(f_1-\overline{f}_1)+f_2\overline{f}_2\frac{\partial\overline{f}_1}{\partial\overline{z}_2}\big)=0$$

i.e.

$$(\overline{f}_1f_1+f_2\overline{f}_2)\big((\overline{f}_1-f_1)\frac{\partial\overline{f}_1}{\partial z_1}-\overline{f}_2\frac{\partial f_2}{\partial z_1}-f_2\frac{\partial\overline{f}_1}{\partial\overline{z}_2}\big)=0$$

$$\overline{f}_2\big(f_2\overline{f}_2\frac{\partial f_1}{\partial z_1}+f_1(\overline{f}_1-f_1)\frac{\partial\overline{f}_1}{\partial z_1}-f_1\overline{f}_2\frac{\partial f_2}{\partial z_1}+f_2\frac{\partial\overline{f}_2}{\partial z_1}(\overline{f}_1-f_1)-f_1f_2\frac{\partial\overline{f}_1}{\partial\overline{z}_2}-f_2f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_2}\big)=0$$

$$f_1\big(+\overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_1}+(f_1-\overline{f}_1)\overline{f}_2\frac{\partial\overline{f}_1}{\partial z_1}+\overline{f}_2\overline{f}_2\frac{\partial f_2}{\partial z_1}-\overline{f}_1\frac{\partial\overline{f}_2}{\partial z_1}(f_1-\overline{f}_1)+f_2\overline{f}_2\frac{\partial\overline{f}_1}{\partial\overline{z}_2}-\overline{f}_1f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_2}\big)=0$$

By sum

$$\overline{f}_2\big(f_2\overline{f}_2\frac{\partial f_1}{\partial z_1}+f_2\frac{\partial\overline{f}_2}{\partial z_1}(\overline{f}_1-f_1)-f_2f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_2}\big)$$

$$+f_1\big(\overline{f}_1\overline{f}_2\frac{\partial f_1}{\partial z_1}-\overline{f}_1\frac{\partial\overline{f}_2}{\partial z_1}(f_1-\overline{f}_1)-\overline{f}_1f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_2}\big)=0$$

i.e.

$$\overline{f}_2\frac{\partial f_1}{\partial z_1}+\frac{\partial\overline{f}_2}{\partial z_1}(\overline{f}_1-f_1)-f_2\frac{\partial\overline{f}_2}{\partial\overline{z}_2}=0$$

□

Corollary 3.2. *If f satisfies the conditions of the Proposition, the same is true for αf with $\alpha\in\mathbb{R}$.*

3.2.4. Let $f = f_1 + 0\mathbf{j}$ be an almost everywhere holomorphic function, then the condition (ii) of Proposition 3.1 is satisfied.

Proposition 3.3. *Let $f = f_1 + f_2\mathbf{j}$ be a quaternionic function such that f_1 and f_2 are real.*

Then, the following conditions are equivalent:

- (i) *the function $f = f_1 + f_2\mathbf{j}$ and its right inverse are hyperholomorphic, when they are defined;*
- (ii) *f_1, f_2 satisfy the equations:*

$$\frac{\partial f_2}{\partial z_1} + \frac{\partial f_1}{\partial \overline{z}_2} = 0$$

$$\frac{\partial f_1}{\partial z_1} - \frac{\partial f_2}{\partial \overline{z}_2} = 0$$

Proof. Assume f_1 and f_2 real: condition (ii) of Proposition 3.1 reduces to:

$$f_2\frac{\partial f_2}{\partial z_1} + f_2\frac{\partial f_1}{\partial \overline{z}_2} = 0; \quad f_2\frac{\partial f_1}{\partial z_1} - f_2\frac{\partial f_2}{\partial \overline{z}_2} = 0$$

i.e.: $f_2 = 0$ and the linear system:

$$\frac{\partial f_2}{\partial z_1} + \frac{\partial f_1}{\partial \overline{z}_2} = 0;$$

$$\frac{\partial f_1}{\partial z_1} - \frac{\partial f_2}{\partial \overline{z}_2} = 0$$

The solution $f_2 = 0$ means $f = f_1$=real; since f_1 is holomorphic and real, then f_1 = real constant: no interest. □

Numerical example. Let $f_1 = z_1 + \overline{z}_1 + z_2 + \overline{z}_2 + A$, $f_2 = -z_1 - \overline{z}_1 + z_2 + \overline{z}_2 + B$, $A, B \in \mathbb{R}$, then: f and f^{-1} outside the zero set of f, are hyperholomorphic.

The *the null set of $f = f_1 + f_2\mathbf{j}$, for f_1, f_2 as above, for $A = B = 0$, is:*

$$z_1 + \overline{z}_1 + z_2 + \overline{z}_2 = 0$$

$$-z_1 - \overline{z}_1 + z_2 + \overline{z}_2 = 0$$

i.e, by difference and sum:

$$\overline{z}_1 + z_1 = 0; z_2 + \overline{z}_2 = 0,$$

i.e.

$$x_1 = 0; x_2 = 0$$

in $\mathbb{R}^4$.

3.3. **Definition.** We will call *w-hypermeromorphic function* (w- for *weak*) any almost everywhere defined hyperholomorphic function whose right inverse is hyperholomorphic almost everywhere.

Recall: the conjugate (in the sense of quaternions) of $f = f_1 + f_2\mathbf{j}$ is $\overline{f} = \overline{f}_1 - f_2\mathbf{j}$, and $f^{-1} = |f|^{-1}\overline{f}$.

4. On the spaces of hypermeromorphic functions.

4.1. **Sum of two w-hypermeromorphic functions.** Let f, g be two almost everywhere hyperholomorphic functions whose inverses have the same property, then:

1) $f + g$ is almost everywhere hyperholomorphic;

2) the inverse of $f+g$, $(f+g)^{-1}$ has to be almost everywhere hyperholomorphic.

Recall $\mathcal{D} = \frac{1}{2}\big(\frac{\partial}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial}{\partial\overline{z}_2}\big)$.

$(f+g)^{-1} = |f+g|^{-1}(\overline{f+g})$; $|f+g|^{-1}$, being real, is transparent with respect to $\mathcal{D}$ because $z_1\mathbf{j} = \mathbf{j}\overline{z}_1$ for $z_1 \in \mathbb{C}$;

$$\mathcal{D}((f+g)^{-1}) = -|f+g|^{-2}\mathcal{D}(|f+g)|)(\overline{f+g}) + |f+g|^{-1}\mathcal{D}(\overline{f+g})$$

We must have:

$$|f+g|^2\mathcal{D}((f+g)^{-1}) = -\mathcal{D}(|f+g)|)(\overline{f+g}) + |f+g|\mathcal{D}(\overline{f+g}) = 0$$

Proposition 4.1. *If f and g are two w-hypermeromorphic functions, then the following conditions are equivalent:*

(i) the sum $h = f + g$ is w-hypermeromorphic;

(ii) h satisfies the following PDE:

$$-\big(\frac{\partial|h|}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial|h|}{\partial\overline{z}_2}\big)(\overline{h}_1 - h_2\mathbf{j}) + |h|\big(\frac{\partial}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial}{\partial\overline{z}_2}\big)(\overline{h}_1 - h_2\mathbf{j}) = 0$$

If f and g are holomorphic, the condition (i) is trivially satisfied.

Proof. Explicit the condition:

$$|h|^2\mathcal{D}(h^{-1}) = -\mathcal{D}(|h)|)(\overline{h}) + |h|\mathcal{D}(\overline{h}) = 0$$

; with $\overline{h} = \overline{h}_1 - h_2\mathbf{j}$

$$2\mathcal{D}\overline{h} = \big(\frac{\partial}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial}{\partial\overline{z}_2}\big)(\overline{h}_1 - h_2\mathbf{j}) = \frac{\partial\overline{h}_1}{\partial\overline{z}_1} + \frac{\partial\overline{h}_2}{\partial z_2} - \big(\frac{\partial h_2}{\partial\overline{z}_1} - \frac{\partial h_1}{\partial z_2}\big)\mathbf{j}$$

$$\mathcal{D}(|h|) = \mathcal{D}(h_1\overline{h}_1 + h_2\overline{h}_2) = \frac{1}{2}\big(\frac{\partial}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial}{\partial\overline{z}_2}\big)(h_1\overline{h}_1 + h_2\overline{h}_2)$$
$$= \frac{1}{2}\big(\overline{h}_1\frac{\partial h_1}{\partial\overline{z}_1} + \overline{h}_2\frac{\partial h_2}{\partial\overline{z}_1} + h_1\frac{\partial\overline{h}_1}{\partial\overline{z}_1} + h_2\frac{\partial\overline{h}_2}{\partial\overline{z}_1}\big)$$
$$+ \frac{1}{2}\big(\overline{h}_1\frac{\partial h_1}{\partial z_2} + \overline{h}_2\frac{\partial h_2}{\partial z_2} + h_1\frac{\partial\overline{h}_1}{\partial z_2} + h_2\frac{\partial\overline{h}_2}{\partial z_2}\big)\mathbf{j} = 0.$$

□

Remark on functions of complex variables z,$\overline{z}$ with real values. We assume such a function $f(z,\overline{z})$ a convergente series in $(z,\overline{z})$, $\sum_{0,0}^{\infty,\infty} a_{kl}z^k\overline{z}^l$. From $f = \overline{f}$, we get: $a_{kl} = \overline{a}_{lk}$. $\frac{\partial f}{\partial\overline{z}} = \sum la_{kl}z^k\overline{z}^{l-1}$; $\frac{\partial\overline{f}}{\partial z} = \sum k\overline{a}_{kl}\overline{z}^{k-1}z^l$; i.e. $\frac{\partial f}{\partial\overline{z}} = \frac{\partial f}{\partial z}$. The result remains valid if f is C^∞.

Proposition 4.2. *If f and g are two w-hypermeromorphic functions whose components are real, then the following propeeties are equivalent:*

(*i*) *the sum $h = f + g$ is w-hypermeromorphic;*
(*ii*) *f and g satisfy the following condition: $h_1 = A_1(z_2,\overline{z}_2)$; $h_2 = A_2(z_1,\overline{z}_1)$ and $A_1 + A_2 = B$, const.*

Consider the full condition: $|h|^2\mathcal{D}(h^{-1}) = -\mathcal{D}(|h|)\overline{h} + |h|\mathcal{D}\overline{h} = 0$

Proof. From Proposition 3.3, we have

$$\frac{\partial f_2}{\partial z_1} + \frac{\partial f_1}{\partial\overline{z}_2} = 0$$
$$\frac{\partial f_1}{\partial z_1} - \frac{\partial f_2}{\partial\overline{z}_2} = 0$$

Recall: $\frac{\partial f_1}{\partial\overline{z}_1}\mathbf{j} = \mathbf{j}\frac{\partial\overline{f}_1}{\partial z_1}$

$$2\mathcal{D}h = \big(\frac{\partial}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial}{\partial\overline{z}_2}\big)(h_1 - h_2\mathbf{j}) = \frac{\partial h_1}{\partial\overline{z}_1} + \frac{\partial h_2}{\partial z_2} - \big(\frac{\partial h_2}{\partial\overline{z}_1} - \frac{\partial h_1}{\partial z_2}\big)\mathbf{j}$$

$$\mathcal{D}(|h|) = \mathcal{D}(h_1^2+h_2^2) = \frac{1}{2}\big(\frac{\partial}{\partial\overline{z}_1}+\mathbf{j}\frac{\partial}{\partial\overline{z}_2}\big)(h_1^2+h_2^2) = h_1\frac{\partial h_1}{\partial\overline{z}_1}+h_2\frac{\partial h_2}{\partial\overline{z}_1}+\big(h_1\frac{\partial h_1}{\partial z_2}+h_2\frac{\partial h_2}{\partial z_2}\big)\mathbf{j},$$

$$|h|^2\mathcal{D}(h^{-1}) = -\mathcal{D}(|h|)\overline{h}+|h|\mathcal{D}(\overline{h}) = -\Big(h_1\frac{\partial h_1}{\partial\overline{z}_1}+h_2\frac{\partial h_2}{\partial\overline{z}_1}+\big(h_1\frac{\partial h_1}{\partial z_2}+h_2\frac{\partial h_2}{\partial z_2}\big)\mathbf{j}\Big)(h_1-h_2\mathbf{j})$$
$$+(h_1^2+h_2^2)\frac{1}{2}\big(\frac{\partial h_1}{\partial\overline{z}_1} + \frac{\partial h_2}{\partial z_2} - \big(\frac{\partial h_2}{\partial\overline{z}_1} - \frac{\partial h_1}{\partial z_2}\big)\mathbf{j}\big) = 0$$

$$\cong \big(- h_1^2\frac{\partial h_1}{\partial \overline{z}_1} + 3h_2^2\frac{\partial h_2}{\partial z_2}\big) + \big(- h_1^2\frac{\partial h_2}{\partial z_2} + h_2^2\frac{\partial h_1}{\partial \overline{z}_1}\big)\mathbf{j} = 0 \tag{2}$$

From (2)

$$-h_1^2\frac{\partial h_1}{\partial \overline{z}_1} + 3h_2^2\frac{\partial h_2}{\partial z_2} = 0; h_2^2\frac{\partial h_1}{\partial \overline{z}_1} - h_1^2\frac{\partial h_2}{\partial z_2} = 0$$

Then: either $h_1^4 - 3h_2^4 = 0$, or:

$$\frac{\partial h_1}{\partial \overline{z}_1} = 0; \frac{\partial h_2}{\partial z_2} = 0$$

and, previously

$$\frac{\partial h_1}{\partial z_2} = -\frac{\partial h_2}{\partial \overline{z}_1}$$

Then: $h_1 = A_1(z_2, \overline{z}_2); h_2 = A_2(z_1, \overline{z}_1)$ and $A_1 + A_2 = B$, const. □

4.2. **Product of two w-hypermeromorphic functions.**

Proposition 4.3. *Let f, g be two w-hypermeromorphic functions on U, then the following conditions are equivalent:*

(*i*) *the product $f * g$ is w-hypermeremorphic;*

(*ii*) *f and g satisfy the system of PDE:*

$$g_1(\frac{\partial f_1}{\partial \overline{z}_1} + \frac{\partial \overline{f}_2}{\partial z_2}) + (f_1 - \overline{f}_1)\frac{\partial g_1}{\partial \overline{z}_1} + \overline{f}_2\frac{\partial g_1}{\partial z_2} - f_2\frac{\partial \overline{g}_2}{\partial \overline{z}_1} = 0$$

$$g_1(\frac{\partial f_1}{\partial \overline{z}_2} - \frac{\partial \overline{f}_2}{\partial z_1}) + (f_1 - \overline{f}_1)\frac{\partial g_1}{\partial \overline{z}_2} - \overline{f}_2\frac{\partial g_1}{\partial z_1} - f_2\frac{\partial \overline{g}_2}{\partial \overline{z}_2} = 0$$

Proof. Let $f = f_1 + f_2\mathbf{j}$ and $g = g_1 + g_2\mathbf{j}$ two hypermeromorphic functions and $f * g = f_1g_1 - f_2\overline{g}_2 + (f_1g_2 - f_2\overline{g}_1)\mathbf{j}$ their product, then

$$\frac{\partial f_1}{\partial \overline{z}_1} - \frac{\partial \overline{f}_2}{\partial z_2} = 0;$$

$$\frac{\partial (f_1g_1 - f_2\overline{g}_2)}{\partial \overline{z}_1} - \frac{\partial (\overline{f}_1\overline{g}_2 - \overline{f}_2g_1)}{\partial z_2}$$

$$= g_1(\frac{\partial f_1}{\partial \overline{z}_1} + \frac{\partial \overline{f}_2}{\partial z_2}) - \overline{g}_2(\frac{\partial \overline{f}_1}{\partial z_2} + \frac{\partial f_2}{\partial \overline{z}_1}) + f_1\frac{\partial g_1}{\partial \overline{z}_1} - \overline{f}_1\frac{\partial \overline{g}_2}{\partial z_2} + \overline{f}_2\frac{\partial g_1}{\partial z_2} - f_2\frac{\partial \overline{g}_2}{\partial \overline{z}_1} = 0$$

$$g_1(\frac{\partial f_1}{\partial \overline{z}_1} + \frac{\partial \overline{f}_2}{\partial z_2}) + f_1\frac{\partial g_1}{\partial \overline{z}_1} - \overline{f}_1\frac{\partial \overline{g}_2}{\partial z_2} + \overline{f}_2\frac{\partial g_1}{\partial z_2} - f_2\frac{\partial \overline{g}_2}{\partial \overline{z}_1} = 0.$$

$$\frac{\partial (f_1g_1 - f_2\overline{g}_2)}{\partial \overline{z}_2} + \frac{\partial (\overline{f}_1\overline{g}_2 - \overline{f}_2g_1)}{\partial z_1}$$

$$= g_1(\frac{\partial f_1}{\partial \overline{z}_2} - \frac{\partial \overline{f}_2}{\partial z_1}) + \overline{g}_2(\frac{\partial \overline{f}_1}{\partial z_1} - \frac{\partial f_2}{\partial \overline{z}_2}) + f_1\frac{\partial g_1}{\partial \overline{z}_2} - \overline{f}_1\frac{\partial \overline{g}_2}{\partial z_1} + \overline{f}_2\frac{\partial g_1}{\partial z_1} - f_2\frac{\partial \overline{g}_2}{\partial \overline{z}_2} = 0$$

$$g_1(\frac{\partial f_1}{\partial \overline{z}_2} - \frac{\partial \overline{f}_2}{\partial z_1}) + f_1\frac{\partial g_1}{\partial \overline{z}_2} + \overline{f}_1\frac{\partial \overline{g}_2}{\partial z_1} - \overline{f}_2\frac{\partial g_1}{\partial z_1} - f_2\frac{\partial \overline{g}_2}{\partial \overline{z}_2} = 0$$

□

Remark. If f and g are holomorphic, or meromorphic, the condition (ii) is trivially satisfied.

Proposition 4.4. *Let f, g be two w-hypermeromorphic functions on U, whose component are real, then the following conditions are equivalent:*

(i) *the product $f * g$ is w-hypermeremorphic;*
(ii) *f and g satisfy the PDE:*

$$\frac{\partial f_1}{\partial z_1}\frac{\partial g_2}{\partial z_2} + \frac{\partial f_1}{\partial z_2}\frac{\partial g_2}{\partial z_1} = 0$$

Proof. Assume that f_1, f_2, g_1, g_2 are real, then:

$$\text{(3)} \qquad \begin{cases} g_1(\dfrac{\partial f_1}{\partial \overline{z}_1} + \dfrac{\partial f_2}{\partial z_2}) + f_2(\dfrac{\partial g_1}{\partial z_2} - \dfrac{\partial g_2}{\partial \overline{z}_1}) = 0 \\ \\ g_1(\dfrac{\partial f_1}{\partial \overline{z}_2} - \dfrac{\partial f_2}{\partial z_1}) - f_2(\dfrac{\partial g_1}{\partial z_1} + \dfrac{\partial g_2}{\partial \overline{z}_2}) = 0 \end{cases}$$

f being w-hypermeromorphic satisfies:

$$\text{(4)} \qquad \begin{cases} \dfrac{\partial f_2}{\partial z_1} + \dfrac{\partial f_1}{\partial \overline{z}_2} = 0; \\ \\ \dfrac{\partial f_1}{\partial z_1} - \dfrac{\partial f_2}{\partial \overline{z}_2} = 0 \end{cases}$$

the same for g. Using (4), we get, from (3)

$$\begin{cases} g_1\dfrac{\partial f_1}{\partial z_1} + f_2\dfrac{\partial g_2}{\partial z_1} = 0 \\ \\ g_1\dfrac{\partial f_1}{\partial z_2} - f_2\dfrac{\partial g_2}{\partial z_2} = 0 \end{cases}$$

To get f_2, g_1 not both 0, , we need the relation:

$$\frac{\partial f_1}{\partial z_1}\frac{\partial g_2}{\partial z_2} + \frac{\partial f_1}{\partial z_2}\frac{\partial g_2}{\partial z_1} = 0$$

□

4.3. **Definition.** We will call *hypermeromorphic* the w-hypermeromorphic functions whose sum and product are w-hypermeromorphic. Their space is nonempty, since the product 1 of a w-hypermeromorphic and its inverse is w-hypermeromorphic; moreover this space contains the right $\mathbb{H}$-algebra of the meromorphic functions.

4.4. **Hypermeromorphic functions whose two components are real.** Let f, g be two hypermeromorphic functions on U.

From Proposition 3.3, they satisfy:

$$\frac{\partial f_2}{\partial z_1} + \frac{\partial f_1}{\partial \overline{z}_2} = 0; \frac{\partial f_1}{\partial z_1} - \frac{\partial f_2}{\partial \overline{z}_2} = 0$$

$$\frac{\partial g_2}{\partial z_1} + \frac{\partial g_1}{\partial \overline{z}_2} = 0; \frac{\partial g_1}{\partial z_1} - \frac{\partial g_2}{\partial \overline{z}_2} = 0$$

The properties: the sum $h = f + g$ is hypermeromorphic, is equivalent to:

$h_1 = A_1(z_2, \overline{z}_2); h_2 = A_2(z_1, \overline{z}_1)$ and $A_1 + A_2 = B$, const.;

the product $f * g$ is hypermeremorphic, is equivalent to:

$$\frac{\partial f_1}{\partial z_1}\frac{\partial g_2}{\partial z_2} + \frac{\partial f_1}{\partial z_2}\frac{\partial g_2}{\partial z_1} = 0$$

Proposition 4.5. *Let f and g be quaternionic functions such that their components be real. The following conditions are equivalent:*

(i) f, g, their sum and their product are w-hypermeromorphic;

(ii) f, g satisfy the following relations:

$$\frac{\partial f_2}{\partial z_1} + \frac{\partial f_1}{\partial z_2} = 0; \frac{\partial f_1}{\partial z_1} - \frac{\partial f_2}{\partial z_2} = 0$$

$$\frac{\partial g_2}{\partial z_1} + \frac{\partial g_1}{\partial z_2} = 0; \frac{\partial g_1}{\partial z_1} - \frac{\partial g_2}{\partial z_2} = 0$$

$$\frac{\partial f_1}{\partial z_1}\frac{\partial g_1}{\partial z_1} - \frac{\partial f_1}{\partial z_2}\frac{\partial g_1}{\partial z_2} = 0$$

$f_1 + g_1 = A_1(z_2, \overline{z}_2); f_2 + g_2 = A_2(z_1, \overline{z}_1)$ where A_1 and A_2 are given and satisfy $A_1 + A_2 = B$, const.

Altogether we have 10 variables and 8 relations.

Proposition 4.6. *The set of hypermeromorphic functions whose two components are real is an $\mathbb{R}$-right algebra.*

Proposition 4.7. *The set $\mathcal{M}$ of hypermeromorphic functions on U is a subalgebra of the algebra of quaternionic functions.*

Proposition 4.8. *The set $\mathcal{A}$ of hypermeromorphic functions on U is a "field" with only associativity of the multiplication.*

4.5. **Meromorphic functions and Hypermeromorphic functions.**

4.5.1. *Motivation.* Meromorphic functions satisfy all PDE satisfied by hypermeromorphic functions.

Meromorphic functions have, classically, behavior on their domain of definition U compatible with the behavior of hypermeromorphic functions on U. In the same way the sum and the product of a hypermeromorphic function and a meromorphic function is a hypermeromorphic function

4.5.2. *Conclusion.* The $\mathbb{H}$-algebra and the field of meromorphic functions are substructures of the set of hypermeromorphic functions.

In particular the union of the set of meromorphic functions and the set of hypermeromorphic functions whose components are real give a large number of examples of hypermeromorphic functions.

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