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Transport equations for superconductors in the presence of spin interaction

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Quasi-classical theory of superconductivity provides a powerful and yet simple description of the superconductivity phenomenology. In particular, the Eilenberger and Usadel equations provide a neat simplification of the description of the superconducting state in the presence of disorder and electromagnetic interaction. However, the modern aspects of superconductivity require a correct description of the spin interaction as well. Here, we generalize the transport equations of superconductivity in order to take into account space-time dependent electromagnetic and spin interactions on equal footing. Using a gauge-covariant Wigner transformation for the Green-Gor'kov correlation functions, we establish the correspondence between the Dyson-Gor'kov equation and the quasi-classical transport equation in the time-dependent phase-space. We give the expressions for the gauge-covariant current and charge densities (quasi-particle, electric and spin) in the transport formulation. The generalized Eilenberger and Usadel limits of the transport equation are given, too. This study is devoted to the formal derivation of the equations of motion in the electromagnetic plus spin plus particle-hole space. The studies of some specific systems are postponed to future works.

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Without doubt, the theory of superconductivity, first established by Bardeen, Cooper and Schrieffer [1, 2], and reformulated by Valatin [3], Bogoliubov [4], Gor'kov [5, 6] and Nambu [7, 8] is a masterpiece of condensed matter in particular, and quantum field theory in general. It consists in a few concepts – a second-order phase transition due to electron-phonon interaction, or a classical gauge-symmetry breaking in high-energy language – together with a predictive power which provided breakthrough discoveries all along the second half of the 20-th century. Among others, the BCS theory and its close parent the Ginzburg-Landau model [9, 10] allow the prediction of the vortex states [11], the Josephson effect [12], the generation of massive boson field at the phase transition [13–15], and the great family of the proximity effects [16, 17], ... all experimentally well-established since.

The balance between a few concepts involved in a large number of novel effects is certainly due to the robustness of the quasi-classical description of superconductivity [18–21]. Indeed, most superconductors are characterized by a relevant energy scale, namely the gap parameter energy, much smaller than the Fermi energy. Then it becomes possible to adapt for superconductors the quasi-classical theory developed for normal metals [22, 23].

Due to its success describing such vast problems as vortex in bulk, Josephson and proximity effect in mesoscopic systems as well as the competition between superconductivity and disorder, the quasi-classical description of superconductivity was naturally extended to discuss the competition between superconductor and magnetic orders. There, the quasi-classical description opened a new era of discoveries, which are too numerous to be listed here. We just mention that their possible domain of applicability ranges from spintronic effects to some proposed fundamental phases in neutron stars and in the early universe, passing through original vortex states and new electronic devices based on novel Josephson effects, see *e.g.* [24–27] and references therein.

Whereas the first studies discussing the competing effect between superconductivity and spin coupling focussed on constant ferromagnetic field, there are emerging interests in the description of superconducting systems having spin texture. The promises these systems carry on arose several fields of research. On one side, there are fundamental questions in bulk systems about the competitions between non-centrosymmetric magnetic order and the superconducting phase, leading to interesting magneto-electric effects, original vortex lattices, helical superconducting phase, ... [28]. On the other side, the presence of spin-orbit interaction in superconducting wires has been predicted to generate topological states of matter, possibly helpful for quantum computation [29, 30]. In these wires hosting Majorana modes, there are still vivid discussions about the role of impurities, the nature of the competition between the proximity effect and the spin-texture, ... see *e.g.* [31]. Moreover, having ferromagnetic plus spin-orbit coupling in a superconducting wire does not seem to be rich enough to provide universal quantum computation, and people are recently discussing spin-texture in quantum-Hall plus superconducting heterostructures in order to generate possible parafermions [32–34].

A quasi-classical description of superconductivity able to take into account spin-texture and impurities is thus highly desirable. Of course, it exists several ways to perturbatively simplify this complicated problem, as dis-

cussing diffusive systems, or perturbatively weak disorder, and/or small spin-orbit effect for Majorana wires for instance [35–37]. Note also the literature associated to the inclusion of spin-orbit effect in bulk superconductor or superfluid without exchange field [38–41], or the alternative possibility to use topological superconductivity (p-wave) as an effective model for spin-textured superconductor [42][1]. Nevertheless, a reliable construction of a general quasi-classical theory should be of interest in several active research fields. This is also the case in normal and semiconducting diffusive systems including spin-texture. There, it has been shown recently that a gauge-theory construction provides a transparent procedure for the derivation of transport equations [44] including spin effects.

This paper is devoted to the question of the inclusion of the spin texture (Zeeman plus spin-orbit interaction say) in the superconductivity phenomenology. Here, we recognize the venerable principle of gauge redundancy (see *e.g.* [45]) as a fruitful tool for the construction of a transport theory of superconductivity, including space-time dependent spin and charge fields. In particular, we generalize the results from [44] in order to include the superconducting correlation functions. We adapt the description of the quark-gluon plasma [46] to the non-relativistic situation of a superconductor in the presence of some generic Abelian (electromagnetic for instance) and non-Abelian (spin and particle-hole) gauge-fields.

I aim this paper to be as pedagogical as possible, especially in the sometimes confusing adoption of the mixed-Fourier transformation [47–51], which is nothing more than a Wigner transformation [52, 53], here fruitfully made gauge-covariant, see Section III. To that purpose, I sum-up the conventions I follow in Section I and Section II, and I provide explicit – though lengthy – calculations in an appendix. Even though the calculation of the appendix can be generalized straightforwardly to higher orders, I discuss in the main text the explicit model of non-relativistic free particles in the quasi-classical limit. This limits the present study to the Rashba-like spin-orbit effect when the spin interaction is linear in momentum.

I generalize the BCS treatment given by Gor'kov [5] to the non-Abelian gauge theory in Section I. I then discuss the equations of motion for the gauge fields in Section II, following the standard treatment [54]. The transport equations at the quasi-classical level are given in Section III. There I establish the main results of this paper, namely expressions (55) and (56). Then I turn to the Eilenberger (Section IV, eq.(76)) and Usadel (Section V, eq.(84)) limits of these equations, when the relevant energies are constrained to the proximity of the Fermi energy in the general and diffusive limit, respectively. Especially, the derivation from the transport equation to the Eilenberger one is treated in full details as well as the so-called normalization condition (Section IV). The two last sections sum up an alternative derivation of the gauge-covariant Eilenberger equation (Section VI), perhaps more comprehensible than the lengthy calculation of Section IV, and a discussion of the usual treatment of a constant exchange field in the quasi-classical limit (Section VII). Some perspectives of the present work are given alongside the conclusion in Section VIII.

[1] Note also that, after completion of this work, I became aware of a similar study by Bergeret and Tokatly, which use the same method as the one in this paper to obtain similar equations [43]. A discussion of the main differences between their paper and mine can be find in Section VII.

I. MATTER FIELD: DYSON-GOR'KOV'S EQUATIONS

We start our discussion with a brief summary of some known results in the theory of superconductivity. In fact, this work starts from the Dyson-Gor'kov equations at zero temperature [5, 6]. These equations represent the evolution of the quantum-field correlation functions in space-time. According to the Gor'kov theory, the superconducting systems are described in the so-called Nambu space, or particle-hole space. Here, I generalize the Gor'kov theory toward a non-Abelian theory including both the Nambu and the spin space, in addition to the usual Abelian electromagnetic space. Reader familiar with the quantum field theory of superconductivity can skip this section up to the equation (7), reader also familiar with the concept of gauge theory can skip this section entirely, as well as Section II.

Since we will discuss gauge properties in space-time, it is convenient to use the relativistic quadri-vectors notations. They are defined as $x^\mu = (ct, \boldsymbol{x})$ and $\partial_\mu \equiv (\partial_{ct}, \boldsymbol{\partial_x})$ with the metric tensor $g_{\mu\nu} = (1, -1, -1, -1)$. Later on we will define an energy-momentum 4-vector $p^\mu = (E/c, \boldsymbol{p})$ with $E = \hbar\omega$ which defines the angular frequency ω, and $\boldsymbol{p} = \hbar\boldsymbol{k}$. Contracted indices are implicitly summed, the greek ones being over the full space-time $\mu, \nu \equiv (0, 1, 2, 3)$, whereas the latin ones are only over the space variables $i, j, k = (1, 2, 3)$. To not confound the indices with the imaginary unit vector, the latter is noted in bold $\mathbf{i}^2 = -1$. As much as possible, I try to avoid using bold symbols for the collection of the components of the vectors, which are preferably written in terms of their components. The bold letters are kept for the symbolic notation of the elements in the Nambu space. When not possible otherwise, I use italic bold letters to describe vectors in space-time. I use also the notation for the central dot to represent the scalar product, either in space-time or in space: for instance $p \cdot x/\hbar = p_\mu x^\mu/\hbar = \omega t - \boldsymbol{k} \cdot \boldsymbol{x} = \omega t + k^i x_i$.

In order to discuss the quantum field theory of superconductivity, I adopt the spinor notation in the Nambu

space

$$\Psi(x) = \begin{pmatrix} \Psi_\uparrow(x) \\ \Psi_\downarrow(x) \end{pmatrix} \text{ and } \tilde{\Psi}(x) = \begin{pmatrix} \Psi_\uparrow(x) & \Psi_\downarrow(x) \end{pmatrix} \quad (1)$$

with the convention $\Psi^\dagger = \tilde{\Psi}^*$, where $\Psi(x)$ annihilates a fermion at space-time position x, whereas $\Psi^\dagger(x)$ creates a fermion at that position. The Green-Gor'kov correlation functions in space-time are defined via a matrix in the Nambu space

$$\begin{aligned} \mathbf{G} &= \frac{\mathbf{i}}{\hbar}\left\langle \hat{T}\left[\begin{pmatrix} -\Psi(x_1) \\ \tilde{\Psi}^\dagger(x_1) \end{pmatrix} \otimes \begin{pmatrix} \Psi^\dagger(x_2) & \tilde{\Psi}(x_2) \end{pmatrix}\right]\right\rangle \\ &= \begin{pmatrix} G(x_1,x_2) & -F(x_1,x_2) \\ F^\dagger(x_1,x_2) & G^\dagger(x_1,x_2) \end{pmatrix} \end{aligned} \quad (2)$$

where $x_{1,2} \equiv x^\mu_{1,2}$, the $\hat{T}$ operator is the time-ordering operator, and the average $\langle \cdots \rangle$ is a quantum average [6]. The definition (2) in terms of a tensor product will simplify the gauge transformation treatment below. Note that we do not describe further the sub-space for the functions G, F, ... Additionally, the spin and charge structure will be entirely defined through the gauge-potentials defined later. In contrary, the Pauli matrices notation will be of importance. I use the τ_i matrices to represent the Nambu algebra, and the σ_i matrices to represent the spin algebra.

We start from the simplest model of a free electron gas interacting through the usual BCS interaction and described by the Hamiltonian $H = H_0 + H_{\text{int}}$ with

$$H_0 = \int dx \left[\Psi^\dagger(x)\left(-\frac{\hbar^2}{2m}\boldsymbol{\partial_x}\cdot\boldsymbol{\partial_x} - \mu\right)\Psi(x)\right] \quad (3)$$

where μ is the chemical potential, and

$$H_{\text{int}} = \int dx \frac{V_0(x)}{2} \times \left[\tilde{\Psi}(x)\,\mathbf{i}\sigma_2\Psi(x)\right]^\dagger \left[\tilde{\Psi}(x)\,\mathbf{i}\sigma_2\Psi(x)\right] \quad (4)$$

is a two-body interaction, with the spin-independent V_0 being an attractive interaction strength in the superconducting regions of space, and otherwise vanishing. The Heisenberg equation of motion $\mathbf{i}\hbar\partial_t\Psi = [\Psi, H]$ leads to

$$\int dy \left[\mathbf{G}^{-1}(x_1,y)\,\mathbf{G}(y,x_2)\right] = \delta(x_1 - x_2) \quad (5)$$

with $\mathbf{G}^{-1}(x_1,y) = \mathbf{G}_0^{-1}(x_1)\,\delta(x_1 - y)$ and

$$\mathbf{G}_0^{-1}(x) = \frac{\hbar^2}{2m}\boldsymbol{\partial_x}\cdot\boldsymbol{\partial_x} + \mu + \begin{pmatrix} \mathbf{i}\hbar\partial_t & \Delta(x) \\ -\Delta^\dagger(x) & -\mathbf{i}\hbar\partial_t \end{pmatrix} \quad (6)$$

is the so-called propagator. The gap parameter

$$\Delta(x) = V_0(x)\left\langle \hat{T}\left[\tilde{\Psi}(x)\,(\mathbf{i}\sigma_2)\,\Psi(x)\right]\right\rangle (\mathbf{i}\sigma_2)^\dagger \quad (7)$$

is defined self-consistently as

$$\Delta_0(x_2) = -\mathbf{i}\hbar \lim_{x_1\to x_2} V_0(x_1)\,\mathrm{Tr}\left\{\mathbf{i}\sigma_2 F(x_1,x_2)\right\} \quad (8)$$

with $\Delta(x) = \Delta_0(x)\,(\mathbf{i}\sigma_2)^\dagger$ and the trace is taken over the sub-space(s) of the $F(x_1,x_2)$ matrix. The gap parameter appears in (5) thanks to a mean-field decoupling in the Cooper pairing channel, see [6] for more details.

It is noteworthy to realize that the interaction Hamiltonian H_{int} is both $U(1)$ and $SU(2)$ gauge invariant, *i.e.* it is invariant under the transformation[2]

$$\begin{aligned} \Psi(x) &\rightsquigarrow R(x)\,\Psi(x) \\ \Psi^\dagger(x) &\rightsquigarrow \Psi^\dagger(x)\,R^\dagger(x) \end{aligned} \quad (9)$$

with $R(x) \in U(1)\otimes SU(2)$. Since it describes singlet spin coupling, any spin rotation will let it unaffected hence it is $SU(2)$ gauge invariant – mathematically speaking this corresponds to the remark that $e^{\mathbf{i}n^i\sigma_i}\sigma_2 e^{\left(\mathbf{i}n^i\sigma_i\right)^*} = \sigma_2$ for any unit vector components n^i. Since it has the same number of $\Psi(x)$ as $\Psi^\dagger(x)$, it is $U(1)$ gauge invariant. We also realize that H_{int} is nothing but the usual s-wave interaction Hamiltonian [6, 16, 17]. Then we could promote the equation of motion (5) to be $U(1)\otimes SU(2)$ gauge covariant in principle.

From its definition (2), a gauge transformation (9) of the Green-Gor'kov matrix $\mathbf{G}$ reads

$$\mathbf{G}(x_1,x_2) \rightsquigarrow \mathbf{R}(x_1)\,\mathbf{G}(x_1,x_2)\,\mathbf{R}^{-1}(x_2) \quad (10)$$

with $\mathbf{R}$ given by

$$\mathbf{R}(x) = \begin{pmatrix} R(x) & 0 \\ 0 & R^*(x) \end{pmatrix} \quad (11)$$

in the particle-hole space. Since we consider some unitary matrices $R^\dagger R = 1$, one has $\mathbf{R}^{-1} = \mathbf{R}^\dagger$. Note nevertheless that the left and right $\mathbf{R}$ transformation matrices are not evaluated at the same point: the Green-Gor'kov functions are two-points correlation functions in space-time. Then the general covariance of the Green-Gor'kov equations is constructed under the demand that the transformation

$$\mathbf{G}_0^{-1}(x_1)\,\mathbf{G}(x_1,x_2) \rightsquigarrow \mathbf{R}(x_1)\,\mathbf{G}_0^{-1}(x_1)\,\mathbf{G}(x_1,x_2)\,\mathbf{R}^\dagger(x_2) \quad (12)$$

works after a proper substitution of the derivatives with some covariant derivatives, the usual minimal or Weyl's substitution [55]. One verifies easily that the correct minimal substitution reads

$$\mathbf{G}_0^{-1}(x) = \mathbf{i}\hbar c\tau_3\mathbf{D}_0(x) - \frac{\hbar^2}{2m}\mathbf{D}^i\mathbf{D}_i(x) + \boldsymbol{\Delta}(x) \quad (13)$$

[2] In this paper, I propose to use the symbol $\rightsquigarrow$ to denote the gauge transformation mapping. No confusion with its definition in symbolic calculation is possible.

with the covariant derivative

$$\mathbf{D}_{\mu}(x)=\frac{\partial}{\partial x^{\mu}}+\mathbf{i}\tau_{3}\begin{pmatrix} A_{\mu} & 0 \\ 0 & A_{\mu}^{*}\end{pmatrix}=\partial_{\mu}+\mathbf{i}\mathbf{A}_{\mu}(x) \quad (14)$$

defining the gauge potential $\mathbf{A}_{\mu}$. It transforms according to

$$\mathbf{A}_{\mu}(x)\rightsquigarrow\mathbf{R}(x)\mathbf{A}_{\mu}(x)\mathbf{R}^{\dagger}(x)-\mathbf{i}\mathbf{R}(x)\partial_{\mu}\mathbf{R}^{\dagger}(x) \quad (15)$$

when the Green-Gor'kov matrix transforms as (10). One associates the gauge field

$$\mathbf{F}_{\mu\nu}(x)=\partial_{\mu}\mathbf{A}_{\nu}-\partial_{\nu}\mathbf{A}_{\mu}+\mathbf{i}\left[\mathbf{A}_{\mu}(x),\mathbf{A}_{\nu}(x)\right] \quad (16)$$

with the gauge potential. It transforms covariantly as well

$$\mathbf{F}_{\mu\nu}(x)\rightsquigarrow\mathbf{R}(x)\mathbf{F}_{\mu\nu}(x)\mathbf{R}^{\dagger}(x) \quad (17)$$

under the gauge transformation (15). We also define

$$\mathbf{\Delta}(x)=\tau_{+}\Delta(x)-\tau_{-}\Delta^{\dagger}(x) \quad (18)$$

for the gap-parameter matrix in the Nambu space, with $\tau_{\pm}=(\tau_{1}\pm\mathbf{i}\tau_{2})/2$. We remark that we could have included the gap parameter in the $\mathbf{D}_{0}$, but there are difficulties with dealing with non-diagonal covariant derivatives in the particle-hole space, see the end of Section III for more details. Note also that the gap parameter is affected by the gauge transformation as

$$\mathbf{\Delta}(x)\rightsquigarrow\mathbf{R}(x)\mathbf{\Delta}(x)\mathbf{R}^{\dagger}(x) \quad (19)$$

though there is no signature of this in the propagator, since its gauge transformation is absorbed by the correct Green-Gor'kov correlation function when writing the $\mathbf{G}_{0}^{-1}(x_{1})\mathbf{G}(x_{1},x_{2})$ product explicitly. According to our general prescription, we do not write explicit expressions for the gauge fields A_{μ} for the moment. The impatient reader who wants to know why the chemical potential disappeared in (13) can check (28).

A similar calculation for the equation of motion

$$\int dy\left[\mathbf{G}(x_{1},y)\left[\mathbf{G}^{-1}(y,x_{2})\right]^{\dagger}\right]=\delta(x_{1}-x_{2}) \quad (20)$$

gives the same propagator (13) now in its adjoint form $\left[\mathbf{G}^{-1}(y,x_{2})\right]^{\dagger}=\left[\mathbf{G}_{0}^{-1}(x_{2})\right]^{\dagger}\delta(y-x_{2})$. One verifies that

$$\mathbf{G}(x_{1},x_{2})\mathbf{G}_{0}^{-1}(x_{2})\rightsquigarrow \mathbf{R}(x_{1})\mathbf{G}(x_{1},x_{2})\left[\mathbf{G}_{0}^{-1}(x_{2})\right]^{\dagger}\mathbf{R}^{\dagger}(x_{2}) \quad (21)$$

under a gauge transformation. Note that $\mathbf{D}_{\mu}^{\dagger}=\partial_{\mu}-\mathbf{i}\mathbf{A}_{\mu}$ where the derivative applies to the left and the gauge potential is supposed Hermitian. When discussing non-Abelian gauge, the gauge-potential A_{μ} in (14) does not commute with the Green-Gor'kov functions.

We conclude this section with a few words about impurities. They are usually accounted for in a self-energy term, which corresponds to defining [6, 51, 56]

$$\mathbf{G}^{-1}(x_{1},x_{2})=\mathbf{G}_{0}^{-1}(x_{1})\delta(x_{1}-x_{2})-\mathbf{\Sigma}(x_{1},x_{2}) \quad (22)$$

in the equations (5) and (20), respectively.

Equations (5) and (20) are the equations of motion for the Green-Gor'kov correlators in the presence of space-time dependent non-Abelian gauge-field and impurities realization. For the moment we did not precise the gauge structure explicitly. This allow the equations (5) and (20) to be of full generality. Note that the corresponding Bogoliubov-de Gennes [16] and Landau-Ginzburg [9] formalisms can be adapted as well to the non-Abelian gauge interaction: it is sufficient to replace the Abelian covariant derivative with the non-Abelian one in (14). We stress one more time that the generalization toward a non-Abelian formalism was here possible thanks to the invariance of the interaction Hamiltonian (4) under a gauge transformation. In the next section we establish the equations of motion for the gauge fields, and connect them to the Green-Gor'kov functions.

II. GAUGE FIELD: MAXWELL'S AND YANG-MILLS'S EQUATIONS

Since the concept of the non-Abelian gauge theory could be new for a few condensed matter physicists, I discuss it in this section. The general form of the gauge potential and the gauge field for the $U(1)\otimes SU(2)\otimes SU(2)$ gauge redundancy in the Nambu $\otimes$ spin $\otimes$ electromagnetic space is discussed, as well as the associated equations of motion. In particular, the gauge formalism allows to define the charge and current densities associated with the different gauge fields. This section ends up with a discussion of the explicit form of the gauge potential in non-trivial situations, namely when Zeeman and/or spin-orbit interactions are participating to the electron dynamics. Reader familiar with the principles of gauge theory can skip this section.

From the discussion of the above section, one can establish an effective classical Lagrangian density

$$L_{\psi}=\boldsymbol{\psi}^{\dagger}\left[\mathbf{i}\hbar c\mathbf{D}_{0}+\frac{\hbar^{2}}{2m}\mathbf{D}_{j}\mathbf{D}_{j}+\mathbf{\Delta}\right]\boldsymbol{\psi} \quad (23)$$

where $\boldsymbol{\psi}^{\dagger}=\begin{pmatrix}\psi^{\dagger} & \tilde{\psi}\end{pmatrix}$ is a classical spinor in the Nambu space, and eventually ψ, its transpose $\tilde{\psi}$ and its adjoint $\psi^{\dagger}$ are spinors themselves. In the following they will indeed be spinors in the spin-space. Whether the $\boldsymbol{\psi}$ classical spinor field makes sense in the Nambu space or not is of no concern to us: our primary interest is in the construction of the currents in this section. By construction, L_{ψ} is invariant under the gauge transform

$$\boldsymbol{\psi}(x)\rightsquigarrow\mathbf{R}(x)\boldsymbol{\psi}(x)\text{ and }\boldsymbol{\psi}^{\dagger}(x)\rightsquigarrow\boldsymbol{\psi}^{\dagger}(x)\mathbf{R}^{\dagger}(x) \quad (24)$$

for the classical spinor, and (15), (17) and (19) for the gauge potential and field, and the gap matrix $\mathbf{\Delta}$ transformations.

The variation of the action $S_\psi = \int dx\,[L_\psi]$ with respect to the matter-field $\boldsymbol{\psi}$ gives the classical equations of motion $\mathbf{G}_0^{-1}(x)\,\boldsymbol{\psi}(x) = 0$ and its adjoint, with $\mathbf{G}_0^{-1}$ in (6). Next, the variation of the matter-field action S_ψ with respect to the gauge potential gives some current and charge densities. Since the gauge potential $\mathbf{A}_\mu$ is a scalar in the particle-hole space, one has

$$\delta S_\psi = \int dx \left[-\frac{c}{\hbar}\rho_n \delta \mathbf{A}_0 - \frac{j_n^i}{\hbar}\delta \mathbf{A}_i\right] \tag{25}$$

(plus the variations with respect to $\boldsymbol{\psi}$ and $\boldsymbol{\psi}^\dagger$ which are discarded here for commodity) with

$$\rho_n(x) = \boldsymbol{\psi}^\dagger(x)\,\boldsymbol{\psi}(x) \tag{26}$$

for the particle density and

$$j_n^i(x) = \frac{\mathbf{i}\hbar}{2m}\boldsymbol{\psi}^\dagger(x)\left[\mathbf{D}_j^\dagger(x) - \mathbf{D}_j(x)\right]\boldsymbol{\psi}(x) \tag{27}$$

for the particle current density. This current is neutral and conserved: $\partial_t \rho_n + \boldsymbol{\partial_x}\cdot\boldsymbol{j}_n = 0$.

To find the microscopic spin and electric currents, we expand the gauge potential thanks to the representation

$$\mathbf{A}_\mu \equiv \left(\frac{e}{\hbar c}\varphi\tau_3 - \frac{\mu}{\hbar c} + \frac{g}{\hbar c}a_0^i\frac{\mathbf{s}_i}{2}, -\frac{e}{\hbar}A_j\tau_3 - \frac{g}{\hbar}a_j^i\frac{\mathbf{s}_i}{2}\right) \tag{28}$$

where we define

$$\mathbf{s}_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i^* \end{pmatrix} \tag{29}$$

the spin matrix in the Nambu space. All the components φ, A_j and a_μ^i are real. We associate the charges e and g to the electric $U(1)$ and the spin $SU(2)$ gauge fields, respectively, and μ is the chemical potential. One then has

$$\delta S_\psi = \int dx\left[-j_e^i\delta A_i - j_s^{ik}\delta a_i^k - \rho_e\delta\varphi - \rho_s^k\delta a_0^k\right] \tag{30}$$

with the electric charge and current densities

$$\rho_e = e\boldsymbol{\psi}^\dagger\tau_3\boldsymbol{\psi} \text{ and } j_e^i = \frac{\mathbf{i}\hbar e}{2m}\boldsymbol{\psi}^\dagger\left[\mathbf{D}_i\tau_3 - \tau_3\mathbf{D}_i^\dagger\right]\boldsymbol{\psi} \tag{31}$$

and the spin charge and current densities

$$\rho_s^k = \frac{g}{2}\boldsymbol{\psi}^\dagger\mathbf{s}_k\boldsymbol{\psi} \text{ and } j_s^{ik} = \frac{\mathbf{i}\hbar g}{4m}\boldsymbol{\psi}^\dagger\left[\mathbf{D}_i\mathbf{s}^k - \mathbf{s}^k\mathbf{D}_i^\dagger\right]\boldsymbol{\psi} \tag{32}$$

in term of the classical spinor $\boldsymbol{\psi}$. One can nevertheless promote the above expressions to operators formula. It consists in promoting the spinor $\boldsymbol{\psi}$ to the second quantized fields $\mathbf{\Psi}$ and to average the above expressions for the densities to obtain

$$\rho_n = -\mathbf{i}\hbar\lim_{x_1\to x_2}\mathrm{Tr}\left\{\tau_3\mathbf{G}(x_1, x_2+0)\right\}$$

$$j_n^i = \frac{\hbar^2}{2m}\lim_{x_1\to x_2}\mathrm{Tr}\left\{\mathbf{D}_i(x_1)\,\tau_3\mathbf{G}(x_1, x_2+0) - \tau_3\mathbf{G}(x_1, x_2+0)\,\mathbf{D}_i^\dagger(x_2)\right\} \tag{33}$$

$$\rho_e = -\mathbf{i}e\hbar\lim_{x_1\to x_2}\mathrm{Tr}\left\{\mathbf{G}(x_1, x_2+0)\right\}$$

$$j_e^i = \frac{e\hbar^2}{2m}\lim_{x_1\to x_2}\mathrm{Tr}\left\{\mathbf{D}_j(x_1)\,\mathbf{G}(x_1, x_2+0) - \mathbf{G}(x_1, x_2+0)\,\mathbf{D}_j^\dagger(x_2)\right\} \tag{34}$$

$$\rho_s^k = -\mathbf{i}\frac{g}{2}\hbar\lim_{x_1\to x_2}\left[\mathrm{Tr}\left\{\tau_3\mathbf{s}_k\mathbf{G}(x_1, x_2+0)\right\}\right]$$

$$j_s^{ik} = \frac{g\hbar^2}{4m}\lim_{x_1\to x_2}\mathrm{Tr}\left\{\mathbf{D}_i(x_1)\,\tau_3\mathbf{s}_k\mathbf{G}(x_1, x_2+0) - \tau_3\mathbf{s}_k\mathbf{G}(x_1, x_2+0)\,\mathbf{D}_i^\dagger(x_2)\right\} \tag{35}$$

in term of the Green functions (2). We did not introduce a new notation for the averaged densities, since we will only use the expressions (33), (34) and (35) for the neutral particle and current densities, and the electric and spin charge and current densities, respectively.

Note that both the electric (34) and spin (35) currents contain some magneto-electric contributions proportional to ge. This feature is a hallmark of the non-relativistic gauge theory we discuss in this section[3], and opens some interesting perspectives in the manipulation of the quantum state via coherent circuits, as well as in the electromagnetic response in spin textured superconductors [28].

An other important property of the gauge theory is its ability to provide the equations of motion for the gauge field itself. To establish them, we have to complement L_ψ with a Lagrangian density for the gauge field. Thus we represent the gauge fields as

$$\mathbf{F}_{\mu\nu} = \frac{e}{\hbar}F_{\mu\nu} + \frac{g}{\hbar}F_{\mu\nu}^k\frac{\sigma_k}{2} \tag{36}$$

which we inject in the definition (16) to get

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{37}$$

for the gauge field in the charge sector, when $A_\mu \equiv (\varphi/c, -\boldsymbol{A})$ and

$$F_{\mu\nu}^k = \partial_\mu a_\nu^k - \partial_\nu a_\mu^k - \frac{g}{\hbar}\varepsilon_{ijk}a_\mu^i a_\nu^j \tag{38}$$

[3] More precisely this is the result of the second order space (covariant) derivative in the non-relativistic model of Section I.

in the spin sector, with the gauge potential components as in (28), and ε_{ijk} the complete antisymmetric symbol. Then the Lagrangian density

$$L_F = -\frac{1}{4\mu_0} F_{\mu\nu} F_{\mu\nu} - \frac{1}{4} F^k_{\mu\nu} F^k_{\mu\nu} \tag{39}$$

is gauge invariant under the transformation (15). This can be easily verified by noting that (39) can be written as some traces of (36) and using the transformation law (17), for more details see [54]. Next the variation of the total action $S = \int dx \left[L_\psi + L_F \right]$ with respect to the gauge-potential gives the usual Maxwell's equations [54]

$$\partial_\mu F_{\mu\nu} = \mu_0 J_\nu \tag{40}$$

and the so-called Yang-Mills's equations [57]

$$\partial_\mu F^k_{\mu\nu} - \frac{g}{\hbar} \varepsilon_{ijk} a^i_\mu F^j_{\mu\nu} = J^k_\nu \tag{41}$$

for the equation of motion of the gauge-fields, with $J^\mu \equiv (c\rho_e, \boldsymbol{j}_e)$ and $J^\mu_k \equiv \left(c\rho^k_s, \boldsymbol{j}^k_s \right)$ the quadri-currents for charge and spin, respectively. Note that the equations of motion (41) are non-linear in term of the gauge potential, due to (38).

To illustrate the gauge formalism, let me give some examples of the gauge-potentials in simple systems. Obviously when there is no spin interaction, there is no need for a non-Abelian gauge potential. Yet injecting the Abelian potential as above is the usual way to generate the interaction between electric charge and fields, at both the classical and quantum level [54]. Next, a Zeeman effect usually appears as the additional $L_{\mathrm{Z}} = -h^i(x)\sigma_i$ term in the Lagrangian density for the otherwise free particles. Then it can be absorbed as a gauge-potential $a^i_0 = h^i(x)$ with $g = 2/\hbar c$ and all the other gauge-potentials are zero. Another example is the case of spin-orbit interactions of the Rashba type. These are usually found from the study of the band structure and some symmetry arguments [58], and they appear generically as linear terms in the momentum $L_{\mathrm{s.o.}} = -\alpha_{ij}(x) p_i \sigma_j$ with some tensor α_{ij} eventually depending in space. In that case, we convert the momentum operator $p_j = -\mathbf{i}\hbar\partial_j$ in the space representation, and $L_{\mathrm{s.o.}}$ can be written as a gauge-potential in the space-sector: $a^j_i = \alpha_{ij}$ with $g = -m/\hbar$ whereas the Abelian gauge-potential reads $\varphi = \alpha^2_{ij}$ with a charge $e = -m/\hbar c$. Obviously, there are freedom in the choice of the charge and the gauge-potential. Above we gave the natural notations, when one restores the usual electromagnetism in the Abelian sector. Contrary to the Abelian situation when a space-time independent gauge-potential leads to the trivial situation without gauge-field, the non-Abelian gauge-potential can be space-time independent, yet the associated gauge-field is non-zero because of the commutator in the definition (16). For instance, suppose α_{ij} to be space-time independent, then $F^k_{ij} = (m/\hbar)^2 \alpha_{im}\alpha_{jn}\epsilon^{mnk}$; note $F_{ij} = 0$ when α_{ij} has only one non-zero value and the one-dimensional spin-orbit problem appears trivial in the gauge formalism. More complicated spin-orbit interaction, like the Dresselhaus one which scales as a cubic momentum [58] will not be discussed in our gauge formalism since we started from the non-relativistic and free quasi-particle model, see (3). In that case, the Fermi surface is isotropic and only linear-in-momentum spin-orbit interaction – *i.e.* Rashba-like – can be described in a gauge covariant way. The following calculations could nevertheless be extended to higher order derivatives in principle, see Section A.

When using the Green functions representations (34) and (35) on their right-hand-side, (40) and (41) constitute a self-consistent set of equations of motion for the gauge-potentials, up to the gauge redundancy. With the Dyson-Gor'kov equations of motion (5) and (20) for the Green-Gor'kov functions, they constitute a *closed* system of non-linear coupled equations of motion, which could serve as a basic set of equations for the study of magnetic superconductivity. Instead of venturing in the perilous – and certainly impossible – task to solve the above system, we will reduce the complexity of the Dyson-Gor'kov equations in the next sections. The strategy is to write some quasi-classical expansion for the Dyson-Gor'kov equations, which will then look like some transport equations, perhaps easier to solve. At least we will cure the Green-Gor'kov formalism from its intrinsic difficulty to deal with the evolution of some two-points correlation functions.

III. TRANSPORT EQUATIONS

We have seen in Section I and Section II that the electronic spin degree of freedom can be properly described in terms of a (non-Abelian) gauge theory. In this formalism, one associates a curvature – the gauge field – with the spin space. To simplify the Dyson-Gor'kov equations, a usual procedure is to transform the two-point correlators in the real space to some correlators in the phase-space, via the Wigner transformation [52, 53]. Nevertheless, the curvature in the spin space alters the Wigner transformation: we need a correct transformation of the covariant derivative. This transformation is pretty lengthy and is given in Section A, in addition to some general recipes for the transformation of the equations of motion. We here introduce the gauge-covariant Wigner transformation, and discuss it at the quasi-classical level. Then we invoke the results of Section A, and we derive a transport-like equation for the quasi-classical propagator in the phase-space.

The Wigner transformation of the Green function $G(x_1, x_2)$ (also called the mixed-Fourier transformation)

is defined as

$$G\left(p,x\right)=\int dz\left[e^{-\mathbf{i}p\cdot z/\hbar}G\left(x-z/2,x+z/2\right)\right] \tag{42}$$

where $p\cdot z=p_{\mu}z^{\mu}=Et-\boldsymbol{p}\cdot\boldsymbol{x}$ in space-time (see the beginning of Section I). To simplify the discussion, we discuss here a generic Green function, not necessarily the Green-Gor'kov ones introduced in Section I. We do not introduce a different notation for the Green function $G\left(p,x\right)$ in the phase-space and the correlation function $G\left(x_{1},x_{2}\right)$ in the real space, since the names of their variables are sufficient to distinguish them. The above definition is obviously not gauge-covariant, since the Green function transforms as $G\left(x_{1},x_{2}\right)\rightsquigarrow R\left(x_{1}\right)G\left(x_{1},x_{2}\right)R^{\dagger}\left(x_{2}\right)$ and the transformations matrices R are not compensated. We need a way to get read off the $x_{1,2}$ dependency of the gauge transformation of the $G\left(x_{1},x_{2}\right)$ function. This is done when one slightly generalizes (42) toward a gauge-covariant Wigner transformation, as we discuss in the next few paragraphs.

The gauge-covariant Wigner transformation has a long and rich history, and appeared in several places and for different purposes [59–63]. Most of the studies are devoted to the Abelian gauge theory, when the gauge-field is supposed classical [64–66] or quantized [63, 67]. To the best of my knowledge, only a few studies are devoted to the non-Abelian problem of finding a correct gauge-covariant Wigner transformation [46, 62, 68, 69], and none of them address the question of non-relativistic systems. We here follow the approach of Elze, Gyulassy and Vasak [62] who rewrite the Wigner transformation as

$$G\left(p,x\right)=\int dz\left[e^{-\mathbf{i}p\cdot z/\hbar}e^{-z\cdot\partial/2}G\left(x,x\right)e^{z\cdot\partial^{\dagger}/2}\right] \tag{43}$$

where $e^{-z\cdot\partial/2}\Psi\left(x\right)=\Psi\left(x-z/2\right)$ and the same for the derivative $\partial^{\dagger}$ applied to the left on $\Psi^{\dagger}$, where the fields Ψ are the fermionic particle-field operators defining the Green function. If one defines $x_{1,2}=x\mp z/2$, the definitions (42) and (43) are equivalent[4]. Additionally, a gauge-covariant Wigner transformation would simply be deduced from (43) by the substitution of the usual derivatives with the covariant ones. Then one defines

$$G\left(p,x\right)=\int dz\left[e^{-\mathbf{i}p\cdot z/\hbar}e^{-z\cdot D/2}G\left(x,x\right)e^{z\cdot D^{\dagger}/2}\right] \tag{44}$$

as a gauge-covariant Wigner transformation [62], with a generic covariant derivative $D_{\mu}=\partial_{\mu}+\mathbf{i}A_{\mu}$ for the moment. In a few paragraphs we will come back to the superconductors, and its bold notations. By definition of D and $D^{\dagger}$, we have $G\left(p,x\right)\rightsquigarrow R\left(x\right)G\left(p,x\right)R^{\dagger}\left(x\right)$ under a gauge transformation, the property we were looking for. When the gauge-field is trivial, the definition (44) obviously reduces to (43), and so we should adopt (44) as the most general definition for the gauge-covariant Wigner transformation [62]. Nevertheless, it is important to realize that the Wigner transformation (44) has nothing to do with a Fourier transformation anymore, except for a trivial gauge-field, when (44) reduces to (42).

Next one has the property demonstrated in [62]

$$e^{-z\cdot D/2}\Psi\left(x\right)=U\left(x,x-z/2\right)\Psi\left(x-z/2\right) \tag{45}$$

with

$$U\left(b,a\right)=\hat{P}\exp\left[-\mathbf{i}\left(b-a\right)^{\mu}\int_{0}^{1}ds\left[A_{\mu}\left(\tau_{s}\right)\right]\right] \tag{46}$$

the parallel transport operator *along a straight line* $\tau_{s}=a+\left(b-a\right)s$ parameterized by s. The operator $\hat{P}$ orders the path. Injecting the definition (46), one rewrites (44) as

$$G\left(p,x\right)=\int dz\left[e^{-\mathbf{i}p\cdot z/\hbar}U\left(x,x_{1}\right)G\left(x_{1},x_{2}\right)U\left(x_{2},x\right)\right] \tag{47}$$

in a mixed notation in term of (x,z) and $(x_{1}=x-z/2,x_{2}=x+z/2)$ for compactness purpose. The main advantage of promoting (44) instead of (47) as the genuine definition of the gauge-covariant Wigner transformation is because (44) is independent of the path chosen to link the different points x and $x_{1,2}$, whereas there always is an ambiguity in the notation (47). Because of the definition of the covariant derivative the paths connecting the points x_{1} to x and from x to x_{2} are straight lines, as demonstrated and discussed in [62, 64]. Note that the expression (47) was used by Gorini *et al.* [44] as a heuristic definition for a gauge-covariant Wigner transformation, where they choose a straight line as the simplest realization of the path connecting the points x to x_{1} or x_{2}.

Suppose for a while that A_{μ} describes a non-trivial Abelian gauge-field instead of the more elaborated situation of a non-Abelian problem. Then A_{μ} commutes with everything, and the definition (46) reduces to a phase-shift which commutes with the Green correlation function in (47). In that case the two phase shifts $U\left(x,x_{1}\right)$ and $U\left(x_{2},x\right)$ combine in a resulting Abelian phase shift

$$U_{\text{Abel.}}\left(x_{1},x_{2}\right)=e^{-\mathbf{i}z^{\mu}\int_{0}^{1}ds[A_{\mu}(x+z(s-1)/2)]} \tag{48}$$

where the path connects now the points x_{1} and x_{2}. This Abelian phase shift has been used in the description of a gauge-invariant Wigner transformation when only electromagnetism is taken into account [63–67]. The above

[4] In practice, we should include the displacement operators $e^{-z\cdot\partial/2}$ inside the averaging brackets in the definition of the Green functions: $G\left(p,x\right)=\int dz\left[e^{-\mathbf{i}p\cdot z/\hbar}\left\langle\hat{T}\left[e^{-z\cdot\partial/2}\Psi\left(x\right)\right]\left[e^{z\cdot\partial/2}\Psi\left(x\right)\right]^{\dagger}\right\rangle\right]$ in order to properly define the Wigner transformation of the Green function. This more rigorous definition nevertheless makes the notations cumbersome, the reason why we adopt the notations in (43).

Abelian phase shift is also sometimes heuristically introduced in order to obtain some gauge-covariant Wigner transformation for both the normal metal [60] and the superconductor [51] situations. Note that $U_{\text{Abel.}}(x_1,x_2)$ is gauge *in*variant, whereas the Wigner transformation (47) is gauge *co*variant. This is the main difference between Abelian and non-Abelian gauge theory: in the later case, there is no gauge invariant quantity in the theory, only the observables which trace out the gauge degrees of freedom are gauge invariant, see also below the construction of the current densities in (63), (64) and (65), and more general literature on this subject [70].

Equipped with the above gauge-covariant Wigner transformation (47), we can now come back to the problem of the obtention of the transport equations for a non-Abelian superconducting plasma. The strategy is to start with the Dyson equations of motion (5) and (20) and the propagator (13). Then we transform the two Dyson equations according to the Wigner transformation

$$\mathbf{G}(p,x)=\int dz\left[e^{-\mathbf{i}p\cdot z/\hbar}\mathbf{U}(x,x_1)\,\mathbf{G}(x_1,x_2)\,\mathbf{U}(x_2,x)\right] \tag{49}$$

with

$$\mathbf{U}(b,a)=\hat{P}\exp\left[-\mathbf{i}(b-a)^{\mu}\int_0^1 ds\left[\mathbf{A}_{\mu}(\tau_s)\right]\right] \tag{50}$$

the parralel transport in the $U(1)\otimes SU(2)\otimes SU(2)$ space, when the gauge-potential $\mathbf{A}_{\mu}$ is defined in (28). The propagator (13) contains the spin texture in a covariant manner, and the associated curvature is properly taken into account in the definition (49). The covariant derivatives in $\mathbf{G}^{-1}(x)$ can be transformed according to the set of rules found in Section A. Since the gauge-potential $\mathbf{A}_{\mu}$ is diagonal in the Nambu space, it commutes with τ_3, as well as the gauge-field (16). One then immediately has that

$$\int dz\left[e^{-\mathbf{i}p\cdot z/\hbar}\mathbf{U}\left(\tau_3\mathbf{D}_0\mathbf{G}(x_1,x_2)\right)\mathbf{U}\right]= \tau_3\int dz\left[e^{-\mathbf{i}p\cdot z/\hbar}\mathbf{U}\left(\mathbf{D}_0\mathbf{G}(x_1,x_2)\right)\mathbf{U}\right] \tag{51}$$

which greatly simplifies the following treatment. The parallel transport operators $\mathbf{U}(x,x_1)$ on the left and $\mathbf{U}(x_2,x)$ on the right always have the same space-time dependencies, so we do not write them explicitly. Finally, the pair potential is treated as a conventional potential in the $SU(2)\otimes SU(2)\otimes U(1)$ space, according to the general recipe

$$\int dz\left[e^{-\mathbf{i}p\cdot z/\hbar}\mathbf{U}\left(\mathbf{M}(x_1)\,\mathbf{G}(x_1,x_2)\right)\mathbf{U}\right]= \mathbf{U}(x,x-\mathbf{i}\hbar\partial_p/2)\,\mathbf{M}(x-\mathbf{i}\hbar\partial_p/2)\,\mathbf{U}(x-\mathbf{i}\hbar\partial_p/2,x)\,\mathbf{G} \approx\mathbf{M}(x)\,\mathbf{G}(p,x)-\mathbf{i}\frac{\hbar}{2}\mathfrak{D}_{\nu}\mathbf{M}(x)\,\partial_p^{\nu}\mathbf{G}(p,x) \tag{52}$$

at first order in $\hbar$, where we defined the covariant derivative

$$\mathfrak{D}_{\nu}\mathbf{M}=\partial_{\nu}\mathbf{M}+\mathbf{i}\left[\mathbf{A}_{\nu},\mathbf{M}\right] \tag{53}$$

applied to any matrix $\mathbf{M}\in SU(2)\otimes SU(2)\otimes U(1)$. We injected some $\mathbf{U}(x_1,x)\,\mathbf{U}(x,x_1)=1$ in the Wigner transformation (52), thanks to the straight path convention in (46). A similar calculation gives

$$\int dz\left[e^{-\mathbf{i}p\cdot z/\hbar}\mathbf{U}\left(\mathbf{G}(x_1,x_2)\,\mathbf{M}(x_2)\right)\mathbf{U}\right]= \approx\mathbf{G}(p,x)\,\mathbf{M}(x)+\mathbf{i}\frac{\hbar}{2}\partial_p^{\nu}\mathbf{G}(p,x)\,\mathfrak{D}_{\nu}\mathbf{M}(x) \tag{54}$$

when the potential is applied on the second variable from the right.

Taking the difference and the sum of the Dyson equations (5) and (20), we finally have:

$$\begin{aligned}&\frac{\mathbf{i}\hbar c}{2}\left[\tau_3,\mathfrak{D}_0\mathbf{G}\right]_{+}+\hbar\omega\left[\tau_3,\mathbf{G}\right]_{-}+\mathbf{i}\hbar v^i\mathfrak{D}_i\mathbf{G}(p,x)\\&\quad+\left[\mathbf{\Delta}(x),\mathbf{G}\right]_{-}+\mathbf{i}\frac{\hbar}{2}\left[\mathfrak{D}_{\mu}\mathbf{\Delta},\partial_p^{\mu}\mathbf{G}\right]_{+}\\&+\frac{\mathbf{i}\hbar}{8}\left[\mathbf{F}_{i0}\left(3\tau_3\partial_p^i\mathbf{G}+\partial_p^i\mathbf{G}\tau_3\right)+\left(\tau_3\partial_p^i\mathbf{G}+3\partial_p^i\mathbf{G}\tau_3\right)\mathbf{F}_{i0}\right]\\&\quad+\mathbf{i}\frac{\hbar}{2}v^i\left[\mathbf{F}_{\mu i}(x),\partial_p^{\mu}\mathbf{G}\right]_{+}-\left(\mathbf{I}_{+}-\mathbf{I}_{-}\right)=0\end{aligned} \tag{55}$$

$$\begin{aligned}&\frac{\mathbf{i}\hbar c}{2}\left[\tau_3,\mathfrak{D}_0\mathbf{G}\right]_{-}+\hbar\omega\left[\tau_3,\mathbf{G}\right]_{+}-\frac{p^2}{m}\mathbf{G}(p,x)\\&\quad+\left[\mathbf{\Delta}(x),\mathbf{G}\right]_{+}-\mathbf{i}\frac{\hbar}{2}\left[\mathfrak{D}_{\mu}\mathbf{\Delta},\partial_p^{\mu}\mathbf{G}\right]_{-}\\&+\frac{\mathbf{i}\hbar}{8}\left[\mathbf{F}_{i0}\left(3\tau_3\partial_p^i\mathbf{G}-\partial_p^i\mathbf{G}\tau_3\right)+\left(\tau_3\partial_p^i\mathbf{G}-3\partial_p^i\mathbf{G}\tau_3\right)\mathbf{F}_{i0}\right]\\&\quad+\mathbf{i}\frac{\hbar}{4}v^i\left[\mathbf{F}_{\mu i}(x),\partial_p^{\mu}\mathbf{G}\right]_{-}-\left(\mathbf{I}_{+}+\mathbf{I}_{-}\right)=2\end{aligned} \tag{56}$$

at first order in $\hbar$. We defined $v^i=p^i/m$ a velocity, and $[A,B]_{\pm}=AB\pm BA$ define the (anti-)commutator. The above transport-like equation (55) is the quasi-classical equation for superconductors in the presence of non-Abelian gauge-fields. The sum-equation (56) helps when discussing the quasi-classical correlation function $\mathbf{G}(p,x)$ and its quantum corrections. The terms

$$\mathbf{I}_{+}(p,x)=\int dz\int dy e^{-\mathbf{i}p\cdot z/\hbar}\times \left[\mathbf{U}(x,x_1)\,\mathbf{\Sigma}(x_1,y)\,\mathbf{G}(y,x_2)\,\mathbf{U}(x_2,x)\right] \tag{57}$$

$$\mathbf{I}_{-}(p,x)=\int dz\int dy e^{-\mathbf{i}p\cdot z/\hbar}\times \left[\mathbf{U}(x,x_1)\,\mathbf{G}(x_1,y)\,\mathbf{\Sigma}^{\dagger}(y,x_2)\,\mathbf{U}(x_2,x)\right] \tag{58}$$

correspond to the impurities scattering terms. They are generically called collision integrals. Here, they are gauge-covariant by construction.

The difference (55) and sum (56) equations are obviously covariant with respect to the gauge-transformation

$$\begin{aligned}\mathbf{G}(p,x) &\rightsquigarrow \mathbf{R}(x)\,\mathbf{G}(p,x)\,\mathbf{R}^{\dagger}(x)\\ \mathbf{A}_{\mu}(x) &\rightsquigarrow \mathbf{R}(x)\,\mathbf{A}_{\mu}(x)\,\mathbf{R}^{\dagger}(x)-\mathbf{i}\mathbf{R}(x)\,\partial_{\mu}\mathbf{R}^{\dagger}(x)\\ \mathbf{\Delta}_{\mu\nu}(x) &\rightsquigarrow \mathbf{R}(x)\,\mathbf{\Delta}_{\mu\nu}(x)\,\mathbf{R}^{\dagger}(x)\end{aligned} \tag{59}$$

since $\mathbf{R}$ and τ_3 commute. Note that the gauge-transformation for the quasi-classical Green function is local, in comparison with (10). This is due to the definition of the gauge-covariant Wigner transformation (47).

The Abelian version of these equations reduces to the usual ones [47, 48, 51, 71, 72]. Since $\mathbf{A}_{\mu}$ is diagonal and A_{μ} is real in the Abelian case, the Abelian limit corresponds to $\mathbf{F}_{\mu\nu}=\tau_3 F_{\mu\nu}$ and $F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}$ commutes with $\mathbf{G}$. Note in this case that the covariant derivative $\mathfrak{D}_{\mu}$ still contains a non-trivial gauge potential part, responsible for the asymmetry between the $F(p,x)$ and the $G(p,x)$ sectors: the space derivative reads $(\boldsymbol{\partial_x}\pm 2\mathbf{i}e\boldsymbol{A}/\hbar)$ in front of the anomalous correlation sector, whereas the $G(p,x)$ correlation function becomes uncharged. Supposing further the absence of gauge field reduces the above equations to the usual transport equations for superconductors, when all the gauge fields vanish, and the covariant derivatives reduce to the usual derivative [51, 73].

The normal metal limit consists in projecting (55) and (56) to the particle sector, *i.e.* choosing $\tau_3=1$ and $\mathbf{\Delta}=0$. We then recover the non-Abelian case [44]. Supposing a pure Abelian gauge field reduces further to the usual transport equation [63, 65–67, 74, 75]. Interestingly enough, the covariant derivatives are reduced to the usual derivatives in this case. Finally, the standard transport equations are recovered when we suppose in addition that the gauge fields disappear [22, 49].

The equations (55) and (56) constitute the fundamental result of this study. They are strictly equivalent to the Dyson-Gor'kov equations (5) and (20) in the low energy sector, characterized by the relation

$$\frac{\hbar}{\tilde{p}\tilde{x}}\ll 1 \tag{60}$$

where $\tilde{p}$ and $\tilde{x}$ stand for the characteristic values of the momentum and the position, respectively. Additionally, the variations of the momentum and/or the position must rely on the approximation (60) which then defines the quasi-classic evolution.

We still have to define the observables associated with the transport equations. The microscopic quantities (33), (34) and (35) are all evaluated in the limit $x_1\to x_2$. Since it corresponds to the limit $z\to 0$, the Wigner transformation (47) is not well defined, and we have to find a work-around to obtain the expressions for the charges and current densities. According to the general recipe in the Abelian case [22], we can suppose the density to be the integrated version of the phase-space density $\mathbf{G}(p,x)$ over the momentum. Then we propose to define

$$\begin{aligned}\rho_n(x) &= \hbar\int\frac{dp}{2\pi\hbar}\mathrm{Tr}\left\{\tau_3\mathbf{G}(p,x)\right\}\\ \rho_e(x) &= e\hbar\int\frac{dp}{2\pi\hbar}\mathrm{Tr}\left\{\mathbf{G}(p,x)\right\}\\ \rho_e(x) &= \frac{g}{2}\hbar\int\frac{dp}{2\pi\hbar}\mathrm{Tr}\left\{\mathbf{s}_k\tau_3\mathbf{G}(p,x)\right\}\end{aligned} \tag{61}$$

for the particle, electric and spin densities, respectively, where we used the notation $dp/2\pi\hbar\equiv dp_x dp_y dp_z d\omega/(2\pi)^4\hbar^3 c$, such that the proposed densities $\rho_{n,e,s}$ depend only on space-time. Our strategy is to manipulate the transport equation (55) in order to obtain the conservation laws $\partial_t\rho_{n,e,s}+\boldsymbol{\partial_x}\cdot\boldsymbol{j}_{n,e,s}=0$, then we identify the conserved current as the correct particle, electric and spin currents, respectively. Before turning to this program, we define

$$\Delta_0(x)=-\hbar V_0(x)\int\frac{dp}{2\pi\hbar}\mathrm{Tr}\left\{\mathbf{i}\sigma_2\tau_+\mathbf{G}(p,x)\right\} \tag{62}$$

for the self-consistent relation (compare with (8)). Next we take the trace of (55), which cancels most of the terms. Then we integrate over the momentum, which cancels all the gauge field terms and the collision integrals. We are left with the integrated trace of $\left(c\mathfrak{D}_0\tau_3+v^i\mathfrak{D}_i\right)\mathbf{G}(p,x)$. The commutators in the covariant derivatives cancel thanks to the cyclic properties of the trace, and we obtain the desired continuity equation, with the conserved current

$$j_n^i=\hbar\int\frac{dp}{2\pi\hbar}\mathrm{Tr}\left\{v^i\mathbf{G}(p,x)\right\} \tag{63}$$

for the quasiparticles current density. A similar calculation after replacement of $\mathbf{G}(p,x)$ by $\tau_3\mathbf{G}(p,x)$ in (55) leads to the conserved current

$$j_e^i=e\hbar\int\frac{dp}{2\pi\hbar}\mathrm{Tr}\left\{\tau_3 v^i\mathbf{G}(p,x)\right\} \tag{64}$$

for the electric current density. To obtain the spin current, we replace $\mathbf{G}(p,x)$ by $\mathbf{s}_k\mathbf{G}(p,x)$ in (55), we take the trace and we integrate over the momentum. Nevertheless, the gap terms remain. They cancel in virtue of the relation (62) and the cyclic property of the trace. We are left with

$$j_s^{ik}=\hbar\frac{g}{2}\int\frac{dp}{2\pi\hbar}\mathrm{Tr}\left\{\mathbf{s}_k v^i\mathbf{G}(p,x)\right\} \tag{65}$$

for the spin current density. Relations (61), (63), (64) and (65) are the observables associated with the equations of motion (55) and (56). Injected in the relations (40) and (41) they constitute a complete set of

self-consistent equations for the superconducting plasma, provided we use the self-consistency relation (62) and the definitions (37) and (38) in addition to the gauge redundancy (59). In comparison with the normal metal when only electromagnetism is present, one realizes that the gauge potential appears alongside the gauge field in the transport equation for the superconductors, a clear hallmark of their quantum behavior. The non-Abelian generalization we provided here even enriches this picture, with possible magneto-electric couplings and non-trivial boundary conditions at the interface between different devices.

Let us now discuss the inclusion of impurities into the transport equation in term of the self-energy via the substitution (22). The explicit form of the self-energy depends on the materials one wants to describe, and on the approximation one develops for it. For a disorder weakly coupled to the particle trajectories and randomly distributed along the sample, the Born approximation might be sufficient. When the disorder is also isotropic, the self-energy term becomes constant in momentum [6, 51]

$$\mathbf{\Sigma}\left(\omega,x\right)=\frac{\hbar}{2\pi N_0^d\tau c}\int\frac{d^d\boldsymbol{p}}{\left(2\pi\hbar\right)^d}\mathbf{G}\left(p,x\right) \tag{66}$$

with N_0 the density of particles in the normal state which depends on the space dimension d, and τ the mean free time. We will not explore the effect of the disorder beyond this simple model in the following.

To conclude this section, we remark that an alternative way toward the quasi-classical superconducting equations is to construct some propagator

$$\mathbf{G}^{-1}\left(x\right)=\mathbf{i}\hbar\tau_3\mathbf{\Pi}_0\left(x\right)-\frac{\hbar^2}{2m}\mathbf{D}_j\mathbf{D}^j\left(x\right) \tag{67}$$

instead of (13), with an alternative covariant derivative in the time sector

$$\mathbf{\Pi}_0=\partial_0+\mathbf{i}\tau_3\begin{pmatrix}A_0 & \Delta\\ \Delta^\dagger & A_0^*\end{pmatrix}=\partial_0+\mathbf{i}\mathfrak{B}_0\left(x\right) \tag{68}$$

where the pair potential has been promoted to be a gauge potential in the time sector of the Nambu space. Working with the $\mathbf{\Pi}_0$ operator, one has to define a parallel displacement as

$$\tilde{\mathbf{U}}\left(b,a\right)=\hat{P}\exp\left[-\mathbf{i}\int_a^b dz^\mu\mathfrak{B}_\mu\left(z\right)\right] \tag{69}$$

with $\mathfrak{B}_\mu\equiv\left(\mathfrak{B}_0,-\mathbf{A}_j\right)$, and we have to use an expression like

$$\begin{aligned}\int dz\left[e^{-\mathbf{i}\boldsymbol{p}\cdot z/\hbar}\tilde{\mathbf{U}}\left(\tau_3\mathbf{\Pi}_0\left(x_1\right)\mathbf{G}\right)\tilde{\mathbf{U}}\right]&=\\ \tilde{\mathbf{U}}\left(x,x-\mathbf{i}\hbar\partial_p\right)\tau_3\tilde{\mathbf{U}}\left(x-\mathbf{i}\hbar\partial_p,x\right)&\\ \times\int dz\left[e^{-\mathbf{i}\boldsymbol{p}\cdot z/\hbar}\tilde{\mathbf{U}}\left(\mathbf{\Pi}_0\left(x_1\right)\mathbf{G}\left(x_1,x_2\right)\right)\tilde{\mathbf{U}}\right]&\end{aligned} \tag{70}$$

which is more complicated to deal with than the convention (49) we used before, since the operator τ_3 does not commute with the parallel displacement operator $\tilde{\mathbf{U}}$ in (69) anymore. Even in the quasi-classical limit, the resulting transport equations will not look like (55) and (56), since now the gap parameter has the property of a gauge potential, and accordingly transforms like $\mathfrak{B}_0\rightsquigarrow\mathbf{R}\mathfrak{B}_0\mathbf{R}^\dagger-\mathbf{i}\mathbf{R}\partial_{ct}\mathbf{R}^\dagger$ under a gauge transformation. This property may have interesting consequences – especially for the symmetry classification of superconducting states for instance – that we keep for future studies. Note also that a further generalization of the gap parameter could make possible its inclusion as some gauge potential in the space sector (some terms in the $\mathbf{\Pi}_i=\partial_i-\mathbf{i}\mathfrak{B}_i\left(x\right)$ which are absent in our present construction), which seems to take into account higher symmetries of the gap parameter (p-wave for instance), see *e.g.* [76] for the usual treatment of such symmetries. This hypothesis is far beyond the scope of this introductory study.

IV. EILENBERGER EQUATION

In this section, we simplify even more the equation of motion for the quasi-classical Green functions, towards the so-called Eilenberger equation, here generalized to include non-Abelian gauge interactions. Reader familiar with the usual derivation of the quasi-classical equation [18, 19, 48, 51, 56] for superconductors can just have a look on the expression (76) and skip the remaining of this section.

The transport equation (55) was valid at first order in $\hbar/\tilde{p}\tilde{x}$, where $\tilde{p}$ and $\tilde{x}$ are characteristic values for the momentum and the space variations. The characteristic values for a superconductor are the Fermi momentum p_F and the coherence length ξ_0, verifying

$$\frac{\hbar}{p_F\xi_0}\sim\frac{\lambda_F}{\xi_0}\sim\frac{\Delta}{E_F}\ll 1 \tag{71}$$

in most of the cases. This means that, for a description in space with resolution ξ_0 at best, one can content ourself with fixing the momentum to be the Fermi one in (55). Thus, the amplitude of the momentum is pinned to the Fermi surface, and we could forget all the momentum derivatives in the transport equations (55) and (56). Nevertheless, the angular dependency of the momentum is still free in principle. For instance, suppose a two-dimensional and circular Fermi surface (a Fermi circle then), we decompose $\boldsymbol{p}=p_F\hat{\boldsymbol{p}}_{\boldsymbol{F}}+p_\varphi\hat{\boldsymbol{\varphi}}$, with unit radial $\hat{\boldsymbol{p}}_{\boldsymbol{F}}$ and tangential $\hat{\boldsymbol{\varphi}}$ vectors. Next, the gradient in the momentum space reads $\boldsymbol{\partial_p}=\hat{\boldsymbol{p}}_{\boldsymbol{F}}\boldsymbol{\partial}_{\hat{\boldsymbol{p}}}+p_F^{-1}\hat{\boldsymbol{\varphi}}\boldsymbol{\partial_\varphi}$ and we suppose that the variation along the radial direction $\hat{\boldsymbol{p}}_{\boldsymbol{F}}$ vanishes. Then we note that the contribution $\hbar/p_F\ll\xi_0$ is small, and the radial derivative should be neglected as well. This situation is generic, and valid for

three-dimensional problems as well as for non symmetrical Fermi surfaces, see [51] for longer discussions.

So all the momentum derivatives (55) are of order of magnitude at least $\hbar p_F^{-1}$ or even higher, and we are left with

$$\frac{\mathbf{i}\hbar c}{2}\left[\tau_3,\mathfrak{D}_0\mathbf{G}\right]_+ + \hbar\omega\left[\tau_3,\mathbf{G}\right]_- + \mathbf{i}\hbar v_F^i\mathfrak{D}_i\mathbf{G} + \left[\mathbf{\Delta}\left(x\right),\mathbf{G}\right]_- = \mathbf{I}_+ - \mathbf{I}_- \quad (72)$$

for the transport equation with relevant energies at the Fermi level. The phase-space dependency now is confined to the Fermi surface in momentum, whereas the frequency is well below the superconducting gap. The gauge-potentials should be of low energy, so their characteristic length should be larger than the coherence one, too. The collision integrals will be discussed later.

Usually one replaces $p/m \approx v_F$, the Fermi velocity, in front of the space derivative, as we naively did in (72). Here one might wonder whether the Fermi surface is a well defined quantity. Indeed, it is well known that adding a spin-orbit and/or a Zeeman effect splits the Fermi surface in two sheets [58]. Nevertheless, the gauge theory is an extension of the free particles model (see (13)), for which the Fermi surface has a single sheet. So the genuine Fermi surface in (72) is the conventional one defined for the free-particles. Despite the absence of a Fermi surface *per se* for superconductors, one can still replace $p/m \approx v_F$ with v_F the Fermi velocity for the free particles defined in the absence of the Cooper pairing. The only momentum dependency which remains is the angular one.

Still because the gap parameter is weak in comparison with the Fermi energy, all the characteristic energies in (72) will be close to the Fermi energy. Then we can define a renormalized quasi-classical function, the so-called ξ-integrated Green function

$$\mathbf{g}\left(\Omega_F,\omega,x\right) = \int \frac{d\xi_p}{\mathbf{i}\pi c}\mathbf{G}\left(p,x\right) \quad (73)$$

with $d\xi_p = v_F dp$, the increment of the linear variation of the energy relative to the Fermi one: $\xi_p = v_F\left(p - p_F\right)$, see [6, 51] for more details about the quantity ξ_p. Note that $\mathbf{g}$ still depends on the solid angle Ω_F in the momentum space at the Fermi surface. In a sense, $\mathbf{g}$ corresponds to the low-energy sector of the quasi-classical Green function, when the high-energies have been integrated out. It sometimes requires some care to explicitely make this integration, see *e.g.* [48, 51]. Once ξ-integrated, the equation (72) is called the Eilenberger equation [18, 19, 51]. The transposition of (72) toward the ξ-integrated representation of the quasi-classical Green function is straightforward, except for the collision integral which we discuss now separately.

If we suppose the disorder to be weakly and isotropically interacting with the electrons and randomly distributed along the sample, a convenient approximation to describe it is the Born approximation (66). There, we substitute $d^3p/\left(2\pi\hbar\right)^3 \approx N_0^{3d}d\xi_p d\Omega_p/4\pi$ in 3D, $d^2p/\left(2\pi\hbar\right)^2 \approx N_0^{2d}d\xi_p d\Omega_p/2\pi$ in 2D or $dp/2\pi\hbar = N_0^{1d}d\xi_p$ in 1D, with Ω_p the solid-angle in the momentum space and $N_0^{3d} = mp_F/2\pi^2\hbar^3$, $N_0^{2d} = m/2\pi\hbar^2$ or $N_0^{1d} = m/p_F 2\pi\hbar$ the density of state in the normal metal in 3D, 2D and 1D, respectively. In the following we treat the 3D case and note it N_0 since the substitution are straightforward for the other dimensions. The integral in (66) then reduces to

$$\mathbf{\Sigma}\left(x,\omega\right) = \frac{\mathbf{i}\hbar}{2\tau}\left\langle\mathbf{g}\right\rangle \quad (74)$$

where $\langle\cdots\rangle$ stands for the averaging of the quasi-classical ξ-integrated functions over the Fermi surface, spanned by the increment $d\Omega_p$:

$$\left\langle\mathbf{g}\right\rangle = \int \frac{d\Omega_p}{4\pi}\mathbf{g}\left(\Omega_F,\omega,x\right) \quad (75)$$

and so on for 2D and 1D, where the average is just the sum over two contributions.

We can then integrate the equation (72) over the energies $d\xi_p$. Since the self-energy is already ξ-integrated by virtue of the relation (74) and more generally by the use of the Born approximation, the integration consists in the replacement of the $\mathbf{G}\left(p,x\right)$ function by the $\mathbf{g}\left(\Omega_F,\omega,x\right)$ one. Then the Eilenberger equation reads

$$\frac{\mathbf{i}\hbar c}{2}\left[\tau_3,\mathfrak{D}_0\mathbf{g}\right]_+ + \mathbf{i}\hbar v^i\mathfrak{D}_i\mathbf{g}\left(\Omega_F,\omega,x\right) + \left[\hbar\omega\tau_3 + \mathbf{\Delta} - \frac{\mathbf{i}\hbar}{2\tau}\left\langle\mathbf{g}\right\rangle,\mathbf{g}\right]_- = 0 \quad (76)$$

in the non-Abelian case. In the case of a simpler Abelian gauge field, one has $\mathbf{A}_\mu = e\tau_3 A_\mu/\hbar$ with A_μ real, and the equation looks exactly the same [51, 77]. The commutator in the covariant derivative distributes the charge asymmetrically among the components of $\mathbf{g}$, and the equation for g looks uncharged in the Abelian case. In the absence of a gauge field, the covariant derivatives disappear, and only normal derivatives remain, see *e.g.* [51] for these two situations.

The generalized Eilenberger equation (76) consistent with a non-Abelian gauge theory is our second important result in this study. It may allow considerable simplifications in the understanding of intricate problems dealing with spin textures and superconductivity.

Being a homogeneous equation, the Eilenberger equation (76) accepts all multiples of $\mathbf{g}$ as solution. In addition, the restriction $\hbar/p_F\xi_0 \ll 1$ makes the sum-equation (56) meaningless, as can be checked easily after cancelation of all the terms we discarded in this section: it gives the classical expression for the ξ-integrated function. The remedy to this curse is the so-called normalization condition [18, 51]. Multiplying (76) from the left by $\mathbf{g}$, or

from the right by $\mathbf{g}$, and then summing the two contributions, we realize that the commutator helps making both $\mathbf{g}$ and $\mathbf{gg}$ solutions of the same equation (76). This means that a generic solution reads $\mathbf{gg} = A\mathbf{g} + B$, with A and B two constants [51]. The normalization condition for $\mathbf{g}(\Omega_F, \omega, x)$ reads then

$$\mathbf{gg} = 1 \tag{77}$$

found as the solution of (76) for large time and space, where there are neither impurities nor gauge-field, and when the ξ-integration can be performed exactly, for which situation we find $A = 1$ and $B = 0$ [51].

We shortly give the definitions of the observables for the ξ-integrated functions. They follow from the substitution of the integration element $dp/2\pi\hbar \approx N_0 d\xi_p (d\Omega_p/4\pi)(d\omega/2\pi\hbar c)$ in the general relations (61), (63), (64) and (65). One has

$$\begin{aligned} \rho_n(x) &= \mathbf{i}\pi\hbar N_0 \int \frac{d\omega}{2\pi} \langle \mathrm{Tr}\{\tau_3 \mathbf{g}\}\rangle \\ j_n^i(x) &= \mathbf{i}\pi\hbar N_0 \int \frac{d\omega}{2\pi} \left\langle \mathrm{Tr}\left\{v_F^i \mathbf{g}\right\}\right\rangle \end{aligned} \tag{78}$$

for the quasiparticle density and current density,

$$\begin{aligned} \rho_e(x) &= \mathbf{i}\pi e\hbar N_0 \int \frac{d\omega}{2\pi} \langle \mathrm{Tr}\{\mathbf{g}\}\rangle \\ j_e^i(x) &= \mathbf{i}\pi e\hbar N_0 \int \frac{d\omega}{2\pi} \left\langle \mathrm{Tr}\left\{\tau_3 v_F^i \mathbf{g}\right\}\right\rangle \end{aligned} \tag{79}$$

for the electric charge and current densities, and

$$\begin{aligned} \rho_s^k(x) &= \mathbf{i}\pi \frac{g}{2}\hbar N_0 \int \frac{d\omega}{2\pi} \langle \mathrm{Tr}\{\mathbf{s}_k \tau_3 \mathbf{g}\}\rangle \\ j_s^{ik}(x) &= \mathbf{i}\pi \frac{g}{2}\hbar N_0 \int \frac{d\omega}{2\pi} \left\langle \mathrm{Tr}\left\{\mathbf{s}_k v_F^i \mathbf{g}\right\}\right\rangle \end{aligned} \tag{80}$$

for the spin charge and current densities. We also have

$$\Delta(x) = -\mathbf{i}\pi\hbar V_0 N_0 \int \frac{d\omega}{2\pi} \langle \mathrm{Tr}\{\mathbf{i}\sigma_2\tau_+ \mathbf{g}\}\rangle \tag{81}$$

for the self-consistent relation of the gap parameter.

The Eilenberger equation (76) constitutes a convenient simplification in the description of the superconductor phenomenology in the clean limit when $\tau \to \infty$. For finite mean free time τ, the self-consistency in $\langle \mathbf{g}\rangle$ might be problematic. Thanks to the normalization condition, one can go further to the diffusive limit, called the Usadel limit.

V. USADEL EQUATION: DIFFUSIVE LIMIT

The final approximation we will give in this paper is the diffusive one, also called Usadel limit [20]. The associated Usadel equation is a restriction of the Eilenberger one for diffusive systems, when the self-consistent impurity contribution in (76) disapears in a resulting diffusion-like equation. For diffusive systems, the ξ-integrated Green function can be expanded as

$$\mathbf{g} = \mathbf{g}_0(\omega, x) + \hat{\boldsymbol{v}}_{\boldsymbol{F}} \cdot \hat{\boldsymbol{g}} \tag{82}$$

with an isotropic component $\mathbf{g}_0$ and a smaller contribution $\hat{\boldsymbol{g}}$ along the Fermi velocity $\hat{\boldsymbol{v}}_{\boldsymbol{F}}$ (the hat marks the unit vector). One then has $\langle \mathbf{g}\rangle = \mathbf{g}_0$. The Usadel equation corresponds to the equation for the isotropic part only. The derivation of the Usadel equation from the Eilenberger one in the Abelian case is well described in [20, 51, 56] for instance, so we just sketch its generalisation below since there is no more difficulty to deal with the non-Abelian situation.

The derivation of the Usadel equation relies on the normalization condition (77), which reads twofold now: $\mathbf{g_0}\cdot\mathbf{g}_0 = 1$ and $\hat{\boldsymbol{g}} \cdot \mathbf{g}_0 + \mathbf{g}_0 \cdot \hat{\boldsymbol{g}} = \mathbf{0}$. After multiplying the Eilenberger equation (76) with $\hat{\boldsymbol{v}}_{\boldsymbol{F}}$ and averaging it as in (75), one obtains

$$-\ell \mathbf{g}_0 \boldsymbol{\mathfrak{D}}_i \mathbf{g}_0 = \hat{\boldsymbol{g}}_i \tag{83}$$

after use of the normalization conditions several times, and with $\ell = v_F\tau$ the mean free path. Next step is to average the Eilenberger equation itself, and to substitute (83) into the resulting equation. One obtains then

$$\begin{aligned} \frac{\mathbf{i}\hbar c}{2}\left[\tau_3, \boldsymbol{\mathfrak{D}}_0 \mathbf{g}_0\right]_+ - \mathbf{i}\hbar D\left(\boldsymbol{\mathfrak{D}}_i \mathbf{g}_0\right)\cdot\left(\boldsymbol{\mathfrak{D}}_i \mathbf{g}_0\right) \\ + \left[\hbar\omega\tau_3 + \boldsymbol{\Delta}, \mathbf{g}_0\right]_- = 0 \end{aligned} \tag{84}$$

for the generalized Usadel equation in the presence of non-Abelian gauge field, with $D = \ell v_F/3$ the diffusion constant.

Since the expressions (78), (79), (80) and (81) already contains the averaging over the Fermi surface angular dependency, it is sufficient to substitute the expansion (82) and the substitution (83) to get

$$\Delta(x) = \mathbf{i}\pi\hbar V_0 N_0 \int \frac{d\omega}{2\pi} \mathrm{Tr}\{\mathbf{i}\sigma_2\tau_+ \mathbf{g}_0(x,\omega)\} \tag{85}$$

for the self-consistency relation,

$$\begin{aligned} \rho_n(x) &= \mathbf{i}\pi\hbar N_0 \int \frac{d\omega}{2\pi} \mathrm{Tr}\{\tau_3 \mathbf{g}_0(\omega, x)\} \\ j_n^i(x) &= -\mathbf{i}\pi D\hbar N_0 \int \frac{d\omega}{2\pi} \mathrm{Tr}\{\mathbf{g}_0 \boldsymbol{\mathfrak{D}}_i \mathbf{g}_0\} \end{aligned} \tag{86}$$

for the quasi-particle density and current,

$$\begin{aligned} \rho_e(x) &= \mathbf{i}\pi e\hbar N_0 \int \frac{d\omega}{2\pi} \mathrm{Tr}\{\mathbf{g}_0(\omega, x)\} \\ j_e^i(x) &= -\mathbf{i}\pi e D\hbar N_0 \int \frac{d\omega}{2\pi} \mathrm{Tr}\{\tau_3 \mathbf{g}_0 \boldsymbol{\mathfrak{D}}_i \mathbf{g}_0\} \end{aligned} \tag{87}$$

for the electric charge and current densities, and

$$\rho_e(x) = \mathbf{i}\pi\frac{g}{2}\hbar N_0 \int \frac{d\omega}{2\pi}\mathrm{Tr}\left\{\mathbf{s}_k\tau_3\mathbf{g}_0(\omega,x)\right\}$$
$$j_e^{ik}(x) = -\mathbf{i}\pi\frac{g}{2}D\hbar N_0 \int \frac{d\omega}{2\pi}\mathrm{Tr}\left\{\mathbf{s}_k\mathbf{g}_0\boldsymbol{\mathfrak{D}}_i\mathbf{g}_0\right\} \tag{88}$$

for the spin charge and densities, respectively. We see that the currents contain magneto-electric contributions: the spin current contains a term proportional to the electric charge, whereas the electric current contains a term proportional to the spin charge, via the non-Abelian covariant derivative (53) with (28).

VI. POOR-MAN DERIVATION OF THE GAUGE-COVARIANT EILENBERGER EQUATION

The two last sections of this paper contain extra materials, shortly discussed. In this section, we discuss the derivation of the usual Eilenberger equation using the so-called gradient expansion, and its generalization to a gauge-covariant set of equations. We then recover the non-Abelian Eilenberger equation (76) in a (perhaps) more direct way. In the next section, we use the result of the present one to discuss the difference between a gauge potential and a usual potential in term of transport equation.

We thus suppose no gauge field for the moment. Then we define the Wigner transformation as in (42) and apply it to the Dyson equation (5) which then reads $\mathbf{G}^{-1}(p,x)\cdot e^{\hbar\Lambda/2\mathbf{i}}\cdot\mathbf{G}(p,x)=1$ with the Moyal operator $F\cdot\Lambda\cdot G = \partial_p F\cdot\partial_x G - \partial_x F\cdot\partial_p G$ for any functions F and G, see [52, 53] for more details. At first order in a $\hbar$ expansion, one has

$$\mathbf{G}^{-1}(p,x)\,\mathbf{G}(p,x) + \frac{\mathbf{i}\hbar}{2}\left(\partial_p\mathbf{G}^{-1}\cdot\partial_x\mathbf{G} - \partial_x\mathbf{G}^{-1}\cdot\partial_p\mathbf{G}\right)\approx 1 \tag{89}$$

with $\mathbf{G}^{-1}(p,x) = \mathbf{G}_0^{-1}(p,x)+\boldsymbol{\Sigma}(p,x)$ in general, but we no more discuss the self-energy in the following. We have $\mathbf{G}_0^{-1}(p,x) = \hbar\omega\tau_3 - p^2/2m + \mu + \boldsymbol{\Delta}(x)$ for a conventional superconductor, with $\boldsymbol{\Delta}$ defined in (18). Taking the difference of the Dyson equation and its adjoint, one ends up with

$$\frac{\mathbf{i}\hbar}{2}\left[\tau_3,\partial_t\mathbf{g}\right]_+ + \mathbf{i}\hbar v_F^i\partial_i\mathbf{g} + \left[\hbar\omega\tau_3+\boldsymbol{\Delta},\mathbf{g}\right]_- = 0 \tag{90}$$

for the ξ-integrated Green functions in the vicinity of the Fermi surface. We once again discard all the terms with momentum derivatives, see Section IV. The expression (90) is the so-called Eilenberger equation, when no gauge-field applies [18, 19, 51].

If one wants (90) to become gauge-covariant with respect to the gauge transformation $\mathbf{g}\rightsquigarrow\mathbf{R}\mathbf{g}\mathbf{R}^\dagger$, one can promote the usual derivatives ∂_μ in (90) to some covariant derivatives (53) which transforms as $\boldsymbol{\mathfrak{D}}_\mu\rightsquigarrow\mathbf{R}\boldsymbol{\mathfrak{D}}_\mu\mathbf{R}^\dagger$ when $\mathbf{A}_\mu\rightsquigarrow\mathbf{R}\mathbf{A}_\mu\mathbf{R}^\dagger - \mathbf{i}\mathbf{R}\partial_\mu\mathbf{R}^\dagger$. Then we recover the gauge covariant Eilenberger equation (76) without the impurities corrections here for simplicity. Adding the isotropic model for the scattering is straightforward, as well as the derivation of the Usadel limit from there.

The above derivation is highly satisfying, since it does not require the lengthy calculations of Section III and Section IV to obtain the gauge-covariant Eilenberger equation. Nevertheless, the general transport equation (55) cannot be obtained using a simple argument of covariance, since the gauge fields are present there.

VII. THE EXCHANGE FIELD AS A USUAL POTENTIAL

One can really wonder whether it makes sense or not to discuss a complicated gauge theory to include magnetic interaction. Indeed, conventional ferromagnetism seems to be properly described when adding the Zeeman term $h_Z\sigma_3$ into the equation of motion as a regular potential, not a gauge potential. Here I clarify a bit the difference between the two approaches.

When writing the propagator

$$\mathbf{G}_0^{-1}(x) = \mathbf{i}\hbar c\tau_3\mathbf{D}_0 + \frac{\hbar^2}{2m}\boldsymbol{\partial}_{\boldsymbol{x}}\cdot\boldsymbol{\partial}_{\boldsymbol{x}} + \boldsymbol{\Delta}(x,t)$$
$$= \mathbf{i}\hbar\tau_3\partial_t - \hbar\tau_3\mathbf{A}_0(x,t) + \frac{\hbar^2}{2m}\boldsymbol{\partial}_{\boldsymbol{x}}\cdot\boldsymbol{\partial}_{\boldsymbol{x}} + \boldsymbol{\Delta}(x,t) \tag{91}$$

one has the choice to express

$$\mathbf{A}_0 = \tau_3\frac{h_Z\sigma_3-\mu}{\hbar} \tag{92}$$

either in term of a gauge-potential (first line of (91)) or as a usual potential (second line of (91)). Reproducing the derivation in the previous section, we find (see [24, 26, 27] for a discussion of the consequences of this equation)

$$\frac{\mathbf{i}\hbar}{2}\left[\tau_3,\partial_t\bar{\mathbf{g}}\right]_+ + \mathbf{i}\hbar v_F^i\partial_i\bar{\mathbf{g}} + \left[\hbar\omega\tau_3 - h_Z\sigma_3 + \boldsymbol{\Delta},\bar{\mathbf{g}}\right]_- = 0 \tag{93}$$

when we suppose (92) to be a usual potential. In contrary, the gauge covariant Eilenberger equation (76) leads to

$$\frac{\mathbf{i}\hbar}{2}\left[\tau_3,\partial_t\tilde{\mathbf{g}} + \frac{\mathbf{i}}{\hbar}\left[h_Z\tau_3\sigma_3,\tilde{\mathbf{g}}\right]_-\right]_+ + \mathbf{i}\hbar v_F^i\partial_i\tilde{\mathbf{g}} + \left[\hbar\omega\tau_3+\boldsymbol{\Delta},\tilde{\mathbf{g}}\right]_- = 0 \tag{94}$$

where the time-covariant-derivative is displayed explicitly.

The two equations (93) and (94) do not look the same, and questions rise up about the correctness of the present approach. To resolve this paradox, note that the two

functions $\mathbf{g}$ in (93) and (94) are not equivalent: $\bar{\mathbf{g}}$ in (93) corresponds to the Wigner transformed correlation function adapted to a situation without gauge field (42) (more explicitly, $\bar{\mathbf{g}}$ is the ξ-integrated, mixed-Fourier transform of the Green-Gor'kov functions $\mathbf{G}(x_1, x_2)$), whereas $\tilde{\mathbf{g}}$ in (94) is the result of the gauge-covariant Wigner transformation with curvature (47). In addition, the gauge field associated to the gauge potential (92) is rather trivial $\mathbf{F}_{i0} = \tau_3 (\sigma_3 \partial_i h_Z - \partial_i \mu)/\hbar$ and contains only the non-Abelian generalization of the electric field. This is obvious since the electric field is the only one having a pure potential contribution. This suggests that one may possibly kill the gauge potential in (94). Indeed, it is always possible to work in a gauge such that the time-sector of the gauge potential $\mathbf{A}_0 = 0$ vanishes, called the temporal, or Weyl gauge [78].

For simplicity we assume in the following that the exchange field h_Z is space-time independent, in which case the non-Abelian gauge field disappears. Then the gauge potential can be canceled explicitly in (94) using the gauge transformation $\mathbf{R}_h = \exp[-\mathbf{i}\tau_3\sigma_3 h_Z t/\hbar]$. Transforming $\tilde{\mathbf{g}} = \mathbf{R}_h \tilde{\mathbf{g}}' \mathbf{R}_h^\dagger$ reduces (94) to (90) for the $\tilde{\mathbf{g}}'$ functions, when no gauge field is present. In particular, the transformation does not alter the gap parameter, since it has a singlet representation in the spin sector. We note that the transformation looks like the replacement $\omega' = \omega - h_Z \sigma_3 \tau_3/\hbar$ for $\tilde{\mathbf{g}}$, to be compared with the expression (93).

This intriguing result is partially consistent with the old-known result that the gap parameter is unaffected by a weak paramagnetic interaction established by Sarma [79]. Sarma invoked the singlet symmetry of the gap parameter as well to understand his result; we simply recast this argument into a gauge invariance in Section I. Nevertheless, Sarma also found that a large exchange field $h_Z > \Delta$ alters the critical line transition. In the Eilenberger formalism, the high energy sector $h_Z > \Delta$ is in principle not reachable, but the gauge transformation $\mathbf{R}_h$ discards the constant paramagnetic interaction for all energies, since it can be applied to the Dyson-Gor'kov's equation (Section I) as well. It is not yet clear whether the Sarma's result can be entirely understood or not in the gauge formalism I propose here. It might well be that the mean field treatment of the superconducting phase imposes some restriction on the use of the gauge redundancy. It is for instance clear that a large enough exchange field compensates the kinetic energy of the quasiparticles. Perhaps the gauge invariance of the interaction Hamiltonian (4) is verified only at low energies, at least for energies smaller than the gap parameter. A detailed study of this effect is postponed to future works, but I fear a complete discussion of the pair-destruction mechanism requires the self-consistent treatment of the Cooper instability mechanism: an effective theory with coupling constants as discussed in this paper might well be not powerful enough.

Thus, expressions (93) and (94) lead to the same conclusion in the low energy sector $\Delta < h_Z$ and for constant exchange field at zero temperature, according to the Sarma's result [79]. For a space-dependent exchange field, the cancelation of the exchange field is not a trivial task, and further discussions are necessary. Also, generalizations to non-zero-temperatures should be done with care, because the Matsubara formalism alters the gauge structure of the theory, see *e.g.* [80].

Obviously a spin-orbit term can not be canceled by a gauge transformation affecting the time-sector only. Additionally, the choice of the temporal gauge will not alter the spin-orbit interaction. Then, to understand the interaction between (at least) Zeeman and spin-orbit interaction, the gauge formalism I developed in this study should be useful.

Note added in proof: An other way to treat the spin interaction is to consider the spin-orbit interaction as a gauge potential, whereas the exchange field is treated as a conventional potential. This method, especially useful in the case of stationary problems, leads then to the Eilenberger equation

$$\mathbf{i}\hbar v_F^i \mathfrak{D}_i \mathbf{g} + [\hbar\omega\tau_3 - h_Z\sigma_3 + \mathbf{\Delta}, \mathbf{g}]_- = 0 \qquad (95)$$

and can be conveniently transformed toward the Matsubara formalism using a Wick rotation [6]. Additionally, using a gauge-covariant Wigner transformation for the space coordinates only allows to include non-stationary effect via the Keldysh space. This last approach has been recently followed by Bergeret and Tokatly [43], who derived essentially the same equations as in this paper, using the same covariant method but transforming only the space components of the Green functions. In the notations of the present paper, they thus transform the Green functions from $\mathbf{G}(\boldsymbol{x}_1, \boldsymbol{x}_2, t_1, t_2)$ to $\mathbf{G}(\boldsymbol{p}, \boldsymbol{x}, t_1, t_2)$ using $\mathbf{G}(\boldsymbol{p}, \boldsymbol{x}, t_1, t_2) = \int d\boldsymbol{z} \left[e^{-\mathbf{i}\boldsymbol{p}\cdot\boldsymbol{z}/\hbar} \mathbf{U}(\boldsymbol{x}, \boldsymbol{x}_1) \mathbf{G}(\boldsymbol{x}_1, \boldsymbol{x}_2, t_1, t_2) \mathbf{U}(\boldsymbol{x}_2, \boldsymbol{x})\right]$ as the gauge-covariant Wigner transformation, the main difference being that $\mathbf{G}(\boldsymbol{x}_1, \boldsymbol{x}_2, t_1, t_2)$ then stands for some matrix in the Keldysh plus particle-hole plus spin plus charge space, a complication I wanted to avoid here. They do not discuss non-stationary problems, though. At the time of writing, it is not clear which of these approaches (to treat the exchange field as the time-sector of the gauge-potential or not, and/or to Wigner-transform the time variables of the Green functions or not) will get the more tractable analysis of relevant situations. For a comprehensible review of the difficulties to deal with non-equilibirum superconductors in the quasi-classical limit, one can consult [51, 81] and references therein.

VIII. CONCLUSION AND PERSPECTIVES

In this study, I focused on the establishment of a family of transport-like equations which are of possible in-

terest for the study of superconductors in the presence of magnetic interaction like space-time dependent Zeeman and/or spin-orbit interaction linear in the momentum. Having in mind the recently discussed spin texture competing with the superconducting order, I proposed to enlarge the usual description of the electromagnetic interaction in a gauge interaction including non-Abelian spin plus Abelian charge sectors. I show how the Gor'kov set of equations can be generalized to a $U(1) \otimes SU(2) \otimes SU(2)$, charge plus spin plus particle-hole gauge theory (Section I).

Thanks to the well established gauge principles, the proposed description takes into account the self-consistent interactions between the gauge degrees of freedom and the superconducting phase (Section II).

This set of equations is nevertheless intrinsically non-linear and self-consistent. To simplify it, I thus proposed to reduce the quantum structure of the Dyson-Gor'kov equations toward a transport-like theory at the quasi-classical level, when the quasi-classical Green function $\mathbf{G}(p,x)$ now describes the normal and anomalous correlation functions in a time-dependent phase-space (Section III, in particular expressions (55) and (56)).

In addition, the superconducting state usually has a clear energy scale separation $\Delta/E_F \ll 1$ between the gap parameter Δ and the Fermi energy E_F, which allows to reduce even further the transport equation into the so-called Eilenberger equation, here generalized to take into account the electronic spin degree of freedom on the same footing as the charge one (Section IV, especially (76)). The diffusive limit of the Eilenberger equation, known as the Usadel equation, is also given (Section V, see (84)). In each case (quasi-classical transport, Eilenberger and Usadel) I provided the associated quasi-particle, electric and spin charges currents. In particular, the charge and the spin currents now exhibit some magneto-electric couplings, which will be discussed in subsequent studies.

The different levels of approximations discussed in this paper may constitute an interesting way for studying the topological problems in condensed matter, when Zeeman plus spin-orbit effects compete with the superconducting proximity effect in disordered structures, as well as to address fundamental questions in bulk magnetic superconductors. Since the equations I derived contain the limiting cases of normal metal interacting or not with a (non-)Abelian gauge field, this study may be of interest for the understanding of quantum Hall effects, for spintronics, and for many more topics in condensed matter when the internal degrees of freedom of the electrons need to be correctly accounted for. In any case, this study should pave the way toward a better understanding of the relations between condensed matter and gauge theories.

Among other urgent problems to be resolved in the gauge formalism proposed in this study are the inclusion of boundary conditions in the transport formalism, the discussion of the lattice symmetry in addition to the gauge redundancy, and the understanding of the gauge properties of the superconducting phase in a statistical field theory. It should be interesting to understand the role of a possible quantization of the gauge field on the superconducting phase as well, and its associated phenomenology (anomaly, confinement, ...). The intrinsic non-linearity of the non-Abelian formalism suggests that the proposed transport-like theory exhibit some sort of instantons, too. This has to be checked as well. In mesoscopic physics terms, the proposed formalism opens the way to discuss the role of impurities in a self-consistent way. Possible applications are in the recently emerging field of topological matter and its relation to quantum information perspectives.

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Appendix A: Correspondence rules

In this appendix, we discuss the generic rules of transformation from the Dyson equation to some quantum transport equations through the gauge-covariant Wigner transformation (47). In particular, we sum up the long algebra required to obtain the first and second order covariant derivatives of the gauge-covariant Wigner-transformation introduced in the main text. The equations below are generic, and can be applied to any gauge-field $F_{\mu\nu}$ defined from a gauge potential A_μ through the definition $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathbf{i}\left[A_\mu, A_\nu\right]$. We intensively use the relativistic tensorial notations in this appendix; they are sum-up at the beginning of Section I. The covariant derivatives are defined as $D_\mu = \partial_\mu + \mathbf{i}A_\mu$ and $D_\mu^\dagger = \partial_\mu - \mathbf{i}A_\mu$ where the derivative applies to the left in $D_\mu^\dagger$ and A_μ is a Hermitian field. Finally the covariant derivative is $\mathfrak{D}^\mu(x) F_{\mu\nu}(x) = \partial F_{\mu\nu}/\partial x_\mu + \mathbf{i}\left[A^\mu(x), F_{\mu\nu}(x)\right]$ when applied to the second-rank gauge-field tensor. In the following, both the quasi-classical Green function $G(p,x)$ and the gauge field $F_{\mu\nu}(x)$ behave as second-rank tensors.

One then defines the geometric differential propagator, applied to a field amplitude or a second-rank tensor $F(x)$ (demonstrated in [62] using an expansion of the exponential)

$$e^{y\cdot\mathfrak{D}(x)} F(x) = U(x, x+y)\, F(x+y)\, U(x+y, x) \quad \text{(A1)}$$

with the path-ordered integral $U(x,y)$ defined in (46),

also called parallel displacement operator, Wilson line or link operator [54]. The parallel transport operator is an operator in gauge-space only (for instance, in the charge $\otimes$ spin-space, or the Nambu $\otimes$ charge $\otimes$ spin-space in the main text), whereas it is just a phase-shift in real-space, as can be seen from its property (45). The last expressions we need to proceed are the covariant derivatives of the parallel shift operator

$$D_\nu(b)\,U(b,a) = \mathbf{i}\,(b-a)^\mu \times \int_0^1 ds\,[sU(b,\tau_s)\,F_{\mu\nu}(\tau_s)\,U(\tau_s,a)] \quad \text{(A2)}$$

$$U(b,a)\,D_\nu^\dagger(a) = \mathbf{i}\,(b-a)^\mu \int_0^1 ds\,[(1-s)\,U(b,\tau_s)\,F_{\mu\nu}(\tau_s)\,U(\tau_s,a)] \quad \text{(A3)}$$

the demonstration of which are in [62, 68], and where $\tau_s = a + (b-a)\,s$ represents the straight line between the extremum points a and b, as discussed in [62, 64]. They include non-trivial curvature effect due to the presence of the gauge field in their right-hand-side. We define the gauge-covariant Wigner-transform (44) adapting the treatment given by Elze, Gyulassy and Vasak [62] to the Green function problem. We then use the property (45) to write the workable representation of the Wigner transformation in (47).

The change of coordinates from the two-points correlators and the phase-space quasi-classical Green function reads $x_{1,2} = x \mp z/2$ and $\partial_{1,2} = \partial_x/2 \mp \partial_z$. Then one obtains – be warn that the notations are mixed in the formulas below (they should display only the variables x and z) for the sake of notational compactness –

$$\frac{\partial}{\partial x_1^\nu} U\left(\frac{x_1+x_2}{2}, x_1\right) = \mathbf{i}U(x,x_1)\,A_\nu(x_1) - \frac{\mathbf{i}}{2}A_\nu(x)\,U(x,x_1) + \frac{\mathbf{i}}{2}\int_0^1 ds\,\left[\mathfrak{F}_\nu^{s-1}(x,z)\right]U(x,x_1) \quad \text{(A4)}$$

$$\frac{\partial}{\partial x_1^\nu} U\left(x_2, \frac{x_1+x_2}{2}\right) = \frac{\mathbf{i}}{2}U(x_2,x)\,A_\nu(x) + \frac{\mathbf{i}}{2}U(x_2,x)\int_0^1 ds\,[\mathfrak{F}_\nu^s(x,z)] \quad \text{(A5)}$$

$$\frac{\partial}{\partial x_2^\nu} U\left(\frac{x_1+x_2}{2}, x_1\right) = -\frac{\mathbf{i}}{2}A_\nu(x)\,U(x,x_1) + \frac{\mathbf{i}}{2}\int_0^1 ds\,\left[\bar{\mathfrak{F}}_\nu^{s-1}(x,z)\right]U(x,x_1) \quad \text{(A6)}$$

$$\frac{\partial}{\partial x_2^\nu} U\left(x_2, \frac{x_1+x_2}{2}\right) = -\mathbf{i}A_\nu(x_2)\,U(x_2,x) + \frac{\mathbf{i}}{2}U(x_2,x)\,A_\nu(x) + \frac{\mathbf{i}}{2}U(x_2,x)\int_0^1 ds\,\left[\bar{\mathfrak{F}}_\nu^s(x,z)\right] \quad \text{(A7)}$$

where we used that the path is a straight line, so we can write some expressions like $U(z,x_1) = U(z,x_1)\,U(x_1,x)\,U(x,x_1) = U(z,x)\,U(x,x_1)$ since $U(x_1,x)\,U(x,x_1) = 1$ is not a Wilson loop, in which case it might be a phase factor. The same applies for $U(x_2,z) = U(x_2,x)\,U(x,z)$, which allows the expressions to be written in terms of the $\mathfrak{D}(x)$ operator (see also (A8) below). We propose the notational simplifications

$$\mathfrak{F}_\nu^s(x,z) = \frac{1-s}{2}e^{sz\cdot\mathfrak{D}(x)/2}z^\mu F_{\mu\nu}(x)$$

$$\bar{\mathfrak{F}}_\nu^s(x,z) = \frac{1+s}{2}e^{sz\cdot\mathfrak{D}(x)/2}z^\mu F_{\mu\nu}(x) \quad \text{(A8)}$$

which keep the following calculations tractable. Essentially, passing from $\mathfrak{F}$ to $\bar{\mathfrak{F}}$ consists in changing, in the associated integral, the direction of propagation from the center-of-mass coordinate x to one of the extremum x_1 or x_2 along a straight line.

We can now evaluate

$$U(x,x_1)\,[D_\nu(x_1)\,G(x_1,x_2)]\,U(x_2,x) = \frac{\partial}{\partial x_1^\nu}\tilde{G} + \frac{\mathbf{i}}{2}\left[A_\nu(x),\tilde{G}\right] - \frac{\mathbf{i}}{2}\int_0^1 ds\,\left[\mathfrak{F}_\nu^{s-1}(x,z)\,\tilde{G} + \tilde{G}\mathfrak{F}_\nu^s(x,z)\right]$$

$$U(x,x_1)\,\left[G(x_1,x_2)\,D_\nu^\dagger(x_2)\right]U(x_2,x) = \frac{\partial}{\partial x_2^\nu}\tilde{G} + \frac{\mathbf{i}}{2}\left[A_\nu(x),\tilde{G}\right] - \frac{\mathbf{i}}{2}\int_0^1 ds\,\left[\bar{\mathfrak{F}}_\nu^{s-1}(x,z)\,\tilde{G} + \tilde{G}\bar{\mathfrak{F}}_\nu^s(x,z)\right] \quad \text{(A9)}$$

with $\tilde{G} = U(x,x_1)\,G(x_1,x_2)\,U(x_2,x)$. Finally, one can calculate easily the gauge-covariant Wigner-transformation of the covariant derivative of the Green function as the Fourier transform of the previous expressions. It gives:

$$\int dz\,\left[e^{-\mathbf{i}p\cdot z/\hbar}U\,[D_\nu(x_1)\,G(x_1,x_2)]\,U\right] = \left(\frac{1}{2}\mathfrak{D}_\nu(x) - \mathbf{i}\frac{p_\nu}{\hbar}\right)G(p,x) - \frac{\mathbf{i}}{2}\left\{\left[\int_0^1 ds\mathfrak{F}_\nu^{s-1}\right]G(p,x) + G(p,x)\left[\int_0^1 ds\mathfrak{F}_\nu^s\right]\right\} \quad \text{(A10)}$$

$$\int dz\left[e^{-\mathbf{i}p\cdot z/\hbar}U\left[G\left(x_1,x_2\right)D_\nu^\dagger\left(x_2\right)\right]U\right]=$$
$$\left(\frac{1}{2}\mathfrak{D}_\nu\left(x\right)+\mathbf{i}\frac{p_\nu}{\hbar}\right)G\left(p,x\right)$$
$$-\frac{\mathbf{i}}{2}\left\{\left[\int_0^1 ds\bar{\mathfrak{F}}_\nu^{s-1}\right]G\left(p,x\right)+G\left(p,x\right)\left[\int_0^1 ds\bar{\mathfrak{F}}_\nu^{s}\right]\right\} \tag{A11}$$

using some integration by part of the ∂_z term, and the symbolic formula $\int dz\left[e^{-\mathbf{i}p\cdot z/\hbar}f\left(z\right)g\left(z\right)\right]=f\left(\mathbf{i}\hbar\partial_p\right)\int dz\left[e^{-\mathbf{i}p\cdot z/\hbar}g\left(z\right)\right]$, so that all the contributions $\mathfrak{F}$ have to be understood as being $\mathfrak{F}\left(x,\mathbf{i}\hbar\partial_p\right)$ dependent. At the end, only the Wigner-Green-function $G\left(p,x\right)$ depends on p, so the momentum derivatives apply to $G\left(p,x\right)$ only.

To calculate the gauge-covariant Wigner-transform of the second order covariant derivative of the Green function, a convenient method is to rewrite

$$\begin{aligned}U\left(x,x_1\right)\left[D_\nu\left(x_1\right)D^\nu\left(x_1\right)G\left(x_1,x_2\right)\right]U\left(x_2,x\right)&=\\ U\left(\partial_\nu\left(x_1\right)D^\nu G\right)U+\mathbf{i}UA_\nu\left(x_1\right)\left(D^\nu G\right)U&\\ =\partial_\nu\left[U\left(D^\nu G\right)U\right]-\left(\partial_\nu U\right)\left(D^\nu G\right)U&\\ -U\left(D^\nu G\right)\left(\partial_\nu U\right)+\mathbf{i}UA_\nu\left(x_1\right)\left(D^\nu G\right)U&\end{aligned} \tag{A12}$$

where we do not write explicitly all the coordinates on the right-hand-side when they are trivially reproduced from the left-hand-side. All the derivatives are with respect to the first argument x_1 of the Green function $G\left(x_1,x_2\right)$. Then, we use the formula (A4) and (A5) such that the last term of (A12) disappears with the first term of (A4) and we are left with

$$\begin{aligned}U\left(x,x_1\right)\left[D_\nu\left(x_1\right)D^\nu\left(x_1\right)G\left(x_1,x_2\right)\right]U\left(x_2,x\right)&=\\ \partial_\nu\left[U\left(D^\nu G\right)U\right]+\frac{\mathbf{i}}{2}\left[U\left(D^\nu G\right)U,A_\nu\left(x\right)\right]&\\ -\frac{\mathbf{i}}{2}\left\{\int ds\left[\mathfrak{F}_\nu^{s-1}\right]U\left(D^\nu G\right)U+U\left(D^\nu G\right)U\int ds\left[\mathfrak{F}_\nu^{s}\right]\right\}&\end{aligned} \tag{A13}$$

then, we just have to evaluate the derivative of (A9) and to produce some algebra. Note the group structure of the covariant derivative, since the above equation is exactly the same as (A9) when we replace $UGU=\tilde{G}$ by $U\left(DG\right)U$. After tedious algebra, one obtains

$$\begin{aligned}\int dz\left[e^{-\mathbf{i}p\cdot z/\hbar}U\left[D_\nu D^\nu\left(x_1\right)G\left(x_1,x_2\right)\right]U\right]&=\\ \left(\frac{1}{2}\mathfrak{D}_\nu\left(x\right)-\mathbf{i}\frac{p_\nu}{\hbar}\right)\left(\frac{1}{2}\mathfrak{D}^\nu\left(x\right)-\mathbf{i}\frac{p^\nu}{\hbar}\right)G\left(p,x\right)&\\ -\mathbf{i}\int_0^1 ds\left[\mathfrak{F}_\nu^{s-1}\right]\left(\frac{1}{2}\mathfrak{D}^\nu\left(x\right)-\mathbf{i}\frac{p^\nu}{\hbar}\right)G\left(p,x\right)&\\ -\mathbf{i}\left(\frac{1}{2}\mathfrak{D}^\nu\left(x\right)-\mathbf{i}\frac{p^\nu}{\hbar}\right)G\left(p,x\right)\int_0^1 ds\left[\mathfrak{F}_\nu^{s}\right]&\\ -\frac{\mathbf{i}}{2}\int_0^1 ds\left[\frac{2-s}{2}\mathfrak{D}^\nu\left(x\right)\mathfrak{F}_\nu^{s-1}\right]G\left(p,x\right)&\\ -\frac{\mathbf{i}}{2}G\left(p,x\right)\int_0^1 ds\left[\frac{1-s}{2}\mathfrak{D}^\nu\left(x\right)\mathfrak{F}_\nu^{s}\right]&\\ +\frac{1}{4}\int_0^1 ds\int_0^1 d\tilde{s}\left(1-s\right)\left[\mathfrak{F}_{(1-s)(1-\tilde{s})}^\nu\mathfrak{F}_\nu^{s-1}\right.&\\ \left.-\mathfrak{F}_\nu^{s-1}\mathfrak{F}_{\tilde{s}(s-1)}^\nu\right]G\left(p,x\right)&\\ +\frac{1}{4}G\left(p,x\right)\int_0^1 ds\int_0^1 d\tilde{s}s\left[\mathfrak{F}_\nu^s\mathfrak{F}_{\tilde{s}s}^\nu-\mathfrak{F}_{s(1-\tilde{s})}^\nu\mathfrak{F}_\nu^s\right]&\\ -\frac{1}{4}\left\{\left[\int_0^1 ds\mathfrak{F}_{s-1}^\nu\right]^2G\left(p,x\right)+G\left(p,x\right)\left[\int_0^1 ds\mathfrak{F}_s^\nu\right]^2\right\}&\\ -\frac{1}{2}\left\{\int_0^1 ds\left[\mathfrak{F}_{s-1}^\nu\right]G\left(p,x\right)\int_0^1 ds\left[\mathfrak{F}_\nu^s\right]\right\}&\end{aligned} \tag{A14}$$

for the gauge-covariant Wigner-transformation of the second covariant derivative of the Green function, where all the $\mathfrak{F}$ contributions are formal formula, to be understood as $\mathfrak{F}\left(x,\mathbf{i}\hbar\partial_p\right)$. We used the symbolic notation

$$\mathfrak{D}^\nu\left(x\right)\mathfrak{F}_\nu^s\left(x,z\right)=\frac{1-s}{2}e^{sz\cdot\mathfrak{D}(x)/2}z^\mu\mathfrak{D}^\nu\left(x\right)F_{\mu\nu}\left(x\right) \tag{A15}$$

for notational convenience. Note that s and $\tilde{s}$ are the path arguments of the $\mathfrak{F}$ functional and their co- or contra-variant position is meaningless, *i.e.* $\mathfrak{F}^s=\mathfrak{F}_s$. In contrary ν is the component of the gauge field strength tensor, so $\mathfrak{F}_\nu\neq\mathfrak{F}^\nu=g^{\nu\mu}\mathfrak{F}_\mu$. Note that the notations could be confusing: to adapt (A14) to the main text, the greek indices ν should be replaced by some latin ones. This is because only the space Laplacian appears in the Dyson equations (5) and (20).

The same calculation as before for the transformation of the second order covariant derivative applied to the second variable x_2 gives the same expression (A14) except for the replacement $p_\nu\to-p_\nu$ and $\mathfrak{F}\to\bar{\mathfrak{F}}$ since the derivative applies to $x_2=x+z/2$ (these are the same differences as between (A10) and (A11) obtained for the first order covariant derivatives).

The method explained above to get the Wigner transformation of the covariant derivative can in principle be continued to the third order covariant derivative, and so on. Nevertheless, having in mind the free particle model

of the main text, we here stop at the second order derivative.

To conclude this set of rules for the transformation toward the transport equation of the Green-Wigner function $G(p,x)$, we give the transformation rule of a scalar potential, independent of the impulsion, which reads $V(x_1)\,G(x_1,x_2)$ in the Dyson equation. Then we apply the rule

$$\int dz\left[e^{-\mathbf{i}p\cdot z/\hbar}UV(x_1)\,G(x_1,x_2)\,U\right] = V(x-\mathbf{i}\hbar\partial_p/2)\,G(p,x) \quad \text{(A16)}$$

since the potential is a scalar, it commutes with the parallel displacement operator, which acts only in the Lie algebra sub-space. We supposed that the potential can be expanded in series, as usual. The same argument gives

$$\int dz\left[e^{-\mathbf{i}p\cdot z/\hbar}UG(x_1,x_2)\,V(x_2)\,U\right] = G(p,x)\,V(x+\mathbf{i}\hbar\partial_p/2) \quad \text{(A17)}$$

for the Dyson equation of the second variable x_2. Above, the potential $V(x)\equiv V(\boldsymbol{x},t)$ could be time-dependent. The substitution $x\pm z/2\to x\pm\mathbf{i}\hbar\partial_p$ is sometimes called a Bopp's rule [53]. Note the difference between the transformation rules (A16) and (52), when in the later the potential does not commute with the parallel shift U, whereas $V(x)$ commutes with U in (A16).

Thanks to their generality, expressions (A10), (A11) (A14), (A16) and (A17) can be used to transform any Dyson equation for a two-point Green function $G(x_1,x_2)$ in the real space into an equation for the associated quasi-classical Green function $G(p,x)$ in the phase-space. Nevertheless, these relations are of purely formal interest, since the expressions for $\mathfrak{F}_s(x,\mathbf{i}\hbar\partial_p)$ sounds hardly to be found in any concrete case. Even for a given gauge-potential A_μ, it is difficult to believe one will be able to find a compact expression for the associated parallel transport operator $U(x_1,x_2)$. Anyways, we do not really need the full expressions, and we would be already happy to find a systematic expansion of the relations (A10), (A14) and (A16). This would indeed induce a systematic expansion of the equation for $G(p,x)$. Such a natural expansion is provided by the parameter $\hbar$, and it is called a quasi-classical expansion [52]. For instance we have, expanding (A16) at first order

$$V(x-\mathbf{i}\hbar\partial_p/2)\,G(p,x) = V(x)\,G(p,x) + \mathbf{i}\frac{\hbar}{2}\frac{\partial V}{\partial x^\mu}\frac{\partial G}{\partial p_\mu} + \mathcal{O}\left(\left(\frac{\hbar}{\tilde{p}\tilde{x}}\right)^2\right) \quad \text{(A18)}$$

and

$$G(p,x)\,V(x+\mathbf{i}\hbar\partial_p/2) = G(p,x)\,V(x) - \mathbf{i}\frac{\hbar}{2}\frac{\partial G}{\partial p_\mu}\frac{\partial V}{\partial x^\mu} + \mathcal{O}\left(\left(\frac{\hbar}{\tilde{p}\tilde{x}}\right)^2\right) \quad \text{(A19)}$$

here $\hbar/\tilde{p}\tilde{x}$ represents a dimensionless phase-space volume, when $\tilde{x}$ and $\tilde{p}$ are characteristic momentum and position of the system. In other words, the characteristic phase-space extension of the system $\tilde{p}\tilde{x}$ should be larger than $\hbar$ for the above expansions to be valid.

In the following we keep only terms linear in $\hbar$, which has to be understood as expansion in $\hbar/\tilde{p}\tilde{x}$. Also, to avoid the calculation of irrelevant terms, we expand the Wigner transformation of $\hbar D_\nu G$ instead of $D_\nu G$, since the covariant derivatives always appear with $\hbar$ factor in the equation of motion. One has then

$$\hbar\int dz\left[e^{-\mathbf{i}p\cdot z/\hbar}U\left[D_\nu(x_1)\,G(x_1,x_2)\right]U\right] = \hbar\left(\frac{1}{2}\mathfrak{D}_\nu(x)-\mathbf{i}\frac{p_\nu}{\hbar}\right)G(p,x) + \frac{\hbar}{2}\left(\frac{3}{4}F_{\mu\nu}(x)\,\partial_p^\mu G+\frac{1}{4}\partial_p^\mu G(p,x)\,F_{\mu\nu}\right) \quad \text{(A20)}$$

(we drop the variables for $U(x,x_1)$ and $U(x_2,x)$ in the following, since there is no more possible confusion) and

$$\hbar\int dz\left[e^{-\mathbf{i}p\cdot z/\hbar}U\left[G(x_1,x_2)\,D_\nu^\dagger(x_2)\right]U\right] = \hbar\left(\frac{1}{2}\mathfrak{D}_\nu(x)+\mathbf{i}\frac{p_\nu}{\hbar}\right)G(p,x) + \frac{\hbar}{2}\left(\frac{1}{4}F_{\mu\nu}(x)\,\partial_p^\mu G+\frac{3}{4}\partial_p^\mu G(p,x)\,F_{\mu\nu}\right) \quad \text{(A21)}$$

for the single covariant derivative. For the second order derivative, only the four first lines of (A14) are kept, since the other ones are of higher order ; even the term $\int ds[\mathfrak{F}]\mathfrak{D}G$ are of $\hbar^2$ order in the expansion of the Wigner transformation of $\hbar^2 D_\mu D_\mu G$ and its adjoint. We then get

$$\hbar^2\int dz\left[e^{-\mathbf{i}p\cdot z/\hbar}U\left[D_\nu D^\nu(x_1)\,G(x_1,x_2)\right]U\right] = -\mathbf{i}\hbar p_\nu\mathfrak{D}^\nu(x)\,G(p,x)-p_\nu p^\nu G(p,x) - \mathbf{i}\hbar p^\nu\left\{\frac{3}{4}F_{\mu\nu}(x)\,\partial_p^\mu G+\frac{1}{4}\partial_p^\mu G(p,x)\,F_{\mu\nu}\right\} \quad \text{(A22)}$$

and

$$\hbar^2\int dz\left[e^{-\mathbf{i}p\cdot z/\hbar}U\left[G(x_1,x_2)\,D_\nu^\dagger D^{\nu\dagger}(x_2)\right]U\right] = +\mathbf{i}\hbar p_\nu\mathfrak{D}^\nu(x)\,G(p,x)-p_\nu p^\nu G(p,x) + \mathbf{i}\hbar p^\nu\left\{\frac{1}{4}F_{\mu\nu}(x)\,\partial_p^\mu G+\frac{3}{4}\partial_p^\mu G(p,x)\,F_{\mu\nu}\right\} \quad \text{(A23)}$$

which complete the set of correspondence rules. In principle, they allow to rewrite any Dyson equation into an equation of motion for the quasi-classical Green function $G(p,x)$ truncated at the quasi-classical order. Nevertheless, one usually sums and subtracts different Dyson equations to obtain transport-like equations, as it is done

in the main text (see also [22] and [51] for similar treatment, or the literature cited in Section III). Note that the above equations can be adapted to any gauge-field, since the only assumption we have done in this appendix is the definition of the covariant derivative $D_\mu = \partial_\mu + \mathbf{i}A_\mu$. Since we treated the generic non-Abelian case, any gauge-theory can be transformed using the rules above. This is what we do in the main text, defining a particle-hole $\otimes$ spin $\otimes$ charge gauge-theory. In the simpler limit of an Abelian gauge theory, when the A_μ components commute among themselves, $G(p,x)$ and its derivatives commute with the gauge-field $F_{\mu\nu}(x)$, and the covariant derivative $\mathfrak{D}_\mu(x)\,G(p,x)$ is just a usual derivative $\partial_\mu G(p,x)$.

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