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On First-Order Definable Colorings

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Abstract

We address the problem of characterizing H -coloring problems that are first-order definable on a fixed class of relational structures. In this context, we give also several characterizations of a homomorphism dualities arising in a class of structure.

Keywords: graph, coloring, first-order logic, bounded expansion class
2000 MSC: 05C15 coloring of graphs and hypergraphs

1. Introduction

Recall that *classical model theory* studies properties of abstract mathematical structures (finite or not) expressible in first-order logic [20], and *finite model theory* is the study of first-order logic (and its various extensions) on finite structures [11], [27].

Constraint Satisfaction Problems (CSPs), and more specifically H -coloring problems, are standard examples of problems which can be expressed in monadic second order logic but usually not in the first-order logic. Of course, expressing a H -coloring problem in first-order logic would be highly appreciable, as it would allow fast checking (in at most polynomial time) although problems expressed in monadic second order logic are usually NP-complete. In this direction, it has been proved by Hell and Nešetřil [18] that in the context of finite undirected graphs the H -coloring problem is NP-complete unless H is

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bipartite, in which case the H -coloring problem is clearly polynomially solvable. This, and a similar dichotomy result of Schaefer [41], led Feder and Vardi [14, 15] to formulate the celebrated Dichotomy Conjecture which asserts that, for every constraint language over an arbitrary finite domain, the corresponding constraint satisfaction problems are either solvable in polynomial time, or are NP-complete. It was soon noticed that this conjecture is equivalent to the existence of a dichotomy for (general) H -coloring problems, and in fact it suffices to prove it for oriented graphs (see [15] and [19]).

Alternatively, the class P of all polynomially solvable problems can be described as the class of problems expressible (on ordered structures) in first-order logic with a least fixed point operator [44, 21]. On the other hand the class NP may be characterized (up to polynomial equivalence) as the class of all problems which have a lift (or expansion) determined by forbidden homomorphisms from a finite set [25]. Hence, we are led naturally to the question of descriptive complexity of classes of structures corresponding to H -coloring problems. A particular case is the question whether a H -coloring problem may be expressed in first-order logic or not.

In this paper, we will consider the relativized version of the problem of first-order definability of H -coloring problems to graphs (or structures) belonging to a fixed class \mathcal{C} :

Problem 1. *Given a fixed class \mathcal{C} of graphs (directed graphs, relational structures), determine which H -coloring problems are first-order definable in \mathcal{C} . Explicitly, determine for which graphs (directed graphs, relational structures) H there exists a first-order sentence ϕ_H such that*

$$\forall G \in \mathcal{C} : \quad (G \models \phi_H) \iff (G \rightarrow H).$$

The case where \mathcal{C} is the whole class of all finite graphs (all finite directed graphs, all finite relational structures with given finite signature) is well understood. Atserias [1, 2] and Rossman [39] proved that in this case first-order definable H -colorings correspond exactly to *finite homomorphism dualities*, and these dualities have been fully characterized (for undirected graphs, by Nešetřil and Pultr [36]; for directed graphs, by Komárek [22]; for general finite structures, by Nešetřil and Tarif [37]) as follows:

Theorem 1 ([37]). *For any signature σ and any finite set \mathcal{F} of σ -structures the following two statements are equivalent:*

1. *There exists D such that \mathcal{F} and D form a finite duality, that is:*

$$\forall \text{ finite } G : \quad (\forall F \in \mathcal{F}, F \rightarrow G) \iff (G \rightarrow D)$$

2. *\mathcal{F} is homomorphically equivalent to a set of finite (relational) trees.*

Note that an example of such a duality for the class of all finite directed graphs is the Gallai-Hasse-Roy-Vitaver theorem [16, 17, 40, 45], which states that for every directed graph \vec{G} it holds:

$$\vec{P}_{k+1} \rightarrow \vec{G} \iff \vec{G} \rightarrow \vec{T}_k.$$

For general classes of graphs the answer is more complicated. For instance, let \mathcal{C} be the class of (undirected) toroidal graphs and let ϕ be the sentence

$$\forall x_0 \forall x_1 \dots \forall x_{10} \bigvee_{i=0}^{10} \neg(x_i \sim x_{i+1}) \vee \neg(x_i \sim x_{i+2}) \vee \neg(x_i \sim x_{i+3}),$$

where additions are considered modulo 11 and where $u \sim v$ denotes that u and v are adjacent. Then, it follows from [43] that a graph $G \in \mathcal{C}$ satisfies ϕ if and only if it is 5-colourable. This property can be alternatively be expressed by the following *restricted duality*:



In fact the class of toroidal graphs has all restricted dualities in the following sense: for every connected F there exists H_F such that $F \not\rightarrow H_F$ and for every toroidal graph G holds

$$F \not\rightarrow G \quad \iff \quad G \rightarrow H_F.$$

For a general class of graphs \mathcal{C} , Problem 1 is very complex. We have to specialize. Hence we first require that the studied class \mathcal{C} has some basic properties: we assume that

- \mathcal{C} is *hereditary* (meaning that every induced subgraph of a graph in \mathcal{C} is in \mathcal{C});
- \mathcal{C} is *addable* (meaning that disjoint unions of graphs in \mathcal{C} are in \mathcal{C});
- \mathcal{C} is *topologically closed* (meaning that every subdivision of a graph in \mathcal{C} is in \mathcal{C}).

We approach the problem of characterizing first-order definable colorings by first discriminating between the cases of sparse and dense classes of graphs using our *class taxonomy* [32, 33, 35], which seems relevant here. The second central ingredient of our study is the notion of *homomorphism preservation theorem* (HPT) for a class \mathcal{C} , which was investigated in [5, 7, 8, 35, 39]. Our approach can be outlined as follows:

If there exists some integer p such that every p -subdivision of a complete graph appears as a subgraph of some graph in \mathcal{C} (meaning that \mathcal{C} is *somewhere dense*) then we prove that every first-order definable H -coloring defines a restricted duality on the subclass $\mathcal{C}' \subseteq \mathcal{C}$ of the p -subdivisions of simple graphs (follows from HPT for \mathcal{C}'). Using a classical construction of Erdős [12], we deduce that if a H -coloring problem is first-order definable on \mathcal{C} then either H is bipartite, or the odd-girth of H is at most $2p + 1$.

Otherwise (meaning that \mathcal{C} is *nowhere dense*) it follows from HPT for nowhere dense classes that every first-order definable H -coloring defines a restricted duality on \mathcal{C} . In the case where there exists some integer p such that p -subdivisions of graphs with arbitrarily large average degree appear as subgraphs of graphs in \mathcal{C} (meaning that \mathcal{C} is not a *bounded expansion* class) we prove that H cannot be a restricted dual of a non-bipartite graph with arbitrarily large odd-girth. Modulo some reasonable conjecture (Conjecture 2), we get that (as in the somewhere dense case) H is either bipartite or has bounded odd-girth.

In the reminding case (where \mathcal{C} is a bounded expansion class), we prove that first-order definable coloring correspond to restricted dualities, hence are exactly defined by

a sentence expressing that no homomorphism exists from one of the connected graphs belonging to a finite set.

This study naturally leads to the following conjecture:

Conjecture 1. Let \mathcal{C} be a hereditary addable topologically closed class of graphs. The following properties are equivalent:

1. for every integer p there is a non-bipartite graph H_p of odd-girth strictly greater than $2p + 1$ and a first order definable class \mathcal{D}_p such that a graph $G \in \mathcal{C}$ is H_p -colorable if and only if $G \in \mathcal{D}_p$. Explicitly, there exists a formula Φ_p such that for every graph $G \in \mathcal{C}$ holds

$$(G \models \Phi_p) \iff (G \rightarrow H_p);$$

2. the class \mathcal{C} has bounded expansion.

In this paper, we make a significant progress toward a solution to Conjecture 1. We state a structural conjecture, Conjecture 2, about nowhere dense classes that fail to be bounded expansion classes, which expresses that such classes are characterized by the existence (as subgraphs of graphs in the class) of p -subdivisions of graphs with arbitrarily large chromatic number and girth. More precisely:

Conjecture 2. Let \mathcal{C} be a monotone nowhere dense class that does not have bounded expansion. Then there exists an integer p such that \mathcal{C} includes p -subdivisions of graphs with arbitrarily large chromatic number and girth.

Our main result toward a characterization of first-order definable colorings is the following reduction.

Theorem 2. *Let \mathcal{C} be a hereditary topologically closed class of graphs that is somewhere dense or has bounded expansion. Then the following properties are equivalent:*

1. for every integer p there is a non-bipartite graph H_p of odd-girth strictly greater than $2p + 1$ such that H_p -coloring is first-order definable on \mathcal{C} ;
2. the class \mathcal{C} has bounded expansion.

Moreover, if Conjecture 2 holds then the statement holds for every hereditary addable topologically closed class of graphs, that is: if Conjecture 2 holds then Conjecture 1 also holds.

In support to Conjecture 2, we prove (Proposition 7) that it would follow from a positive solution to any of the following two well known conjectures.

Conjecture 3 (Erdős and Hajnal [13]). For every integers g and n there exists an integer $N = f(g, n)$ such that every graph G with chromatic number at least N has a subgraph H with girth at least g and chromatic number at least n .

The case $g = 4$ of the conjecture was proved by Rödl [38], while the general case is still open. Remark that the existence of graphs of both arbitrarily high chromatic number and high girth is a well known result of Erdős [12].

Conjecture 4 (Thomassen [42]). For all integers c, g there exists an integer $f(c, g)$ such that every graph G of average degree at least $f(c, g)$ contains a subgraph of average degree at least c and girth at least g .

The case $g = 4$ of this conjecture is a direct consequence of the simple fact that every graph can be made bipartite by deleting at most half of its edges. The case $g = 6$ has been proved in [24].

Although we do not settle Conjecture 1, our study led us to the two following characterization theorems of classes that have all restricted dualities.

Theorem 3. *Let \mathcal{C} be a topologically closed class of graphs.*

The following properties are equivalent:

1. *the class \mathcal{C} has bounded expansion;*
2. *the class \mathcal{C} has all restricted dualities;*
3. *for every odd integer g there exists a graph H_g with odd-girth greater than g such that every graph $G \in \mathcal{C}$ with odd-girth greater than g has a homomorphism to H_g .*

These results motivate more general studies of classes of relational structures having all restricted dualities, which includes classes of structures whose Gaifman graphs form a class with bounded expansion. In a very general setting, we obtain the following characterization:

Theorem 4. *Let \mathcal{C} be a class of σ -structures. Then \mathcal{C} is bounded and has all restricted dualities if and only if for every integer t there is an integer $N(t)$ such that for every $\mathbf{A} \in \mathcal{C}$ there exists a σ -structure \mathbf{A}_t (called t -approximation of \mathbf{A}) such that*

- \mathbf{A}_t has order $|A_t|$ at most $N(t)$,
- $\mathbf{A} \rightarrow \mathbf{A}_t$,
- every substructure \mathbf{F} of \mathbf{A}_t with order $|F| < t$ has a homomorphism to \mathbf{A} .

This paper is organized as follows:

In Section 2, we recall the notions needed in the development of our study, in particular class taxonomy and basics on relational structures.

In Section 3, we discuss classes satisfying a homomorphism preservation theorem (HPT). On these classes, first-order definable H -colorings correspond to finite restricted dualities. Moreover, when the considered class is addable, they correspond to finite restricted dualities with connected templates.

In Section 5, we discuss classes having all restricted dualities, that is classes such that to every finite set \mathcal{F} of connected templates correspond a graph H such that a graph G in the class is H -colorable if and only if none of the templates in \mathcal{F} has a homomorphism to G .

2. Taxonomy of Classes of Graphs

In the following, we denote by \mathcal{G} the class of all finite graphs. A class of graphs \mathcal{C} is *monotone* (resp. *hereditary*, *topologically closed*) if every subgraph (resp. every induced subgraph, every subdivision) of a graph in \mathcal{C} also belongs to \mathcal{C} . Notice that if a class \mathcal{C} is both hereditary and topologically closed it is also monotone: If H is a subgraph of a graph $G \in \mathcal{C}$, the graph H is an induced subgraph of the graph $G' \in \mathcal{C}$ obtained from G by subdividing every edge not in H , hence $H \in \mathcal{C}$. For a graph G , we denote by $\omega(G)$

its clique number, by $\chi(G)$ its chromatic number, and by $\bar{d}(G)$ the average degree of its vertices. By extension, for a class of graphs \mathcal{C} we define

$$\begin{aligned}\omega(\mathcal{C}) &= \sup\{\omega(G), G \in \mathcal{C}\} \\ \chi(\mathcal{C}) &= \sup\{\chi(G), G \in \mathcal{C}\} \\ \bar{d}(\mathcal{C}) &= \sup\{\bar{d}(G), G \in \mathcal{C}\}\end{aligned}$$

We proposed in [31, 34, 29] a general classification scheme for graph classes which is based on the density of shallow (topological) minors (we refer the interested reader to the monography [35]). This classification can be defined in several very different ways and we give here one of the simplest definitions, which relates to subdivision:

A *subdivision* (resp. a *k-subdivision*) of an edge $e = \{u, v\}$ of a graph G consists in replacing the edge e by a path (resp. by a path of length $k + 1$) with endpoints u and v . A *subdivision* of a graph G is a graph H resulting from G by subdividing edges; the graph H is the *k-subdivision* of G if it has been obtained from G by k -subdividing all the edges (i.e. if all the edges of G have been replaced by paths of length $k + 1$). The graph H is a $\leq k$ -*subdivision* of H if it has been obtained by replacing each edge of G by a path of length at most $k + 1$.

Let p be a half-integer. A graph H is a *shallow topological minor* of a graph G at depth p if some $\leq 2p$ -subdivision of H is a subgraph of G ; the set of all shallow topological minors of G at depth p is denoted by $G \tilde{\nabla} p$ and, more generally, $\mathcal{C} \tilde{\nabla} p$ denotes the class of all shallow topological minors at depth p of graphs in \mathcal{C} .

A class of undirected graphs \mathcal{C} is *somewhere dense* if there exists an integer p such that the p -th subdivision of every finite graph H may be found as a subgraph of some graph in \mathcal{C} ; it is *nowhere dense* otherwise.

In other words, the class \mathcal{C} is nowhere dense if

$$\forall p \in \mathbb{N}, \quad \omega(\mathcal{C} \tilde{\nabla} p) < \infty.$$

A particular type of nowhere dense classes will be of particular importance in this paper: A class \mathcal{C} has *bounded expansion* [29] if

$$\forall p \in \mathbb{N}, \quad \bar{d}(\mathcal{C} \nabla p) < \infty.$$

Among the numerous equivalent characterizations that can be given for the property of having bounded expansion, we will make use of a characterization based on the chromatic numbers of the shallow topological minors of the graphs in the class. This characterization can be deduced from the following result of Dvořák [9, 10] (see also [35]):

Lemma 5. *Let $c \geq 4$ be an integer and let G be a graph with minimum degree $d > 56(c-1)^2 \frac{\log(c-1)}{\log c - \log(c-1)}$. Then the graph G contains a subgraph G' that is the 1-subdivision of a graph with chromatic number c . \square*

Hence the following characterization of classes having bounded expansion:

Theorem 6. *A class \mathcal{C} has bounded expansion if and only if it holds*

$$\forall p \in \mathbb{N}, \quad \chi(\mathcal{C} \tilde{\nabla} p) < \infty. \tag{1}$$

Proof. According to Lemma 5, for every graph G and every integer p there exists an integer C such that:

$$\bar{d}(G \tilde{\vee} p) \leq C \chi(G \tilde{\vee} (2p + 1/2))^4.$$

Moreover, as every graph G is $(\lfloor \bar{d}(G \tilde{\vee} 0) \rfloor + 1)$ -colorable, every graph in $G \tilde{\vee} p$ is $(\lfloor \bar{d}(G \tilde{\vee} p) \rfloor + 1)$ -colorable, that is:

$$\bar{d}(G \tilde{\vee} p) \geq \chi(G \tilde{\vee} p) - 1.$$

The result follows from these two inequalities. \square

Thus we see that parameters \bar{d} and χ can be used to define bounded expansion classes, although nowhere dense classes are defined by means of the parameter ω .

Characterizing nowhere dense classes that do not have bounded expansion in a structural way is challenging, and thus we proposed Conjecture 2, from which Conjecture 1 would follow.

We now prove that Conjecture 2 follows from any of the Conjecture 3 (by Erdős and Hajnal) or Conjecture 4 (by Thomassen):

Proposition 7. *If either Conjecture 3 or Conjecture 4 holds, then so does Conjecture 2.*

Proof. Let \mathcal{C} be a nowhere dense class of graphs which is not a bounded expansion class. By definition of a bounded expansion class, there exists an integer q such that \mathcal{C} includes $\leq q$ -subdivisions of graphs with arbitrarily large average degree hence (by standard pigeon-hole argument) there is an integer p such that \mathcal{C} includes exact q -subdivisions of graphs with arbitrarily large average degree.

Assume Conjecture 3 holds. Define $p = 2q + 1$, and let $g, n \in \mathbb{N}$. According to the statement of the conjecture, there exists N such that every graph with chromatic number at least N has a subgraph with girth at least g and chromatic number at least n . We can assume $N \geq 4$. Let $d \geq 56(N - 1)^2 \frac{\log(N-1)}{\log N - \log(N-1)}$. Let $G \in \mathcal{C}$ be such that G includes the q -subdivision of a graph with average degree at least $2d$ hence the q -subdivision of a graph H with minimum degree at least d . According to Lemma 5, H has a subgraph H' that is the 1-subdivision of a graph with chromatic number N . It follows that G has a subgraph which is a p -subdivision of a graph K with chromatic number N . According to Conjecture 3, the graph K has a subgraph K' which has chromatic number at least n and girth at least g . It follows that G contains the p -subdivision of a graph with chromatic number at least n and girth at least g . Thus Conjecture 2 holds.

Assume that Conjecture 4 holds. Define $p = 2q + 1$, and let $g, n \in \mathbb{N}$. Let $d \geq 56(n - 1)^2 \frac{\log(n-1)}{\log n - \log(n-1)}$. According to the statement of the conjecture, there exists N such that every graph with average degree at least N has a subgraph with girth at least $2g + 1$ and average degree at least $2d$ hence a subgraph with girth at least $2g + 1$ and minimum degree at least d . Let $G \in \mathcal{C}$ be such that G includes the q -subdivision of a graph with average degree at least N . Then G has a subgraph which is the q -subdivision of a graph H with minimum degree at least d and girth at least $2g + 1$. According to Lemma 5, H has a subgraph which is the 1-subdivision of a graph with chromatic number at least n . This subgraph is a p -subdivision of a graph with girth at least g and chromatic number at least n . Thus Conjecture 2 holds. \square

3. Homomorphism Preservation Theorems

Suppose that an H -coloring problem is first-order definable. By this we mean that there is a first-order sentence Φ such that

$$G \rightarrow H \quad \iff \quad G \models \Phi.$$

It immediately follows that $\neg\Phi$ is preserved by homomorphisms:

$$G \models \neg\Phi \quad \text{and} \quad G \rightarrow G' \quad \implies \quad G' \models \neg\Phi$$

(for otherwise $G \rightarrow G' \rightarrow H$ hence $G \models \Phi$, a contradiction).

Such a property suggests that such a formula Φ could be equivalent to a formula with a specific syntactic form. Indeed the classical *Homomorphism Preservation Theorem* (HPT) asserts that a first-order formula is preserved under homomorphisms on all structures if, and only if, it is logically equivalent to an existential-positive formula. The terms “all structures”, which means finite and infinite structures, is crucial in the statement of these theorems.

3.1. Finite Structures

It was not known until recently whether HPT would hold when relativized to the finite. In fact other well known theorems relating preservation under some specified algebraic operation and certain syntactic forms, like Łoś-Tarski theorem or Lyndon’s theorem, fail in the finite.

However, the finite relativization of the homomorphism preservation has been proved to hold by B. Rossman [39] for general relational structures.

Theorem 8 ([39]). *Let ϕ be a first order formula. Then,*

$$\mathbf{G} \rightarrow \mathbf{H} \text{ and } \mathbf{G} \models \phi \quad \implies \quad \mathbf{H} \models \phi$$

holds for all finite relational structures \mathbf{G} and \mathbf{H} if and only if for finite relational structures ϕ is equivalent to an existential first-order formula.

It follows that for finite relational structures, the only \mathbf{H} -coloring problems which are expressible in first-order logic are those for which there exists a finite family \mathcal{F} of finite structures with the property that for every structure \mathbf{G} the following finite homomorphism duality holds:

$$\exists \mathbf{F} \in \mathcal{F} \quad \mathbf{F} \rightarrow \mathbf{G} \quad \iff \quad \mathbf{G} \dashv \mathbf{H}. \tag{2}$$

In this paper, we will be mostly interested by graphs, although relational structures will be considered in Section 5. Definitions and constructions concerning relational structures are particularly discussed in Section 5.1.

3.2. Nowhere dense classes

If we want to relativize Theorem 8, we should consider each relativization as a new problem. The Łoś-Tarski theorem, for instance, holds in general, yet fails when relativized to the finite, but holds when relativized to hereditary classes of structures with bounded degree which are closed under disjoint union [3]. These examples stress again that some properties of structures (in general) and graphs (in particular) need, at times, to be studied in the context of a fixed class, in order to state a relativized version of a general statement which could fail in general.

In this context Atserias, Dawar and Kolaitis defined classes of graphs called *wide*, *almost wide* and *quasi-wide* (cf. [6] for instance). It has been proved in [3] that the extension preservation theorem holds in any class \mathcal{C} that is wide, hereditary (i.e. closed under taking substructures) and closed under disjoint unions, for instance hereditary classes with bounded degree that are closed under disjoint unions. Also, it has been proved in [4] [5] that the homomorphism preservation theorem holds in any class \mathcal{C} that is almost wide, hereditary and closed under disjoint unions. Almost wide classes of graphs include classes of graphs which exclude a minor [23]. In [8] Dawar proved that the homomorphism preservation theorem holds in any hereditary quasi-wide class that is closed under disjoint unions.

Theorem 9 ([8]). *Let \mathcal{C} be a hereditary addable quasi-wide class of graphs. Then the homomorphism preservation theorem holds for \mathcal{C} .*

Moreover, we have proved that hereditary quasi-wide classes of graphs are exactly hereditary nowhere dense classes [31]:

Theorem 10. *A hereditary class of graphs \mathcal{C} is quasi-wide if and only if it is nowhere dense.*

Thus it follows from Theorems 9 and 10 that the relativization of the homomorphism preservation theorem holds for every hereditary addable nowhere dense class of graphs. But nowhere dense classes are not the only classes with relativized homomorphism preservation theorem. In the next section we show HPT also holds for some nowhere dense classes.

3.3. Somewhere dense classes

We now show that relativized homomorphism preservation theorems are preserved by particular interpretations, from which will deduce that relativized homomorphism preservation theorems hold for the classes $\text{Sub}_q(\text{Graph})$ of all q -subdivisions of (simple) finite graphs. This is of particular interest as somewhere dense classes (i.e. classes which fail to be nowhere dense) are characterized by containment of classes $\text{Sub}_q(\text{Graph})$ for some q .

In the framework of the model theoretical notion of *interpretation* (see, for instance [26, pp. 178-180]), we can construct the q -subdivision $\mathfrak{I}(G)$ of a graph G by means of first-order formulas on the q -tuples of vertices of G :

- vertices of $\mathfrak{I}(G)$ are the equivalence classes x of the $(q + 1)$ -tuples (v_1, \dots, v_{q+1}) with form

$$\underbrace{(u, \dots, u)}_j, \underbrace{(v, \dots, v)}_{q+1-j}$$

where u and v are adjacent vertices in G (and $0 \leq j \leq q+1$), where tuples of the form

$$\overbrace{(u, \dots, u)}^j \overbrace{(v, \dots, v)}^{q+1-j} \text{ and } \overbrace{(v, \dots, v)}^{q+1-j} \overbrace{(u, \dots, u)}^j$$

are identified;

- edges of $\mathbb{I}(G)$ are those pairs $\{x, y\}$ where x and y have representative of the form

$$\overbrace{(u, \dots, u)}^j \overbrace{(v, \dots, v)}^{q+1-j} \text{ and } \overbrace{(u, \dots, u)}^{j+1} \overbrace{(v, \dots, v)}^{q-j}$$

(for some $u, v \in G$ and $0 \leq j \leq q$).

The main interest of such a logical construction (called *interpretation*) lies in the following property:

Proposition 11 (See, for instance [26], p. 180). *For every first-order formula $\phi[v_1, \dots, v_k]$ there exists a formula $\mathbb{I}(\phi)[\bar{w}_1, \dots, \bar{w}_k]$ with $k(q+1)$ free variables (each \bar{w}_i represents a succession of $(q+1)$ free variables) such that for every graph G and every $(x_1, \dots, x_k) \in \mathbb{I}(G)^k$ the three following conditions are equivalent:*

1. $\mathbb{I}(G) \models \phi[x_1, \dots, x_k]$;
2. *there exist $\bar{b}_1 \in x_1, \dots, \bar{b}_k \in x_k$ such that $G \models \mathbb{I}(\phi)[\bar{b}_1, \dots, \bar{b}_k]$;*
3. *for all $\bar{b}_1 \in x_1, \dots, \bar{b}_k \in x_k$ it holds $G \models \mathbb{I}(\phi)[\bar{b}_1, \dots, \bar{b}_k]$.*

In particular, it holds:

Corollary 12. *For every sentence (i.e. closed first order formula) ϕ (in the language of graphs) there exists a sentence ψ such that for every graph G we have*

$$G \models \psi \iff \text{Sub}_{2p}(G) \models \phi. \quad (3)$$

Lemma 13. *If the homomorphism preservation theorem holds for a hereditary class of graphs \mathcal{C} , it also holds for the class $\text{Sub}_q(\mathcal{C})$ of all q -subdivisions of the graphs in \mathcal{C} .*

Proof. If q is odd then the property is obvious as \mathcal{C} contains at most two homomorphism equivalence classes, the one of K_1 and the one of K_2 . Hence we can assume q is even and we define $p = q/2$.

Let ϕ be a sentence preserved by homomorphisms on $\text{Sub}_{2p}(\mathcal{C})$, where \mathcal{C} is a hereditary class of graphs on which the homomorphism preservation theorem holds. Then we shall prove that there exists a finite family of $2p$ -subdivided graphs \mathcal{F} , all of which satisfy ϕ , and such that for any graph G it holds

$$\text{Sub}_{2p}(G) \models \phi \iff \exists F \in \mathcal{F} \quad \text{Sub}_{2p}(F) \rightarrow \text{Sub}_{2p}(G). \quad (4)$$

According to Corollary 12 there exists a sentence ψ such that for every graph G it holds

$$G \models \psi \iff \text{Sub}_{2p}(G) \models \phi.$$

Assume that $G \models \psi$ and $G \rightarrow H$, with $G, H \in \mathcal{C}$. Then $\text{Sub}_{2p}(G) \models \phi$ and $\text{Sub}_{2p}(G) \rightarrow \text{Sub}_{2p}(H)$. As ϕ is preserved by homomorphisms on $\text{Sub}_{2p}(\mathcal{C})$ we get $\text{Sub}_{2p}(H) \models \phi$

hence $H \models \psi$. Thus ψ is preserved by homomorphisms on \mathcal{C} . As the homomorphism preservation theorem holds by assumption on \mathcal{C} , ψ is equivalent on \mathcal{C} with a positive first-order formula, that is: there exists a finite family \mathcal{F}_0 of finite graphs such that for every $G \in \mathcal{C}$ it holds:

$$G \models \psi \iff \exists F \in \mathcal{F}_0 \quad F \rightarrow G.$$

Moreover, by considering the subgraphs induced by the homomorphic images of the graphs $F \in \mathcal{F}_0$ and as \mathcal{C} is hereditary, we can assume $\mathcal{F}_0 \subseteq \mathcal{C}$. Thus every $F \in \mathcal{F}_0$ satisfies ψ hence the $2p$ -subdivision of the graphs in \mathcal{F}_0 satisfy ϕ . Let \mathcal{F} be the set of the $2p$ -subdivisions of the graphs in \mathcal{F}_0 . As ϕ is preserved by homomorphisms on $\text{Sub}_{2p}(\mathcal{C})$ it follows that for every graph $G \in \mathcal{C}$ if there exists $F \in \mathcal{F}$ such that $F \rightarrow \text{Sub}_{2p}(G)$ then $\text{Sub}_{2p}(G)$ satisfies ϕ . Conversely, if $\text{Sub}_{2p}(G)$ satisfies ϕ for some $G \in \mathcal{C}$ then G satisfies ψ , thus there exists $F \in \mathcal{F}_0$ such that $F \rightarrow G$ hence $\text{Sub}_{2p}(F) \rightarrow \text{Sub}_{2p}(G)$. \square

We deduce this extension of Rossman's theorem to the class of p -subdivided graphs:

Corollary 14. *For every integer p , the homomorphism preservation theorem holds for $\text{Sub}_p(\text{Graph})$.*

For a discussion on relativization of the homomorphism preservation theorem, we refer the reader to [35, Chapter 10].

We summarize below the results obtained in this section:

Lemma 15. *Let \mathcal{C} be a hereditary class of graphs. Assume H -coloring is first-order definable on \mathcal{C} .*

- *If \mathcal{C} is topologically closed and somewhere dense, then there exist an integer p (independent of H) and a finite set \mathcal{F} of finite graphs such that for every graph G it holds*

$$\text{Sub}_{2p}(G) \not\models H \iff \exists F \in \mathcal{F} : F \rightarrow \text{Sub}_{2p}(G). \quad (5)$$

- *If \mathcal{C} is addable and nowhere dense, then there exists a finite set \mathcal{F} of finite graphs such that for every graph $G \in \mathcal{C}$ it holds*

$$G \not\models H \iff \exists F \in \mathcal{F} : F \rightarrow G. \quad (6)$$

Proof. If \mathcal{C} is topologically closed and somewhere dense, then there exist an integer p such that the class \mathcal{S}_{2p} of all $2p$ -subdivisions of finite graphs is a subclass of \mathcal{C} . As H -coloring is first-order definable on \mathcal{C} (thus on \mathcal{S}_{2p}) and as the homomorphism preservation theorem holds for \mathcal{S}_{2p} (according to Corollary 14), H -coloring may be expressed on \mathcal{S}_{2p} by an existential first-order formulas, that is there exists a finite set \mathcal{F} of finite graphs such that for every graph G equivalence (5) holds.

If \mathcal{C} is addable and nowhere dense, then the existence of a finite set \mathcal{F} of finite graphs such that for every graph $G \in \mathcal{C}$ equivalence (6) holds immediately follows from Theorem 9. \square

4. Connectivity of Forbidden Graphs

Homomorphism preservation theorems allow to reduce the study of first-order colorings of a class \mathcal{C} to the study of finite restricted dualities of \mathcal{C} , that is of pairs (\mathcal{F}, H) , where \mathcal{F} is a finite set of finite graphs, where H is a graph, and where it holds

$$\forall G \in \mathcal{C} : \quad (\exists F \in \mathcal{F} : F \twoheadrightarrow G) \iff (G \rightarrow H). \quad (7)$$

In general, it is not required that $F \twoheadrightarrow H$ when $F \in \mathcal{F}$, nor is it required that the graphs in \mathcal{F} are connected. We shall see that if the class \mathcal{C} is addable or monotone then we can require the graphs in \mathcal{F} to be connected, and that if \mathcal{C} is monotone we can further require every $F \in \mathcal{F}$ belongs to \mathcal{C} (hence cannot be homomorphic to H).

For a graph G , we define $\text{Pre}(G)$ has the set of all the graphs G' obtained from G by identifying two vertices of G . Note that we have the following property:

Lemma 16. *Let F, G be graphs. Then $F \rightarrow G$ if and only if either F is isomorphic to a subgraph of G , or there is $F' \in \text{Pre}(F)$ such that $F' \rightarrow G$.*

Proof. Assume $f : F \rightarrow G$ is a homomorphism. Either f is injective and F is isomorphic to a subgraph of G , or at least two vertices of F are identified by f and thus there is $F' \in \text{Pre}(F)$ such that $F' \rightarrow G$. Conversely, if F is isomorphic to a subgraph of G or if there is $F' \in \text{Pre}(F)$ such that $F' \rightarrow G$ then obviously $F \rightarrow G$. \square

Let \mathcal{C} be a class of graphs and let (\mathcal{F}, H) be a restricted duality of \mathcal{C} . We say that the set \mathcal{F} is *minimal* if

- every element of \mathcal{F} is a *core* (that is a graph F such that every homomorphism $F \rightarrow F$ is an automorphism);
- for any proper subset \mathcal{F}' of \mathcal{F} , the pair (\mathcal{F}', H) is not a restricted duality of \mathcal{C} ,
- and for any $F \in \mathcal{F}$, the pair $(\mathcal{F} \setminus \{F\} \cup \text{Pre}(F), H)$ is not a restricted duality of \mathcal{C} .

It is clear that we can restrict our attention to minimal restricted dualities, as if (\mathcal{F}, H) is a restricted duality of a class \mathcal{C} then there exists minimal \mathcal{F}' such that (\mathcal{F}', H) is a restricted duality of \mathcal{C} .

Lemma 17. *Let (\mathcal{F}, H) be a restricted duality of \mathcal{C} , with \mathcal{F} minimal.*

- *If the class \mathcal{C} is addable, then every graph in \mathcal{F} is connected.*
- *If the class \mathcal{C} is monotone, then every graph in \mathcal{F} is connected and $\mathcal{F} \subseteq \mathcal{C}$ (hence $F \twoheadrightarrow H$ holds for every $F \in \mathcal{F}$).*

Proof. Assume \mathcal{C} is addable, and assume for contradiction that $F_1 + F_2 \in \mathcal{F}$. By minimality of \mathcal{F} , neither $(\mathcal{F} \setminus \{F_1 + F_2\} \cup \{F_1\}, H)$ nor $(\mathcal{F} \setminus \{F_1 + F_2\} \cup \{F_2\}, H)$ are restricted dualities of \mathcal{C} . Hence there exist $G_1, G_2 \in \mathcal{C}$ such that $F_1 \twoheadrightarrow G_1$, $F_2 \twoheadrightarrow G_1$ (hence $G_1 \rightarrow H$), $F_1 \twoheadrightarrow G_2$, $F_2 \twoheadrightarrow G_2$ (hence $G_2 \rightarrow H$). As \mathcal{C} is addable, $G_1 + G_2 \in \mathcal{C}$. But $F_1 + F_2 \twoheadrightarrow G_1 + G_2$, what contradicts $G_1 + G_2 \rightarrow H$.

Assume \mathcal{C} is monotone, and assume $F \in \mathcal{F}$. By minimality of \mathcal{F} , there exists $G \in \mathcal{C}$ such that $F \rightarrow G$ but no graph $F' \in \text{Pre}(F)$ is homomorphic to G . Thus, according to Lemma 16, F is isomorphic to a subgraph of G hence, as \mathcal{C} is monotone, $F \in \mathcal{C}$. Assume for contradiction that $F = F_1 + F_2$. Then $F_1, F_2 \in \mathcal{C}$ and none of F_1, F_2 is homomorphic to the other. By minimality of \mathcal{F} it follows that $F_1 \rightarrow H$, $F_2 \rightarrow H$ hence $F_1 + F_2 \rightarrow H$, contradicting $F_1 + F_2 \in \mathcal{F}$. \square

5. Restricted Dualities

As restricted dualities appear as the central notion when dealing with first-order definable coloring, we take time to define and characterize classes with all restricted dualities in the more general framework of relational structures.

5.1. Classes of Relational Structures

We recall some basic definitions, notations and result of model theory. Our terminology is standard, cf [11, 26]:

A *signature* σ is a finite set of relation symbols, each with a specified arity. A σ -*structure* \mathbf{A} consists of a *universe* A , or *domain*, and an *interpretation* which associates to each relation symbol $R \in \sigma$ of some arity r , a relation $R^{\mathbf{A}} \subseteq A^r$.

A σ -structure \mathbf{B} is a *substructure* of \mathbf{A} if $B \subseteq A$ and $R^{\mathbf{B}} \subseteq R^{\mathbf{A}}$ for every $R \in \sigma$. It is an *induced substructure* if $R^{\mathbf{B}} = R^{\mathbf{A}} \cap B^r$ for every $R \in \sigma$ of arity r . Notice the analogy with the graph-theoretical concept of subgraph and induced subgraph. A substructure \mathbf{B} of \mathbf{A} is *proper* if $\mathbf{A} \neq \mathbf{B}$. If \mathbf{A} is an induced substructure of \mathbf{B} , we say that \mathbf{B} is an *extension* of \mathbf{A} . If \mathbf{A} is a proper induced substructure, then \mathbf{B} is a *proper extension*. If \mathbf{B} is the disjoint union of \mathbf{A} with another σ -structure, we say that \mathbf{B} is a *disjoint extension* of \mathbf{A} . If $S \subseteq A$ is a subset of the universe of \mathbf{A} , then $\mathbf{A} \cap S$ denotes the *induced substructure generated by S* ; in other words, the universe of $\mathbf{A} \cap S$ is S , and the interpretation in $\mathbf{A} \cap S$ of the r -ary relation symbol R is $R^{\mathbf{A}} \cap S^r$.

The *Gaifman graph* $\text{Gaifman}(\mathbf{A})$ of a σ -structure \mathbf{A} is the graph with vertex set A in which two vertices $x \neq y$ are adjacent if and only if there exists a relation R of arity $k \geq 2$ in σ and $v_1, \dots, v_k \in A$ such that $\{x, y\} \subseteq \{v_1, \dots, v_k\}$ and $(v_1, \dots, v_k) \in R^{\mathbf{A}}$.

A *block* of a σ -structure \mathbf{A} is a tuple (R, x_1, \dots, x_k) such that $R \in \sigma$ has arity k and $(x_1, \dots, x_k) \in R^{\mathbf{A}}$. The *incidence graph* $\text{Inc}(\mathbf{A})$ is the bipartite graph (A, B, E) where A is the universe of \mathbf{A} , B is the set of all *blocks* of \mathbf{A} , and E is the set of the pairs $\{(R, x_1, \dots, x_k), y\} \subseteq B \times A$ such that $y \in \{x_1, \dots, x_k\}$. Thus for us $\text{Inc}(\mathbf{A})$ is a simple graph. No multiple edges are needed for our purposes.

A *homomorphism* $\mathbf{A} \rightarrow \mathbf{B}$ between two σ -structure is defined as a mapping $f : A \rightarrow B$ which satisfies for every relational symbol $R \in \sigma$ the following:

$$(x_1, \dots, x_k) \in R^{\mathbf{A}} \implies (f(x_1), \dots, f(x_k)) \in R^{\mathbf{B}}.$$

The class of all σ -structures is denoted by $\text{Rel}(\sigma)$.

The definition of bounded expansion extends to classes of relational structures: a class \mathcal{C} of relational structures has *bounded expansion* if the class of the Gaifman graphs of the structures in \mathcal{C} has bounded expansion. It is immediate that two relational structures have the same Gaifman graph if they have the same incidence graph, but that the converse does not hold in general. For a class of relational structures \mathcal{C} , denote by $\text{Inc}(\mathcal{C})$ the class of all the incidence graphs $\text{Inc}(\mathbf{A})$ of the relational structures $\mathbf{A} \in \mathcal{C}$.

Proposition 18 ([35]). *Assume that the arities of the relational symbols in σ are bounded, and let \mathcal{C} be an infinite class of σ -structures. Then the class \mathcal{C} has bounded expansion if and only if the class $\text{Inc}(\mathcal{C})$ has bounded expansion.*

5.2. Classes with all restricted dualities

A class of σ -structures \mathbf{A} has *all restricted dualities* if every non-empty connected σ -structure has a restricted dual for \mathcal{C} , that is: for every non-empty connected σ -structure \mathbf{F} there exists a σ -structure \mathbf{D} such that $\mathbf{F} \rightarrow \mathbf{D}$ and

$$\forall \mathbf{A} \in \mathcal{C} : \quad (\mathbf{F} \rightarrow \mathbf{A}) \quad \iff \quad (\mathbf{A} \rightarrow \mathbf{D}).$$

Note that this definition implies that also for any finite set $\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_t$ of connected σ -structures there exists a σ -structure \mathbf{D} such that $\mathbf{F}_i \rightarrow \mathbf{D}$ (for $1 \leq i \leq t$) and

$$\forall \mathbf{A} \in \mathcal{C} : \quad (\exists i \leq t : \mathbf{F}_i \rightarrow \mathbf{A}) \quad \iff \quad (\mathbf{A} \rightarrow \mathbf{D}).$$

For a structure \mathbf{A} and an integer t , define $\Theta^t(\mathbf{A})$ as the minimum order of a structure \mathbf{B} such that

- $\mathbf{A} \rightarrow \mathbf{B}$,
- every substructure \mathbf{F} of \mathbf{B} with order $|F| < t$ has a homomorphism to \mathbf{A} .

Intuitively, such a structure \mathbf{B} can be seen as approximate core of \mathbf{A} : For $t \geq |B|$, \mathbf{A} and \mathbf{B} are homomorphism-equivalent and \mathbf{B} is the core of \mathbf{A} (alternately, \mathbf{B} is the minimal retract of \mathbf{A}). A structure \mathbf{B} with the above properties and order $\Theta^t(\mathbf{A})$ is called a *t-approximation* of (the homomorphism equivalence class of) \mathbf{A} .

It appears that existence of a uniform approximation is equivalent for a class to having all restricted dualities. This is formalized by Theorem 4, stated in the introduction. Theorem 4 will be proved now:

Proof of Theorem 4. Assume \mathcal{C} is bounded and has all restricted dualities and let $t \in \mathbb{N}$ be an integer. Let \mathbf{Z} be a strict bound of \mathcal{C} , that is a structure such that for every $\mathbf{A} \in \mathcal{C}$ it holds $\mathbf{A} \rightarrow \mathbf{Z}$ but $\mathbf{Z} \not\rightarrow \mathbf{A}$. As the sequence $(\Theta^t(\mathbf{A}))_{t \in \mathbb{N}}$ is obviously non-decreasing, we may assume without loss of generality that $t \geq |Z|$. For a structure $\mathbf{A} \in \mathcal{C}$, let $\mathcal{F}_t(\mathbf{A})$ be the set of all connected cores \mathbf{T} of order at most t such that $\mathbf{T} \rightarrow \mathbf{A}$. This set is not empty as it contains the core of \mathbf{Z} . For $\mathbf{T} \in \mathcal{F}_t(\mathbf{A})$, let $\mathbf{D}_{\mathbf{T}}$ be the dual of \mathbf{T} relative to \mathcal{C} and let \mathbf{A}' be the product of all the $\mathbf{D}_{\mathbf{T}}$ for $\mathbf{T} \in \mathcal{F}_t(\mathbf{A})$. First notice that for every $\mathbf{T} \in \mathcal{F}_t(\mathbf{A})$ we have $\mathbf{T} \rightarrow \mathbf{A}$ hence $\mathbf{A} \rightarrow \mathbf{D}_{\mathbf{T}}$. It follows that $\mathbf{A} \rightarrow \mathbf{A}'$. Let \mathbf{T}' be a connected substructure of order at most t of \mathbf{A}' . Suppose for a contradiction that $\mathbf{T}' \not\rightarrow \mathbf{A}$. Then $\text{Core}(\mathbf{T}') \in \mathcal{F}_t(\mathbf{A})$ hence $\mathbf{A}' \rightarrow \mathbf{D}_{\mathbf{T}'}$ thus $\mathbf{T}' \rightarrow \mathbf{A}'$ (as for otherwise $\mathbf{T}' \rightarrow \mathbf{D}_{\mathbf{T}'}$), a contradiction. Thus $\mathbf{T}' \rightarrow \mathbf{A}$. It follows that $\Theta^t(\mathbf{A}) \leq |A'| \leq C(t)$ for some suitable finite constant $C(t)$ independent of \mathbf{A} (for instance, one can choose $C(t)$ to be the product of the orders of all the duals relative to \mathcal{C} of connected cores of order at most t).

Conversely, assume that we have $\sup_{\mathbf{A} \in \mathcal{C}} \Theta^t(\mathbf{A}) < \infty$ for every $t \in \mathbb{N}$. The class \mathcal{C} is obviously bounded by the disjoint union of all non-isomorphic minimal order 1-approximations of the structures in \mathcal{C} . Let \mathbf{F} be a connected σ -structure, let $t \geq |F|$, and let \mathcal{D} be a set of t -approximations of all the structures $\mathbf{A} \in \mathcal{C}$ such that $\mathbf{F} \rightarrow \mathbf{A}$. As all the $\Theta^t(\mathbf{A})$ are bounded by some constant $C(t)$, the set \mathcal{D} is finite. If \mathcal{D} is empty, let $D_t(\mathbf{F})$ be the empty substructure. Otherwise, let $D_t(\mathbf{F})$ be the disjoint union of all the graphs in \mathcal{D} . First notice that $\mathbf{F} \rightarrow D_t(\mathbf{F})$ as for otherwise \mathbf{F} would have a homomorphism to some structure in \mathcal{D} (as \mathbf{F} is connected), that is to some t -approximation \mathbf{B}' of a structure \mathbf{B}

such that $\mathbf{F} \twoheadrightarrow \mathbf{B}$ (this would contradict $\mathbf{F} \rightarrow \mathbf{B}'$). Also, if $\mathbf{F} \rightarrow \mathbf{A}$ then $\mathbf{A} \twoheadrightarrow D_t(\mathbf{F})$ (for otherwise $\mathbf{F} \rightarrow D_t(\mathbf{F})$) and if $\mathbf{F} \twoheadrightarrow \mathbf{A}$ then \mathcal{D} contains a t -approximation \mathbf{A}' of \mathbf{A} thus $\mathbf{A} \rightarrow D_t(\mathbf{F})$. Altogether, $D_t(\mathbf{F})$ is a dual of \mathbf{F} relative to \mathcal{C} . \square

We proved in [30] that bounded expansion classes have all restricted dualities:

Theorem 19. *Let \mathcal{C} be a class with bounded expansion. Then for every connected graph F there exists a graph D such that (F, D) is a restricted homomorphism duality for \mathcal{C} :*

$$\forall G \in \mathcal{C} \quad (F \rightarrow G) \iff (G \twoheadrightarrow H). \quad (8)$$

Theorem 19 naturally extends to relational structures by considering Gaifman graphs. We sketch a proof of this generalization, which is based on the above Theorem 4.

Theorem 20. *Let \mathcal{K} be a class of relational structures. If the class of the Gaifman graphs of the structures in \mathcal{K} has bounded expansion then the class \mathcal{K} has all restricted dualities.*

Sketch of the proof. Let \mathcal{K} be a class of relational structures. Assume the class of the Gaifman graphs of the structures in \mathcal{K} has bounded expansion. Let $\mathbf{A} \in \mathcal{K}$, and let $t \in \mathbb{N}$ be at least as large as the maximum arity of a relation in the signature of \mathbf{A} .

The *tree-depth* $\text{td}(G)$ of a graph G is the minimum height of a rooted forest whose closure includes G as a subgraph. One of the most interesting properties of tree-depth is that there exists a function $F : \mathbb{N} \rightarrow \mathbb{N}$ with the property that if the Gaifman graph of a structure \mathbf{B} has tree-depth at most t then there exists a homomorphism $f : \mathbf{B} \rightarrow \mathbf{B}$ such that $|f(\mathbf{B})| \leq F(t)$ [28]. For integer t , we defined in [28] the graph invariant χ_t as follows: for a graph G , $\chi_t(G)$ is the minimum number of colors needed in a coloring of G such that the union of every subset of $k \leq t$ color classes induces a subgraph with tree-depth at most k (such colorings are called *low tree-depth colorings*). It has been proved in [29] that a class of graphs \mathcal{C} has bounded expansion if and only if for every integer t it holds $\sup\{\chi_t(G) : G \in \mathcal{C}\} < \infty$ (this is related to Theorem 6 above).

Consider a coloring c of the Gaifman graph of \mathbf{A} by $N = \chi_t(\text{Gaifman}(\mathbf{A}))$ colors, which is such that the union of every subset of $k \leq t$ color classes induces a subgraph with tree-depth at most k . It follows that for each $I \in \binom{[N]}{t}$ there exists a homomorphism $f_I : \mathbf{A}_I \rightarrow \mathbf{A}_I$ such that $|f_I(\mathbf{A}_I)| \leq F(t)$, where \mathbf{A}_I denotes the substructure of \mathbf{A} induced by elements with color in I . Define the equivalence relation \sim on the domain of \mathbf{A} by

$$x \sim y \iff c(x) = c(y) \text{ and } \forall I \in \binom{[N]}{t} f_I(x) = f_I(y).$$

Define the structure $\hat{\mathbf{A}}$ (with same signature as \mathbf{A}) whose domain is the set of the equivalence classes $[x] \in A/\sim$, and relations are defined by

$$([x_1], \dots, [x_{k_i}]) \in R_i^{\hat{\mathbf{A}}} \iff \forall I \in \binom{[N]}{t} (f_I(x_1), \dots, f_I(x_{k_i})) \in R_i^{\mathbf{A}}.$$

We also define a N -coloration of $\hat{\mathbf{A}}$ by $\hat{c}([x]) = c(x)$. One checks easily that $\hat{\mathbf{A}}$ and \hat{c} are well defined. By construction, $x \mapsto [x]$ is a homomorphism $\mathbf{A} \rightarrow \hat{\mathbf{A}}$. Moreover, for

every $I \in \binom{[N]}{t}$ the mapping $[x] \mapsto f_I(x)$ is a homomorphism $\hat{\mathbf{A}}_I \rightarrow \mathbf{A}_I$ (where $\hat{\mathbf{A}}_I$ is the substructure of $\hat{\mathbf{A}}$ induced by colors in I). It follows that

$$|\Theta^t(\mathbf{A})| \leq |\hat{A}| \leq F(t)^{N^t} \leq F(t)^{\chi_t(\text{Gaifman}(\mathcal{K}))^t}.$$

According to Theorem 4, this implies that the class \mathcal{K} has all restricted dualities. \square

For an alternate proof of this Theorem, we refer the reader to [30, 35].

5.3. Topologically closed classes of graphs with all restricted dualities

The special case of topologically closed classes of graphs is of particular interest here, and we have in this case a much simpler characterization of the classes that have all restricted dualities. We are now ready to prove Theorem 3.

Proof of Theorem 3. The proof follows from the next three implications:

- (1) \Rightarrow (2) is a direct consequence of Theorem 19.
- (2) \Rightarrow (3) is straightforward (consider for H_g a dual of C_g for \mathcal{C}).
- (3) \Rightarrow (1) is proved by contradiction: assume that (3) holds and that \mathcal{C} does not have bounded expansion. According to Theorem 6 there exists an integer p such that $\mathcal{C} \tilde{\nabla} p$ has unbounded chromatic number. As \mathcal{C} is topologically closed there exists an odd integer $g \geq p$ and a graph $G_0 \in \mathcal{C}$ such that G_0 is the $(g-1)$ -subdivision of a graph H_0 with chromatic number $\chi(H_0) > |H_g|$. According to (3), there exists a homomorphism $f : G_0 \rightarrow H_g$. As $C_g \rightarrow H_g$, the ends of a path of length g cannot have the same image by f . It follows that any two adjacent vertices in H_0 correspond to branching vertices of G_0 which are mapped by f to distinct vertices of H_g . It follows that $\chi(H_0) \leq |H_g|$, a contradiction.

\square

6. On First-Order Definable H -colorings

In this section we prove our main characterization result on first-order definable colorings, stated in the introduction as Theorem 2.

Proof of Theorem 2. Assume that the class \mathcal{C} is somewhere dense. As \mathcal{C} is topologically closed, there exists an integer p such that $\text{Sub}_{2p}(\text{Graph}) \subseteq \mathcal{C}$.

Assume for contradiction that there exists a non-bipartite graph H (different from K_1) of odd-girth strictly greater than $2p+1$ and a first-order formula Φ such that for every graph $G \in \mathcal{C}$ holds

$$(G \models \Phi) \iff (G \rightarrow H).$$

According to Corollary 14, as $\neg\Phi$ is preserved by homomorphisms on \mathcal{C} (hence on $\text{Sub}_{2p}(\text{Graph})$) it is equivalent on $\text{Sub}_{2p}(\text{Graph})$ with an existential first-order formula, that is: there exists a finite family \mathcal{F} such that for every graph G it holds:

$$\forall F \in \mathcal{F} \ F \rightarrow \text{Sub}_{2p}(G) \iff \text{Sub}_{2p}(G) \rightarrow H.$$

Clearly, the graphs in \mathcal{F} are non-bipartite. Let g be the maximum of girth of graphs in \mathcal{F} and let G be a graph with chromatic number $\chi(G) > |H|$ and odd-girth $\text{odd-girth}(G) > g$. Then for every $F \in \mathcal{F}$ we have $F \twoheadrightarrow \text{Sub}_{2p}(G)$ hence $\text{Sub}_{2p}(G) \rightarrow H$. However, the odd-girth of H is strictly greater than $2p + 1$ two branching vertices of $\text{Sub}_{2p}(G)$ corresponding to adjacent vertices of G cannot be mapped to a same vertex. It follows that $|H| \geq \chi(G)$, a contradiction.

To the opposite, if \mathcal{C} has bounded expansion, there exists for every integer p a non-bipartite graph H_p of odd-girth strictly greater than $2p + 1$ and a first order formula Φ_p such that for every graph $G \in \mathcal{C}$ holds

$$(G \models \Phi_p) \iff (G \rightarrow H_p).$$

Indeed, consider for Φ_p the formula asserting that G contains an odd cycle of length at most $2p + 1$, and for H_p the restricted dual of the cycle C_{2p+1} with respect to \mathcal{C} (whose existence follows from Theorem 19).

Now assume that Conjecture 2 holds, and that \mathcal{C} is a hereditary topologically closed addable nowhere dense class that is not a bounded expansion class. Then there exists an integer p such that \mathcal{C} includes $2p$ -subdivisions of graph with arbitrarily large chromatic number and girth. Assume for contradiction that there is a non-bipartite graph H_p of odd-girth strictly greater than $2p + 1$ such that H_p -coloring is first-order definable on \mathcal{C} , and let Φ be a formula such that for every $G \in \mathcal{C}$ it holds $G \rightarrow H_p$ if and only if $G \models \Phi$. According to Theorem 9, there exists a finite family \mathcal{F} of finite graphs such that $G \models H_p$ if and only if no graph in \mathcal{F} is homomorphic to G . The graphs in \mathcal{F} are non-bipartite (\mathcal{C} contains long odd cycles homomorphic to H_p). Let g be the maximum of the odd-girths of the graphs in \mathcal{F} . Let $G \in \mathcal{C}$ be a $2p$ -subdivision of a graph H with girth greater than g and chromatic number $\chi(H) > |H_p|$. As the girth of G is greater than g , no graph in \mathcal{F} is homomorphic to G hence there exists a homomorphism $f : G \rightarrow H_p$. As $C_{2p+1} \twoheadrightarrow G$ it follows that vertices of G linked by a path of length $2p + 1$ are mapped to distinct vertices of H_p hence f defines a homomorphism $H \rightarrow K_{|H_p|}$, contradicting $\chi(H) > |H_p|$. \square

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