



**HAL**  
open science

# Sticking and nonsticking orbits for a two-degree-of-freedom oscillator excited by dry friction and harmonic loading

Madeleine Pascal

► **To cite this version:**

Madeleine Pascal. Sticking and nonsticking orbits for a two-degree-of-freedom oscillator excited by dry friction and harmonic loading. *Nonlinear Dynamics*, 2014, 77, pp.267–276. 10.1007/s11071-014-1291-7 . hal-00956375

**HAL Id: hal-00956375**

**<https://hal.science/hal-00956375v1>**

Submitted on 30 Mar 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Sticking and nonsticking orbits for a two-degree-of-freedom oscillator excited by dry friction and harmonic loading

Madeleine Pascal

**Abstract** We consider a system composed of two masses connected by linear springs. One of the masses is in contact with a rough surface. Friction force, with Coulomb's characteristics, acts between the mass and the surface. Moreover, the mass is also subjected to a harmonic external force. Several periodic orbits are obtained in closed form. A first kind of orbits involves sticking phases: during these parts of the orbit, the mass in contact with the rough surface remains at rest for a finite time. Another kind of orbits includes one or more stops of the mass with zero duration. Normal and abnormal stops are obtained. Moreover, for some of these periodic solutions, we prove that symmetry in space and time occurs.

**Keywords** Dry friction · Periodic orbits · Coupled oscillator · Sticking and nonsticking motion

## 1 Introduction

This paper is a continuation of several investigations [14–17] related to vibrating systems excited by dry friction. Friction is present in many technical applications, including turbine blade joints, robot

joints, electric motor drives, wheel rail coupling of mass transit systems and brake systems. Dry friction systems are strongly nonlinear and they are usually modelled as spring mass oscillator. Friction-induced vibration has received considerable attention from a number of researchers [1–13, 18, 19] with analytical, numerical and experimental approach, and with several models of the friction force. The simple case of a one-degree-of-freedom system [6, 7, 9–12] was more particularly investigated. The case of multi-degree of freedom systems was mainly investigated with a numerical approach [2, 4, 8]. However, in [5] analytical methods were used to predict chaotic orbits in mechanical systems with friction. In [14–17], a two-degree-of-freedom oscillator is considered. Assuming Coulomb's laws of dry friction, the corresponding dynamical model is a piecewise linear system, and some analytical results about the existence of periodic solutions including stick-slip phases have been obtained.

In this work, we consider the same model of dry friction oscillator subjected to a harmonic external force considered in [17], but instead of a moving belt, we assume that one of the masses is in contact with a fixed rough surface. We investigate the existence of periodic solutions including phases, where the mass remains at rest for a finite time (sticking orbits) or of periodic solutions with one or several zero duration stops (non-sticking orbits). Moreover, for some of these periodic solutions, we prove that symmetry in space and time occurs.

---

M. Pascal (✉)  
IBISC, Université d'Evry Val d'Essonne,  
40 rue du Pelvoux, 91020 Evry Cedex, France  
e-mail: mpascal@iup.univ-evry.fr;  
madeleine.pascal3@wanadoo.fr

## 2 Problem formulation

The system (Fig. 1) is composed of two masses  $m_1$  and  $m_2$  connected by two linear springs of stiffness  $k_1$  and  $k_2$ . The second mass is in contact with a fixed rough surface. A friction force  $\tilde{F}$  acts between the mass and the surface. Moreover, the second mass is also subjected to an external force  $\tilde{R}$  given by

$$\tilde{R} = \tilde{p} \cos(\tilde{\omega}t' + \phi) \quad (\tilde{p}, \tilde{\omega}, \phi \text{ are constant parameters}) \quad (1)$$

The equations of motion related to this system are written as

$$\begin{aligned} x_1'' + x_1 - \chi x_2 &= 0, \\ x_2'' + \chi \eta (x_2 - x_1) &= \eta u + \cos(\omega t + \phi) \quad (2) \\ \eta &= \frac{m_1}{m_2}, \quad \chi = \frac{k_2}{k_1 + k_2}, \\ u &= \frac{\tilde{F}}{\eta \tilde{p}}, \quad x_i = (k_1 + k_2) \frac{y_i}{\eta \tilde{p}}, \quad (i = 1, 2) \\ t &= \Omega t', \quad \Omega = \sqrt{\frac{k_1 + k_2}{m_1}}, \quad (O)' = \frac{d(O)}{dt}, \quad \omega = \frac{\tilde{\omega}}{\Omega} \quad (3) \end{aligned}$$

$y_1$  and  $y_2$  are the displacements of the masses.

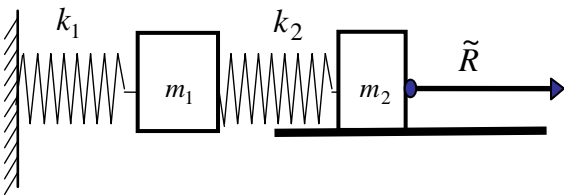
The dry friction force  $u$  is obtained from Coulomb's laws:

- $u = -u_s \text{sign}(x_2')$  if  $x_2' \neq 0$
- $u = \chi(x_2 - x_1) - \frac{1}{\eta} \cos(\omega t + \phi)$

if  $x_2' = 0$ ,  $\left| \chi(x_2 - x_1) - \frac{1}{\eta} \cos(\omega t + \phi) \right| < u_r$

- $u = u_s$  if  $x_2' = 0$ ,  
 $\chi(x_2 - x_1) - \frac{1}{\eta} \cos(\omega t + \phi) > u_r$
- $u = -u_s$  if  $x_2' = 0$ ,  
 $\chi(x_2 - x_1) - \frac{1}{\eta} \cos(\omega t + \phi) < -u_r$

$0 < u_s < u_r$  (4)



**Fig. 1** Dry friction oscillator

$u_r$  is the static friction force, and  $u_s$  is the slipping friction force.

## 3 Prediction of the oscillations exhibited by the system

The dynamical behaviour of this oscillator includes several phases of slip ( $x_2' \neq 0$ ) and stick ( $x_2' = 0$ ) motion of  $m_2$ . For each kind of motion, a close form solution is available.

### 3.1 Slip motion of $m_2$ with $x_2' < 0$

The solution is obtained from a modal analysis of (2) where  $u = u_s$

$$Z(t) = H(t)(Z_0 - F_0) + F(t), \quad F(t) = \begin{pmatrix} R(t) \\ R'(t) \end{pmatrix} \quad (5)$$

$$\begin{aligned} R(t) &= Q \cos(\omega t + \phi), \quad F_0 = F(0) \\ Z &= \begin{pmatrix} z \\ z' \end{pmatrix}, \quad Z_0 = Z(0), \quad z = X - d_0, \quad X = (x_1, x_2)^t \\ H(t) &= \begin{pmatrix} H_1(t) & H_2(t) \\ H_3(t) & H_1(t) \end{pmatrix}, \quad d_0 = (d_{01}, d_{02})^t \quad (6) \end{aligned}$$

$$\begin{aligned} d_{01} &= \frac{u_s}{1 - \chi}, \quad d_{02} = \frac{d_{01}}{\chi} \\ Q &= (q_1, q_2)^t, \quad q_1 = \frac{\chi}{(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)} \\ q_2 &= q_1 \frac{(1 - \omega^2)}{\chi} \quad (7) \end{aligned}$$

The two by two matrices  $H_i(t)$  ( $i = 1, 2, 3$ ) and the natural frequencies ( $\omega_1, \omega_2$ ) are obtained in analytical form (See Appendix 1).

### 3.2 Slip motion of $m_2$ with $x_2' > 0$ (overshooting)

The solution is obtained from (2) where  $u = -u_s$

$$\begin{aligned} \bar{Z}(t) &= H(t)(\bar{Z}_0 - F_0) + F(t) \\ \bar{Z}(t) &= Z(t) + 2\Delta_0, \quad \bar{Z}_0 = Z_0 + 2\Delta_0 \\ \Delta_0 &= \begin{pmatrix} d_0 \\ 0 \end{pmatrix} \quad (8) \end{aligned}$$

### 3.3 Stick motion of $m_2$ ( $x_2' = 0$ )

This motion is related to the following dynamical system:

$$x_1'' + x_1 - \chi x_2 = 0, \quad x_2'' = 0 \quad (9)$$

The solution is given by

$$Z(t) = \Gamma(t)Z_0, \quad \Gamma(t) = \begin{pmatrix} \Gamma_1(t) & \Gamma_2(t) \\ \Gamma_3(t) & \Gamma_1(t) \end{pmatrix} \quad (10)$$

The two by two matrices  $\Gamma_i(t)$  ( $i = 1, 2, 3$ ) are given in Appendix 1. Moreover, during all this kind of motion, the following constraint holds:

$$|\chi \eta (x_2 - x_1) - \cos(\omega t + \phi)| < \eta u_r \quad (11)$$

## 4 Nonsticking periodic solutions

### 4.1 Nonsticking periodic orbits with two transitions per period

In industrial applications, avoiding sticking phases of motion is sometimes necessary. In the past, several authors [12] investigated the existence of periodic nonsticking solutions of a one-degree-of-freedom oscillator subjected to simple harmonic loading. In [17] this problem is revisited for the two-degree-of-freedom oscillator considered in Fig. 1. The nonsticking orbit involves for each period  $\Theta = 2\pi/\omega$ , a slipping motion with a negative mass velocity and a slipping motion with a positive mass velocity (overshooting motion). We proved [17] that each part of the motion (part with negative velocity and part with positive velocity) has the same duration and that the phase portraits  $(x_i, x_i')$ ,  $i = (1, 2)$  of the two masses are symmetric with respect to the origin.

An example of these periodic orbits is obtained for the following values of the data:

$$\begin{aligned} \chi &= 0.3, \quad \eta = 4, \quad \omega = 0.6, \quad u_s = 0.1, \quad u_r = 0.2996, \\ \Theta &= 10.472 \end{aligned}$$

The corresponding values of the initial conditions and of the time lag  $\phi$  are obtained

$$\begin{aligned} x_{10} &= 1.5608, \quad x_{20} = 3.3295, \quad x'_{10} = 0.1523, \\ \phi &= 0.3925 \end{aligned}$$

The phase portraits  $(x_i, x_i')$ ,  $i = (1, 2)$  of the two masses are shown in Fig. 2 (the thick parts of the curves are related to the overshooting motion). These curves are symmetrical with respect to the origin.

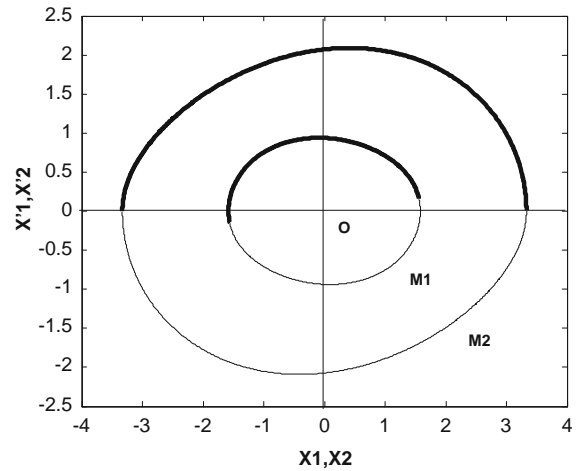


Fig. 2 Phase portrait of the nonsticking solution

### 4.2 Nonsticking orbits with abnormal stops

The nonsticking orbit investigated in the last paragraph involves for each period two normal stops:

$$x_2'(0) = x_2'(\pi/\omega) = 0$$

These stops occur when the displacement of the second mass reaches a local extremum and the mass reverses its direction of motion at the turning point. In [11], for a one-degree-of-freedom oscillator, with dry friction and harmonic load, the set of periodic orbits, including abnormal stops, has been obtained. Abnormal stops occur when at the turning point, the mass moves in the same direction as its motion prior to the stop. At these points, the second mass velocity reaches a local extremum. The same phenomenon can be observed for the two-degree-of-freedom oscillator investigated in this paper. At  $t = 0$ , the following initial conditions are assumed:

$$x_2'0 = 0, \quad f \equiv \chi \eta (x_{20} - x_{10}) - \cos \phi - \eta u_r > 0 \quad (12)$$

For  $0 < t < \tau$ , the system undergoes a slip motion ( $x_2' < 0$ ) given by

$$Z(t) - F(t) = H(t)(Z_0 - F_0) \quad (13)$$

Let us assume that, at  $t = \tau$

$$\begin{aligned} x_2'(\tau) &= 0, \quad f_1 \equiv \chi \eta (x_{2B} - x_{1B}) - \cos \phi_B + \eta u_r < 0 \\ x_{iB} &= x_i(\tau) \quad (i = 1, 2), \quad \phi_B = \omega \tau + \phi \end{aligned} \quad (14)$$

For  $t > \tau$ , the system undergoes an overshooting motion ( $x'_2 > 0$ ) given by

$$\begin{aligned}\bar{Z}(t) - F(t) &= H(t - \tau)(\bar{Z}_B - F_B) \\ \bar{Z}_B &= \bar{Z}(\tau), \quad F_B = F(\tau)\end{aligned}\quad (15)$$

Let us assume that, for  $t = \tau + \tau_1$

$$\begin{aligned}x'_2(\tau + \tau_1) &= 0, \\ f_2 &\equiv \chi\eta(x_{2C} - x_{1C}) - \cos\phi_C + \eta u_r < 0 \\ x_{iC} &= x_i(\tau + \tau_1) \quad (i = 1, 2), \quad \phi_C = \omega(\tau + \tau_1) + \phi\end{aligned}\quad (16)$$

For  $t > \tau + \tau_1$ , the system undergoes a new phase of overshooting motion ( $x'_2 > 0$ ) given by

$$\begin{aligned}\bar{Z}(t) - F(t) &= H(t - \tau - \tau_1)(\bar{Z}_C - F_C) \\ \bar{Z}_C &= \bar{Z}(\tau + \tau_1), \quad F_C = F(\tau + \tau_1) \\ \bar{Z}_C - F_C &= H(\tau_1)(\bar{Z}_B - F_B)\end{aligned}\quad (17)$$

It results that the stop obtained is an abnormal stop and that at this time

$$x''_2(\tau + \tau_1) = 0 \quad (18)$$

A periodic orbit of period  $\Theta = 2\pi/\omega$  is obtained if

$$\bar{Z}_0 - F_0 = H(T)(\bar{Z}_C - F_C), \quad T = \Theta - \tau - \tau_1 \quad (19)$$

From (17) and (19), it results

$$\bar{Z}_0 - F_0 = H(T + \tau_1)(\bar{Z}_B - F_B) \quad (20)$$

and from (13), we deduce

$$\begin{aligned}Z_B - F_B &= H(\tau)(Z_0 - F_0) \\ Z_B &= Z(\tau)\end{aligned}\quad (21)$$

In [17], we proved that the solution of (20), (21) is

$$\begin{aligned}\tau &= \tau_1 + T = \pi/\omega, \quad x'_B = -x'_0, \\ x_0 &= -x_B = Q \cos\phi, \quad \phi_B = \pi + \phi\end{aligned}\quad (22)$$

For each period, the motion is composed of a part with a positive velocity of the second mass for  $0 < t < \pi/\omega$  and a second part with a negative velocity of the second mass for  $\pi/\omega < t < 2\pi/\omega$ . Moreover, it has been proved in [17] that these two parts are symmetrical with respect to the origin. It results that for  $t = \tau_2$ ,  $0 < \tau_2 < \pi/\omega$ , a new abnormal stop exists:

$$\begin{aligned}x'(\tau_2) &= x''(\tau_2) = 0 \\ f_4 &\equiv \chi\eta(x_2(\tau_2) - x_1(\tau_2)) - \cos(\omega\tau_2 + \phi) - \eta u_r > 0\end{aligned}\quad (23)$$

Due to the symmetry of the two parts of the orbit, it results that

$$\begin{aligned}\tau_2 &= \tau_1, \quad x(\tau_1) = -x(\pi/\omega + \tau_1), \\ x'(\tau_1) &= -x'(\pi/\omega + \tau_1)\end{aligned}\quad (24)$$

From (22), we deduce  $f_1 = -f$ : if the condition (12) is fulfilled, then the condition (14) holds. From (18) and the equation of motion (2) with  $u = -u_s$  (overshooting motion), it results  $f_2 = \eta(u_r - u_s)$  and the condition (16) is fulfilled only if

$$u_r = u_s \quad (25)$$

Moreover, due to the symmetry, the condition (23) is fulfilled ( $f_4 = 0$ ).

From (21) and (22), we deduce the following system:

$$\begin{aligned}(H_1 + I)(x'_{10} + Q\omega \sin\phi) - H_3 d_0 &= 0 \\ H_2(x'_{10} + Q\omega \sin\phi) - (H_1 - I)d_0 &= 0 \\ H_i &= H_i(\pi/\omega), \quad (i = 1, 2, 3), \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\end{aligned}\quad (26)$$

From the property of the  $H_i$  matrices

$$H_1^2 - H_2 H_3 = I$$

it was shown [17] that the system (26) reduces to two scalar equations:

$$(H_1 + I)(x'_{10} + Q\omega \sin\phi) - H_3 d_0 = 0 \quad (27)$$

From (23) results two other scalar equations:

$$\begin{aligned}h_{21}(x'_{10} + q_1\omega \sin\phi) + h_{22}q_2\omega \sin\phi - h_{41}d_{01} \\ - h_{42}d_{02} - q_2\omega \sin(\omega\tau_1 + \phi) &= 0 \\ (h_{23} - h_{13})(x'_{10} + q_1\omega \sin\phi) + (h_{24} - h_{14})q_2\omega \sin\phi \\ - (h_{21} - h_{11})d_{01} - (h_{22} - h_{12})d_{02} \\ + [(q_2 - q_1) - (1/\chi\eta)] \cos(\omega\tau_1 + \phi) &= 0 \\ h_i &= H_i(\tau_1), \quad (i = 1, 2, 3), \quad h_1 = (h_{kl}), \\ h_2 &= (h_{kn}), \quad h_3 = (h_{nk}) \\ (k, l = 1, 2), \quad n &= (3, 4)\end{aligned}\quad (28)$$

Assuming that  $(\chi, \eta, \omega)$  are given data, the system (27), (28) gives the values of  $(\tau_1, \phi, x'_{10}, u_s)$ . A numerical example is obtained for

$$\chi = 0.7, \quad \eta = 3.8, \quad \omega = 2\pi/11$$

The other parameters are computed:

$$\begin{aligned}\tau &= 5.5, \quad \tau_1 = 4, \quad \phi = 2.6455, \quad u_s = u_r = 0.1717 \\ x_{10} &= 2.1249, \quad x_{20} = 2.0451, \quad x'_{10} = -0.7267\end{aligned}\quad (29)$$

The phase portrait of the system is shown in Fig. 3 (the thick parts of the curves are related to the overshooting motion).

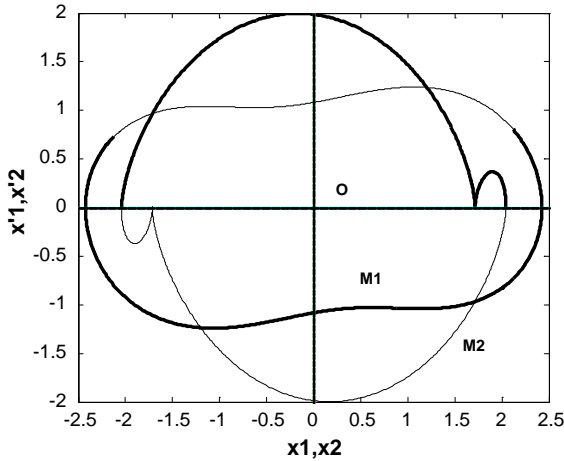


Fig. 3 Phase portrait of the system (abnormal stops)

## 5 Sticking orbits

### 5.1 Sticking orbits with one stop per period

Let us consider another periodic solutions beginning with the initial conditions (12), followed for  $(0 < t < \tau)$  by a slip motion of  $m_2$  given by (13). Let us assume that at  $t = \tau$ , the condition (14) holds. For  $t > \tau$ , the system undergoes an overshooting motion defined by (15). Let us assume that at  $t = \tau + \tau_1$ , instead of (16), the following conditions hold:

$$\begin{aligned} x'_{2c} &\equiv x'_2(\tau + \tau_1) = 0, \\ |\chi\eta(x_{2c} - x_{1c}) - \cos\phi_C| &< \eta u_r \\ x_{ic} &= x_i(\tau + \tau_1) \quad (i = 1, 2), \quad \phi_C = \phi + \omega(\tau + \tau_1) \end{aligned} \quad (30)$$

For  $t > \tau + \tau_1$ , the system undergoes a stick motion ( $x'_2 = 0$ ) defined by

$$\bar{Z}(t) = \Gamma(t - \tau - \tau_1)\bar{Z}_C \quad (31)$$

where  $\bar{Z}_C, F_C$  are deduced from (17). This motion ends if at  $t = \tau + \tau_1 + T$ , the following conditions hold:

$$\begin{aligned} -\eta u_r &= \chi\eta(x_{2D} - x_{1D}) - \cos\phi_D \\ \phi_D &= \omega(\tau + \tau_1 + T) + \phi \end{aligned} \quad (32)$$

For  $(0 < t - (\tau + \tau_1 + T) < \tau_2)$ , the system undergoes a new phase of overshooting motion defined by

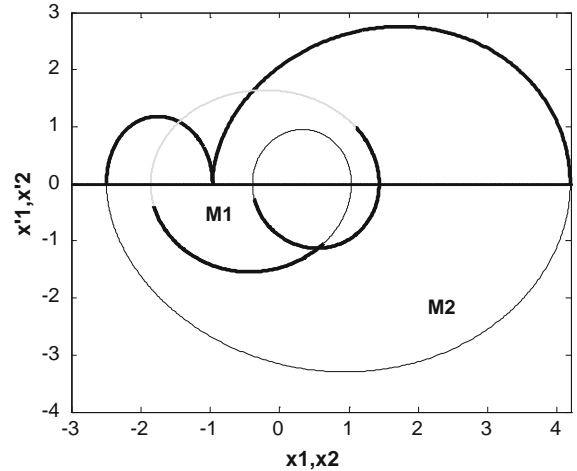


Fig. 4 Phase portraits of the sticking motion

$$\begin{aligned} \bar{Z}(t) - F(t) &= H(t - \tau - \tau_1 - T)(\bar{Z}_D - F_D) \\ \bar{Z}_D &\equiv \bar{Z}(\tau + \tau_1 + T) = \Gamma(T)\bar{Z}_C \\ F_D &= F(\tau + \tau_1 + T) \end{aligned} \quad (33)$$

A periodic solution is obtained if

$$\begin{aligned} \tau + \tau_1 + T + \tau_2 &= 2\pi/\omega \\ \bar{Z}_0 - F_0 &= H(\tau_2)(\bar{Z}_D - F_D) \end{aligned} \quad (34)$$

This solution depends of seven parameters ( $x_{10}, x_{20}, x'_{10}, \tau, \tau_1, T, \phi$ ). On the other hand, the conditions ( $x'_{2B} = x'_{2C} = 0$ ), (32) and (34) give seven scalar equations for the determination of these parameters. However, several conditions must be fulfilled:  $f > 0$ (12),  $f_1 < 0$ (14) and during the sticking motion

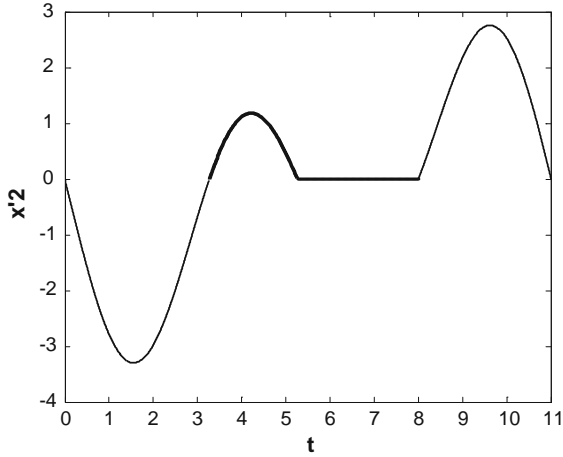
$$\begin{aligned} G &\equiv \chi\eta(x_2 - x_1) - \cos(\omega t + \phi) + \eta u_r \geq 0 \\ G_1 &\equiv -\chi\eta(x_2 - x_1) + \cos(\omega t + \phi) + \eta u_r > 0 \\ 0 &< t - \tau - \tau_1 < T \end{aligned} \quad (35)$$

A numerical example of this sticking solution is obtained for the following data:  $\chi = 0.22$ ,  $\eta = 4.2$ ,  $\omega = 2\pi/11$ ,  $u_r = 0.4962$ ,  $u_s = 0.0138$

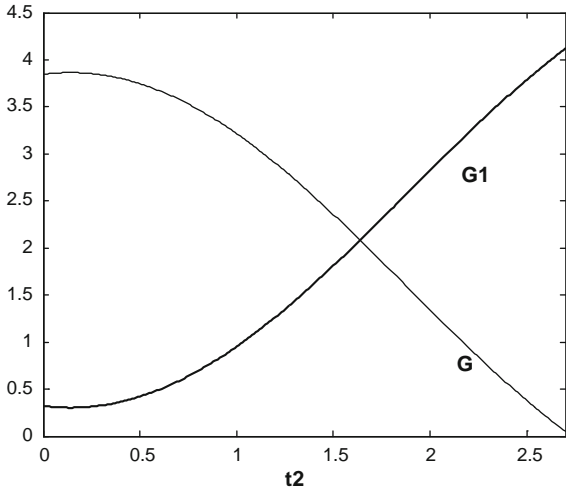
The other parameters are computed:

$$\begin{aligned} \tau &= 3.2663, \quad \tau_1 = 2, \quad \tau_2 = 3, \quad T = 2.7337, \quad \phi = 0.3177 \\ x_{10} &= -0.3643, \quad x_{20} = 4.1887, \quad x'_{10} = -0.2578 \end{aligned} \quad (36)$$

The phase portrait of the system is shown in Fig. 4. (The heavy black curves show the overshooting motion, the thin black curves show the slipping motion with  $x'_2 < 0$  and the grey curve shows the motion of  $m_1$  during the sticking phase).



**Fig. 5** Time behaviour of  $x'_2$

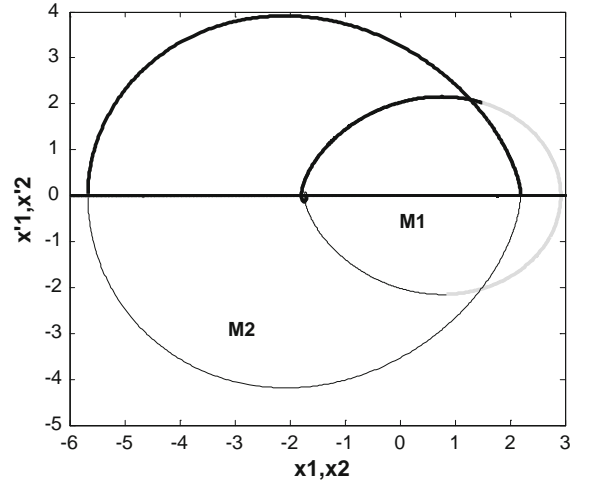


**Fig. 6** Constraints during the sticking part of the orbit

The time behaviour of  $x'_2$  is shown in Fig. 5, and the constraints during the sticking part ( $0 < t_2 < T$ ,  $t_2 = t - \tau - \tau_1$ ) of the solution are shown in Fig. 6.

## 5.2 Another sticking orbits with one stop per period

A second set of periodic orbits of period  $2\pi/\omega$  is obtained. This kind of orbits is a particular case (for a nonmoving belt) of the periodic solutions including an overshooting part investigated in [17]. For each period, the motion is composed of three parts. The first one is a slip motion of  $m_2$  with a negative velocity for ( $0 < t < \tau$ ); the next part ( $0 < t - \tau < \tau_1$ ) is an overshooting motion of the mass ( $x'_2 > 0$ ) and the



**Fig. 7** Phase portrait of the system

last part ( $0 < t - \tau - \tau_1 < T$ ) is a sticking motion of  $m_2$  ( $x'_2 = 0$ ,  $T = \Theta - \tau - \tau_1$ ). At the beginning of the orbit ( $t = 0$ ), instead of (12), the following condition is fulfilled:

$$x'_{20} = 0, \quad f \equiv \chi \eta (x_{20} - x_{10}) - \cos \phi - \eta u_r = 0 \quad (37)$$

At  $t = \tau$ , the condition (14) holds. At the beginning of the sticking motion ( $t = \tau + \tau_1$ ), the condition (30) is fulfilled. A periodic solution of period  $\Theta = \tau + \tau_1 + T$  is obtained if

$$\bar{Z}(\Theta) = \Gamma(T) \bar{Z}_C = \bar{Z}_0, \quad \bar{Z}_C = \bar{Z}(\tau + \tau_1) \quad (38)$$

Moreover, during the sticking motion, instead of (35), the following constraints hold:

$$G > 0, \quad G_1 \geq 0, \quad 0 < t_2 < T \quad (t_2 = t - \tau - \tau_1) \quad (39)$$

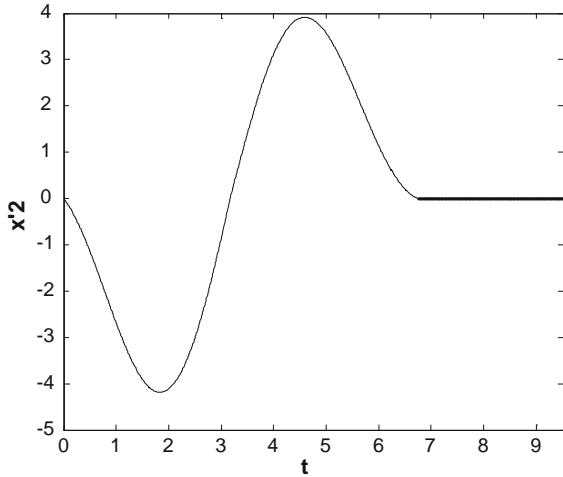
This kind of periodic solutions depends on six parameters ( $\tau$ ,  $\tau_1$ ,  $\phi$ ,  $x_{10}$ ,  $x_{20}$ ,  $x'_{10}$ ) and the conditions (14), (30), (37) and (38) give six scalar equations for the determination of these parameters. A numerical example is obtained for the data:

$$\chi = 0.35, \quad \eta = 4, \quad \omega = 0.66, \quad u_s = 0.1, \quad u_r = 0.4701$$

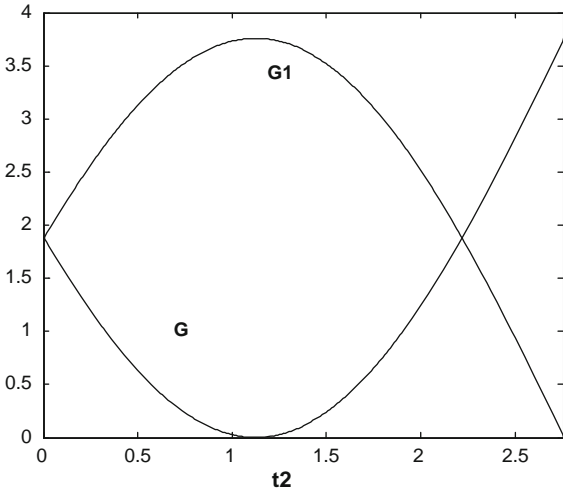
The other parameters are computed:

$$\tau = 3.18, \quad \tau_1 = 3.58, \quad T = 2.76, \quad \phi = 1.5708 \\ x_{10} = 0.8432, \quad x_{20} = 2.1864, \quad x'_{10} = -2.1511 \quad (40)$$

The phase portrait of the system is shown in Fig. 7. (The heavy black curves show the overshooting motion, the



**Fig. 8** Time behaviour of  $x'_2$



**Fig. 9** Constraints during sticking

thin black curves show the slipping motion with  $x'_2 < 0$  and the grey curve shows the motion of  $m_1$  during the sticking phase).

The time behaviour of  $x'_2$  is shown in Fig. 8, and the constraints during the sticking part ( $0 < t_1 < T$ ,  $t_1 = t - \tau - \tau_1$ ) of the solution are shown in Fig. 9.

## 6 Periodic solutions with several stops

In [11], for a one-degree-of-freedom oscillator excited by dry friction and harmonic loading, it is found many types of steady-state behaviour with several stops per

cycles. All these results were obtained by numerical methods. In the following, for the two-degree-of-freedom oscillator considered in this paper, we investigate the existence of periodic solutions including two stops per period. Moreover, we assume that the orbit is composed of two parts. The first part ( $0 < t < \pi/\omega$ ) involves for ( $0 < t < \tau_1$ ) a slip motion ( $x'_2 < 0$ ), then for ( $0 < t - \tau_1 < T$ ), a stick motion ( $x'_2 = 0$ ) and at last for ( $0 < t - \tau_1 - T < \tau_2$ ) a new phase of slip motion. The following relations hold

$$\begin{aligned} Z_B - F_B &= H(\tau_1)(Z_0 - F_0), \quad Z_B = Z(\tau_1), \\ F_B &= F(\tau_1) \\ Z_C &= \Gamma(T)Z_B, \quad Z_C = Z(\tau_1 + T), \\ Z_D - F_D &= H(\tau_2)(Z_C - F_C) \\ F_C &= F(\tau_1 + T), \quad Z_D = Z(\tau_1 + T + \tau_2), \\ F_D &= F(\tau_1 + T + \tau_2) \end{aligned} \quad (41)$$

From (41), it results

$$\begin{aligned} Z_D - F_D &= A(Z_0 - F_0) + NF_B - hF_C \\ h &= H(\tau_2), \quad N = h\Gamma(T), \quad A = NH(\tau_1) \end{aligned} \quad (42)$$

Let us assume that  $\tau_1 + T + \tau_2 = \pi/\omega$ .

The other part ( $0 < t_1 < \pi/\omega$ ,  $t_1 = t - \pi/\omega$ ) of the orbit is obtained by symmetry in time. For  $0 < t_1 < \tau_1$ , the system undergoes an overshooting motion ( $x'_2 > 0$ ), then for ( $0 < t_1 - \tau_1 < T$ ) a stick motion and at last for ( $0 < t_1 - \tau_1 - T < \tau_2$ ) a new phase of overshooting motion. The following relations hold:

$$\begin{aligned} \bar{Z}_E - F_E &= H(\tau_1)(\bar{Z}_D - F_D), \quad Z_E = Z(\pi/\omega + \tau_1), \\ F_E &= F(\pi/\omega + \tau_1) \\ \bar{Z}_G &= \Gamma(T)\bar{Z}_E, \quad \bar{Z}_G = \bar{Z}(\pi/\omega + \tau_1 + T) \\ \bar{Z}_0 - F_0 &= H(\tau_2)(\bar{Z}_G - F_G), \quad F_G = F(\pi/\omega + \tau_1 + T) \end{aligned} \quad (43)$$

From (43), it results

$$\bar{Z}_D - F_D = A(\bar{Z}_D - F_D) + NF_E - hF_G \quad (44)$$

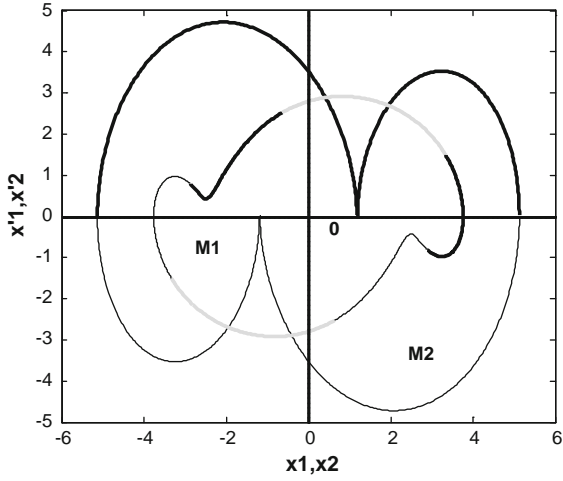
Taking into account (42), (44) and the relations

$$\begin{aligned} F_D &= -F_0, \quad F_E = -F_B, \quad F_G = -F_C \\ Z_D + \bar{Z}_0 &= \bar{Z}_D + Z_0 = X_D + X_0 \\ X_0 &= \begin{bmatrix} x_0 \\ x'_0 \end{bmatrix}, \quad X_D = \begin{bmatrix} x_D \\ x'_D \end{bmatrix}, \end{aligned} \quad (45)$$

we obtain the following result:

$$(A - I_4)(X_D + X_0) = 0, \quad I_4 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad (46)$$





**Fig. 10** Phase portrait of the orbit with two stops

If  $\det(A - I_4) \neq 0$ , the solution of (46) is  $X_D + X_0 = 0$ , hence

$$x_D = -x_0, \quad x'_D = -x'_0 \quad (47)$$

Several constraints must be fulfilled. For the first half period, we have

$$\begin{aligned} x'_{20} = x'_{2B} = 0 \\ \chi\eta(x_{2C} - x_{1C}) - \cos\phi_C - \eta u_r = 0 \\ \phi_C = \omega(\tau_1 + T) + \phi \end{aligned} \quad (48)$$

During the sticking motion, we have the constraints:

$$G > 0, \quad G_1 \geq 0, \quad (0 < t - \tau_1 < T) \quad (49)$$

The conditions during the second half period are the following:

$$\begin{aligned} x'_{2E} = 0, \quad \chi\eta(x_{2G} - x_{1G}) - \cos\phi_G + \eta u_r = 0 \\ \phi_G = \pi + \phi_C \end{aligned} \quad (50)$$

and the constraints during the sticking motion are

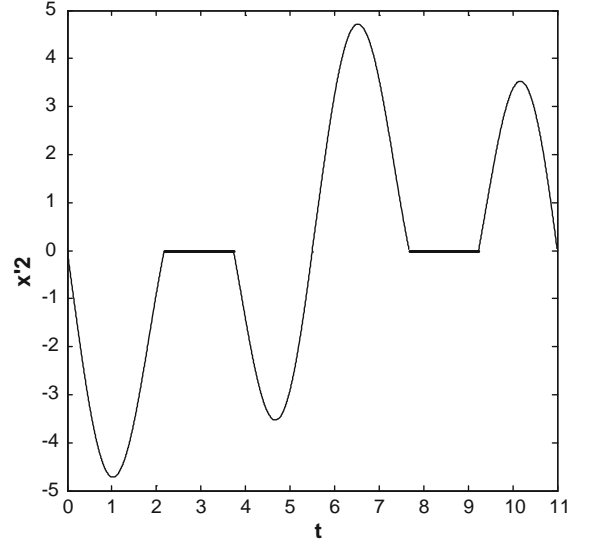
$$G \geq 0, \quad G_1 > 0, \quad (0 < t - \pi/\omega - \tau_1 < T) \quad (51)$$

From (45), (47), it follows

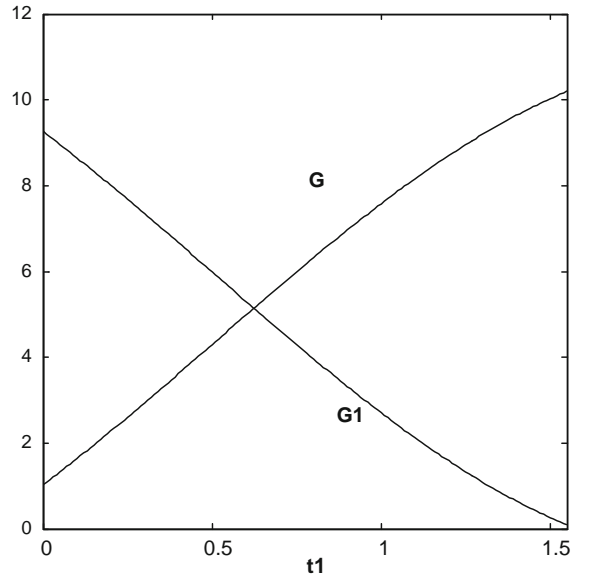
$$\begin{aligned} \bar{Z}_E - F_E = H(\tau_1)(\bar{Z}_D - F_D) = -H(\tau_1)(Z_0 - F_0) \\ = -(Z_B - F_B) \end{aligned} \quad (52)$$

From (52), we obtain  $\bar{Z}_E = -Z_B$ , hence  $X_E = -X_B$ . Similarly,  $\bar{Z}_G = -Z_C$ ,  $X_G = -X_C$ .

From these results, it follows that the solution is symmetrical in time and also in space. Hence, it is sufficient to take into account the relations (48) and (49). On



**Fig. 11** Time behaviour of  $x'_2$



**Fig. 12** Constraints during the sticking part of the orbit

the other hand, the periodicity conditions are deduced from (44) and (47)

$$(A + I_4)(Z_0 - F_0) + NF_B - hF_C + 2\Delta_0 = 0 \quad (53)$$

The solution depends on six parameters ( $\tau_1$ ,  $\tau_2$ ,  $\phi$ ,  $x_{10}$ ,  $x_{20}$ ,  $x'_{10}$ ) which are defined by 6 scalar equations deduced from (48) and (53). A numerical example is obtained for the following data:

$$\chi = 0.7, \eta = 3.8, \omega = 2\pi/11, u_s = 0.06, \\ u_r = 1.4041$$

The other parameters are computed:

$$\tau_1 = 2.175, \tau_2 = 1.765, T = 1.56, \phi = 3 \\ x_{10} = 2.8901, x_{20} = 5.1306, x'_{10} = -0.8108 \quad (54)$$

The phase portrait of the system is shown in Fig. 10. (The heavy black curves show the overshooting motion, the thin black curves show the slipping motion with  $x'_2 < 0$  and the grey curve shows the motion of  $m_1$  during sticking phase). The phase portrait is symmetrical with respect to the origin.

The time behaviour of  $x'_2$  is shown in Fig. 11, and the constraints during the sticking part ( $0 < t_1 < T$ ,  $t_1 = t - \tau_1$ ) of the solution are shown in Fig. 12.

## 7 Conclusion

In this work, the steady-state response of a two-degree-of-freedom oscillator subjected to dry friction and harmonic load is considered. The system consists of two masses connected by linear springs; one of the masses is in contact with a rough surface and is subjected to a harmonic external force. Assuming Coulomb's laws of dry friction, it is found several kinds of periodic orbits, including one or more stops per cycles. Tacking into account that the dynamical model of the system is a piecewise linear system, several interesting analytical results related to the existence and the properties of the orbits have been obtained. It was shown that among the periodic solutions obtained, some of them include stops with zero duration time (nonsticking orbits), the other ones involve phases of sticking motions during which the mass in contact with the rough surface did not move. Moreover, some of these orbits are obtained assuming that symmetry in space and time occurs; for other orbits, the property of symmetry in space and time is proven. At last, a third set of orbits is found without any symmetry in space or in time.

## Appendix 1

$$H_i(t) = \Lambda B_i(t) \Lambda^{-1}, \quad (i = 1, 2, 3) \\ B_2(t) = \begin{pmatrix} s_1/\omega_1 & 0 \\ 0 & s_2/\omega_2 \end{pmatrix}, \quad s_j = \sin(\omega_j t) \quad (j = 1, 2) \\ B_1(t) = B'_2(t), \quad B_3(t) = B''_2(t) \quad (55)$$

The natural frequencies ( $\omega_1, \omega_2$ ) are the roots of the characteristic equation:

$$D(s^2) \equiv \det(K - I s^2) = 0, \quad K = \begin{pmatrix} 1 & -\chi \\ -\chi\eta & \chi\eta \end{pmatrix} \quad (56)$$

The eigenvectors  $\psi_j = \begin{pmatrix} 1 \\ \lambda_j \end{pmatrix}$ , ( $j = 1, 2$ ) are defined by  $(K - I\omega_j^2)\psi_j = 0$

These matrices fulfil the following property

$$H_1^2(t) - H_2(t)H_3(t) = 0 \quad (57)$$

$$\Gamma_i(t) = \Sigma \gamma_i(t) \Sigma^{-1}, \quad (i = 1, 2, 3)$$

$$\gamma_2(t) = \begin{pmatrix} \sin t & 0 \\ 0 & t \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & \chi \\ 0 & 1 \end{pmatrix}$$

$$\gamma_1(t) = \gamma'_2(t), \quad \gamma_3(t) = \gamma''_2(t) \quad (58)$$

The matrices  $\Gamma_i(t)$  fulfil also the property

$$\Gamma_1^2(t) - \Gamma_2(t)\Gamma_3(t) = 0 \quad (59)$$

## References

1. Andreaus, U., Casini, P.: Dynamics of friction oscillators excited by a moving base and/or driving force. *J. Sound Vib.* **245**(4), 685–699 (2001)
2. Awrejcewicz, J., Olejnik, P.: Stick-slip dynamics of a two-degree-of-freedom system. *Int. J. Bifurc. Chaos* **13**(4), 843–861 (2003)
3. Awrejcewicz, J., Fečkan, M., Olejnik, P.: On continuous approximation of discontinuous systems. *Nonlinear Anal.* **62**(7), 1317–1331 (2005)
4. Awrejcewicz, J., Olejnik, P.: Friction pair modeling by a 2-dof system: numerical and experimental investigations. *Special Issue Int. J. Bifurc. Chaos* **15**(6), 1931–1944 (2005)
5. Awrejcewicz, J., Holické, M.: Smooth and nonsmooth high dimensional chaos and the Melnikov-type methods. *World Scientific, Singapore* (2007)
6. Csernak, G., Stepan, G.: On the periodic response of a harmonically excited dry friction oscillator. *J. Sound Vib.* **295**(4), 649–658 (2006)
7. Galvanetto, U., Bishop, S.R.: Stick-slip vibrations of a 2-degree-of-freedom geophysical fault model. *Int. J. Mech. Sci.* **36**(8), 683–698 (1994)
8. Guo, K., Zhang, X., Li, H., Meng, G.: Non-reversible friction modeling and identification. *Arch. Appl. Mech.* **78**(10), 795–809 (2008)
9. Hetzler, H., Schwarzer, D., Seemann, W.: Analytical investigation of steady-state stability and Hopf-bifurcations occurring in sliding friction oscillators with application to low frequency disc brake noise. *Commun. Nonlinear Sci. Numer. Simul.* **12**(1), 83–99 (2007)
10. Hoffmann, N., Fisher, M., Allgaier, R., Gaiul, L.: A minimal model of studying properties of the mode-occurring type instability in friction induced oscillations. *Mech. Res. Commun.* **29**, 197–205 (2002)

- 
11. Hong, H.K., Liu, C.S.: Coulomb friction oscillator: modelling and responses to harmonic loads and base excitations. *J. Sound Vib.* **229**(5), 1171–1192 (2000)
  12. Hong, H.K., Liu, C.S.: Non-sticking formulae for Coulomb friction under harmonic loading. *J. Sound Vib.* **244**(5), 883–898 (2001)
  13. Kirillov, O.N.: Subcritical flutter in the acoustics of friction. *Proc. Royal Soc. A* **464**, 2321–2339 (2008)
  14. Pascal, M.: Dynamics of coupled oscillators excited by dry friction. *ASME J. Comput. Nonlinear Dyn.* **3**(3), 20–26 (2008)
  15. Pascal, M.: Two models of non smooth dynamical systems. *Int. J. Bifurc. Chaos* **21**(10), 2853–2860 (2011)
  16. Pascal, M.: New events in stick-slip oscillators behaviour. *J. Appl. Math. Mech.* **75**(3), 402–409 (2011)
  17. Pascal, M.: New limit cycles of dry friction oscillators under harmonic load. *Nonlinear Dyn.* **70**, 1435–1443 (2012)
  18. Sinou, J.-J., Jezequel, L.: Mode coupling instability in friction-induced vibrations and its dependency on system parameters including damping. *Eur. J. Mech. A* **256**(1), 106–122 (2007)
  19. von Wagner, U., Hochlenert, D., Hagedorn, P.: Minimal models for disc brake squeal. *J. Sound Vib.* **302**, 527–539 (2007)