Functional identities for L-series values in positive characteristic

Bruno Angles, Federico Pellarin

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Abstract. In this paper we show the existence of functional relations for a class of L-series introduced by the second author in [13]. Our results will be applied to obtain a new class of congruences for Bernoulli-Carlitz fractions, and an analytic conjecture is stated, implying an interesting behavior of such fractions modulo prime ideals of $\mathbb{F}_q[\theta]$.

1 Introduction, results

Let $\mathbb{F}_q$ be a finite field having q elements, let θ be an indeterminate over $\mathbb{F}_q$, $A = \mathbb{F}_q[\theta]$ and $K = \mathbb{F}_q(\theta)$. In all the following, we denote by v_∞ the θ^{-1}-adic valuation normalized by setting $v_\infty(\theta) = -1$. Let K_∞ be the completion of K for v_∞, and let us consider the completion $\mathbb{C}_\infty$ of an algebraic closure of K_∞ for the unique extension of this valuation, in which we embed an algebraic closure K^{alg} of K. Carlitz's exponential function is the surjective, $\mathbb{F}_q$-linear, rigid analytic entire function

$$\exp_C : \mathbb{C}_\infty \to \mathbb{C}_\infty$$

defined by

$$\exp_C(z) = \sum_{n \geq 0} \frac{z^{q^n}}{d_n},$$

where

$$d_0 = 1, \quad d_n = (\theta^{q^n} - \theta)(\theta^{q^n} - \theta^q) \cdots (\theta^{q^n} - \theta^{q^{n-1}}), \quad n > 0.$$

The kernel of this function, surjective, is equal to $\widetilde{\pi} A$, where the period $\widetilde{\pi}$, unique up to multiplication by an element of $\mathbb{F}_q^\times$, can be computed by using the following product expansion

$$\widetilde{\pi} := \theta(-\theta)^{\frac{1}{q-1}} \prod_{i=1}^{\infty} (1 - \theta^{1-q^i})^{-1} \in (-\theta)^{\frac{1}{q-1}} K_\infty, \tag{1}$$

once a $(q-1)$-th root of $-\theta$ is chosen.

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Anderson-Thakur function. This function, introduced in [2, Proof of Lemma 2.5.4 p. 177], is defined by the infinite product

$$\omega(t) = (-\theta)^{\frac{1}{q-1}} \prod_{i\geq 0} \left(1 - \frac{t}{\theta^{q^i}}\right)^{-1}, \tag{2}$$

with the same choice of the $(q-1)$-th root as in (1), converges for $t \in \mathbb{C}_\infty$ such that $|t| \leq 1$ (where $|\cdot|$ is an absolute value associated to v_∞) and can be extended to a non-zero rigid analytic function over

$$\mathbb{C}_\infty \setminus \{\theta^{q^k}; k \geq 0\}$$

with simple poles at θ^{q^k}, $k \geq 0$.

Let t be a variable in $\mathbb{C}_\infty$ and let us consider the $\mathbb{F}_q$-algebra homomorphism

$$\chi_t : A \to \mathbb{F}_q[t]$$

defined by formal replacement of θ by t. In other words, χ_t may be viewed as the unique $\mathbb{F}_q$-algebra homomorphism from A to the ring of rigid analytic functions $\mathbb{C}_\infty \to \mathbb{C}_\infty$ such that $\chi_t(\theta) = t$. More generally, we shall consider s independent variables and the $\mathbb{F}_q$-algebra homomorphisms

$$\chi_{t_i} : A \to \mathbb{F}_q[t_1, \ldots, t_s], \quad i = 1, \ldots, s$$

defined respectively by $\chi_{t_i}(\theta) = t_i$. To simplify our notations, we will write $\chi_\xi(a)$ for the evaluation at $t = \xi$ of the polynomial function $\chi_t(a)$ at a given element $\xi \in \mathbb{C}_\infty$. Let α be a positive integer and let $\beta_1, \ldots \beta_s$ be non-negative integers. The following formal series was introduced in [13]:

$$L(\chi_{t_1}^{\beta_1} \cdots \chi_{t_s}^{\beta_s}, \alpha) = \sum_{d\geq 0} \sum_{a \in A^+(d)} \chi_{t_1}(a)^{\beta_1} \cdots \chi_{t_s}(a)^{\beta_s} a^{-\alpha} \in K_\infty[[t_1, \ldots, t_s]]. \tag{3}$$

Here and in all the following, $A^+(d)$ denotes the set of monic polynomials of A of degree d. It is easy to see that this series is well defined. As claimed in [13, Remark 7], this series converges for all $(t_1, \ldots, t_s) \in \mathbb{C}_\infty^s$ to a rigid analytic entire function of s variables $t_1, \ldots, t_s$; see Proposition 6.

The residue of $\omega(t)$ at $t = \theta$ is $-\widetilde{\pi}$:

$$\widetilde{\pi} = -\lim_{t\to\theta}(t-\theta)\omega(t).$$

In [13, Theorem 1], it is proved that

$$L(\chi_t, 1) = \frac{\widetilde{\pi}}{(\theta - t)\omega(t)}.$$

Taking into account the functional equation

$$\omega(t)^q = (t^q - \theta)\omega(t^q)$$

apparent in (2), this implies that, for $m \geq 0$ integer,

$$V_{q^m,1}(t) := \widetilde{\pi}^{-q^m}\omega(t)L(\chi_t, q^m) = \frac{1}{(\theta^{q^m} - t)(\theta^{q^{m-1}} - t)\cdots(\theta - t)}.$$

This result provided an awaited connection between the function ω of Anderson and Thakur and the "even" values of the *Goss zeta function* (or *Carlitz zeta values*)

$$\zeta(n) = \frac{BC_n \widetilde{\pi}^n}{\Pi(n)}, \quad n > 0, \quad n \equiv 0 \pmod{q-1}$$

where BC_n and $\Pi(n)$ denote respectively the n-th Bernoulli-Carlitz fraction and Carlitz's factorial of n, see Goss' book [11, Section 9.1]. Indeed, evaluating at $t = \theta$, we get

$$L(\chi_\theta, q^m) = \zeta(q^m - 1), \quad m \geq 1.$$

More generally, it is proved in [13, Theorem 2] that, if $\alpha \equiv 1 \pmod{q-1}$ and $\alpha \geq 1$, then

$$\mu_\alpha = \widetilde{\pi}^{-\alpha} L(\chi_t, \alpha)\omega(t)$$

is a rational function in $\mathbb{F}_q(\theta, t)$. In [13], it is suggested that this result could be a source of information in the study of the arithmetic properties of the Bernoulli-Carlitz fractions. However, the methods of loc. cit. (based on *deformations of vectorial modular forms* and Galois descent) are only partially explicit.

More recently, Perkins [14] investigated the properties of certain *special polynomials* associated to variants of the functions $L(\chi_t^\beta, \alpha)$ with $\alpha \leq 0$ which turn out to be polynomial. He notably studied the growth of their degrees. Moreover, by using *Wagner's interpolation theory* for the map χ_t, Perkins [15] obtained explicit formulas for the series

$$L(\chi_{t_1} \cdots \chi_{t_s}, \alpha), \quad \alpha > 0, \quad 0 \leq s \leq q, \quad \alpha \equiv s \pmod{q-1}.$$

We quote here a particular case of Perkins' formulas for the functions $L(\chi_t, \alpha)$ with $\alpha \equiv 1 \pmod{q-1}$

$$L(\chi_t, \alpha) = \sum_{j=0}^{\mu} d_j^{-1}(t-\theta)(t-\theta^q)\cdots(t-\theta^{q^{j-1}})\zeta(\alpha - q^j)L(\chi_t, q^j), \tag{4}$$

where μ is the biggest integer such that $q^\mu \leq \alpha$. It seems difficult to overcome the threshold $s \leq q$ giving at once expressions for $L(\chi_{t_1} \cdots \chi_{t_s}, \alpha)$ with the effectiveness of Perkins' results.

In the next Theorem, we extend the previous results beyond the mentioned threshold, providing at once new quantitative information. Let n be a non-negative integer, let us consider the expansion $n = \sum_{i=0}^{r} n_i q^i$ in base q, where $n_i \in \{0, \cdots, q-1\}$. We denote by $\ell_q(n)$ the sum of the digits n_i of this expansion: $\ell_q(n) = \sum_{i=0}^{r} n_i$. We have:

Theorem 1 *Let α, s be positive integers, such that $\alpha \equiv s \pmod{q-1}$. Let δ be the smallest non-negative integer such that, simultaneously, $q^\delta - \alpha \geq 0$ and $s + \ell_q(q^\delta - \alpha) \geq 2$. The formal series:*

$$V_{\alpha,s}(t_1, \ldots, t_s) = \widetilde{\pi}^{-\alpha} L(\chi_{t_1} \cdots \chi_{t_s}, \alpha)\omega(t_1)\cdots\omega(t_s) \prod_{i=1}^{s}\prod_{j=0}^{\delta-1}\left(1 - \frac{t_i}{\theta^{q^j}}\right) \in K_\infty[[t_1, \ldots, t_s]] \tag{5}$$

is in fact a symmetric polynomial of $K[t_1, \ldots, t_s]$ of total degree $\delta(\alpha, s)$ such that

$$\delta(\alpha, s) \leq s\left(\frac{s + \ell_q(q^\delta - \alpha)}{q-1}\right) - s.$$

This statement holds if $\alpha = q^m$ and $s \geq 2$ (so that $\delta = m$) assuming that empty products are equal to one by convention. In this case, since $s \equiv \alpha \pmod{q-1}$, we have $s + \ell_q(q^\delta - \alpha) \equiv 0 \pmod{q-1}$ so that in fact, $s \geq \max\{2, q-1\}$. The reader may have noticed that the choice $\alpha = q^m$ and $s = 1$ is not allowed in Theorem 1. However, as mentioned above, the computation of $V_{q^m,1}$ is completely settled in [13]. This discrimination of the case $\alpha = q^m, s = 1$ should not be surprising neither; similarly, the Goss zeta function associated to A has value 1 at zero, but vanishes at all negative integers divisible by $q-1$.

In Section 3, we will be more specifically concerned with Bernoulli-Carlitz numbers. A careful investigation of the polynomials $V_{1,s}$ and an application of the digit principle ([8]) to the function ω will allow us to show that, for $s \geq 2$ congruent to one modulo $q-1$,

$$\mathbb{B}_s = \Pi(s)^{-1} V_{1,s}(\theta, \ldots, \theta)$$

is a polynomial of $\mathbb{F}_q[\theta]$ (Proposition 24) ([1]). We don't know wether $\mathbb{B}_s$ vanishes or not for general s. In all the following, a *prime* is by definition a monic irreducible polynomial in A. We shall then show the next Theorem, which highlights the interest of these polynomials in θ.

Theorem 2 *Let $s \geq 2$, $s \equiv 1 \pmod{q-1}$. Let us consider the expansion $s = \sum_{i=0}^{r} s_i q^i$ of s in base q. Let d be an integer such that $q^d > s$ and let $\mathfrak{p}$ be a prime of degree d. Then:*

$$\mathbb{B}_s \equiv \frac{(-1)^s BC_{q^d - s} \prod_{i=0}^{r} l_{d-i-1}^{s_i q^i}}{\Pi(q^d - s)} \pmod{\mathfrak{p}}.$$

In this result, l_d denotes the polynomial $(-1)^d \prod_{i=1}^{d} (\theta^{q^i} - \theta)$; we observe that the latter polynomial is invertible modulo $\mathfrak{p}$ just as $\Pi(q^d - s)$. The non-vanishing of $\mathbb{B}_s$ for fixed s signifies the existence of an explicit constant $c > 0$, depending on s and q, such that for all $d \geq c$,

$$BC_{q^d - s} \not\equiv 0 \pmod{\mathfrak{p}}, \quad \text{for all } \mathfrak{p} \text{ such that } \deg \mathfrak{p} = d. \tag{6}$$

However, the non-vanishing of $\mathbb{B}_s$ is also equivalent to the fact that the function

$$L(\chi_{t_1} \cdots \chi_{t_s}, 1) \prod_{i=1}^{s} (t_i - \theta)^{-1},$$

entire of s variables as we will see, is a unit when identified to an element of $\mathbb{C}_\infty[[t_1 - \theta, \ldots, t_s - \theta]]$; we presently do not know how to prove this property for all s. Therefore, the property (6) is linked with the following conjecture of nature analogue of classical results on the simplicity of the zeroes of Goss zeta functions and L-series, which should be, we believe, true.

Conjecture 3 *Let $s \geq 2$ be congruent to one modulo $q-1$. Then, locally at $t_1 = \cdots = t_s = \theta$, the set of the zeroes of the function $L(\chi_{t_1} \cdots \chi_{t_s}, 1)$ is equal to the set of zeroes of the polynomial $\prod_i (t_i - \theta)$.*

Numerical computations on Bernoulli-Carlitz fractions made by Taelman provide some evidence to support this hypothesis. The Conjecture follows from Perkins results [15] in the case $s \leq q$ and $\alpha = s$. The conjecture is also verified if $\ell_q(s) = q$ and $\alpha = 1$, thanks to Corollary 26.

[1] Note that $\mathbb{B}_1$ is not well defined

2 Functional identities for L-series

Let d, s be non-negative integers. We begin with the study of the vanishing of the sums

$$S_{d,s} = S_{d,s}(t_1, \dots, t_s) = \sum_{a \in A^+(d)} \chi_{t_1}(a) \cdots \chi_{t_s}(a) \in \mathbb{F}_q[t_1, \dots, t_s],$$

which are symmetric polynomials in $t_1, \dots, t_s$ of total degree $\leq ds$, with the standard conventions on empty products. We recall that, for $n \geq 0$,

$$\sum_{a \in \mathbb{F}_q} a^n$$

equals -1 if $n \equiv 0 \pmod{q-1}$ and $n \geq 1$, and equals 0 otherwise. We owe the next Lemma to D. Simon [7]. We give the proof here for the sake of completeness.

Lemma 4 (Simon's Lemma) *We have $S_{d,s} \neq 0$ if and only if $d(q-1) \leq s$.*

Proof. Since

$$S_{d,s} = \sum_{a_0 \in \mathbb{F}_q} \cdots \sum_{a_{d-1} \in \mathbb{F}_q} \prod_{i=1}^{s} (a_0 + a_1 t_i + \cdots + a_{d-1} t_i^{d-1} + t_i^d),$$

the coefficient $c_{v_1,\dots,v_s}$ of $t_1^{v_1} \cdots t_s^{v_s}$ with $v_i \leq ds$ $(i = 1, \dots, s)$ is given by the sum:

$$\sum_{a_0 \in \mathbb{F}_q} \cdots \sum_{a_{d-1} \in \mathbb{F}_q} a_{v_1} \cdots a_{v_s}.$$

The last sum can be rewritten as:

$$c_{v_1,\dots,v_s} = \left(\sum_{a_0 \in \mathbb{F}_q} a_0^{\mu_0} \right) \cdots \left(\sum_{a_{d-1} \in \mathbb{F}_q} a_{d-1}^{\mu_{d-1}} \right), \tag{7}$$

where μ_i is the cardinality of the set of the indices j such that $v_j = i$, from which one notices that

$$\sum_{i=0}^{d-1} \mu_i \leq s$$

(notice also that $s - \sum_i \mu_i$ is the cardinality of the set of indices j such that $v_j = d$). For any choice of $\mu_0, \dots, \mu_{d-1}$ such that $\sum_i \mu_i \leq s$, there exists $(v_1, \dots, v_s)$ such that (7) holds.

If $s < d(q-1)$, for all $(v_1, \dots, v_s)$ as above, there exists i such that, in (7), $\mu_i < q-1$ so that $S_{d,s} = 0$. On the other hand, if $s \geq d(q-1)$, it is certainly possible to find $(v_1, \dots, v_s)$ such that, in (7), $\mu_0 = \cdots = \mu_{d-1} = q-1$ so that the sum does not vanish in this case. □

As an immediate corollary of Lemma 4, we see that the series

$$F_s = F_s(t_1, \dots, t_s) = \sum_{d \geq 0} S_{d,s} = \sum_{d \geq 0} \sum_{a \in A^+(d)} \chi_{t_1}(a) \cdots \chi_{t_s}(a)$$

defines a symmetric polynomial of $\mathbb{F}_q[t_1, \dots, t_s]$ of total degree at most $\frac{s^2}{q-1}$. In the next Lemma, we provide a necessary and sufficient condition for the vanishing of the polynomial F_s.

Lemma 5 *If $s \geq 1$, then, $F_s = 0$ if and only if $s \equiv 0 \pmod{q-1}$.*

Proof. We will use several times the following elementary observation : let $G \in \mathbb{C}_\infty[t_1, \ldots, t_s]$ and let $S_1, \ldots, S_s \subset \mathbb{F}_q^{\mathrm{alg}}$ ($\mathbb{F}_q^{\mathrm{alg}}$ denotes the algebraic closure of $\mathbb{F}_q$, embedded in $\mathbb{C}_\infty$) be infinite sets such that G vanishes on $S_1 \times \cdots \times S_s$. Then $G = 0$.

Let us assume first that $s \equiv 0 \pmod{q-1}$. The hypothesis on s implies that

$$\sum_{a \in A, \deg_\theta(a) = d} \chi_{t_1}(a) \cdots \chi_{t_s}(a) = -S_{d,s}.$$

We denote by $A(\leq d)$ the set of polynomials of A of degree $\leq d$ and we write

$$G_{d,s} = \sum_{a \in A(\leq d)} \chi_{t_1}(a) \cdots \chi_{t_s}(a).$$

We then have:

$$G_{\frac{s}{q-1}, s} = -F_s.$$

Let us choose now distinct primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ of respective degrees $d_1, \ldots, d_s \geq s/(q-1)$ and $\mathfrak{f} = \mathfrak{p}_1 \cdots \mathfrak{p}_s$. For all $i = 1, \ldots, s$, we choose a root $\zeta_i \in \mathbb{F}_q^{\mathrm{alg}}$ of $\mathfrak{p}_i$. Let us then consider the Dirichlet character of the first kind $\chi = \chi_{\zeta_1} \cdots \chi_{\zeta_s}$. We have:

$$\begin{aligned} F_s(\zeta_1, \ldots, \zeta_s) &= -G_{\frac{s}{q-1}, s}(\zeta_1, \ldots, \zeta_s) \\ &= -\sum_{a \in A(\leq s/(q-1))} \chi(a) \\ &= -\sum_{a \in A(\leq d_1 + \cdots + d_s)} \chi(a) \\ &= -\sum_{a \in (A/\mathfrak{f}A)^\times} \chi(a) \\ &= 0, \end{aligned}$$

by [16, Proposition 15.3]. By the observation at the beginning of the proof, this implies the vanishing of F_s. On the other hand, if $s \not\equiv 0 \pmod{q-1}$, then $F_s(\theta, \ldots, \theta) = \zeta(-s)$ the s-th Goss' zeta value which is non zero, see [10, Remark 8.13.8 1].

□

2.1 Analyticity

The functions $L(\chi_{t_1} \cdots \chi_{t_s}, \alpha)$ are in fact rigid analytic entire functions of s variables. This property, mentioned in [13], can be deduced from the more general Proposition 6 that we give here for convenience of the reader.

Let a be a monic polynomial of A. we set:

$$\langle a \rangle = \frac{a}{\theta^{\deg_\theta(a)}} \in 1 + \theta^{-1}\mathbb{F}_q[\theta^{-1}].$$

Let $y \in \mathbb{Z}_p$, where p is the prime dividing q. Since $\langle a \rangle$ is a 1-unit of K_∞, we can consider its exponentiation by y:

$$\langle a \rangle^y = \sum_{j \geq 0} \binom{y}{j} (\langle a \rangle - 1)^j \in \mathbb{F}_q[[\theta^{-1}]].$$

Here, the binomial $\binom{y}{j}$ is defined, for j a positive integer, by extending Lucas formula. Writing the p-adic expansion $\sum_{i\geq 0} y_i p^i$ of y ($y_i \in \{0,\dots,p-1\}$) and the p-adic expansion $\sum_{i=0}^{r} j_i p^i$ of j ($j_i \in \{0,\dots,p-1\}$), we are explicitly setting:

$$\binom{y}{j} = \prod_{i=0}^{r} \binom{y_i}{j_i}.$$

Note that, for $a \in A_+$, we have a continuous function: $\mathbb{Z}_p \to K_\infty, y \mapsto \langle a\rangle^y$. We also recall, from [11, Chapter 8], the topological group $\mathbb{S}_\infty = \mathbb{C}_\infty^\times \times \mathbb{Z}_p$. For $(x,y) \in \mathbb{S}_\infty$ and d, s non-negative integers, we define the sum

$$S_{d,s}(x,y) = S_{d,s}(x,y)(t_1,\dots,t_s) = x^{-d} \sum_{a\in A^+(d)} \chi_{t_1}(a)\cdots\chi_{t_s}(a)\langle a\rangle^y \in x^{-d}K_\infty[t_1,\dots,t_s],$$

which is, for all x, y, a symmetric polynomial of total degree $\leq ds$.

Let us further define, more generally, for variables $t_1,\dots,t_s \in \mathbb{C}_\infty$ and $(x,y) \in \mathbb{S}_\infty$, the series:

$$L(\chi_{t_1}\dots\chi_{t_s}; x, y) = \sum_{d\geq 0} S_{d,s}(x,y)(t_1,\dots,t_s).$$

For fixed choices of $(x,y) \in \mathbb{S}_\infty$, it is easy to show that

$$L(\chi_{t_1}\dots\chi_{t_s}; x, y) \in \mathbb{C}_\infty[[t_1,\dots,t_s]],$$

and with a little additional work, one also verifies that this series defines an element of the standard Tate $\mathbb{C}_\infty$-algebra $\mathbb{T}_{t_1,\dots,t_s}$ in the variables $t_1,\dots,t_s$. Of course, if $(x,y) = (\theta^\alpha, -\alpha)$ with $\alpha > 0$ integer, we find

$$L(\chi_{t_1}\dots\chi_{t_s}; \theta^\alpha, -\alpha) = L(\chi_{t_1}\dots\chi_{t_s}, \alpha).$$

The next Proposition holds, and improves results of Goss; see [12, Theorems 1, 2]).

Proposition 6 *The series $L(\chi_{t_1},\dots,\chi_{t_s}; x, y)$ converges for all $(t_1,\dots,t_s) \in \mathbb{C}_\infty^s$ and for all $(x,y) \in \mathbb{S}_\infty$, to an entire function on $\mathbb{C}_\infty^s \times \mathbb{S}_\infty$ in the sense of Goss [11, Definition 8.5.1].*

The proof of this result is a simple consequence of the Lemma below. The norm $\|\cdot\|$ used in the Lemma is the supremum norm of $\mathbb{T}_{t_1,\dots,t_s}$.

Lemma 7 *Let (x,y) be in $\mathbb{S}_\infty$ and let us consider an integer $d > (s-1)/(q-1)$, with $s > 0$. Then:*

$$\|S_{d,s}(x,y)\| \leq |x|^{-d} q^{-q^{[d-\frac{s}{q-1}]-1}},$$

where for $x \in \mathbb{R}$, $[x]$ denotes the integer part of x.

Proof. Let us write the p-adic expansion $y = \sum_{n\geq 0} c_n p^n$, with $c_n \in \{0,\dots,p-1\}$ for all n. Collecting blocks of e consecutive terms (where $q = p^e$), this yields a "q-adic" expansion, from which we can extract partial sums:

$$y_n = \sum_{k=0}^{en-1} c_k p^k = \sum_{i=0}^{n} u_i q^i \in \mathbb{Z}_{\geq 0},$$

where

$$u_i = \sum_{j=ei}^{e(i+1)-1} c_j p^{j-ei} \in \{0, \dots, q-1\}.$$

In particular, for $n \geq 0$, we observe that $\ell_q(y_n) \leq (n+1)(q-1)$. Since

$$\begin{aligned} S_{d,s}(x, y_n) &= \frac{1}{x^d \theta^{dy_n}} \sum_{a \in A^+(d)} \chi_{t_1}(a) \cdots \chi_{t_s}(a) a^{y_n} \\ &= \frac{1}{x^d \theta^{dy_n}} S_{d,r}(t_1, \dots, t_s, \underbrace{\theta, \dots, \theta}_{u_0 \text{ times}}, \underbrace{\theta^q, \dots, \theta^q}_{u_1 \text{ times}}, \dots, \underbrace{\theta^{q^n}, \dots, \theta^{q^n}}_{u_n \text{ times}}) \end{aligned}$$

with $r = s + \ell_q(y_n)$, if $d(q-1) > s + \ell_q(y_n)$, we have by Simon's Lemma 4:

$$S_{d,s}(x, y_n) = 0.$$

This condition is ensured if $d(q-1) > s + (n+1)(q-1)$.

Now, we claim that

$$\|S_{d,s}(x, y) - S_{d,s}(x, y_n)\| \leq |x|^{-d} q^{-q^{n+1}}.$$

Indeed,

$$S_{d,s}(x, y) - S_{d,s}(x, y_n) = x^{-d} \sum_{a \in A^+(d)} \chi_{t_1}(a) \cdots \chi_{t_s}(a) \sum_{j \geq 0} \left(\binom{y}{j} - \binom{y_n}{j} \right) (\langle a \rangle - 1)^j,$$

and $\binom{y}{j} = \binom{y_n}{j}$ for $j = 0, \dots, q^{n+1} - 1$ by Lucas' formula and the definition of the binomial, so that

$$\left| \sum_{j \geq 0} \left(\binom{y}{j} - \binom{y_n}{j} \right) (\langle a \rangle - 1)^j \right| \leq q^{-q^{n+1}}.$$

The Lemma follows by choosing $n + 2 = [d - \frac{s}{q-1}]$. □

In particular, we have the following Corollary to Proposition 6 which generalizes [12, Theorem 1], the deduction of which, easy, is left to the reader.

Corollary 8 *For any choice of an integer $\alpha > 0$ and non-negative integers $M_1, \dots, M_s$, the function*

$$L(\chi_{t_1}^{M_1} \cdots \chi_{t_s}^{M_s}, \alpha) = \sum_{d \geq 0} \sum_{a \in A^+(d)} \chi_{t_1}(a)^{M_1} \cdots \chi_{t_s}(a)^{M_s} a^{-\alpha}$$

defines a rigid analytic entire function $\mathbb{C}_\infty^s \to \mathbb{C}_\infty$.

2.2 Computation of polynomials with coefficients in K_∞

Lemma 9 *For all $d \geq 0$, we have:*

$$S_d(-\alpha) = \sum_{a \in A^+(d)} a^{-\alpha} \neq 0.$$

Proof. This follows from [11, proof of Lemma 8.24.13]. □

We introduce, for d, s, α non-negative integers, the sum:

$$S_{d,s}(-\alpha) = \sum_{a \in A^+(d)} \chi_{t_1}(s) \cdots \chi_{t_s}(a) a^{-\alpha} \in K[t_1, \ldots, t_s],$$

representing a symmetric polynomial of $K[t_1, \ldots, t_s]$ of exact total degree ds by Lemma 9. We have, with the notations of Section 2.1:

$$S_{d,s}(-\alpha) = S_{d,s}(\theta^\alpha, -\alpha).$$

From the above results, we deduce the following Proposition.

Proposition 10 *Let α be a fixed positive integer. Let $l \geq 0$ be an integer such that $q^l - \alpha \geq 0$ and $\ell_q(q^l - \alpha) + s \geq 2$. If $\ell_q(q^l - \alpha) + s \leq d(q-1)$, then:*

$$S_{d,s}(-\alpha) \equiv 0 \pmod{\prod_{j=1}^{s}(t_j - \theta^{q^l})}.$$

Furthermore, let us assume that $s \equiv \alpha \pmod{q-1}$. With l as above, let k be an integer such that $k(q-1) \geq \ell_q(q^l - \alpha) + s$. Then:

$$\sum_{d=0}^{k} S_{d,s}(-\alpha) \equiv 0 \pmod{\prod_{j=1}^{s}(t_j - \theta^{q^l})}.$$

Proof. Let us write $m = \ell_q(q^l - \alpha)$. We have $s - 1 + m < d(q-1)$ so that, by Simon's Lemma 4, $S_{d,s-1+m} = 0$. Now, let us write the q-ary expansion $q^l - \alpha = n_0 + n_1 q + \cdots + n_r q^r$ with $n_i \in \{0, \ldots, q-1\}$ and let us observe that, since $q^l - \alpha \geq 0$,

$$\begin{aligned}
S_{d,s}(-\alpha)(t_1, \ldots, t_{s-1}, \theta^{q^l}) &= \sum_{a \in A^+(d)} \chi_{t_1}(a) \cdots \chi_{t_{s-1}}(a) a^{q^l - \alpha} \\
&= \sum_{a \in A^+(d)} \chi_{t_1}(a) \cdots \chi_{t_{s-1}}(a) \chi_\theta(a)^{n_0} \chi_{\theta^q}(a)^{n_1} \cdots \chi_{\theta^{q^r}}(a)^{n_r} \\
&= S_{d,s-1+m}(t_1, \ldots, t_{s-1}, \underbrace{\theta, \ldots, \theta}_{n_0 \text{ times}}, \underbrace{\theta^q, \ldots, \theta^q}_{n_1 \text{ times}}, \ldots, \underbrace{\theta^{q^r}, \ldots, \theta^{q^r}}_{n_r \text{ times}}) \\
&= 0.
\end{aligned}$$

Therefore $t_s - \theta^{q^l}$ divides $S_{d,s}(-\alpha)$. The first part of the Proposition follows from the fact that this polynomial is symmetric. For the second part, we notice by the first part, that the condition on k is sufficient for the sum $S_{d,s}(-\alpha)(t_1, \ldots, t_s)$ to be congruent to zero modulo $\prod_{i=1}^{s}(t_i - \theta^{q^l})$ for all $d \geq k$. It remains to apply Lemma 5 to conclude the proof. □

Proposition 11 *Let $s, \alpha \geq 1$, $s \equiv \alpha \pmod{q-1}$. Let δ be the smallest non-negative integer such that $q^\delta \geq \alpha$ and $s + \ell_q(q^\delta - \alpha) \geq 2$. Then, the function of Theorem 1*

$$V_{\alpha,s}(t_1, \ldots, t_s) = L(\chi_{t_1} \cdots \chi_{t_s}, \alpha) \omega(t_1) \cdots \omega(t_s) \widetilde{\pi}^{-\alpha} \left(\prod_{i=1}^{s} \prod_{j=0}^{\delta-1} \left(1 - \frac{t_i}{\theta^{q^j}}\right) \right)$$

is in fact a symmetric polynomial of $K_\infty[t_1,\dots,t_s]$. *Moreover, its total degree* $\delta(\alpha,s)$ *is not bigger than* $s\left(\frac{s+\ell_q(q^\delta-\alpha)}{q-1}\right)-s$.

Proof. Let δ be the smallest non-negative integer such that $q^\delta-\alpha\geq 0$ and $s+\ell_q(q^\delta-\alpha)\geq 2$. We fix an integer k such that

$$k(q-1)\geq s+\ell_q(q^\delta-\alpha). \tag{8}$$

We also set:

$$N(k)=\delta+k-\frac{s+\ell_q(q^\delta-\alpha)}{q-1}.$$

Obviously, $N(k)\geq\delta$. Let l be an integer such that

$$\delta\leq l\leq N(k).$$

We claim that we also have

$$k(q-1)\geq s+\ell_q(q^l-\alpha).$$

Indeed, let us write the q-ary expansion $\alpha=\alpha_0+\alpha_1 q+\cdots+\alpha_m q^m$ with $\alpha_m\neq 0$. Then, $\delta=m$ if $\alpha=q^m$ and $s\geq 2$ and $\delta=m+1$ otherwise. If l is now an integer $l\geq\delta$, we have

$$\begin{aligned} q^l-\alpha &= q^l-q^\delta+q^\delta-\alpha \\ &= q^\delta(q-1)\left(\sum_{i=0}^{l-\delta-1}q^i\right)+q^\delta-\alpha, \end{aligned}$$

where the sum over i is zero if $l=\delta$, and

$$\ell_q(q^l-\alpha)=(q-1)(l-\delta)+\ell_q(q^\delta-\alpha)$$

because there is no carry over in the above sum. Now, the claim follows from (8).

By Proposition 10 we have, with k as above, that the following expression

$$W_{k,s,\alpha}:=\left(\prod_{i=1}^{s}\prod_{j=\delta}^{N(k)}\left(1-\frac{t_i}{\theta^{q^j}}\right)^{-1}\right)\sum_{d=0}^{k}S_{d,s}(-\alpha)$$

is in fact a symmetric polynomial in $K[t_1,\dots,t_s]$. By Lemma 9, $S_{d,s}(-\alpha)\in K[t_1,\dots,t_s]$ is a symmetric polynomial of total degree ds; indeed, the coefficient of $t_1^d\cdots t_s^d$ is exactly $S_d(-\alpha)$. Hence, the total degree of $\sum_{d=0}^{k}S_{d,s}(-\alpha)$ is exactly ks. The total degree of the product

$$\prod_{i=1}^{s}\prod_{j=\delta}^{N(k)}\left(1-\frac{t_i}{\theta^{q^j}}\right)$$

is equal to $s(1+N(k)-\delta)$ so that, by the definition of $N(k)$:

$$\begin{aligned} \deg(W_{k,s,\alpha}) &= sk-s-sN(k)+s\delta \\ &= sk-sk-s\delta+s\delta-s+s\left(\frac{s+\ell_q(q^\delta-\alpha)}{q-1}\right) \\ &= s\left(\frac{s+\ell_q(q^\delta-\alpha)}{q-1}\right)-s, \end{aligned}$$

independent on k. We now let k tend to infinity. The Proposition follows directly from the definition (2) of ω as an infinite product, the fact that, in (2), $\widetilde{\pi}\theta^{-1/(q-1)} \in K_\infty$, and the definition of $L(\chi_{t_1}\cdots\chi_{t_s},\alpha)$. □

2.3 Preliminaries on Gauss-Thakur sums

We review quickly the theory of Gauss-Thakur sums, introduced by Thakur in [17].

Let $\mathfrak{p}$ be an irreducible monic polynomial of A of degree d, let $\Delta_\mathfrak{p}$ be the Galois group of the $\mathfrak{p}$-cyclotomic function field extension $K_\mathfrak{p} = K(\lambda_\mathfrak{p})$ of K, where $\lambda_\mathfrak{p}$ is a non zero $\mathfrak{p}$-torsion element of K^{alg} (the algebraic closure of K in $\mathbb{C}_\infty$). Gauss-Thakur sums can be associated to the elements of the dual character group $\widehat{\Delta}_\mathfrak{p}$ via the Artin symbol (see [11, Sections 7.5.5 and 9.8]). If χ is in $\widehat{\Delta}_\mathfrak{p}$, we denote by $g(\chi)$ the associated Gauss-Thakur sum. In particular, we have the element $\vartheta_\mathfrak{p} \in \widehat{\Delta}_\mathfrak{p}$ obtained by reduction of the *Teichmüller character* [11, Definition 8.11.2], uniquely determined by a choice of a root ζ of $\mathfrak{p}$, and the Gauss-Thakur sums $g(\vartheta_\mathfrak{p}^{q^j})$ associated to its q^j-th powers, with $j = 0,\ldots,d-1$, which can be considered as the building blocks of the Gauss-Thakur sums $g(\chi)$ for general $\chi \in \widehat{\Delta}_\mathfrak{p}$.

Definition 12 With $\mathfrak{p}$, d, $\vartheta_\mathfrak{p}$ as above, the *basic Gauss-Thakur sum* $g(\vartheta_\mathfrak{p}^{q^j})$ associated to this data is the element of K^{alg} defined by:

$$g(\vartheta_\mathfrak{p}^{q^j}) = \sum_{\delta\in\Delta_\mathfrak{p}} \vartheta_\mathfrak{p}(\delta^{-1})^{q^j}\delta(\lambda_\mathfrak{p}) \in \mathbb{F}_\mathfrak{p}[\lambda_\mathfrak{p}].$$

The same sum is denoted by g_j in [11, 17]. The basic Gauss-Thakur sums are used to define *general Gauss-Thakur sums* associated to arbitrary elements of $\widehat{\Delta}_\mathfrak{p}$. For instance, if $\chi = \chi_0$ is the trivial character, then $g(\chi) = 1$.

The group $\widehat{\Delta}_\mathfrak{p}$ being isomorphic to $\Delta_\mathfrak{p}$, it is cyclic; it is in fact generated by $\vartheta_\mathfrak{p}$. Let χ be an element of $\widehat{\Delta}_\mathfrak{p}$. There exists a unique integer i with $0 < i < q^d$, such that $\chi = \vartheta_\mathfrak{p}^i$. Let us expand i in base q, that is, let us write $i = i_0 + i_1 q + \cdots + i_{d-1}q^{d-1}$ with $i_j \in \{0,\ldots,d-1\}$. Then, $\chi = \prod_{j=0}^{d-1}(\vartheta_\mathfrak{p}^{q^j})^{i_j}$.

Definition 13 The *general Gauss-Thakur sum* $g(\chi)$ associated to $\chi \in \widehat{\Delta}_\mathfrak{p}$ as above, is defined by:

$$g(\chi) = \prod_{j=0}^{d-1} g(\vartheta_\mathfrak{p}^{q^j})^{i_j}.$$

More generally, let us now consider a non-constant monic polynomial $\mathfrak{a} \in A$. We denote by $\widehat{\Delta}_\mathfrak{a}$ the dual character group $\mathbf{Hom}(\Delta_\mathfrak{a}, (\mathbb{F}_q^{\mathrm{alg}})^\times)$, where $\Delta_\mathfrak{a}$ is the Galois group of the extension $K_\mathfrak{a}$ of K generated by the $\mathfrak{a}$-torsion of the Carlitz module.

If χ is in $\widehat{\Delta}_\mathfrak{a}$, we set: $\mathbb{F}_q(\chi) = \mathbb{F}_q(\chi(\delta); \delta \in \Delta_\mathfrak{a}) \subset \mathbb{F}_q^{\mathrm{alg}}$. We also write

$$\mathbb{F}_\mathfrak{a} = \mathbb{F}_q(\chi; \chi \in \widehat{\Delta}_\mathfrak{a})$$

and we recall that $\mathbf{Gal}(K_\mathfrak{a}(\mathbb{F}_\mathfrak{a})/K(\mathbb{F}_\mathfrak{a})) \simeq \Delta_\mathfrak{a}$. We observe that $\widehat{\Delta}_\mathfrak{a}$ is isomorphic to $\Delta_\mathfrak{a}$ if and only if $\mathfrak{a}$ is squarefree. If $\mathfrak{a} = \mathfrak{p}_1\cdots\mathfrak{p}_n$ with $\mathfrak{p}_1,\ldots,\mathfrak{p}_n$ distinct primes, then

$$\widehat{\Delta}_\mathfrak{a} \simeq \prod_{i=1}^{n} \widehat{\Delta}_{\mathfrak{p}_i}.$$

Let us then assume that $\mathfrak{a}$ is non-constant and square-free. We want to extend the definition of the Gauss-Thakur sums to characters in $\widehat{\Delta}_{\mathfrak{a}}$. For $\chi \in \widehat{\Delta}_{\mathfrak{a}}$, $\chi \neq \chi_0$, there exist r distinct primes $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ and characters $\chi_1, \ldots, \chi_r$ with $\chi_j \in \widehat{\Delta}_{\mathfrak{p}_j}$ for all j, with

$$\chi = \chi_1 \cdots \chi_r.$$

Definition 14 The *Gauss-Thakur sum* associated to χ is the product:

$$g(\chi) = g(\chi_1) \cdots g(\chi_r).$$

The polynomial $\mathfrak{f}_\chi = \mathfrak{p}_1 \cdots \mathfrak{p}_r$ is called the *conductor* of χ; it is a divisor of $\mathfrak{a}$. The degree of $\mathfrak{f}_\chi$ will be denoted by d_χ. If $\mathfrak{a}$ itself is a prime $\mathfrak{p}$ of degree d, then $F_\chi = \mathfrak{p}$ and $d_\chi = d$.

The following result collects the basic properties of the sums $g(\chi)$ that we need, and can be easily deduced from Thakur's results in [17, Theorems I and II].

Proposition 15 *Let $\mathfrak{a} \in A$ be monic, square-free of degree d. The following properties hold.*

1. *For all $\delta \in \Delta_{\mathfrak{a}}$, we have $\delta(g(\chi)) = \chi(\delta) g(\chi)$.*
2. *If $\chi \neq \chi_0$, then $g(\chi) g(\chi^{-1}) = (-1)^{d_\chi} \mathfrak{f}_\chi$.*

By the normal basis theorem, $K_{\mathfrak{a}}$ is a free $K[\Delta_{\mathfrak{a}}]$-module of rank one. Gauss-Thakur's sums allow to determine explicitly generators of this module:

Lemma 16 *Let us write $\eta_{\mathfrak{a}} = \sum_{\chi \in \widehat{\Delta}_{\mathfrak{a}}} g(\chi) \in K_{\mathfrak{a}}$. Then :*

$$K_{\mathfrak{a}} = K[\Delta_{\mathfrak{a}}] \eta_{\mathfrak{a}},$$

and

$$A_{\mathfrak{a}} = A[\Delta_{\mathfrak{a}}] \eta_{\mathfrak{a}},$$

where $A_{\mathfrak{a}}$ is the integral closure of A in $K_{\mathfrak{a}}$.

Moreover, let χ be in $\widehat{\Delta}_{\mathfrak{a}}$. Then, the following identity holds:

$$K_{\mathfrak{a}}(\mathbb{F}_{\mathfrak{a}}) g(\chi) = \{x \in K_{\mathfrak{a}}(\mathbb{F}_{\mathfrak{a}}) \text{ such that for all } \delta \in \Delta_{\mathfrak{a}}, \delta(x) = \chi(\delta) x\}. \tag{9}$$

Proof. Let us expand $\mathfrak{a}$ in product $\mathfrak{p}_1 \cdots \mathfrak{p}_n$ of distinct primes $\mathfrak{p}_i$. To show that $A_{\mathfrak{a}} = A[\Delta_{\mathfrak{a}}]\eta_{\mathfrak{a}}$ (this yields the identity $K_{\mathfrak{a}} = K[\Delta_{\mathfrak{a}}]\eta_{\mathfrak{a}}$) one sees that

$$A_{\mathfrak{a}} \simeq A_{\mathfrak{p}_1} \otimes_A \cdots \otimes_A A_{\mathfrak{p}_n},$$

because the discriminants of the extensions $A_{\mathfrak{p}_i}/A$ are pairwise relatively prime and the fields $K_{\mathfrak{p}_i}$ are pairwise linearly disjoint (see [9, Property (2.13)]). One then uses [3, Théorème 2.5] to conclude with the second identity.

We now prove the identity (9). We recall that if we set, for $\chi \in \widehat{\Delta}_{\mathfrak{a}}$,

$$e_\chi = \frac{1}{|\Delta_{\mathfrak{a}}|} \sum_{\delta \in \Delta_{\mathfrak{a}}} \chi(\delta) \delta^{-1} \in \mathbb{F}_q(\chi)[\Delta_{\mathfrak{a}}]$$

(well defined because p, the rational prime dividing q, does not divide $|\Delta_{\mathfrak{a}}|$), then the following identities hold:

- $e_\chi e_\psi = \delta_{\chi,\psi} e_\chi$ (where $\delta_{\chi,\psi}$ denotes Kronecker's symbol),
- for all $\delta \in \Delta_{\mathfrak{a}}$, $\delta e_\chi = \chi(\delta) e_\chi$,
- $\sum_{\chi \in \widehat{\Delta}_{\mathfrak{a}}} = 1$.

This yields $e_\chi \eta_{\mathfrak{a}} = g(\chi)$. Now, by $K_{\mathfrak{a}}(\mathbb{F}_{\mathfrak{a}}) = K_{\mathfrak{a}}(\mathbb{F}_{\mathfrak{a}})\eta_{\mathfrak{a}}$, we get $e_\chi K_{\mathfrak{a}}(\mathbb{F}_{\mathfrak{a}}) = K_{\mathfrak{a}}(\mathbb{F}_{\mathfrak{a}}) g(\chi)$. The second part of the Lemma then follows by observing that if M is an $\mathbb{F}_{\mathfrak{a}}[\Delta_{\mathfrak{a}}]$-module, then

$$e_\chi M = \{m \in M \text{ such that for all } \delta \in \Delta_{\mathfrak{a}}, \delta m = \chi(\delta) m\}.$$

□

2.4 An intermediate result on special values of Goss L-functions

Let χ be a Dirichlet character of the first kind, that is a character

$$\chi : (A/\mathfrak{a}A)^\times \to (\mathbb{F}_q^{\text{alg}})^\times,$$

where $\mathfrak{a}$ is a non-constant squarefree monic element of A which we identify, by abuse of notation, to a character of $\widehat{\Delta}_{\mathfrak{a}}$ still denoted by χ, of conductor $\mathfrak{f} = \mathfrak{f}_\chi$, and degree $d = \deg_\theta f$.

Let $s(\chi)$ be the *type* of χ, that is, the unique integer $s(\chi) \in \{0, \cdots, q-2\}$ such that:

$$\chi(\zeta) = \zeta^{s(\chi)} \quad \text{for all } \zeta \in \mathbb{F}_q^\times.$$

We now consider the special value of Goss' *abelian L-function* [11, Section 8]:

$$L(\alpha, \chi) = \sum_{a \in A^+} \chi(a) a^{-\alpha}, \quad \alpha \geq 1.$$

The following result is inspired by the proofs of [1, Proposition 8, VII] and [4, Proposition 8.2]:

Proposition 17 *Let $\alpha \geq 1$, $\alpha \equiv s(\chi) \pmod{q-1}$. Then there exists an element $B_{\alpha,\chi^{-1}}$ in $\mathbb{F}_q(\chi)(\theta)$ such that :*

$$\frac{L(\alpha,\chi) g(\chi)}{\widetilde{\pi}^\alpha} = \frac{B_{\alpha,\chi^{-1}}}{\Pi(\alpha)},$$

where $\Pi(\alpha)$ is the Carlitz factorial of α (see [11, chapter 9, section 1]).

Proof. The proposition is known to be true for the trivial character (see [11, Section 9.2]); in this case, we notice that:

$$B_{\alpha,\chi_0^{-1}} = BC_\alpha, \quad \alpha \geq 1, \quad \alpha \equiv 0 \pmod{q-1},$$

where we recall that BC_α is the α-th Bernoulli-Carlitz number (see [11, Definition 9.2.1]. We now assume that $\chi \neq \chi_0$. Since:

$$\exp_C(z) = z \prod_{a \in A \setminus \{0\}} \left(1 - \frac{z}{\widetilde{\pi} a}\right),$$

We have:

$$\frac{1}{\exp_C(z)} = \sum_{a \in A} \frac{1}{z - \widetilde{\pi} a}.$$

Let $b \in A$ be relatively prime with $\mathfrak{f}$ and let $\sigma_b \in \mathbf{Gal}(K_\mathfrak{f}/K)$ be the element such that $\sigma_b(\lambda_\mathfrak{f}) = \phi_b(\lambda_\mathfrak{f})$. We have:

$$\frac{1}{\exp_C(z) - \sigma_b(\lambda_\mathfrak{f})} = -\sum_{n\geq 0} \frac{\mathfrak{f}^{n+1}}{\widetilde{\pi}^{n+1}} \left(\sum_{a\in A} \frac{1}{(b+a\mathfrak{f})^{n+1}} \right) z^n.$$

Therefore, we obtain:

$$\sum_{b\in(A/\mathfrak{f}A)^\times} \frac{\chi(b)}{\exp_C(z) - \sigma_b(\lambda_\mathfrak{f})} = -\sum_{n\geq 0} \frac{\mathfrak{f}^{n+1}}{\widetilde{\pi}^{n+1}} \left(\sum_{a\in A\setminus\{0\}} \frac{\chi(a)}{a^{n+1}} \right) z^n.$$

If $n+1 \not\equiv s(\chi) \pmod{q-1}$, we get:

$$\sum_{a\in A\setminus\{0\}} \frac{\chi(a)}{a^{n+1}} = 0,$$

and if $n+1 \equiv s(\chi) \pmod{q-1}$, we have:

$$\sum_{a\in A\setminus\{0\}} \frac{\chi(a)}{a^{n+1}} = -L(n+1,\chi).$$

Thus:

$$\sum_{b\in(A/\mathfrak{f}A)^\times} \frac{\chi(b)}{\exp_C(z) - \sigma_b(\lambda_\mathfrak{f})} = \sum_{i\geq 1,\, i\equiv s(\chi) \pmod{(q-1)}} \frac{\mathfrak{f}^i L(i,\chi)}{\widetilde{\pi}^i} z^{i-1}. \tag{10}$$

But note that by the second part of Lemma 16:

$$\sum_{b\in(A/\mathfrak{f}A)^\times} \frac{\chi(b)}{\exp_C(z) - \sigma_b(\lambda_\mathfrak{f})} \in g(\chi^{-1}) F_q(\chi)(\theta)[[z]].$$

Since by Proposition 15,

$$g(\chi)g(\chi^{-1}) = (-1)^d \mathfrak{f},$$

where $d = \deg_\theta f_\chi$, we get the result by comparison of the coefficients of the series expansion of both sides of (10). □

Remark 18 In the above proof of Proposition 17, if we set $\alpha = 1$ and we assume $s(\chi) \equiv 1 \pmod{q-1}$, we have, by comparison of the constant terms in the series expansions in powers of z in (10):

$$\widetilde{\pi}^{-1}\mathfrak{f}L(1,\chi) = -\sum_{b\in(A/\mathfrak{f}A)^\times} \frac{\chi(b)}{\sigma_b(\lambda_\mathfrak{f})} \in g(\chi^{-1})\mathbb{F}_q(\chi)(\theta).$$

Assuming that $\mathfrak{f}$ is not a prime, by [16, Proposition 12.6], $\lambda_\mathfrak{f}$ is a unit in the integral closure $A_\mathfrak{f}$ of A in $K_\mathfrak{f}$. Therefore,

$$\sum_{b\in(A/\mathfrak{f}A)^\times} \frac{\chi(b)}{\sigma_b(\lambda_\mathfrak{f})} \in g(\chi^{-1})\mathbb{F}_q(\chi)[\theta]$$

and we deduce that

$$\widetilde{\pi}^{-1}L(1,\chi)g(\chi) \in \mathbb{F}_q(\chi)[\theta].$$

This remark will be crucial in the proof of Corollary 21.

2.5 Proof of Theorem 1

In [5, Theorem 3] (see also [6]), we noticed that the function ω can be viewed as a *universal Gauss-Thakur sum*. We review this result, which will be used a little later.

Theorem 19 *Let $\mathfrak{p}$ be a prime element of A of degree d and ζ a root of $\mathfrak{p}$ as above. We have:*

$$g(\vartheta_{\mathfrak{p}}^{q^j}) = \mathfrak{p}'(\zeta)^{-q^j}\omega(\zeta^{q^j}), \quad j = 0, \dots, d-1.$$

In this theorem, $\mathfrak{p}'$ denotes the derivative of $\mathfrak{p}$ with respect to θ.

The next Lemma provides a rationality criterion for a polynomial a priori with coefficients in K_∞, again based on evaluation at roots of unity.

Lemma 20 *Let $F(t_1, \dots, t_s) \in K_\infty[t_1, \dots, t_s]$ such that for all $\zeta_1, \dots, \zeta_s \in \mathbb{F}_q^{alg}$, pairwise not conjugate over $\mathbb{F}_q$,*

$$F(\zeta_1, \dots, \zeta_s) \in K(\zeta_1, \dots, \zeta_s).$$

Then $F(t_1, \dots, t_s) \in K[t_1, \dots, t_s]$.

Proof. We begin by pointing out that if elements $a_1, \dots, a_s \in K_\infty$ are $K \otimes_{\mathbb{F}_q} \mathbb{F}_q^{\mathrm{alg}}$-linearly dependent, then they also are K-linearly dependent. The proof proceeds by induction on $s \geq 1$. For $s = 1$, this is obvious. Now, let

$$\sum_{i=1}^{s} \lambda_s a_s = 0 \tag{11}$$

be a non-trivial relation of linear dependence with the $\lambda_i \in K \otimes \mathbb{F}_q^{\mathrm{alg}} \setminus \{0\}$. We may assume that $\lambda_s = 1$ and that there exists $i \in \{1, \dots, s-1\}$ such that $\lambda_i \notin K$. Then, there exists

$$\sigma \in \mathbf{Gal}(K_\infty \otimes \mathbb{F}_q^{\mathrm{alg}}/K_\infty) \simeq \mathbf{Gal}(K \otimes \mathbb{F}_q^{\mathrm{alg}}/K) \simeq \mathbf{Gal}(\mathbb{F}_q^{\mathrm{alg}}/\mathbb{F}_q)$$

such that $\sigma(\lambda_i) \neq \lambda_i$. Applying σ on both left- and right-hand sides of (11) and subtracting, yields a non-trivial relation involving at most $s-1$ elements of K_∞ on which we can apply the induction hypothesis.

We can now complete the proof of the Lemma. Let F be a polynomial in $K_\infty[t_1, \dots, t_s]$ not in $K[t_1, \dots, t_s]$. It is easy to show that there exist $a_1, \dots, a_m \in K_\infty$, linearly independent over K, such that

$$F = a_1 P_1 + \cdots + a_m P_m,$$

where $P_1, \dots, P_m$ are non-zero polynomials of $K[t_1, \dots, t_s]$. Let us suppose by contradiction that there exists $F \in K_\infty[t_1, \dots, t_s] \setminus K[t_1, \dots, t_s]$ satisfying the hypotheses of the Lemma. By the observation at the beginning of the proof of Lemma 5, there exists a choice of such roots of unity $\zeta_1, \dots, \zeta_s$ and $i \in \{1, \dots, m\}$ such that $P_i(\zeta_1, \dots, \zeta_m) \neq 0$. This means that $a_1, \dots, a_m$ are $K \otimes \mathbb{F}_q^{\mathrm{alg}}$-linearly dependent, thus K-linearly dependent by the previous observations; a contradiction. □

Proof of Theorem 1. In view of Lemma 20, we want to show that the polynomial

$$V_{\alpha,s} = \widetilde{\pi}^{-\alpha} L(\chi_{t_1} \cdots \chi_{t_s}, \alpha)\omega(t_1) \cdots \omega(t_s) \left(\prod_{i=1}^{s} \prod_{j=0}^{\delta-1} \left(1 - \frac{t_i}{\theta^{q^j}}\right) \right) \in K_\infty[t_1, \cdots, t_s]$$

of Proposition 11 takes values in $K(\zeta_1,\dots,\zeta_s)$ for all $\zeta_1,\dots,\zeta_s \in \mathbb{F}_q^{\mathrm{alg}}$ pairwise non conjugate over $\mathbb{F}_q$. Let $(\zeta_1,\dots,\zeta_s)$ be one of such s-tuples of roots of unity and, for $i=1,\dots,s$, let $\mathfrak{p}_i \in A$ be the minimal polynomial of ζ_i, so that $\mathfrak{p}_1,\dots,\mathfrak{p}_s$ are pairwise relatively prime. We choose the characters $\vartheta_{\mathfrak{p}_i}$ so that $\vartheta_{\mathfrak{p}_i}(\sigma_\theta)=\zeta_i$ for all i. We construct the Dirichlet character of the first kind χ defined, for $a \in A$, by

$$\chi(a) = \chi_{\zeta_1}(a)\cdots\chi_{\zeta_s}(a).$$

By Proposition 17, we have

$$\frac{L(\alpha,\chi)g(\chi)}{\widetilde{\pi}^\alpha} = (-1)^{d_\chi}\frac{B_{\alpha,\chi^{-1}}}{\mathfrak{f}_\chi^{\alpha-1}} \in \mathbb{F}_q(\chi)(\theta).$$

Since

$$L(\alpha,\chi) = L(\chi_{\zeta_1}\cdots\chi_{\zeta_s},\alpha),$$

we get:

$$\begin{aligned} V_{\alpha,s}(\zeta_1,\dots,\zeta_s) &= L(\alpha,\chi)\omega(\zeta_1)\cdots\omega(\zeta_s)\widetilde{\pi}^{-\alpha} \\ &= \frac{L(\alpha,\chi)g(\chi)}{\widetilde{\pi}^\alpha}\frac{\omega(\zeta_1)\cdots\omega(\zeta_s)}{g(\chi)} \\ &= (-1)^{d_\chi}\frac{B_{\alpha,\chi^{-1}}}{\mathfrak{f}_\chi^{\alpha-1}}\chi_{\zeta_1}(\mathfrak{p}_1')\cdots\chi_{\zeta_s}(\mathfrak{p}_s') \\ &\in K(\zeta_1,\dots,\zeta_s), \end{aligned}$$

where in the next to last step, we have used Theorem 19. The proof of Theorem 1 now follows from Lemma 20. □

3 Congruences for Bernoulli-Carlitz numbers

In this Section, we shall prove Theorem 2. This is possible because in Theorem 1, more can be said when $\alpha=1$. In this case, one sees that the integer δ of Theorem 1 is equal to zero and $s \geq q$, so that, with the notations of that result,

$$V_{1,s} = \widetilde{\pi}^{-1}L(\chi_{t_1}\cdots\chi_{t_s},1)\omega(t_1)\cdots\omega(t_s).$$

In the next Subsection we will show that the above is a polynomial of $A[t_1,\dots,t_s]$.

3.1 Functional identities with $\alpha = 1$

We begin with the following Corollary of Theorem 1. The main result of this subsection is Proposition 24.

Corollary 21 *Let $s \geq 2$ be such that $s \equiv 1 \pmod{q-1}$. Then the symmetric polynomial $V_{1,s} \in K[t_1,\dots,t_s]$ of Theorem 1 is in fact a polynomial of $A[t_1,\dots,t_s]$ of total degree $\leq s^2/(q-1)-s$ in the variables $t_1,\dots,t_s$.*

Proof. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be distinct primes in A, let us write $\mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_s$ and let us consider the Dirichlet character of the first kind χ associated to $\vartheta_{\mathfrak{p}_1} \cdots \vartheta_{\mathfrak{p}_s}$ that we also loosely identify with the corresponding element of $\widehat{\Delta}_{\mathfrak{a}}$. Since $\mathfrak{a}$ is not a prime power, Remark 18 implies that

$$\widetilde{\pi}^{-1} L(1, \chi) g(\chi) \in \mathbb{F}_q(\chi)[\theta]. \tag{12}$$

Now, specializing at $t_i = \zeta_i$ the root of $\mathfrak{p}_i$ associated to the choice of characters $\vartheta_{\mathfrak{p}_i}$ for all $i = 1, \dots, s$, we obtain

$$V_{1,s}(\zeta_1, \dots, \zeta_s) = \widetilde{\pi}^{-1} L(1, \chi) g(\chi) \in \mathbb{F}_q(\zeta_1, \dots, \zeta_s)[\theta],$$

and the result follows from an idea similar to that of Lemma 20, the bound on the degree agreeing with that of Theorem 1. □

3.1.1 Digit principle for the function ω and the L-series

We denote by $\mathbb{T}_t$ the standard Tate $\mathbb{C}_\infty$-algebra in the variable t. Let $\varphi : \mathbb{T}_t \to \mathbb{T}_t$ be the $\mathbb{C}_\infty$-linear map defined by

$$\varphi\left(\sum_{n \geq 0} c_n t^n\right) = \sum_{n \geq 0} c_n t^{qn}, \quad c_n \in \mathbb{C}_\infty.$$

We also set, for N a non-negative integer with its expansion in base q, $N = N_0 + N_1 q + \cdots + N_r q^r$, $N_i \in \{0, \dots, q-1\}$:

$$\omega_N(X) = \prod_{i=0}^{r} \varphi^i(\omega(X))^{N_i}.$$

We then have the next Lemma.

Lemma 22 *Let $\mathfrak{p}$ be a prime of A of degree d and let N be an integer such that $1 \leq N \leq q^d - 1$. The following identity holds:*

$$\omega_N(\zeta) = \vartheta_{\mathfrak{p}}(\sigma_{\mathfrak{p}'})^N g(\vartheta_{\mathfrak{p}}^N),$$

where ζ is the root of $\mathfrak{p}$ that determines the character $\vartheta_{\mathfrak{p}}$.

Proof. This is a direct application of Theorem 19. Indeed,

$$\omega_N(\zeta) = \prod_{i=0}^{d-1} \omega(\zeta^{q^i})^{N_i} = \prod_{i=0}^{d-1} \vartheta_{\mathfrak{p}}(\sigma_{\mathfrak{p}'})^{q^i N_i} g(\vartheta_{\mathfrak{p}}^{q^i})^{N_i}.$$

□

Let X, Y be two indeterminates over K. We introduce a family of polynomials $(G_d)_{d \geq 0}$ in $\mathbb{F}_q[X, Y]$ as follows. We set $G_0(X, Y) = 1$ and

$$G_d(X, Y) = \prod_{i=0}^{d-1} (X - Y^{q^i}), \quad d \geq 1.$$

This sequence is closely related to the sequence of polynomials $G_n(y)$ of [2, Section 3.6]: indeed, the latter can be rewritten in terms of the former:

$$G_d(y) = G_d(T^{q^d}, y^q), \quad d \geq 1,$$

in both notations of loc. cit. and ours ([2]). The polynomial G_d is monic of degree d in the variable X, and $(-1)^d G_d$ is monic in the variable Y of degree $(q^d-1)/(q-1)$. We now define, for $N = N_0 + N_1 q + \cdots + N_r q^r$ a non-negative integer expanded in base q, the polynomial

$$H_N(t) = \prod_{i=0}^{r} G_i(t^{q^i}, \theta)^{N_i} = \prod_{i=0}^{r} \prod_{j=0}^{i-1} (t^{q^i} - \theta^{q^j})^{N_i}.$$

We also define the quantities associated to N and q:

$$\begin{aligned} \mu_q(N) &= \sum_{i=0}^{r} N_i i q^i, \\ \mu_q^*(N) &= \frac{N}{q-1} - \frac{\ell_q(N)}{q-1}, \\ \ell_q'(N) &= \sum_{i=0}^{r} N_i i. \end{aligned}$$

Lemma 23 *Let N be a non-negative integer. The following properties hold.*

1. *The polynomial $H_N(t)$, as a polynomial of the indeterminate t, is monic of degree $\mu_q(N)$.*
2. *As a polynomial of the indeterminate θ, $H_N(t)$ has degree $\mu_q^*(N)$ and the leading coefficient is $(-1)^{\ell_q'(N)}$.*
3. *We have $H_N(\theta) = \Pi(N)$ and $v_\infty(H_N(\theta)) = -\mu_q(N)$, where v_∞ is the ∞-adic valuation of $\mathbb{C}_\infty$.*
4. *We also have, for all $\zeta \in \mathbb{F}_q^{alg}$, $v_\infty(H_N(\zeta)) = -\mu_q^*(N)$.*

Proof. Easy and left to the reader. □

We observe that:

$$\varphi^d \omega(t) = \frac{1}{G_d(t^{q^d}, \theta)} \omega(t)^{q^d} = \omega_{q^d N}(t), \quad d \geq 0$$

so that, with N as above,

$$\omega_N(t) = \frac{\omega(t)^N}{\prod_{i=0}^{r} G_i(t^{q^i}, \theta)^{N_i}} = \frac{\omega(t)^N}{H_N(t)}. \tag{13}$$

The following Proposition was inspired by a discussion with D. Goss.

[2] As an aside remark, we also notice that we recover in this way the coefficients of the formal series in $K[[\tau]]$ associated to Carlitz's exponential and logarithm

$$\mathfrak{e} = \sum_{i \geq 0} d_i^{-1} \tau^i, \quad \mathfrak{l} = \sum_{i \geq 0} l_i^{-1} \tau^i,$$

because $d_i = G_i(\theta^{q^i}, \theta)$ and $l_i = G_i(\theta, \theta^q)$. Moreover, if $\mathfrak{p}$ is a prime of A of degree d, we observe that

$$\mathfrak{p} = \prod_{i=1}^{d} (\theta - \zeta_i) = \prod_{j=0}^{d-1} (\theta - \vartheta_{\mathfrak{p}}(\sigma_{\theta^{q^j}})) = G_d(\theta, \vartheta_{\mathfrak{p}}(\sigma_\theta)).$$

Proposition 24 *Let $s \geq 2$ be an integer. Let $M_1, \dots, M_s$ be positive integers such that $M_1 + \cdots + M_s \equiv 1 \pmod{q-1}$. Then:*

$$W(t_1, \dots, t_s) = \widetilde{\pi}^{-1} L(\chi_{t_1}^{M_1} \cdots \chi_{t_s}^{M_s}, 1) \omega_{M_1}(t_1) \cdots \omega_{M_s}(t_s) \in A[t_1, \dots, t_s].$$

For all i, the degree in t_i of W satisfies

$$\deg_{t_i}(W) \leq M_i \left(\frac{\sum_j M_j}{q-1} - 1 \right) - \mu_q(M_i)$$

Proof. We shall write

$$H = \prod_{i=1}^{s} H_{M_i}(t_i).$$

We know from Lemma 23 that $\deg_{t_i}(H) = \mu_q(M_i)$. Let us consider the function

$$V = \widetilde{\pi}^{-1} L(\chi_{t_1}^{M_1} \cdots \chi_{t_s}^{M_s}, 1) \omega^{M_1}(t_1) \cdots \omega^{M_s}(t_s),$$

so that by (13),

$$V = WH.$$

Corollary 21 implies that:

$$V \in \mathbb{F}_q[\theta, t_1, \dots, t_s]$$

and we are done if we can prove that H divides V in $\mathbb{F}_q[\theta, t_1, \dots, t_s]$.

Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be distinct primes of A such that $|\mathfrak{p}_i| - 1 > M_i$, and let $\zeta_1, \dots, \zeta_s$ be respective roots of these polynomials chosen in compatibility with the characters $\vartheta_{\mathfrak{p}_1}, \dots, \vartheta_{\mathfrak{p}_s}$. Let us also write

$$\chi = \vartheta_{\mathfrak{p}_1}^{M_1} \cdots \vartheta_{\mathfrak{p}_s}^{M_s}.$$

By Lemma 22,

$$\omega_{M_1}(\zeta_1) \cdots \omega_{M_s}(\zeta_s) = \vartheta_{\mathfrak{p}_1}(\sigma_{\mathfrak{p}_1'})^{M_1} \cdots \vartheta_{\mathfrak{p}_s}(\sigma_{\mathfrak{p}_s'})^{M_s} g(\chi).$$

Therefore,

$$W(\zeta_1, \dots, \zeta_s) = \widetilde{\pi}^{-1} L(1, \chi) g(\chi) \vartheta_{\mathfrak{p}_1}(\sigma_{\mathfrak{p}_1'})^{M_1} \cdots \vartheta_{\mathfrak{p}_s}(\sigma_{\mathfrak{p}_s'})^{M_s}.$$

By (12), $\widetilde{\pi}^{-1} L(1, \chi) g(\chi) \in \mathbb{F}_q(\chi)[\theta]$, while $\prod_{i=1}^{s} \vartheta_{\mathfrak{p}_i}(\sigma_{\mathfrak{p}_i'})^{M_i} \in \mathbb{F}_q(\chi)$ so that

$$W(\zeta_1, \dots, \zeta_s) \in \mathbb{F}_q(\chi)[\theta] = \mathbb{F}_q(\zeta_1, \dots, \zeta_s)[\theta].$$

Now, H is a polynomial in θ with leading coefficient in $\mathbb{F}_q^\times$ (see Lemma 23). Dividing V by H as polynomials in θ we find

$$V = HQ + R,$$

where Q, R are polynomials in $\mathbb{F}_q[\theta, t_1, \dots, t_s]$, and $\deg_\theta R < \deg_\theta H = \sum_i \mu_q^*(M_i)$ (the last inequality by Lemma 23). But for $\zeta_1, \dots, \zeta_s$ as above, we must have $Q(\theta, \zeta_1, \dots, \zeta_s) = W(\zeta_1, \dots, \zeta_s)$ and

$$R(\zeta_1, \dots, \zeta_s) = 0.$$

This implies $R = 0$ and thus $W = Q \in \mathbb{F}_q[\theta, t_1, \dots, t_s]$. □

3.1.2 The polynomials W_s

By Proposition 24, the function

$$W_s(t) = \widetilde{\pi}^{-1} L(\chi_t^s, 1)\omega_s(t) = \frac{L(\chi_t^s, 1)\omega(t)^s}{\widetilde{\pi} H_s(t)}$$

is a polynomial of $\mathbb{F}_q[t, \theta]$. Furthermore, we have:

Proposition 25 *Assuming that $s \geq 2$ is an integer congruent to 1 modulo $q-1$ and is not a power of q, the following properties hold.*

1. *The degree in t of W_s does not exceed $s(s-1)/(q-1) - s - \mu_q(s)$,*
2. *the degree in θ of W_s is equal to $(\ell_q(s) - q)/(q-1)$.*

By the remarks in the introduction, we know how to handle the case of $s = q^i$; we then have

$$W_{q^i}(t) = \frac{1}{\theta - t^{q^i}}.$$

Proof of Proposition 25. The bound for the degree in t is a simple consequence of Proposition 24 and Lemma 23. To show the property of the degree in θ, we first notice that, by Lemma 23, for all $\zeta \in \mathbb{F}_q^{\mathrm{alg}}$,

$$v_\infty(W_s(\zeta)) = -\frac{\ell_q(s) - q}{q-1}. \tag{14}$$

The computation of $W_s(\zeta)$ is even explicit if $\zeta \in \mathbb{F}_q$. Indeed, with the appropriate choice of a $(q-1)$-th root of $(\zeta - \theta)$, the fact that $\chi_\zeta = \chi_\zeta^s$, [5, Lemma 12] and [13, Theorem 1],

$$\begin{aligned} W_s(\zeta) &= \frac{L(\chi_\zeta^s, 1)\omega(\zeta)^s}{\widetilde{\pi} H_s(\zeta)} \\ &= \frac{L(\chi_\zeta, 1)\omega(\zeta)^s}{\widetilde{\pi} H_s(\zeta)} \\ &= \frac{L(\chi_\zeta, 1)\omega(\zeta)^s}{\widetilde{\pi}(\zeta - \theta)^{\frac{s - \ell_q(s)}{q-1}}} \\ &= (\zeta - \theta)^{-\frac{1}{q-1}} (\theta - \zeta)^{-1} (\zeta - \theta)^{\frac{s}{q-1}} (\zeta - \theta)^{\frac{\ell_q(s) - s}{q-1}} \end{aligned}$$

and

$$W_s(\zeta) = -(\zeta - \theta)^{\frac{\ell_q(s) - q}{q-1}}. \tag{15}$$

Let us write:

$$W_s(t) = \sum_{i=0}^{g} a_i t^i, \quad a_i \in A.$$

By (15), we have

$$a_0 = W_s(0) = -(-\theta)^{\frac{\ell_q(s) - q}{q-1}} \tag{16}$$

and for all $\zeta \in \mathbb{F}_q^{\mathrm{alg}}$ we have, by (14),

$$|W_s(\zeta)| = |a_0|.$$

This means that for $i = 1, \ldots, g$, $|a_i| < |a_0|$, and the identity on the degree in θ follows as well. □

Corollary 26 *If $\ell_q(s) = q$, then $W_s = -1$.*

Proof. It follows from (16) and the fact that $|a_i| < |a_0|$ for $i = 1, \dots, g$. □

By Corollary 21, the function

$$V_{1,s}(t_1, \dots, t_s) = \widetilde{\pi}^{-1} L(\chi_{t_1} \cdots \chi_{t_s}, 1)\omega(t_1) \cdots \omega(t_s)$$

is, for $s \equiv 1 \pmod{q-1}$ and $s \geq 2$, a polynomial of $A[t_1, \dots, t_s]$. Since

$$\omega(t) = \frac{\widetilde{\pi}}{\theta - t} + o(1),$$

where $o(1)$ represents a function locally analytic at $t = \theta$, the function $L(\chi_{t_1} \cdots \chi_{t_s}, 1)$ vanishes on the set

$$D = \bigcup_{i=1}^{s} D_i,$$

where

$$D_i = \{(t_1, \dots, t_{i-1}, \theta, t_{i+1}, \dots, t_s) \in \mathbb{C}_\infty\}.$$

In other words, in $\mathbb{C}_\infty[[t_1 - \theta, \dots, t_s - \theta]]$, we have

$$L(\chi_{t_1} \cdots \chi_{t_s}, 1) = \sum_{i_1, \dots, i_s \geq 1} c_{i_1, \dots, i_s} (t_1 - \theta)^{i_1} \cdots (t_s - \theta)^{i_s}, \quad c_{i_1, \dots, i_s} \in \mathbb{C}_\infty, \tag{17}$$

where on both sides, we have entire analytic functions (see Corollary 8). This can also be seen, alternatively, by considering the function F_{s-1} of Lemma 5, which vanishes, and observing that

$$L(\chi_{t_1} \cdots \chi_{t_s}, 1)|_{t_i = \theta} = F_{s-1}(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_s).$$

Let us focus on the coefficient $c_{1,\dots,1}$ in the expansion (17). We then have

$$c_{1,\dots,1} = \left(\frac{d}{dt_1} \cdots \frac{d}{dt_s} L(\chi_{t_1} \cdots \chi_{t_s}, 1) \right) \Big|_{t_1 = \cdots = t_s = \theta}$$

so that

$$V_{1,s}(\theta, \dots, \theta) = (-1)^s \widetilde{\pi}^{s-1} \sum_{d \geq 0} \sum_{a \in A^+(d)} \frac{a'^s}{a} = (-1)^s \widetilde{\pi}^{s-1} c_{1,\dots,1} \in \mathbb{F}_q[\theta]$$

(by Corollary 8, the series on the right-hand side is convergent). Now, by Proposition 24, $\Pi(s)$ divides the polynomial $V_{1,s}(\theta, \dots, \theta)$ in A. We then set, as in the introduction:

$$\mathbb{B}_s = \frac{V_{1,s}(\theta, \dots, \theta)}{\Pi(s)} = G_s(\theta) \in A.$$

3.2 Proof of Theorem 2

Let τ be the unique $\mathbb{F}_q[t]$-linear automorphism of $\mathbb{T}_t$ which extends the automorphism of $\mathbb{C}_\infty$ defined, for $c \in \mathbb{C}_\infty$, by $c \mapsto c^q$. If B is a polynomial of $A[t]$ and if $\mathfrak{p}$ is a prime of degree $d > 0$, then

$$\tau^d B \equiv B \pmod{\mathfrak{p}}.$$

The reason for this is that $\mathfrak{p}$ divides the polynomial $\theta^{q^d} - \theta$. In particular,

$$(\tau^d B)(\theta) \equiv B(\theta) \pmod{\mathfrak{p}}. \tag{18}$$

Recalling the $\mathbb{C}_\infty$-linear operator φ of subsection 3.1.1, we have

$$\tau\varphi = \varphi\tau = \rho,$$

where ρ is the operator defined by $\rho(x(t)) = x(t)^q$ for all $x \in \mathbb{C}_\infty((t))$. In particular, if $s = \sum_{i=0}^{r} s_i q^i$ is expanded in base q and if $d \geq r \geq i$, from

$$\tau^d \varphi^i = \tau^{d-i}\tau^i\varphi^i = \tau^{d-i}\rho^i$$

we deduce

$$(\tau^d \omega_s)(t) = \prod_{i=0}^{r} ((\tau^{d-i}\omega)(t))^{s_i q^i},$$

so that

$$(\tau^d \omega_s)(t) = \omega(t)^s \prod_{i=0}^{r} G_{d-i}(t,\theta)^{s_i q^i}. \tag{19}$$

We can finish the proof of Theorem 2. By (18),

$$\mathbb{B}_s \equiv (\tau^d W_s)(\theta) \pmod{\mathfrak{p}}.$$

We shall now compute $(\tau^d W_s)(\theta)$. If $d > r$, then for $i = 0, \ldots, r$ we can write

$$G_{d-i}(t,\theta)^{s_i q^i} = (t-\theta)^{s_i q^i} \prod_{j=1}^{d-i-1} (t - \theta^{q^j})^{s_i q^i},$$

and

$$\prod_{i=0}^{r} G_{d-i}(t,\theta)^{s_i q^i} = (t-\theta)^s F(t),$$

where $F(t)$ is a polynomial such that

$$F(\theta) = \prod_{i=0}^{r} l_{d-i-1}^{s_i q^i}.$$

Since

$$(\tau^d W_s)(t) = \widetilde{\pi}^{-q^d} L(\chi_t^s, q^d)(t-\theta)^s \omega(t)^s F(t)$$

and $\lim_{t\to\theta}(t-\theta)\omega(t) = -\widetilde{\pi}$, we get

$$\begin{aligned}\lim_{t\to\theta}(\tau^d W_s)(t) &= (-1)^s \widetilde{\pi}^{-q^d}\zeta(q^d-s)\widetilde{\pi}^s \prod_{i=0}^{r} l_{d-i-1}^{s_i q^i} \\ &= (-1)^s \frac{BC_{q^d-s}}{\Pi(q^d-s)} \prod_{i=0}^{r} l_{d-i-1}^{s_i q^i}.\end{aligned}$$

Our Theorem 2 follows at once. □

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