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LARGE-TIME ASYMPTOTICS OF SOLUTIONS TO THE KRAMERS-FOKKER-PLANCK EQUATION WITH A SHORT-RANGE POTENTIAL

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Abstract. In this work, we use scattering method to study the Kramers-Fokker-Planck equation with a potential whose gradient tends to zero at the infinity. For short-range potentials in dimension three, we show that complex eigenvalues do not accumulate at low-energies and establish the low-energy resolvent asymptotics. This combined with high energy pseudospectral estimates valid in more general situations gives the large-time asymptotics of the solution in weighted L^2 spaces.

1. Introduction

The Kramers equation, also called the Kramers-Fokker-Planck (or KFP in short) equation, is the evolution equation for the distribution functions describing the Brownian motion of particles in an external field ([16]):

$$\frac{\partial W}{\partial t} = \left(-v\cdot\nabla_x + \nabla_v\cdot(\gamma v - \frac{F(x)}{m}) - \frac{\gamma kT}{m}\Delta_v\right) W, \tag{1.1}$$

where $W = W(t;x,v)$, $x, v \in \mathbb{R}^n$, $t \geq 0$ and $F(x) = -m\nabla V(x)$ is the external force. In this equation, x and v represent position and velocity of particles, m the mass, k the Boltzmann constant, γ the friction coefficient and T the temperature of the media. This equation is a special case of the more general Fokker-Planck equation. After a change of unknowns and for appropriate values of physical constants, the equation (1.1) can be written into the form ([9, 16])

$$\partial_t u(t;x,v) + Pu(t;x,v) = 0,\ (x,v) \in \mathbb{R}^n \times \mathbb{R}^n, n \geq 1, t > 0 \tag{1.2}$$

with the initial data

$$u(0;x,v) = u_0(x,v). \tag{1.3}$$

Here P is the Kramers-Fokker-Planck operator:

$$P = -\Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2} + v\cdot\nabla_x - \nabla V(x)\cdot\nabla_v, \tag{1.4}$$

where the potential $V(x)$ is supposed to be a real-valued C^1 function.

The large-time asymptotics of the solution is motivated by the trend to equilibrium in statistical physics and is studied by several authors in the case where $|\nabla V(x)| \to \infty$

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as $x \to \infty$. See [1, 6, 8, 9, 10, 11, 17] and references quoted therein. Note that in the work [11] on low-temperature analysis of the tunnelling effect, only the condition $|\nabla V(x)| \geq C > 0$ outside some compact set is needed. In these cases, the spectrum of P is discrete in a neighbourhood of zero and nonzero eigenvalues of P are of positive real parts. If in addition $V(x) > 0$ outside some compact set, the Maxiwilliam

$$\mathfrak{m}(x, v) = \frac{1}{(2\pi)^{\frac{n}{4}}} e^{-\frac{1}{2}(\frac{v^2}{2} + V(x))}, \tag{1.5}$$

is an eigenfunction of P associated with the eigenvalue zero. It is proven in these situations that the solution $u(t)$ to (1.2) with initial data u_0 converges exponentially rapidly to the Maxwillian in the sense that there exists some $\sigma > 0$ such that for any nice initial data u_0, one has in appropriate spaces

$$u(t) = \langle \mathfrak{m}, u_0 \rangle \mathfrak{m} + O(e^{-\sigma t}), \quad t \to +\infty, \tag{1.6}$$

where σ can be evaluated in some regimes in terms of the spectral gap in the real part of the spectra between the eigenvalue zero and the other eigenvalues. Since the change of unknowns from (1.1) to (1.2) is essentially $u = \mathfrak{m}W$ for appropriate physical constants, this result shows that the density of distribution functions described by (1.1) converges to some constant exponentially rapidly at large times and describes the well-known phenomenon of the return to equilibrium in statistical physics. Since the KFP operator is neither elliptic nor selfadjoint, the proof of such result is highly nontrivial and is realized first by the entropy method in [6] with a reminder estimate $O(t^{-\infty})$ (see also [17]) and later on by microlocal and spectral methods in [9, 10, 11]. If $V(x)$ is slowly increasing so that $|\nabla V(x)| \to 0$ as $|x| \to \infty$, (for example, $V(x) \sim c\langle x \rangle^{\mu}$ or $V(x) \sim a \ln |x|$ as $|x| \to \infty$ for some $0 < \mu < 1$ and $a, c > 0$), the Maxwillian is still an eigenfunction of P associated with the eigenvalue zero (in the second case it may be an eigenfunction for some values of a and a resonant state for some other values of a), but now the essential spectrum of P is equal to $[0, +\infty[$ and there is no spectral gap between the eigenvalue zero and the other part of the spectrum of P. A natural question to ask at this connection is whether there still exists some phenomenon of the return to the equilibrium in such cases.

The goal of this work is to study spectral properties of the KFP operator and large-time asymptotics of the solutions to the KFP equation with a potential $V(x)$ such that $|\nabla V(x)| \to 0$ as $x \to \infty$. Although our final result concerns only short-range potentials in dimension three, some intermediate results hold for slowly increasing potentials in any dimension. Throughout this work, we assume that V is C^1 on $\mathbb{R}^n$ and

$$|\nabla V(x)| \leq C\langle x \rangle^{-1-\rho}, \quad x \in \mathbb{R}^n, \tag{1.7}$$

for some $\rho \geq -1$, where $\langle x \rangle = (1 + |x|^2)^{1/2}$. The KFP operator P with the maximal domain in L^2 is a closed, accretive ($\Re P \geq 0$) and hypoelliptic operator. Denote

$$P = P_0 + W, \tag{1.8}$$

with $P_0 = v \cdot \nabla_x - \Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2}$ and $W = -\nabla V(x) \cdot \nabla_v$. If $\rho > -1$, W is a relatively compact perturbation of the free KFP operator P_0: $W(P_0 + 1)^{-1}$ is a compact operator in L^2. One can check that the essential spectrum of P is equal to $\sigma(P_0) = [0, +\infty[$ and the non-zero complex eigenvalues of P have strictly positive real parts and may

accumulate towards any point in the essential spectrum.

The main results of this paper can be summarized as follows.

Theorem 1.1. *(a). Assume $n \geq 1$ and the condition (1.7) $\rho \geq -1$. Then there exists $C > 0$ such that $\sigma(P) \cap \{z; |\Im z| > C, |\Re z| \leq \frac{1}{C}|\Im z|^{\frac{1}{3}}\} = \emptyset$ and the resolvent $R(z) = (P - z)^{-1}$ satisfies the estiamtes*

$$\|R(z)\| \leq \frac{C}{|z|^{\frac{1}{3}}}, \tag{1.9}$$

and

$$\|(1 - \Delta_v + v^2)^{\frac{1}{2}} R(z)\| \leq \frac{C}{|z|^{\frac{1}{6}}}, \tag{1.10}$$

for $|\Im z| > C$ and $|\Re z| \leq \frac{1}{C}|\Im z|^{\frac{1}{3}}$.

(b). Assume $n = 3$ and $\rho > 1$. Then P has no eigenvalues near 0. Let $S(t) = e^{-tP}$ be the semigroup of contractions generated by $-P$. One has

$$\|S(t)\|_{\mathcal{L}^{2,s} \to \mathcal{L}^{2,-s}} \leq Ct^{-\frac{3}{2}}, \quad t > 0. \tag{1.11}$$

for any $s > \frac{3}{2}$. Here $\mathcal{L}^{2,s} = L^2(\mathbb{R}^{2n}_{x,v}; \langle x \rangle^{2s} dxdv)$.

(c). Assume $n = 3$ and $\rho > 2$. Then for any $s > \frac{3}{2}$, there exists $\epsilon > 0$ such that one has the following asymptotics

$$S(t) = \frac{1}{8(\pi t)^{\frac{3}{2}}} \langle \mathfrak{m}, \cdot \rangle \mathfrak{m} + O(t^{-\frac{3}{2}-\epsilon}), \quad t \to +\infty, \tag{1.12}$$

as operators from $\mathcal{L}^{2,s}$ to $\mathcal{L}^{2,-s}$.

It may be interesting to compare (1.6) with (1.12). (1.12) shows that for any initial datum $u_0 \in \mathcal{L}^{2,s}(\mathbb{R}^6_{x,v})$, $s > \frac{3}{2}$, the density of distribution functions $W(x, v; t)$ of particles governed by the equation (1.1) tends to be constant in $\mathbb{R}^{2n}_{x,v}$, but this density decays in time like $t^{-\frac{3}{2}}$. It is well-known for Schrödinger operator with a potential $H = -\Delta_x + U(x)$ that the space-decay rates of the stationary solutions of $Hu = 0$ determine the singularities of the resolvent at the threshold zero, which in turn determine time-decay rates of solutions to the evolution equation (see, for example, [18]). From this point of views, one may expect that the solution to the KFP equation behaves like $t^{-\alpha}$ with α depending on a if $V(x) \sim a \ln |x|$, $a \in]\frac{n-2}{2}, \frac{n}{2}[$. Notice that the KFP operator is a differential operator of the first order in x-variables, but the large-time behavior of solutions looks like those to the heat equation described by $e^{t\Delta_x}$, $t \to +\infty$. This is due to the interplay between the diffusion part and the transport part of the KFP operator P. Under stronger assumption on ρ, one can calculate the second term in large-time asymptotics of solution which is of the order $O(t^{-\frac{5}{2}})$.

The method used in this work is scattering in nature: we regard the full KFP operator P as a perturbation of the free KFP operator P_0 without potential. A large part of this work is devoted to a detailed analysis of the free KFP operator. Several basic questions remain open for P_0 such as the high energy estimates for the free resolvent

near the positive real axis. A key step in the proof of the large-time asymptotics of solutions to the full KFP equation is to show that the complex eigenvalues of P do not accumulate towards the threshold zero. So far as the author knows, such a statement is not yet proven for non-selfadjoint Schrödinger operators $-\Delta + U(x)$ with a general complex-valued potential satisfying $|U(x)| = O(|x|^{-\rho}), \rho > 2$. (See however [14, 19] for dissipative potentials ($\Im U(x) \leq 0$)). Results like (1.11) and (1.12) may fail if there is a sequence of eigenvalues tending to zero tangentially to the imaginary axis. For the KFP operator P with a short-rang potential in dimension three, we prove that there are no complex eigenvalues in neighbourhood of 0 by making use of the method of threshold spectral analysis and the symmetries of the equation.

The organisation of this work is as follows. In Section 2, we study in detail the spectral properties of the model operator P_0 and establish some dispersive estimate for the semigroup generated by $-P_0$. We also prove the limiting absorption principles and the low-energy asymptotics of the resolvent $R_0(z) = (P_0 - z)^{-1}$, as well as some high energy resolvent estimates inside the pseudospectra. The threshold spectral properties of the full KFP operator P with a short-range potential is analyzed in Section 3. We prove that there is no eigenvalue of P in a neighbourhood of zero and calculate the lower-energy resolvent asymptotics. For technical reasons, we only prove these results when the dimension n is equal to three, but we believe that they remain true when $n \geq 4$. Finally, in Section 4, we prove a high energy pseudospectral estimate in general situation and this combined with the low energy resolvent estimates allows to prove the time-decay and the large-time asymptotics of solutions. In Appendix A, we study a family of nonselfadjoint harmonic oscillators which may be regarded as complex translation in variables of selfadjoint harmonic oscillators. We prove some quantitative estimates with respect to the parameters of translation, establish a spectral decomposition formula and prove some uniform time-decay estimates of the semigroup, which are used in Section 2 to analyze the free KFP operator.

2. The free Kramers-Fokker-Planck operator

Denote by P_0 the free Kramers-Fokker-Planck (KFP, inshort) operator (with $\nabla V = 0$):

$$P_0 = v \cdot \nabla_x - \Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2}. \tag{2.1}$$

P_0 is a non-selfadjoint and hypoelliptic operator with loss of $\frac{1}{3}$ derivative in x variables. The following result is known. See [9, 15].

Proposition 2.1. *One has*

$$\|\Delta_v u\|^2 + \||v|^2 u\|^2 + \||D_x|^{\frac{2}{3}} u\|^2 \leq C(\|P_0 u\|^2 + \|u\|^2), \quad u \in \mathcal{S}(\mathbb{R}^{2n}_{x,v}) \tag{2.2}$$

P_0 defined on $\mathcal{S}(\mathbb{R}^{2n}_{x,v})$ is essentially maximally accretive, i.e., the closure of P_0 in $L^2(\mathbb{R}^{2n}_{x,v})$ with core $\mathcal{S}(\mathbb{R}^{2n}_{x,v})$ is of maximal domain $D(P_0) = \{u \in L^2(\mathbb{R}^{2n}_{x,v}); P_0 u \in L^2(\mathbb{R}^{2n}_{x,v})\}$ and $\Re\langle P_0 u, u\rangle \geq 0$ for $u \in D(P_0)$.

Henceforth we still denote by P_0 its maximal extension in L^2 with domain $D(P_0) = \{u \in L^2(\mathbb{R}^{2n}_{x,v}); P_0 u \in L^2(\mathbb{R}^{2n}_{x,v})\}$.

In terms of Fourier transform in x-variables, we have for $u \in D(P_0)$

$$P_0 u(x,v) = \mathcal{F}^{-1}_{x\to\xi} \hat{P}_0(\xi) \hat{u}(\xi, v), \quad \text{where} \tag{2.3}$$

$$\hat{P}_0(\xi) = -\Delta_v + \frac{1}{4}\sum_{j=1}^{n}(v_j + 2i\xi_j)^2 - \frac{n}{2} + |\xi|^2 \tag{2.4}$$

$$\hat{u}(\xi, v) = (\mathcal{F}_{x\to\xi} u)(\xi, v) \triangleq \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x,v)\, dx. \tag{2.5}$$

Denote

$$D(\hat{P}_0) = \{f \in L^2(\mathbb{R}^{2n}_{\xi,v}); \hat{P}_0(\xi) f \in L^2(\mathbb{R}^{2n}_{\xi,v})\}. \tag{2.6}$$

Then $\hat{P}_0 \triangleq \mathcal{F}_{x\to\xi} P_0 \mathcal{F}^{-1}_{x\to\xi}$ is a direct integral of the family of complex harmonic operators $\{\hat{P}_0(\xi); \xi \in \mathbb{R}^n\}$ which is studied in Appendix.

The following abstract result may be useful to determine the spectrum of operators which are direct integral of a family of nonselfadjoint operators. See also Theorem 8.3 of [4] for bounded nonselfadjoint operators.

Theorem 2.2. *Let H be a separable Hilbert space, X a non empty open set of $\mathbb{R}^n$ and $\mathcal{H} = \{f : X \to H; \|f\| = (\int_X \|f(x)\|^2_H dx\}^{\frac{1}{2}} < +\infty\}$. Here dx is the Lebesgue measure of $\mathbb{R}^n$. Suppose that both $\{Q(x); x \in X\}$ and the adjoints $\{Q(x)^*; x \in X\}$ are strongly continuous families of closed, densely defined operators with constant domains $D, D^* \subset H$, respectively. Suppose in addition that for each $z \in \mathbb{C} \setminus \overline{\cup_{x\in X}\sigma(Q(x))}$, one has*

$$\sup_{x\in X} \|(Q(x) - z)^{-1}\| < +\infty. \tag{2.7}$$

Let Q be a closed, densely defined operator in $\mathcal{H}$ such that for any f in the domain of Q one has $f(x) \in D$ about everywhere and

$$Qf = Q(x)f(x) \quad \text{in } \mathcal{H}. \tag{2.8}$$

Then one has

$$\sigma(Q) = \overline{\cup_{x\in X}\sigma(Q(x))} \tag{2.9}$$

Proof. If $z \notin \overline{\cup_{x\in X}\sigma(Q(x))}$, define $R_z : \mathcal{H} \to \mathcal{H}$ by $(R_z f)(x) = (Q(x) - z)^{-1} f(x)$, $f \in \mathcal{H}$. Then (2.7) shows that R_z is bounded on $\mathcal{H}$. One can check that $(Q - z)R_z = 1$ on $\mathcal{H}$ and $R_z(Q - z) = 1$ on $D(Q)$. Thus z is in resolvent set of Q. This shows that $\sigma(Q) \subset \overline{\cup_{x\in X}\sigma(Q(x))}$.

Conversely, if $z \in \sigma(Q(x_0))$ for some $x_0 \in X$, then either $\|(Q(x_0) - z)u_n\| \to 0$ or $\|(Q(x_0) - z)^* v_n\| \to 0$ for some sequences $\{u_n\}$, $\{v_n\}$ in H with $\|u_n\| = \|v_n\| = 1$. In fact if one had both $\|(Q(x_0) - z)u\| \geq c\|u\|$ and $\|(Q(x_0) - z)^* v\| \geq c\|v\|$ for some $c > 0$ and for all $u \in D$ and $v \in D^*$, one might conclude that z belongs to the resolvent set of $Q(x_0)$. Since $x \to Q(x)$ is strongly continuous on D, for any $\epsilon > 0$, we can find $f \in \mathcal{H}$ in the form $f = \chi(x)u_{n_0}$ (or $g = \chi(x)v_{n_0} \in \mathcal{H}$) for some n_0 and for some numerical

function χ supported in a sufficiently small neighborhood of x_0 such that $\|f\| = 1$ and $\|(Q-z)f\| < \epsilon$ (resp., $\|g\| = 1$ and $\|(Q-z)^*g\| < \epsilon$). This shows that $z \in \sigma(Q)$. Consequently, $\sigma(Q) \supset \overline{\cup_{x\in X}\sigma(Q(x))}$ □

Remark that if the condition (2.7) is not satisfied, the equality (2.9) may fail. See Theorem 8.3 of [4].

Proposition 2.3. *Let P_0 denote the free KFP operator with the maximal domain. Then one has: $\sigma(P_0) = [0, +\infty[$.*

Proof. We want to apply Theorem 2.2. One sees that P_0 is unitarily equivalent with $\hat{P}_0$ which is a direct integral of a family of operators $\{\hat{P}_0(\xi)\}$ with constant domain D. Lemma A.1 shows that

$$\cup_{\xi\in\mathbb{R}^n}\sigma(\hat{P}_0(\xi)) = [0, +\infty[.$$

It is clear that $x \to \hat{P}_0(\xi)$ is strongly continuous on D. To apply Theorem 2.2, it remains to check the condition (2.7): for each $z \notin [0, +\infty[$,

$$\|(\hat{P}_0(\xi) - z)^{-1}\| \le C_z \tag{2.10}$$

uniformly in $\xi \in \mathbb{R}^n$. For ξ in a compact set, this follows from the fact that since $\hat{P}_0(\xi)$ forms a holomorphic family of type (A) in sense of Kato, the resolvent $(\hat{P}_0(\xi) - z)^{-1}$ is locally bounded in $\xi \in \mathbb{R}^n$ for each $z \notin [0, +\infty[$. For $|\xi|$ large ($|\xi|^2 > |\Re z| + 1$), using the representation

$$(\hat{P}_0(\xi) - z)^{-1} = -\int_0^T e^{-t(\hat{P}_0(\xi)-z)}\, dt - \int_T^\infty e^{-t(\hat{P}_0(\xi)-z)}\, dt$$

with $T \ge 3$ fixed, one deduces from Corollary A.4 that there exists $C = C(\Re z)$ independent of ξ such that

$$\|(\hat{P}_0(\xi) - z)^{-1}\| \le C + \frac{C}{\xi^2 - \Re z} \tag{2.11}$$

for $\xi^2 > |\Re z| + 1$. The desired result follows from Theorem 2.2 and Lemma A.1. □

From Proposition A.3, one can deduce some time-decay estimates for e^{-tP_0} in appropriate spaces. Denote

$$\mathcal{L}^{2,s}(\mathbb{R}^{2n}) = L^2(\mathbb{R}^{2n}; \langle x\rangle^{2s} dxdv).$$

and

$$\mathcal{L}^p = L^p(\mathbb{R}^n_x; L^2(\mathbb{R}^n_v)), \quad p \ge 1,$$

equipped with their natural norms.

Theorem 2.4. *(a). One has the following dispersive type estimate: $\exists C > 0$ such that*

$$\|e^{-tP_0}u\|_{\mathcal{L}^\infty} \le \frac{C}{t^{\frac{n}{2}}}\|u\|_{\mathcal{L}^1}, \quad t \ge 3, \tag{2.12}$$

for $u \in \mathcal{L}^1$.

(b). For $s > \frac{n}{2}$, one has for some $C_s > 0$

$$\|e^{-tP_0}u\|_{\mathcal{L}^{2,-s}} \le \frac{C_s}{t^{\frac{n}{2}}}\|u\|_{\mathcal{L}^{2,s}}, \tag{2.13}$$

for $t \ge 3$ and $u \in L^{2,s}$.

Proof. For $u \in \mathcal{S}(\mathbb{R}^{2n}_{x,v})$, we denote by $\hat{u}$ the Fourier transform of u in x-variables. Proposition A.3 shows that

$$\|e^{-tP_0}u\|_{\mathcal{L}^\infty} \leq \frac{1}{(2\pi)^n}\|e^{-t\hat{P}_0(\xi)}\hat{u}\|_{\mathcal{L}^1} \tag{2.14}$$

Corollary A.4 gives

$$\|e^{-t\hat{P}_0(\xi)}\hat{u}\|_{L^2_v} \leq \frac{e^{-\xi^2(t-2-\frac{4}{e^t-1})}}{(1-e^{-t})^n}\|\hat{u}(\xi,\cdot)\|_{L^2_v}. \tag{2.15}$$

Since $t-2-\frac{4}{e^t-1} \geq c_0 > 0$ when $t \geq 3$, one obtains that

$$\begin{aligned}\|e^{-t\hat{P}_0(\xi)}\hat{u}\|_{\mathcal{L}^1} &\leq \int_{\mathbb{R}^n_\xi} \frac{e^{-\xi^2(t-2-\frac{4}{e^t-1})}}{(1-e^{-t})^n}d\xi\|u\|_{\mathcal{L}^1} \\ &\leq Ct^{-\frac{n}{2}}\|u\|_{\mathcal{L}^1} \end{aligned}\tag{2.16}$$

for $t \geq 3$ and for any $u \in \mathcal{S}$. An argument of density proves (a) for any $u \in \mathcal{L}^1$.

Part (b) of Theorem 2.4 is a consequence of Part (a). □

As another application of the spectral decomposition for the semigroup e^{-tP_0}, we can prove the following result on large-time approximation of solutions to the free KFP equation. For a symbol $a(x,v;\xi,\eta)$, denote by $a^w(x,v,D_x,D_v)$ the associated Weyl pseudodifferential operator defined by

$$\begin{aligned}&a^w(x,v,D_x,D_v)u \\ &= \frac{1}{(2\pi)^{2n}}\int\int e^{i(x-x')\cdot\xi+i(v-v')\cdot\eta}a(\frac{x+x'}{2},\frac{v+v'}{2},\xi,\eta)u(x'v')dx'dv'd\xi d\eta\end{aligned}\tag{2.17}$$

for $u \in \mathcal{S}(\mathbb{R}^{2n}_{x,v})$.

Proposition 2.5. *Let*

$$a_0(v,\eta,\xi) = 2^{\frac{n}{2}}e^{-v^2-\eta^2+2iv\cdot\xi}$$

and $a_0^w \triangleq a_0^w(v,D_v,D_x)$ be the pseudodifferential operator with Weyl symbol $a_0(v,\eta,\xi)$. Then one has

$$\|e^{-tP_0}u - e^{(t-2)\Delta_x}a_0^w u\|_{\mathcal{L}^\infty} \leq C\frac{e^{-t}}{t^{\frac{n}{2}}}\|u - e^{-2\Delta_x}a_0^w u\|_{\mathcal{L}^1}, \tag{2.18}$$

for $t \geq 3$ and for any $u \in \mathcal{L}^1$ with $e^{-2\Delta_x}u \in \mathcal{L}^1$.

Proof. By a direction calculation, one can check that

$$\mathcal{F}^{-1}_{x\to\xi}\Pi_0^\xi\mathcal{F}_{x\to\xi} = e^{-2\Delta_x}a_0^w, \tag{2.19}$$

$$\mathcal{F}^{-1}_{x\to\xi}e^{-t\hat{P}_0(\xi)}\Pi_0^\xi\mathcal{F}_{x\to\xi} = e^{(t-2)\Delta_x}a_0^w. \tag{2.20}$$

a_0^w is continuous on $\mathcal{L}^1$. In fact, let $\tau : u(x,v) \to u(x+2v,v)$. Then one has

$$a_0^w u(x,v) = \langle\psi_0, \tau u\rangle_{L^2(\mathbb{R}^n_v)}(x)\psi_0(v),$$

where $\psi_0 = \frac{1}{(2\pi)^{\frac{n}{4}}} e^{-\frac{v^2}{4}}$ is the first eigenfunction of $-\Delta_v + \frac{v^2}{4}$. It follows that

$$\|a_0^w u\|_{\mathcal{L}^1} \leq \int_{\mathbb{R}^{2n}} \psi_0(v)|u(x+2v,v)|dxdv = \int_{\mathbb{R}^{2n}} \psi_0(v)|u(y,v)|dydv \leq \|u\|_{\mathcal{L}^1}.$$

Denote $u_0 = e^{-2\Delta_x} a_0^w u$ for $u \in \mathcal{L}^1$ with $\mathrm{supp}_\xi \hat{u}$ compact. Following the proof of Theorem 2.4 with u replaced by $u - u_0$, one has

$$\begin{aligned} \|e^{-tP_0}(u-u_0)\|_{\mathcal{L}^\infty} &\leq C \sum_{k=1}^{\infty} \|e^{-t(k+\xi^2)} \Pi_k^\xi (\hat{u} - \hat{u}_0)\|_{\mathcal{L}^1} \\ &\leq C' \frac{e^{-t}}{(t-2)^{\frac{n}{2}}} \|u - u_0\|_{\mathcal{L}^1}, \quad t > 2. \end{aligned} \tag{2.21}$$

Since $e^{-tP_0} u_0 = e^{(t-2)\Delta_x} a_0^w u$, an argument of density proves the desired result. □

The time-decay of e^{-tP_0} is governed by the first eigenvalue of the harmonic oscillator in v-variables and propagation of energy due to the transport term $v \cdot \nabla_x$. Although this term is of the first order in ξ, Theorem 2.4 shows that solutions to the free Fokker-Planck equation decay like those to the heat equation in space variables. A natural question is to see if results of Theorem 2.4 are still true for the full Fokker-Planck operator with a potential $V(x)$ such that $|\nabla V(x)|$ tends to zero sufficiently rapidly. This will be the subject of the next two Sections.

In order to study the spectral properties of P, we establish here the limiting absorption principle for the resolvent of P_0 and its low-energy asymptotics. Different from the limiting absorption principle for selfadjopint operators, the problem we want to study here is pseudospectral in nature, because $\mathbb{R}_+$ is located in the interior of the numerical range of P_0. Set $R_0(z) = (P_0 - z)^{-1}$, $\hat{R}_0(z) = (\hat{P}_0 - z)^{-1}$ and $\hat{R}_0(z,\xi) = (\hat{P}_0(\xi) - z)^{-1}$ for $z \notin \mathbb{R}_+$. Then $R_0(z) = \mathcal{F}_{x\to\xi}^{-1} \hat{R}_0(z) \mathcal{F}_{x\to\xi}$. Note that $\hat{R}_0(z)$ is multiplication in ξ-variables by $\hat{R}_0(z,\xi)$.

Proposition 2.6. *Let $l \in \mathbb{N}$ and $l < a < l+1$ be fixed. Take $\chi \geq 0$ and $\chi \in C_0^\infty(\mathbb{R}^n_\xi)$ with supp $\chi \subset \{\xi, |\xi| \leq a+4\}$, $\chi(\xi) = 1$ when $|\xi| \leq a+3$ and $0 \leq \chi(\xi) \leq 1$. Then one has*

$$\hat{R}_0(z,\xi) = \sum_{k=0}^{l} \chi(\xi) \frac{\Pi_k^\xi}{\xi^2 + k - z} + r_l(z,\xi), \tag{2.22}$$

for any $\xi \in \mathbb{R}^n$ and $z \in \mathbb{C}$ with $\Re z < a$ and $\Im z \neq 0$. Here $r_l(z,\xi)$ is holomorphic in z with $\Re z < a$ and verifies the estimate

$$\sup_{\Re z < a, \xi \in \mathbb{R}^n} \|r_l(z,\xi)\|_{\mathcal{L}(L^2(\mathbb{R}^n_v))} < \infty. \tag{2.23}$$

Proof. Let $\chi_1 = 1 - \chi$. For $\Re z < 0$, one has

$$\begin{aligned}\hat{R}_0(z,\xi) &= \int_0^\infty e^{-t(\hat{P}_0(\xi)-z)}\,dt \\ &= \int_0^\infty \chi_1(\xi)e^{-t(\hat{P}_0(\xi)-z)}\,dt + \int_0^\infty \chi(\xi)e^{-t(\hat{P}_0(\xi)-z)}\,dt \\ &\triangleq I_1(z,\xi) + I_2(z,\xi). \end{aligned} \tag{2.24}$$

Since $\Re P_0(\xi) \geq 0$, it is clear that for each fixed T, $\int_0^T \chi_1(\xi)e^{-t(\hat{P}_0(\xi)-z)}\,dt$ is uniformly bounded in ξ and z with $\Re z \leq a$:

$$\| \int_0^T \chi_1(\xi)e^{-t(\hat{P}_0(\xi)-z)}\,dt\|_{\mathcal{B}(L^2(\mathbb{R}^n_v))} \leq a^{-1}e^{aT},$$

for all $\xi \in \mathbb{R}^n$. Corollary A.4 with $T = 3$ shows that

$$\|e^{-t(\hat{P}_0(\xi)-z)}\|_{\mathcal{B}(L^2(\mathbb{R}^n_v))} \leq Ce^{-t(\xi^2-2-e^{-1}-\Re z)} \tag{2.25}$$

for $t \geq 3$. Since χ_1 is supported in $\xi^2 \geq a+3$, $I_1(z,\xi)$ is holomorphic in z with $\Re z < a$ and verifies the estimate (2.23). To study $I_2(z,\xi)$, we write

$$e^{-t\hat{P}_0(\xi)} = J_1(t,\xi) + J_2(t,\xi),$$

where $J_j(t,\xi) = e^{-t\hat{P}_0(\xi)}S_j^\xi$ with $S_1^\xi = \sum_{k=0}^l \Pi_k^\xi$ and $S_2^\xi = 1 - S_1^\xi$. For $\Re z < 0$, the contribution of $J_1(t,\xi)$ to $\hat{R}_0(z,\xi)$ is

$$\int_0^\infty e^{tz}J_1(t,\xi)dt = \sum_{k=0}^l \frac{\Pi_k^\xi}{\xi^2 + k - z}.$$

By (A.4), one has for $t \geq T > 0$

$$\|J_2(t,\xi)\|_{\mathcal{B}(L^2(\mathbb{R}^n_v))} \leq \sum_{k=l+1}^\infty e^{-t(k+\xi^2)+2\xi^2}\sum_{j=0}^k \frac{C_k^j}{j!}(4\xi^2)^j \triangleq J_{21}(t,\xi) + J_{22}(t,\xi) \tag{2.26}$$

where

$$\begin{aligned} J_{21}(t,\xi) &= e^{-\xi^2(t-2)}\sum_{j=0}^{l+1}\frac{(4\xi^2)^j}{j!}\sum_{k=l+1}^\infty C_k^j e^{-tk} \\ J_{22}(t,\xi) &= e^{-\xi^2(t-2)}\sum_{j=l+2}^{\infty}\frac{(4\xi^2)^j}{j!}\sum_{k=j}^\infty C_k^j e^{-tk}. \end{aligned}$$

$J_{21}(t,\xi)$ and $J_{22}(t,\xi)$ can be evaluated as in the proof of Proposition A.3 and we omit the details here. One has

$$J_{21}(t,\xi) \leq e^{-\xi^2(t-2)-(l+1)t}\sum_{j=0}^{l+1}\frac{(4\xi^2)^j l!}{j!(1-e^{-t})^{l+1}} \tag{2.27}$$

$$J_{22}(t,\xi) \leq Ce^{-\xi^2(t-4)-(l+1)t} \tag{2.28}$$

Since $|\xi|$ is bounded on the support of χ, this implies that there exists some constant C such that

$$\|J_2(t,\xi)\|_{\mathcal{B}(L^2(\mathbb{R}^n_v))} \leq Ce^{-(l+1)t} \tag{2.29}$$

uniformly in $\xi \in supp\chi$ and $t \geq T$. We obtain a representation formula for $\hat{R}_0(z,\xi)$ when $\Re z < 0$:

$$\hat{R}_0(z,\xi) = \sum_{k=0}^{l} \chi(\xi) \frac{\Pi_k^\xi}{\xi^2 + k - z} + r_l(z,\xi), \tag{2.30}$$

where

$$r_l(z,\xi) = I_1(z,\xi) + \int_0^\infty e^{tz} J_2(t,\xi) dt$$

is holomorphic in z with $\Re z < a$ and verifies the estimate (2.23). Since the both sides of (2.30) are holomorphic in $z \in \mathbb{C} \setminus \mathbb{R}_+$ with $\Re z < a$, this representation formula remains valid for z in this region. □

For $r, s \in \mathbb{R}$, introduce the weighted Sobolev space

$$\mathcal{H}^{r,s} = \{u \in \mathcal{S}'(\mathbb{R}^{2n}); (1 + |D_v|^2 + |v|^2 + |D_x|^{\frac{2}{3}})^{\frac{r}{2}} \langle x \rangle^s u \in L^2\}.$$

Denote $\mathcal{B}(r,s;r',s')$ the space of continuous linear operators from $\mathcal{H}^{r,s}$ to $\mathcal{H}^{r',s'}$. Proposition 2.1 shows that $(P_0+1)^{-1} \in \mathcal{B}(0,0;2,0)$. A commutator argument shows that $(P_0+1)^{-1} \in \mathcal{B}(0,s;2,s)$ for any $s \in \mathbb{R}$.

Corollary 2.7. *Set $R_0(z) = (P_0 - z)^{-1}$, $z \notin \mathbb{R}_+$.*

(a). Let I be a compact interval in $\mathbb{R}$ which does not contain any non negative integer. Then for any $s > \frac{1}{2}$, one has

$$\sup_{\lambda \in I; \epsilon \in]0,1]} \|R_0(\lambda \pm i\epsilon)\|_{\mathcal{B}(0,s;2,-s)} < \infty \tag{2.31}$$

The boundary values of the resolvent $R_0(\lambda \pm i0) = \lim_{\epsilon \to 0_+} R_0(\lambda \pm i\epsilon)$ exists in $\mathcal{B}(0,s;2,-s)$ for $\lambda \in I$ and is continuous in λ.

(b). Assume $n \geq 3$. Let I be a compact interval containing some non negative integer. Then for any $s > 1$, one has

$$\sup_{\lambda \in I; \epsilon \in]0,1]} \|R_0(\lambda \pm i\epsilon)\|_{\mathcal{B}(0,s;2,-s)} < \infty \tag{2.32}$$

for any $k \in \mathcal{N}$, the limite $R_0(k \pm i0) = \lim_{z \to k, z \notin \mathbb{R}_+} R_0(z)$ exists in $\mathcal{B}(0,s;2,-s)$ for any $s > 1$. One has $R_0(0+i0) = R_0(0-i0)$ and $R_0(k+i0) - R_0(k-i0) \in \mathcal{B}(0,s;2,-s)$ for any $s > \frac{1}{2}$ if $k \geq 1$.

Proof. Proposition 2.6 shows that

$$R_0(z) = \sum_{k=0}^{l} \chi(D_x) \Pi_k^{D_x} (-\Delta_x + k - z)^{-1} + r_l(z), \tag{2.33}$$

for $z \in \mathbb{C}$ with $\Re z < a$ and $\Im z \neq 0$ and that $r_l(z)$ is bounded on L^2 and holomorphic in z with $\Re z < a$. $\chi(D_x)\Pi_k^{D_x}$ is a Weyl pseudodifferential operator with nice symbol given by

$$b_k(v,\xi,\eta) = \int_{\mathbb{R}^n} e^{-iv'\cdot\eta/2} \left(\sum_{|\alpha|=k} \chi(\xi) \psi_\alpha(v + v' + 2i\xi) \psi_\alpha(v - v' + 2i\xi) \right) dv'. \tag{2.34}$$

In particular,

$$b_0(v,\xi,\eta) = 2^{\frac{n}{2}}\chi(\xi)e^{-v^2-\eta^2+2iv\cdot\xi+2\xi^2}. \tag{2.35}$$

These operators belong to $\mathcal{B}(r,s;r',s)$ for any $r,r',s\in\mathbb{R}$.

Since for any compact interval $I'\subset\mathbb{R}$, one has

$$\sup_{\lambda\in I',\epsilon\in]0,1]} \|\langle x\rangle^{-s}(-\Delta_x-(\lambda\pm i\epsilon))^{-1}\langle x\rangle^{-s}\|_{\mathcal{B}(L^2(\mathbb{R}^n_x))}<\infty \tag{2.36}$$

for any $s>1/2$ if I' does not contain 0 and for any $s>1$ and $n\geq 3$ if I' contains 0, it follows from (2.33) that for $I\subset]-\infty,a[$

$$\sup_{\lambda\in I;\epsilon\in]0,1]} \|R_0(\lambda\pm i\epsilon)\|_{\mathcal{B}(0,s;0,-s)}<\infty \tag{2.37}$$

for $s>\frac{1}{2}$ if $I\cap\mathbb{N}=\emptyset$ or $s>1$ and $n\geq 3$ if $I\cap\mathbb{N}\neq\emptyset$. Estimates (2.31) and (2.32) can be deduced from (2.37) and the resolvent equation

$$R_0(z) = R_0(-1)+(1+z)R_0(-1)R_0(z)$$

by noticing that $R_0(-1)\in\mathcal{B}(0,s;2,s)$ for any $s\in\mathbb{R}$. In fact, Proposition 2.1 shows that $R_0(-1)\in\mathcal{B}(0,0;2,0)$ and the case for general s follows from an easy commutator technique. The other assertions of Corollary 2.7 can be proven by making use of the properties of $(-\Delta_x-(\lambda\pm i0))^{-1}$. □

The formula (2.33) can also be used to study the threshold asymptotics of the resolvent $R_0(z)$ as $z\to k$, $\Im z\neq 0$. To be simple, we only consider the threshold zero in the case $n=3$.

Proposition 2.8. *Let $n=3$. One has the following low-energy resolvent expansion for $R_0(z)$: for $s,s'>\frac{1}{2}$ and $s+s'>2$, there exists $\epsilon>0$ such that*

$$R_0(z) = G_0+O(|z|^\epsilon),\ \text{as } z\to 0, z\notin\mathbb{R}_+, \tag{2.38}$$

as operators in $\mathcal{B}(-1,s;1,-s')$. More generally, for any integer $N\geq 1$ and $s>N+\frac{1}{2}$, there exists $\epsilon>0$

$$R_0(z) = \sum_{j=0}^{N} z^{\frac{j}{2}}G_j + O(|z|^{\frac{N}{2}+\epsilon}),\ \text{as } z\to 0, z\notin\mathbb{R}_+, \tag{2.39}$$

as operators in $\mathcal{B}(-1,s;1,-s)$. Here the branch of $z^{\frac{1}{2}}$ is chosen such that its imaginary part is positive when $z\notin\mathbb{R}_+$ and $G_j\in\mathcal{B}(-1,s;1,-s)$ for $s>j+\frac{1}{2}$, $j\geq 1$. In particular,

$$G_0 = F_0+F_1, \tag{2.40}$$

where F_0 is the operator with integral kernel

$$F_0(x,v;x',v') = \frac{\psi_0(v)\psi_0(v')}{4\pi|x-x'|} \tag{2.41}$$

and $F_1\in\mathcal{B}(-1,s;1,-s')$ for any $s,s'\geq 0$ and $s+s'>\frac{3}{2}$. $G_1:\mathcal{H}^{-1,s}\to\mathcal{H}^{1,-s}$, $s>\frac{3}{2}$, is an operator of rank one with integral kernel given by

$$K_1(x,x';v,v') = \frac{i}{4\pi}\psi_0(v)\psi_0(v'). \tag{2.42}$$

Here $\psi_0 = (2\pi)^{-\frac{3}{4}} e^{-\frac{v^2}{4}}$ is the first eigenfunction of the harmonic oscillator $-\Delta_v + \frac{1}{4}v^2$.

Proof. Note that by a complex interpolation, the results on the resolvent $R_0(z)$ in Corollary 2.7 also hold in $\mathcal{B}(-1, s;\ 1, -s)$.

For $z \notin \mathbb{R}_+$, (2.33) with $l = 0$ shows that

$$R_0(z) = \chi(D_x)\Pi_0^{D_x}(-\Delta_x - z)^{-1} + r_0(z), \tag{2.43}$$

with $r_0(z) \in \mathcal{B}(-1, 0; 1, 0)$ holomorphic in z when $\Re z < a$ for some $a \in]0, 1[$. Here the cut-off $\chi(\xi)$ is chosen such that $\chi \in C_0^\infty$ and $\chi(\xi) = 1$ in a neighbourhood of $\{|\xi|^2 \leq a\}$. Therefore $r_0(z)$ admits a convergent expansion in powers of z for z near 0

$$r_0(z) = r_0(0) + zr_0'(0) + \cdots$$

in $\mathcal{B}(-1, 0; 1, 0)$. It is sufficient to analyze the lower-energy expansion of $\chi(D_x)\Pi_0^{D_x}(-\Delta_x - z)^{-1}$.

The integral kernel of $\chi(D_x)\Pi_0^{D_x}(-\Delta_x - z)^{-1}$, $z \notin \mathbb{R}_+$, is given by

$$K(x, x'; v, v'; z) = \int_{\mathbb{R}^3} e^{i\sqrt{z}|y-(x-x')|} \frac{1}{4\pi|y - (x - x')|} \Phi(v, v', y)\, dy \tag{2.44}$$

with

$$\Phi(v, v', y) = (2\pi)^{-\frac{9}{2}} e^{-\frac{1}{4}(v^2+v'^2)} \int_{\mathbb{R}^3} e^{i(y-v-v')\cdot\xi + 2\xi^2} \chi(\xi)\, d\xi.$$

Since $\chi \in C_0^\infty$, one has the following asymptotic expansion for $K(x, x'; v, v'; z)$: for any $\epsilon \in [0, 1]$ and $N \geq 0$

$$|K(x, x'; v, v'; z) - \sum_{j=0}^{N} z^{\frac{j}{2}} K_j(x, x', v, v')| \leq C_{N,\epsilon}|z|^{\frac{N+\epsilon}{2}} |x - x'|^{N-1+\epsilon} e^{-\frac{1}{4}(v^2+v'^2)}) \tag{2.45}$$

where

$$K_j(x, x') \ = \ \frac{i^j}{4\pi} \int_{\mathbb{R}^3} |y - (x - x')|^{j-1} \Phi(v, v', y) dy. \tag{2.46}$$

Remark that for $N \geq 1$, $s', s > N + \frac{1}{2}$ and $0 < \epsilon < \min\{s, s'\} - N - \frac{1}{2}$,

$$\langle x\rangle^{-s} \langle x'\rangle^{-s'} |x - x'|^{N-1+\epsilon} e^{-\frac{1}{4}(v^2+v'^2)} \in L^2(\mathbb{R}^{12})$$

and the same is true if $N = 0$ and $s, s' > \frac{1}{2}$ with $s + s' > 2$. We obtain the asymptotic expansion for $\chi(D_x)\Pi_0^{D_x}(-\Delta_x - z)^{-1}$ in powers of $z^{\frac{1}{2}}$ for z near 0 and $z \notin \mathbb{R}_+$.

$$\chi(D_x)\Pi_0^{D_x}(-\Delta_x - z)^{-1} = \sum_{j=0}^{N} z^{\frac{j}{2}} K_j + O(|z|^{\frac{N}{2}+\epsilon}), \text{ as} \tag{2.47}$$

as operators in $\mathcal{B}(0, s'; 0, -s)$, $s', s > N + \frac{1}{2}$ (and $s + s' > 2$ if $N = 0$). By the hypoelltpticity of P_0 (Proposition 2.1), this expansion still holds in $\mathcal{B}(-1, s'; 1, -s)$. This proves (2.39) with

$$G_{2j} = K_{2j} + \frac{r_0^{(j)}(0)}{j!}, \quad G_{2j+1} = K_{2j+1}, \quad j \geq 0. \tag{2.48}$$

To show (2.40) and (2.8), note that since $\chi(0) = 1$, one has

$$\int_{\mathbb{R}^3} \Phi(v, v', y)\, dy = \chi(0)\psi_0(v)\psi_0(v') = \psi_0(v)\psi_0(v').$$

One can then calculate that

$$\begin{aligned} K_0(x, x', v, v') &= \frac{1}{4\pi}\int_{\mathbb{R}^3} \Phi(v, v', y)\frac{1}{|y-(x-x')|}dy \\ &= \frac{1}{4\pi|x-x'|}\psi_0(v)\psi_0(v') \\ &\quad +\frac{1}{4\pi}\int_{\mathbb{R}^3} \Phi(v, v', y)(\frac{1}{|y-(x-x')|} - \frac{1}{|x-x'|})dy \end{aligned} \tag{2.49}$$

and

$$K_1(x, x', v, v') = \frac{i}{4\pi}\int_{\mathbb{R}^3} \Phi(v, v', y)dy = \frac{i}{4\pi}\psi_0(v)\psi_0(v'). \tag{2.50}$$

This shows (2.8) and that $G_0 = F_0 + F_1$ with $F_1 = K_{0,1} + r_0(0)$, $K_{0,1}$ being the operator with the integral kernel

$$K_{0,1}(x, x', v, v') = \frac{1}{4\pi}\int_{\mathbb{R}^3} \Phi(v, v', y)(\frac{1}{|y-(x-x')|} - \frac{1}{|x-x'|})dy.$$

$K_{0,1}(x, x', v, v')$ is a smooth function and for $|x - x'|$ large one has

$$K_{0,1}(x, x', v, v') = O(\psi_0(v)\psi_0(v')|x-x'|^{-2}).$$

Therefore $K_{0,1}$ is bounded in $B(-1, s; 1, -s')$ for any $s, s' \geq 0$ and $s + s' > \frac{3}{2}$. This proves that $F_1 = K_{0,1} + r_0(0)$ has the same continuity property. □

Note that the numerical range of P_0 is equal to the right-half complexe plane. The following high-energy pseudo-spectral estimate of the resolvent is of interest in itself.

Proposition 2.9. *Let $n \geq 1$. Then for every $\delta > 0$, there exists $M > 0$ such that*

$$\|R_0(z)\| \leq \frac{M}{|z|^{\frac{1}{3}}}, \tag{2.51}$$

and

$$\|(1 - \Delta_v + v^2)^{\frac{1}{2}}R_0(z)\| \leq \frac{M}{|z|^{\frac{1}{6}}}, \tag{2.52}$$

for $|\Im z| > \delta$ and $\Re z \leq \frac{1}{M}|\Im z|^{\frac{1}{3}}$.

Proof. We firstly prove that for some constant $C > 0$

$$\|R_0(z)\| \leq \frac{C}{|z|^{\frac{1}{3}}}, \tag{2.53}$$

for $z = -\frac{n}{2} + i\mu$ with $\mu \in \mathbb{R}$. It suffices to show that

$$\|\hat{R}_0(-\frac{n}{2} + i\mu, \xi)\|_{\mathcal{B}(L^2(\mathbb{R}^n_v))} \leq \frac{C}{|z|^{\frac{1}{3}}}, \tag{2.54}$$

uniformly in ξ. Proposition 2.1 shows that for some $C_1 > 0$

$$(|\xi|^{\frac{2}{3}} - C_1(|z| + 1))\|u\| \leq C_1\|(\hat{P}_0(\xi) - z)u\|, \quad u \in D.$$

which gives

$$\|\hat{R}_0(z,\xi)\|_{\mathcal{B}(L^2_v)} \leq \frac{C_1}{1+|z|}, \tag{2.55}$$

if $|\xi| \geq [(C_1+1)(|z|+1)]^{\frac{3}{2}}$. Making use of the resolvent equation

$$\hat{R}_0(z,\xi) = \hat{R}_0(z,0) - \hat{R}_0(z,\xi) iv\cdot\xi \hat{R}_0(z,0)$$

and the estimates $\|\hat{R}_0(z,0)\| \leq \frac{1}{|\Im z|}$ and $\|v\cdot\xi\hat{R}_0(z,0)\| \leq \frac{C'|\xi|}{\sqrt{|\Im z|}}$, $\Im z \neq 0$, one obtains

$$\|\hat{R}_0(z,\xi)\|_{\mathcal{B}(L^2_v)} \leq \frac{C}{|\Im z|}, \tag{2.56}$$

if $|\xi| \leq c_1\sqrt{|z|}$ for some $c_1 > 0$. For $c|z|^{\frac{1}{2}} \leq |\xi| \leq c^{-1}|z|^{\frac{3}{2}}$ with $c > 0$ small enough, since we are concerned with the estimates with $|z|$ large, we can use a rescaling to reduce to a semiclassical problem. Set $A(h) = -h^2\Delta_v + \frac{v^2}{4} + iv_1$. Then

$$\|\hat{R}_0(z,\xi)\| = |\xi|^{-2}\|(A(h)-z')^{-1}\|$$

where $h = |\xi|^{-2}$ and $z' = |\xi|^{-2}(\frac{n}{2}+z)$. According to Theorem 1.4 of [5], one has

$$\|(A(h)-z')^{-1}\| \leq Ch^{-\frac{2}{3}}, \tag{2.57}$$

if $0 < h \leq h_0$, $|z'| \leq C$ and $\Re z' \leq \frac{|\Im z'|^2}{4}$. In particular, for $z = -\frac{n}{2} + i\mu$ with μ real, one has

$$\|\hat{R}_0(z,\xi)\| \leq C'h^{\frac{1}{3}} \leq C''|z|^{-\frac{1}{3}}, \tag{2.58}$$

for $c|z|^{\frac{1}{2}} \leq |\xi| \leq c^{-1}|z|^{\frac{3}{2}}$. This proves (2.53). Now to prove (2.51), set $z = \lambda + i\mu$ with $\lambda, \mu \in \mathbb{R}$ and write

$$R_0(\lambda+i\mu) = R_0(-\frac{n}{2}+i\mu) - (\lambda+\frac{n}{2})R_0(-\frac{n}{2}+i\mu)R_0(\lambda+i\mu).$$

According to (2.53),

$$\|(\lambda+\frac{n}{2})R_0(-\frac{n}{2}+i\mu)\| \leq \frac{C|\lambda+\frac{n}{2}|}{|z|^{\frac{1}{3}}} \leq \frac{1}{2}$$

if $|\lambda| \leq \frac{1}{M}|\mu|^{\frac{1}{3}}$ and $|\mu| \geq M$ for some $M > 1$ large enough. (2.51) follows from (2.53) and the equation $R_0(\lambda+i\mu) = (1+(\lambda+\frac{n}{2})R_0(-\frac{n}{2}+i\mu))^{-1}R_0(-\frac{n}{2}+i\mu)$ when $|\lambda|\frac{1}{M}|\Im z|^{\frac{1}{3}}$ with $M > 0$ sufficiently large. The estimate (2.51) for $\lambda < -\frac{1}{M}|\Im z|^{\frac{1}{3}}$ follows from the accretivity of P_0

To show (2.52), notice that for $z = \lambda + i\mu$ with $\lambda, \mu \in \mathbb{R}$, one has the identity

$$\|\nabla_v u\|^2 + \frac{1}{4}\||v|u\|^2 = (\lambda+\frac{n}{2})\|u\|^2 + \Re\langle(P_0-z)u,u\rangle$$

for $u \in D$. One obtains from (2.51) that

$$\begin{aligned} &\|\nabla_v R_0(z)u\|^2 + \frac{1}{4}\||v|R_0(z)u\|^2 \\ \leq\ & |\lambda + \frac{n}{2}|\|R_0(z)u\|^2 + \|u\|\|R_0(z)u\| \\ \leq\ & C(\frac{|\lambda|}{|z|^{\frac{2}{3}}} + \frac{1}{|z|^{\frac{1}{3}}})\|u\|^2 \end{aligned}$$

for $|\Im z| > M$ and $\Re z \leq \frac{1}{M}|\Im z|^{\frac{1}{3}}$. (2.52) is proven. □

From Propositions 2.8 and 2.9, we deduce the following

Corollary 2.10. *Let $n = 3$. Let $S_0(t)$, $t \geq 0$, denote the semigroup generated by $-P_0$. Then for any integer $N \geq 1$ and $s > N + \frac{1}{2}$, the following asymptotic expansion holds for some $\epsilon > 0$*

$$e^{-tP_0} = \sum_{k\in\mathbb{N}, 2k+1\leq N} t^{-\frac{2k+3}{2}}\beta_k G_{2k+1} + O(t^{-\frac{N+2}{2}-\epsilon}), \quad t \to +\infty, \tag{2.59}$$

in $\mathcal{B}(0, s, 0_s)$. Here β_k is some non zero constant. In particular, $\beta_1 G_1$ is a rank-one operator given by

$$\beta_1 G_1 = \frac{1}{8\pi^{\frac{3}{2}}}\langle 1 \otimes \psi_0,\ \cdot\rangle \mathfrak{m}_0 : \mathcal{L}^{2,s} \to \mathcal{L}^{2,-s} \tag{2.60}$$

for any $s > \frac{3}{2}$. Here $\mathfrak{m}_0(x, v) = 1 \otimes \psi_0(v)$.

The proof of Corollary 2.10 uses a representation of the semigroup as contour integral of the resolvent in the right half complex plane. See Section 5 for more details in the case $V \neq 0$. In Section 4, we will prove a similar result or the full KFP operator P (cF (1.12)). In the final step of its proof, we need the following Proposition 2.12 to compute the leading term. As a preparation for its proof, we first establish some formulai on the evolution of observables which may be of interest in themselves.

Lemma 2.11. *For $t \geq 0$ and $0 \leq s \leq t$, one has the following equalities as operators from $\mathcal{S}(\mathbb{R}^{2n}_{x,v})$ to $L^2(\mathbb{R}^{2n}_{x,v})$, $n \geq 1$.*

$$e^{-(t-s)P_0} v_j e^{-sP_0} = e^{-tP_0}(v_j \cosh s - 2\sinh s\partial_{v_j} + 2(\cosh s - 1)\partial_{x_j}) \tag{2.61}$$

$$e^{-(t-s)P_0}\partial_{v_j} e^{-sP_0} = -\frac{1}{2}e^{-tP_0}((v_j \sinh s - 2\cosh s\partial_{v_j} + 2\sinh s\partial_{x_j})) \tag{2.62}$$

$$\begin{aligned} e^{-(t-s)P_0} x_j e^{-sP_0} &= e^{-tP_0}(x_j + v_j \sinh s - 2(\cosh s - 1)\partial_{v_j} \\ &\quad +2(\sinh s - s)\partial_{x_j}) \end{aligned} \tag{2.63}$$

Proof. Let $f(s) = e^{-(t-s)P_0} v_j e^{-sP_0}$. Proposition 2.1 shows that for $u \in \mathcal{S}$, $A^k e^{-tP_0} u \in L^2$ for any $k \in \mathbb{N}$, where A may be any one of the operators $v_j, \partial_{v_j}, \partial_{x_j}$. As operators from $\mathcal{S}$ to L^2, one has:

$$f'(s) = e^{-(t-s)P_0}[P_0, v_j]e^{-sP_0} = -2e^{-(t-s)P_0}\partial_{v_j} e^{-sP_0} \tag{2.64}$$

$$f''(s) = -2e^{-(t-s)P_0}[\frac{v^2}{4} + v \cdot \partial_x, \partial_{v_j}]e^{-sP_0} = f(s) + 2\partial_{x_j} e^{-tP_0} \tag{2.65}$$

This shows that $f(s) = C_1 e^s + C_2 e^{-s} - 2\partial_{x_j} e^{-tP_0}$. C_1, C_2 can be determined by the initial data $f(0) = e^{-tP_0} v_j$ and $f'(0) = -2e^{-tP_0}\partial_{v_j}$:

$$C_1 = e^{-tP_0}(\frac{1}{2}v_j - \partial_{v_j} + \partial_{x_j}), \quad C_2 = e^{-tP_0}(\frac{1}{2}v_j + \partial_{v_j} + \partial_{x_j}).$$

This proves (2.61). (2.62) follows from (2.61) and the equality

$$e^{-(t-s)P_0}\partial_{v_j} e^{-sP_0} = -\frac{1}{2}f'(s).$$

To prove (2.63), one can check the following commutator relation:

$$\begin{aligned} [e^{-tP_0}, x_j] &= -\int_0^t e^{(t-s)P_0} v_j e^{-sP_0} ds \\ &= -e^{-tP_0}\left(v_j \sinh t - 2(\cosh t - 1)\partial_{v_j} + 2(\sinh t - t)\partial_{x_j}\right). \end{aligned} \tag{2.66}$$

This means that the commutator initiall defined as forms on $\mathcal{S}\times\mathcal{S}$ extends to operators from $\mathcal{S}$ to L^2 and the equality (2.66) holds. A successive application of this commutator relation shows that if $u \in \mathcal{S}$, then $\langle x\rangle^s e^{-tP_0} u \in L^2$ for any s. It follows from (2.61) that

$$\begin{aligned} e^{-(t-s)P_0} x_j e^{-sP_0} &= e^{-tP_0} x_j + \int_0^s e^{-(t-\tau)P_0} v_j e^{-\tau P_0}\, d\tau \\ &= e^{-tP_0}(x_j + v_j \sinh s - 2(\cosh s - 1)\partial_{v_j} + 2(\sinh s - s)\partial_{x_j}). \end{aligned}$$

This proves (2.63) □

Proposition 2.12. *Assume that $u \in \mathcal{L}^{2,-s}$ for some $\frac{3}{2} < s < 2$ such that there exists some constant c_0 and $\psi \in L^2_v$ with $(-\Delta_v + v^2)\psi \in L^2_v$ such that*

$$u(x,v) - c_0(1\otimes\psi) \in \mathcal{L}^{2,\delta}(\mathbb{R}^6_{x,v})$$

for some $\delta > 0$. Then on has

$$\lim_{\lambda\to 0_-} \lambda R_0(\lambda) u = -c_0\langle\psi_0, \psi\rangle_{L^2_v}\mathfrak{m}_0 \tag{2.67}$$

in $\mathcal{L}^{2,-s}$ for any $s > \frac{3}{2}$.

Proof. For $\lambda < 0$, $R_0(\lambda)$ maps $\mathcal{L}^{2,-s}$ to $\mathcal{L}^{2,-s}$ for any s and one has

$$\|R_0(\lambda)\|_{\mathcal{B}(0,0;0,0)} \le \frac{C}{|\lambda|}, \quad \|R_0(\lambda)\|_{\mathcal{B}(0,s';0,-s)} \le C_{s,s'}$$

if $s, s' > \frac{1}{2}$ with $s + s' > 2$, uniformly in $\lambda \in]-1, 0[$. An argument of complex interpolation shows that for any $s, s' > 0$, there exists $\epsilon > 0$ such that

$$\|R_0(\lambda)\|_{\mathcal{B}(0,s';0,-s)} \le C|\lambda|^{-1+\epsilon}. \tag{2.68}$$

This shows that

$$\lambda R_0(\lambda)(u - c_0(1\otimes\psi)) = o(1), \quad \text{as } \lambda \to 0_-$$

in $\mathcal{L}^{2,-s}$ for any $s > 0$. To prove Proposition 2.12, it suffices to study the limit $\lim_{\lambda\to 0_-} \lambda R_0(\lambda)(1\otimes\psi)$. By Proposition 2.6, the resolvent $R_0(\lambda)$ can be decomposed as

$$R_0(\lambda) = \chi(D_x)\Pi_0^{D_x}(-\Delta_x - \lambda)^{-1} + r_0(\lambda) \tag{2.69}$$

where χ is a smooth cut-off around 0 with compact support and $r_0(\lambda)$ is uniformly bounded as operators in L^2 for $\lambda < a$ for some $a \in]0,1[$.

We claim that the following estimate holds uniformly in $\lambda < a$:

$$\|\langle x\rangle^{-2} r_0(\lambda)\langle x\rangle^2 f\| \leq C(\|f\| + \|H_0 f\|) \tag{2.70}$$

for any $f \in D(H_0)$, where $H_0 = -\Delta_v + v^2 - \Delta_x$. Remark that $r_0(\lambda)$ is a pseudodifferential operator in x-variables: $r_0(\lambda) = r_0(\lambda, D_x)$, with operator-valued symbol $r_0(\lambda,\xi) \in \mathcal{B}(L^2_v)$ (see (2.30)). The proof of Proposition 2.6 shows that $r_0(\lambda,\xi)$ can be decomposed as

$$r_0(\lambda,\xi) = r_{0,1}(\lambda,\xi) + r_{0,2}(\lambda,\xi)$$

with

$$r_{0,1}(\lambda,\xi) = \int_0^T \chi_1(\xi) e^{-t(\hat{P}_0(\xi)-\lambda)} dt$$

for some fixed $T \geq 3$ and $r_{0,2}(\lambda,\xi)$ is smooth and rapidly decreasing in ξ, uniformly for $\lambda < a$ (see (2.25), (2.27) and (2.28)). Since $r_{0,2}(\lambda, D_x)$ is a convolution in x-variables with a smooth and rapidly decreasing kernel, one has $\langle x\rangle^{-s} r_{0,2}(\lambda, D_x)\langle x\rangle^s$ is uniformly bounded for any s. To study $r_{0,1}(\lambda, D_x)$, we use commutator techniques. One writes

$$\langle x\rangle^{-2} r_{0,1}(\lambda, D_x) x^2 = \langle x\rangle^{-2} \left(x^2 r_{0,1}(\lambda, D_x) + \sum_{j=1}^n [\, [r_{0,1}(\lambda, D_x), x_j], x_j] \right).$$

Making use of (2.66), one can calculate

$$\begin{aligned}
&[r_{0,1}(\lambda, D_x), x_j] \\
&= -i(\partial_{\xi_j}\chi_1)(D_x) \int_0^T e^{-t(\hat{P}_0(\xi)-\lambda)} dt + \chi_1(D_x) \int_0^T [e^{-t(\hat{P}_0(\xi)-\lambda)}, x_j] dt \\
&= -i(\partial_{\xi_j}\chi_1)(D_x) \int_0^T e^{-t(\hat{P}_0(\xi)-\lambda)} dt \qquad (2.71)\\
&\quad + \chi_1(D_x) \int_0^T e^{-t(P_0-\lambda)} (v_j \sinh t - 2(\cosh t - 1)\partial_{v_j} + 2(\sinh t - t)\partial_{x_j}) dt
\end{aligned}$$

Since P_0 is accretive, one obtains

$$\|[r_{0,1}(\lambda, D_x), x_j] f\| \leq C(\|f\| + \|v_j f\| + \|\partial_{v_j} f\| + \|\partial_{x_j} f\|), \quad f \in \mathcal{S}.$$

Similarly, one can check that the second commutator $[\, [r_{0,1}(\lambda, D_x), x_j], x_j]$ verifies

$$\|[\, [r_{0,1}(\lambda, D_x), x_j], x_j] f\| \leq C(\|f\| + \|H_0 f\|).$$

This proves that

$$\|\langle x\rangle^{-2} r_{0,1}(\lambda)\langle x\rangle^2 f\| \leq C(\|f\| + \|H_0 f\|) \tag{2.72}$$

for any $f \in D(H_0)$, which gives (2.70). It follows from (2.70) and the uniform boundedness of $\|r_0(\lambda)\|$ that for any $s \in [0,2]$

$$\|\langle x\rangle^{-s} r_0(\lambda)\langle x\rangle^s f\| \leq C(\|f\| + \|H_0 f\|), \tag{2.73}$$

Since $1 \otimes \psi = \langle x\rangle^s f_s$ with $f_s \in D(H_0)$ for any $s > \frac{3}{2}$, it follows that

$$\lambda r_0(\lambda)(1 \otimes \psi) = O(|\lambda|) \tag{2.74}$$

in $\mathcal{L}^{2,-s}$ for any $s > \frac{3}{2}$.

Finally we calculate $\lambda\chi(D_x)\Pi_0^{D_x}(-\Delta_x - \lambda)^{-1}(1 \otimes \psi)$ which is independent of $\lambda < 0$. In fact, according to (2.44) one has for $\lambda < 0$

$$\begin{aligned}
&\lambda\chi(D_x)\Pi_0^{D_x}(-\Delta_x - \lambda)^{-1}(1 \otimes \psi) \qquad (2.75)\\
&= \int_{\mathbb{R}^9} \frac{\lambda e^{-\sqrt{|\lambda|}|y-(x-x')|}}{4\pi|y-(x-x')|}\Phi(v,v',y)\psi(v')\, dydx'dv'\\
&= \int_{\mathbb{R}^3} \frac{\lambda e^{-\sqrt{|\lambda|}|x'|}}{4\pi|x'|}dx' \int_{\mathbb{R}^6} \Phi(v,v',y)\psi(v')\, dydv'\\
&= -\int_{\mathbb{R}^3_{v'}} (\int_{\mathbb{R}^3} \Phi(v,v',y)\, dy)\psi(v')\, dv'\\
&= -\chi(0)\langle\psi_0,\psi\rangle_{L^2_v}\psi_0(v) = -\langle\psi_0,\psi\rangle_{L^2_v}\psi_0(v).
\end{aligned}$$

This finishes the proof of Proposition 2.12. □

3. Threshold spectral properties of the Kramers-Fokker-Planck operator

Consider the Kramers-Fokker-Planck operator $P = v\cdot\nabla_x - \nabla_x V(x)\cdot\nabla_v - \Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2}$ with a C^1 potential $V(x)$ satisfying

$$|\nabla_x V(x)| \le C\langle x\rangle^{-\rho-1}, \quad x \in \mathbb{R}^n, \tag{3.1}$$

for some $\rho \ge -1$. In fact, it is sufficient to suppose that $V(x)$ is a Lipschitz function satisfying (3.1) about everywhere and one can even include some mild local singularities in $\nabla V(x)$, by using hypoellipiticiy of the operator. But we will not care about such conditions. P defined on $D(P) = D(P_0)$ is maximally dissipative. By Proposition 2.1, if $\rho > -1$, $\nabla_x V(x)\cdot\nabla_v$ is relatively compact with respect to P_0. Consequently, the spectrum of P is discrete outside $\mathbb{R}_+$ and the complex eigenvalues of P may only accumulate towards points in $\mathbb{R}_+$. The thresholds of P are the eigenvalues of $-\Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2}$ which are equal to $\mathbb{N}$. To be simple, we study only the threshold zero in dimension $n = 3$.

Denote $W = -\nabla_x V(x)\cdot\nabla_v$. One has $W^* = -W$ and

$$(P - z)R_0(z) = 1 + R_0(z)W = 1 + G_0W + O(|z|^\epsilon), \quad \epsilon > 0,$$

in $\mathcal{B}(0,-s;0,-s)$ for $1 < s < (1+\rho)/2$ and z near 0 and $z \notin \mathbb{R}_+$.

Lemma 3.1. *Assume $n = 3$ and (3.1) with $\rho > 1$ (i.e., the potential is of short-range). G_0W is a compact operator on $\mathcal{H}^{1,-s}$ for $1 < s < (1+\rho)/2$. and*

$$\ker_{\mathcal{L}^{2,-s}}(1 + G_0W) = \{0\}. \tag{3.2}$$

Proof. For $1 < s < (1+\rho)/2$, take $1 < s' < s$. Proposition 2.8 (and the arguments used in the proof of Corollary 2.7) shows that $G_0W \in \mathcal{B}(0,-s;1,-s')$. The injection $\mathcal{H}^{1,-s'}$ into $\mathcal{L}^{2,-s}$ is compact. Therefore G_0W is a compact operator on $\mathcal{L}^{2,-s}$. Let $u \in \mathcal{L}^{2,-s}$ with $u + G_0Wu = 0$. Then one can check that $u \in \mathcal{H}^{2,-s}$ and $Pu = 0$. According to (2.40) $u = -F_0Wu - F_1Wu$. Since $Wu \in \mathcal{H}^{-1,\rho+1-s}$ and $F_0 \in \mathcal{B}(-1,s;1,-s')$ for

any $s, s' > \frac{1}{2}$ and $s + s' > 2$, il follows that u is in fact in $\mathcal{H}^{2,-s}$ for any $s > \frac{1}{2}$ and consequently $F_1 W u \in L^2$. By (2.41), one can calculate the asymptotic behavior of $u(x, v)$ for $|x|$ large and deduces that

$$u(x, v) = w(x, v) + r(x, v), \tag{3.3}$$

where $\langle v\rangle^2 r \in L^2(\mathbb{R}^6_{x,v})$ and $w(x, v) = C(u)\frac{e^{-\frac{v^2}{4}}}{|x|}$ with

$$C(u) = \int\int_{\mathbb{R}^6} \frac{e^{-\frac{v^2}{4}}}{2(2\pi)^{\frac{5}{2}}} \nabla_x V(x) \cdot \nabla_v u(x, v)\, dxdx. \tag{3.4}$$

Let $\chi \in C_0^\infty(\mathbb{R})$ be a cut-off with $\chi(\tau) = 1$ for $|\tau| \leq 1$ and $\chi(\tau) = 0$ for $|\tau| \geq 2$ and $0 \leq \chi(\tau) \leq 1$. Set $\chi_R(x) = \chi(\frac{|x|}{R}), R > 1$ and $x \in \mathbb{R}^3$ and $u_R(x) = \chi_R(x)u(x, v)$. Then one has

$$Pu_R = \frac{v \cdot \hat{x}}{R}\chi'(\frac{|x|}{R})u.$$

Taking the real part of the equality $\langle Pu_R, u_R\rangle = \langle \frac{v\cdot\hat{x}}{R}\chi'(\frac{|x|}{R})u, u_R\rangle$, one obtains

$$\int\int_{\mathbb{R}^6} |(\partial_v + \frac{v}{2})u(x, v)|^2 \chi(\frac{|x|}{R})^2\, dxdv = \langle \frac{v \cdot \hat{x}}{R}\chi'(\frac{|x|}{R})u, u_R\rangle \tag{3.5}$$

Since w is even in v, one has

$$\langle \frac{v \cdot \hat{x}}{R}\chi'(\frac{|x|}{R})w, \chi_R w\rangle = 0$$

and the right hand side of the equality (3.5)

$$\begin{aligned} &|\langle \frac{v \cdot \hat{x}}{R}\chi'(\frac{|x|}{R})u, u_R\rangle| \qquad (3.6)\\ &= |2\Re\langle \frac{v \cdot \hat{x}}{R}\chi'(\frac{|x|}{R})w, \chi_R r\rangle + \langle \frac{v \cdot \hat{x}}{R}\chi'(\frac{|x|}{R})r, \chi_R r\rangle|\\ &\leq CR^{-(1-s)}(\|w\|_{\mathcal{L}^{2,-s}} + \|r\|_{L^2})\|\langle v\rangle r\|_{L^2} \end{aligned}$$

for some $\frac{1}{2} < s < 1$. Taking the limit $R \to +\infty$ in (3.5), one obtains that $(\partial_v + \frac{v}{2})u(x, v) \in L^2$ and

$$\int\int_{\mathbb{R}^6} |(\partial_v + \frac{v}{2})u(x, v)|^2\, dxdv = 0. \tag{3.7}$$

This shows that $(\partial_v + \frac{v}{2})u(x, v) = 0$, a.e. in x, v. Since $u \in \mathcal{L}^{2,-s}$ for any $s > \frac{1}{2}$, one sees that u is of the form $u(x, v) = C(x)e^{-\frac{v^2}{4}}$ for some $C \in L^{2,-s}(\mathbb{R}^3_x)$ verifying the equation

$$v \cdot \nabla_x C(x) + \frac{1}{2} v \cdot \nabla V(x) C(x) = 0 \tag{3.8}$$

a.e. in x for all $v \in \mathbb{R}^3$. This proves that $C(x) = c_0 e^{-\frac{V(x)}{2}}$ a.e. for some constant c_0 and

$$u(x, v) = c_0 e^{-\frac{1}{2}(\frac{v^2}{2} + V(x))}.$$

Since $u \in \mathcal{L}^{2,-s}$ for any $s > \frac{1}{2}$ and $V(x)$ is bounded, one deduces that $c_0 = 0$. This proves that $u = 0$ and $\ker_{\mathcal{L}^{2,-s}}(1 + G_0 W) = \{0\}$. □

Lemma 3.1 shows that 0 is neither eigenvalue nor resonance of P. This makes easier the zero-threshold spectral analysis for Kramers-Fokker-Planck operators.

Theorem 3.2. *Assume $n = 3$ and $\rho > 1$. Then zero is not an accumulation point of the eigenvalues of P and one has for any $s, s' > \frac{1}{2}$ with $s + s' > 2$, $\exists \epsilon > 0$ such that*

$$R(z) = A_0 + O(|z|^\epsilon), \quad z \to 0, z \notin \mathbb{R}_+, \tag{3.9}$$

in $\mathcal{B}(-1, s;\ 1, -s')$, where

$$A_0 = (1 + G_0 W)^{-1} G_0. \tag{3.10}$$

There exists $\delta > 0$ such that boundary values of the resolvent

$$R(\lambda \pm i0) = \lim_{\epsilon \to 0_+} R(\lambda \pm i\epsilon), \quad \lambda \in]0, \delta]$$

exist in $\mathcal{B}(-1, s;\ 1, -s)$ for any $s > \frac{1}{2}$ and is continuous in λ.

Proof. Remark that $G_0 \in \mathcal{B}(-1, s; 1, -s')$ for any $s, s' > \frac{1}{2}$ and $s + s' > 2$. Lemma 3.1 implies that $\ker_{\mathcal{H}^{1,-s}}(1 + G_0 W) = \{0\}$ and $G_0 W$ is compact on $\mathcal{H}^{1,-s}$ for $1 < s < (1 + \rho)/2$. Therefore $1 + G_0 W$ is invertible on $\mathcal{H}^{1,-s}$ with bounded inverse. This shows that $1 + R_0(z)W$ is invertible on $\mathcal{H}^{1,-s}$ for $|z|$ small and $z \notin \mathbb{R}_+$ and that

$$(1 + R_0(z)W)^{-1} = (1 + G_0 W)^{-1} + O(|z|^\epsilon), \text{ as } |z| \to 0, z \notin \mathbb{R}_+. \tag{3.11}$$

Since $R_0(z)(P - z) = 1 + R_0(z)W$, this shows that P has no eigenvalues for $|z| < \delta$ for some $\delta > 0$ small enough and that

$$R(z) = (1 + R_0(z)W)^{-1} R_0(z) = A_0 + O(|z|^\epsilon), z \notin \mathbb{R}_+, \tag{3.12}$$

with $A_0 = (1 + G_0 W)^{-1} G_0$. The existence of $R(\lambda \pm i0)$ for $0 < \lambda < \delta$ follows from the first equality in (3.12). □

Theorem 3.3. *Assume $n = 3$ and $\rho > 2$. Then for any $s > \frac{3}{2}$, there exists $\epsilon > 0$ such that*

$$R(z) = A_0 + z^{\frac{1}{2}} A_1 + O(|z|^{\frac{1}{2}+\epsilon}), \tag{3.13}$$

in $\mathcal{B}(-1, s;\ 1, -s)$ for $|z|$ small and $z \notin \mathbb{R}_+$, where A_1 is an operator of rank one given by

$$A_1 = (1 + G_0 W)^{-1} G_1 (1 - W A_0). \tag{3.14}$$

Proof. For $s > \frac{3}{2}$, one has

$$R_0(z) = G_0 + z^{\frac{1}{2}} G_1 + O(|z|^{\frac{1}{2}+\epsilon})$$

in $\mathcal{B}(-1, s; 1, -s)$. If $\rho > 2$, one has $W \in \mathcal{B}(0, -r; 1, \rho + 1 - r)$. Therefore for $\frac{3}{2} < s < (\rho + 1)/2$, it

$$R_0(z)W - (G_0 + z^{\frac{1}{2}} G_1)W = O(|z|^{\frac{1}{2}+\epsilon})$$

in $\mathcal{B}(0, -s; 0, -s)$. Since $B_0 \triangleq (1 + G_0 W)^{-1} \in \mathcal{B}(0, -s; 0, -s)$, it follows that

$$(1 + R_0(z)W)^{-1} = B_0 - z^{\frac{1}{2}} B_0 G_1 W B_0 + O(|z|^{\frac{1}{2}+\epsilon}).$$

From the resolvent equation $R(z) = (1 + R_0(z)W)^{-1} R_0(z)$, we obtain that

$$R(z) = B_0 G_0 + z^{\frac{1}{2}} B_0 G_1 (1 - W B_0 G_0) + O(|z|^{\frac{1}{2}+\epsilon})$$

in $\mathcal{B}(0, s; 0, -s)$. An argument of hypoellticity shows that the same asymptics holds in $\mathcal{B}(-1, s; 1, -s)$. Remark that $A_1 = B_0 G_1 (1 - W B_0 G_0)$ is a rank one operator, because G_1 is of rank one. □

4. Large-time behaviors of solutions

The large-time behaviors of solutions to the KFP equation with a potential will be deduced from resolvent asumptotics. To establish a useful representation formula for the semigroup $S(t) = e^{-tP}$ in terms of the resolvent, we need the following high energy pseudospectral estimate.

Theorem 4.1. *Let $n \geq 1$ and assume (3.1) with $\rho \geq -1$. Then there exists $C > 0$ such that $\sigma(P) \cap \{z; |\Im z| > C, \Re z \leq \frac{1}{C}|\Im z|^{\frac{1}{3}}\} = \emptyset$ and*

$$\|R(z)\| \leq \frac{C}{|z|^{\frac{1}{3}}}, \tag{4.1}$$

and

$$\|(1 - \Delta_v + v^2)^{\frac{1}{2}} R(z)\| \leq \frac{C}{|z|^{\frac{1}{6}}}, \tag{4.2}$$

for $|\Im z| > C$ and $\Re z \leq \frac{1}{C}|\Im z|^{\frac{1}{3}}$.

Proof. Let $W = -\nabla_x V(x) \cdot \nabla_v$. (2.52) shows that $\|WR_0(z)\| + \|R_0(z)W\| = O(|z|^{-\frac{1}{6}})$ for z in the region $\{z; |\Im z| > M, \Re z \leq \frac{1}{M}|\Im z|^{\frac{1}{3}}\}$. Therefore $(1+R_0(z)W)^{-1}$ exists and is uniformly bounded if $\{z; |\Im z| > M, \Re z \leq \frac{1}{M}|\Im z|^{\frac{1}{3}}\}$ with M sufficiently large. Theorem 4.1 follows from Proposition 2.9 and the resolvent equation $R(z) = (1+R_0(z)W)^{-1}R_0(z)$ for $|\Im z| > C$ and $\Re z \leq \frac{1}{C}|\Im z|^{\frac{1}{3}}$ with $C \geq M$ sufficiently large. □

Lemma 4.2. *Let $n \geq 1$ and assume (3.1) with $\rho \geq -1$. Then*

$$S(t)f = \frac{1}{2\pi i} \int_\gamma e^{-tz} R(z) f dz \tag{4.3}$$

for $f \in L^2$ and $t > 0$, where the contour γ is chosen such that

$$\gamma = \gamma_- \cup \gamma_0 \cup \gamma_+$$

with $\gamma_\pm = \{z; z = \pm iC + \lambda \pm iC\lambda^3, \lambda \geq 0\}$ and γ_0 is a curve in the left-half complexe plane joining $-iC$ and iC for some $C > 0$ sufficiently large, γ being oriented from $-i\infty$ to $+i\infty$.

Proof. The spectrum of P is void in the left side of γ. By Theorem 4.1, for $C > 0$ sufficiently large, γ is contained in the resolvent set of P and

$$\|R(z)\| \leq \frac{C}{|z|^{\frac{1}{3}}}, \quad z \in \gamma.$$

Therefore, the integral $\tilde{S}(t) = \frac{1}{2\pi i}\int_\gamma e^{-tz} R(z) dz$ is norm convergent. In addition, one can check as in the standard case (see, for example, [13]) that $\tilde{S}'(t)f = -P\tilde{S}(t)f$ for $f \in D(P_0)$ and that $\lim_{t\to 0+} \tilde{S}(t) = I$ strongly. The uniqueness of solution to the evolution equation $u'(t) + Pu(t) = 0$ for $t > 0$ and $u(0) = u_0$ implies that $\tilde{S}(t) = S(t)$, $t > 0$. □

Corollary 4.3. *Assume that $n = 3$ and $\rho > 1$. One has*

$$\langle S(t)f, g\rangle = \frac{1}{2\pi i} \int_\Gamma e^{-tz} \langle R(z)f, g\rangle dz, \quad t > 0, \tag{4.4}$$

for any $f, g \in \mathcal{L}^{2,s}$, $s > 1$. *Here*

$$\Gamma = \Gamma_- \cup \Gamma_0 \cup \Gamma_+$$

with $\Gamma_\pm = \{z; z = \delta + \lambda \pm i\delta^{-1}\lambda^3, \lambda \geq 0\}$ *for* $\delta > 0$ *small enough and* $\Gamma_0 = \{z = \lambda \pm i0; \lambda \in [0, \delta]\}$. Γ *is oriented from* $-i\infty$ *to* $+i\infty$.

Proof. P has no eigenvalues with real part equal to zero (see the arguments used in the proof of Lemma 3.1) and their only possible accumulation points are in $\mathbb{R}_+$. If $n = 3$ and $\rho > 1$, Theorem 3.2 and Theorem 4.1 imply that there exists some $\delta_0 > 0$ such that

$$\sigma_p(P) \cap \{z; \Re z \leq \delta_0\} = \emptyset. \tag{4.5}$$

Consequently if $0 < \delta < \delta_0$ is small enough, there is no spectrum of P in the interior of the region between γ and Γ and the resolvent is holomorphic there. In addition, the limiting absorption principles at low energies ensure that the integral

$$\int_\Gamma e^{-tz}\langle R(z)f, g\rangle dz, \quad t > 0,$$

is convergent for any $f, g \in \mathcal{L}^{2,s}$ with $s > 1$. By analytic deformation, we conclude from (4.1) that

$$\langle S(t)f, g\rangle = \frac{1}{2\pi i}\int_\gamma e^{-tz}\langle R(z)f, g\rangle dz = \frac{1}{2\pi i}\int_\Gamma e^{-tz}\langle R(z)f, g\rangle dz, \quad t > 0,$$

for any $f, g \in \mathcal{L}^{2,s}$ with $s > 1$. □

The formula (4.4) is useful for studying the time-decay of solutions to the Fokker-Planck equation with a short-range potential ($\rho > 1$).

Theorem 4.4. *Assume* $n = 3$.

(a). If $\rho > 1$, *one has for any* $s > \frac{3}{2}$

$$\|S(t)\|_{\mathcal{B}(0,s;0,-s)} \leq C_s t^{-\frac{3}{2}}, \quad t > 0. \tag{4.6}$$

(b). If $\rho > 2$, *then for any* $s > \frac{3}{2}$, *there exists some* $\epsilon > 0$ *such that*

$$S(t) = t^{-\frac{3}{2}}B_1 + O(t^{-\frac{3}{2}-\epsilon}) \tag{4.7}$$

in $\mathcal{B}(0, s; 0, -s)$ *as* $t \to +\infty$, *where*

$$B_1 = \frac{1}{2i\sqrt{\pi}}A_1 \tag{4.8}$$

is an operator of rank one.

Proof. Assume that $n = 3$ and $\rho > 1$. By Corollary 4.3, one has for $f, g \in \mathcal{L}^{2,s}$ with $s > 1$ and for $t > 0$

$$\begin{aligned}\langle S(t)f, g\rangle &= \frac{1}{2\pi i}\int_0^\delta e^{-t\lambda}\langle (R(\lambda + i0) - R(\lambda - i0))f, g\rangle d\lambda \\ &+ \frac{e^{-t\delta}}{2\pi i}\int_0^\infty e^{-t(\lambda + i\delta^{-1}\lambda^3)}\langle R(\delta + \lambda + i\delta^{-1}\lambda^3)f, g\rangle(1 + 3i\delta^{-1}\lambda^2)d\lambda \quad (4.9)\\ &- \frac{e^{-t\delta}}{2\pi i}\int_0^\infty e^{-t(\lambda - i\delta^{-1}\lambda^3)}\langle R(\delta + \lambda - i\delta^{-1}\lambda^3)f, g\rangle(1 - 3i\delta^{-1}\lambda^2)d\lambda \\ &\triangleq I_1 + I_2 + I_3.\end{aligned}$$

For I_2 and I_3, one can apply Theorem 4.1 to estimate

$$|\langle R(\delta + \lambda \pm i\delta^{-1}\lambda^3)f, g\rangle| \leq C_M \|f\|_{\mathcal{H}^{-1,s}}\|g\|_{\mathcal{H}^{-1,s}}$$

for $s > \frac{1}{2}$ and $\lambda \in]0, M]$ for each fixed $M > 0$ and

$$|\langle R(\delta + \lambda \pm i\delta^{-1}\lambda^3)f, g\rangle| \leq C_M \lambda^{-\frac{1}{3}}\|f\|_{L^2}\|g\|_{L^2}$$

for $\lambda > M$ with $M > 1$ sufficiently large. Therefore, if $\rho > 1$

$$|I_k| \leq Ce^{-t\delta}\|f\|_{\mathcal{H}^{0,s}}\|g\|_{\mathcal{H}^{0,s}} \quad (4.10)$$

for $k = 2, 3$ and for any $s > \frac{1}{2}$.

To show (4.6), it remains to prove that if $\rho > 1$ and $n = 3$, one has

$$|\int_{\Gamma_0} e^{-tz}\langle R(z)f, g\rangle dz| \leq Ct^{-\frac{3}{2}}\|f\|_{\mathcal{L}^{2,s}}\|g\|_{\mathcal{L}^{2,s}} \quad (4.11)$$

for any $f, g \in \mathcal{L}^{2,s}$, $s > \frac{3}{2}$. For $1 < s < (\rho + 1)/2$, one has for some $\epsilon_0 > 0$

$$W(R_0(z) - G_0) = O(|z|^{\epsilon_0}), \quad \text{in } \mathcal{B}(0, s; 0, s)$$

for z near 0 and $z \notin \mathbb{R}_+$. By Lemma 3.1, $1 + WG_0$ is invertible in $\mathcal{B}(0, s; 0, s)$ for any $1 < s < (\rho + 1)/2$. One obtains

$$R(z) = R_0(z)(1 + WR_0(z))^{-1} = \sum_{j=0}^N R_0(z)T(z)^j(1 + WG_0)^{-1} + O(|z|^{(N+1)\epsilon_0}) \quad (4.12)$$

in $\mathcal{B}(0, s; 0, s)$ with $s > 1$, where N is taken such that $(N + 1)\epsilon_0 > \frac{1}{2}$ and

$$T(z) = (1 + WG_0)^{-1}W(R_0(z) - G_0). \quad (4.13)$$

Consequently

$$|\int_{\Gamma_0} e^{-tz}\langle (R(z) - R_0(z)\sum_{j=0}^N T(z)^j(1 + WG_0)^{-1})f, g\rangle dz| \leq Ct^{-\frac{3}{2}-\epsilon}\|f\|_{\mathcal{L}^{2,s}}\|g\|_{\mathcal{L}^{2,s}}, \quad \epsilon > 0, \quad (4.14)$$

if $s > 1$. (4.11) follows from the following lemma which achieves the proof of (4.6). □

Lemma 4.5. *For each $j \geq 0$ and $\frac{3}{2} < s < \rho + \frac{1}{2}$, there exists some $C > 0$ such that*

$$|\int_{\Gamma_0} e^{-tz}\langle R_0(z)T(z)^j(1+WG_0)^{-1}f,g\rangle dz| \leq Ct^{-\frac{3}{2}}\|f\|_{\mathcal{L}^{2,s}}\|g\|_{\mathcal{L}^{2,s}} \tag{4.15}$$

for any $f, g \in \mathcal{L}^{2,s}$ and $t > 0$.

Proof. We want to show that

$$|\langle (R_0(z)T(z)^j - R_0(\overline{z})T(\overline{z})^j)(1+WG_0)^{-1}f,g\rangle| \leq C\sqrt{\lambda}\|f\|_{\mathcal{L}^{2,s}}\|g\|_{\mathcal{L}^{2,s}} \tag{4.16}$$

for $z = \lambda + i0$ with $\lambda \in [0,\delta]$ and for $f, g \in \mathcal{L}^{2,s}$ with $\frac{3}{2} < s < \rho + \frac{1}{2}$. Recall that $R_0(z) = b_0^w(v, D_x, D_v)(-\Delta_x - z)^{-1} + r_1(z)$ with $r_1(z)$ a bounded operator-valued function holomorphic in z when $\Re z < 1$ (see (2.43)). In particular,

$$R_0(z) - R_0(\bar{z}) = b^w(v, D_x, D_v)((-\Delta_x - z)^{-1} - (-\Delta_x - \bar{z})^{-1}) \tag{4.17}$$

for $z = \lambda + i0$ with $\lambda \in [0,\delta]$, $0 < \delta < 1$. Making use of the explicit formula for the integral kernel of $(-\Delta_x - z)^{-1}$, one obtains

$$R_0(z) - R_0(\bar{z}) = O(\sqrt{\lambda}), \quad \lambda \in [0,\delta] \tag{4.18}$$

in $\mathcal{B}(-1, s;\ 1, -s)$ for any $s > \frac{3}{2}$. By Proposition 2.8, $R_0(z)$ is uniformly bounded in $\mathcal{B}(-1, s; 1, -s')$ for $s, s' > \frac{1}{2}$ and $s + s' > 2$. It follows that WG_0 and $(1 + WG_0)^{-1}$ belongs to $\mathcal{B}(0, s;\ 0, s)$ and $T(z) = O(|z|^{\epsilon_0})$ in $\mathcal{B}(0, s; 0, s)$ for any $\frac{1}{2} < s < \rho + \frac{1}{2}$. Notice that for any $\frac{3}{2} < s < \rho + \frac{1}{2}$,

$$T(z) - T(\bar{z}) = (1 + WG_0)^{-1}W(R_0(z) - R_0(\bar{z})) = O(\sqrt{\lambda})$$

in $\mathcal{B}(0, s;\ 0, s)$ for $z = \lambda + i0$. Since

$$\begin{aligned} & R_0(z)T(z)^j - R_0(\overline{z})T(\overline{z})^j \\ &= (R_0(z) - R_0(\bar{z}))T(z)^j + R_0(\overline{z})(T(z) - T(\overline{z}))\sum_{k=0}^{j-1} T(z)^k T(\overline{z})^{j-k-1} \end{aligned} \tag{4.19}$$

one can estimate $|\langle (R_0(z)T(z)^j - R_0(\overline{z})T(\overline{z})^j)(1 + WG_0)^{-1}f, g\rangle|$ by

$$\begin{aligned} & |\langle (R_0(z)T(z)^j - R_0(\overline{z})T(\overline{z})^j)(1 + WG_0)^{-1}f, g\rangle| \\ &\leq C\{\|(R_0(z) - R_0(\bar{z}))\|_{\mathcal{B}(0,s;\,0,-s)}\|T(z)\|^j_{\mathcal{B}(0,s;\,0,s)} \\ &\quad + \|R_0(\overline{z})\|_{\mathcal{B}(0,s;\,0,-s)}\|(T(z) - T(\overline{z}))\|_{\mathcal{B}(0,s;\,0,s)}\|T(z)\|^{j-1}_{\mathcal{B}(0,s;\,0,s)}\}\|f\|_{\mathcal{L}^{2,s}}\|g\|_{\mathcal{L}^{2,s}} \\ &\leq C\sqrt{\lambda}\|f\|_{\mathcal{L}^{2,s}}\|g\|_{\mathcal{L}^{2,s}} \end{aligned} \tag{4.20}$$

for $\frac{3}{2} < s < \rho + \frac{1}{2}$. This proves (4.16) which implies (4.15). (4.6) is proven.

Assume now $\rho > 2$. Theorem 3.3 gives that if $\rho > 2$,

$$R(z) = A_0 + \sqrt{z}A_1 + O(|z|^{\frac{1}{2}+\epsilon}), \quad z \to 0, z \notin \mathbb{R}_+, \tag{4.21}$$

in $\mathcal{B}(-1, s;\ 1, -s)$, $s > \frac{3}{2}$. According to (4.21), I_1 can be evaluated by

$$I_1 = \frac{1}{\pi i}\int_0^\delta e^{-t\lambda}\langle (\sqrt{\lambda}A_1 + O(\lambda^{\frac{1}{2}+\epsilon}))f, g\rangle d\lambda$$

and

$$|I_1 - \frac{1}{t^{\frac{3}{2}}}\langle B_1 f, g\rangle| \leq Ct^{-(\frac{3}{2}+\epsilon)}\|f\|_{\mathcal{H}^{-1,s}}\|g\|_{\mathcal{H}^{-1,s}} \tag{4.22}$$

with

$$B_1 = \frac{1}{\pi i}A_1 \int_0^\infty e^{-s}\sqrt{s}\, ds = \frac{1}{2i\sqrt{\pi}}A_1. \tag{4.23}$$

This proves (4.7). □

Proof of Theorem 1.1. Part (a) of Theorem 1.1 is Theorem 4.1, Part(b) is contained in Theorem 3.2 and Theorem 4.4 (a). See Theorem 4.4 (b), to prove Theorem 1.1 (c), it remains to calculate B_1 given in Theorem 4.4.

For $u \in \mathcal{L}^{2,s}$ with $s > \frac{3}{2}$, one can write

$$\begin{aligned} B_1 u &= \frac{1}{8\pi^{\frac{3}{2}}}\langle \mathfrak{m}_0, (1 - WA_0)u\rangle(1 + G_0W)^{-1}\mathfrak{m}_0 \\ &= \frac{1}{8\pi^{\frac{3}{2}}}\langle (1 - WA_0)^*\mathfrak{m}_0, u\rangle(1 + G_0W)^{-1}\mathfrak{m}_0 \\ &= \frac{1}{8\pi^{\frac{3}{2}}}\langle \nu_0, u\rangle\mu_0 \end{aligned} \tag{4.24}$$

where

$$\mu_0 = (1 + G_0W)^{-1}\mathfrak{m}_0, \quad \nu_0 = (1 - WA_0)^*\mathfrak{m}_0. \tag{4.25}$$

Since $\mathfrak{m}_0 \in \mathcal{L}^{2,-s}$ for any $s > \frac{3}{2}$ and $(1+G_0W)^{-1} \in \mathcal{B}(0, -s; 0, -s)$, G_0W and $(WA_0)^* \in \mathcal{B}(0, -s, ; 0, -s')$ for any $s' > \frac{1}{2}$ and $\frac{3}{2} < s < \frac{\rho+1}{2}$ (if $\rho > 2$), ν_0, μ_0 belong to $\mathcal{L}^{2,-s}$ for any $s > \frac{3}{2}$. In addition, μ_0 satisfies the equation

$$\lim_{z\to 0, z\notin\mathbb{R}_+}(1 + R_0(z)W)\mu_0 = (1 + G_0W)\mu_0 = \mathfrak{m}_0$$

in $\mathcal{L}^{2,-s}$. It follows that

$$P\mu_0 = P_0\mathfrak{m}_0 = 0. \tag{4.26}$$

Similarly, since $A_0 = \lim_{z\to 0, z\notin\mathbb{R}_+} R(z)$ in $\mathcal{B}(-1, s; 1, -s')$ for any $s, s' > \frac{1}{2}$ with $s+s' > 2$ and $(1 - WA_0)^* = 1 + A_0^*W$ (W being skew-adjoint), one can check that

$$\begin{aligned} P^*\nu_0 &= (P_0 + W)^* + W)\mathfrak{m}_0 \\ &= P_0^*\mathfrak{m}_0 = 0. \end{aligned} \tag{4.27}$$

To prove that $\mu_0 = \mathfrak{m}$, we remark that the solution of the equation $(1+G_0W)\mu = \mathfrak{m}_0$ is unique in $\mathcal{L}^{2,-s}$ for any $s > \frac{3}{2}$. Therefore it suffices to check that $\mathfrak{m}$ also verifies the equation $(1 + G_0W)\mathfrak{m} = \mathfrak{m}_0$. To show this, we notice that since $(P_0 + W)\mathfrak{m} = 0$, one has for $\lambda < 0$

$$(1 + R_0(\lambda)W)\mathfrak{m} = -\lambda R_0(\lambda)\mathfrak{m} \tag{4.28}$$

For $\rho > 2$, one has $\mathfrak{m} - \mathfrak{m}_0 \in \mathcal{L}^{2,s}$ for any $0 < s < \rho - \frac{3}{2}$. Proposition 2.12 with $\psi = \psi_0$ shows that

$$\lim_{\lambda\to 0_-} \lambda R_0(\lambda)\mathfrak{m} = -\mathfrak{m}_0. \tag{4.29}$$

Taking the limit $\lambda \to 0_-$ in (4.28), one obtains $(1+G_0W)\mathfrak{m} = \mathfrak{m}_0 = (1+G_0W)\mu_0$. The invertibility of the operator $1+G_0W$ in $\mathcal{L}^{2,-s}$ for any $s > \frac{3}{2}$ gives $\mu_0 = \mathfrak{m}$.

To show that $\nu_0 = \mathfrak{m}$, we notice $1 + WG_0 \in \mathcal{B}(0, s; 0, s)$ for any $\frac{3}{2} < s < \frac{\rho+1}{2}$ and is invertible:

$$(1 + WG_0)^{-1} = 1 - W(1 + G_0W)^{-1}G_0 = 1 - WA_0. \tag{4.30}$$

Therefore $\nu_0 = (1 - WA_0)^*\mathfrak{m}_0 = (1 - G_0^*W)^{-1}\mathfrak{m}_0$. Since $\mathfrak{m}$ verifies also the equation $P^*\mathfrak{m} = (P_0^* - W)\mathfrak{m} = 0$, the argument used above allows to conclude that $(1-G_0^*W)\mathfrak{m} = \mathfrak{m}_0$ which shows $\nu_0 = \mathfrak{m} = (1 - G_0^*W)^{-1}\mathfrak{m}_0$. This shows

$$B_1 u = \frac{1}{8\pi^{\frac{3}{2}}}\langle \mathfrak{m}, u\rangle \mathfrak{m} \quad \text{ for } u \in \mathcal{L}^{2,s} \text{ with } s > \frac{3}{2}, \tag{4.31}$$

which proves (1.12) of Theorem 1.1. □

Remarks. (a). Several interesting questions remain open. In particular, we do not know if results like (1.11) and (1.12) hold for $S(t) = e^{-tP}$ as operators from $\mathcal{L}^1$ to $\mathcal{L}^\infty$. See Theorem 2.4 for the free KFP operator.

(b). The assumptions on dimension and the decay rate of the potentials are only used in low-energy resolvent asymptotics. While we think that the condition $n = 3$ is technical, the condition on the decay rate ρ is more essential for our method based on the model operator P_0. To study the low-energy resolvent asymptotics for slowly increasing potentials, one may try to use other models such as the Witten Laplacian

$$-\Delta_V = (-\nabla_x + \nabla V(x)) \cdot (\nabla_x + \nabla V(x))$$

When $V(x)$ is slowly increasing, under some additional reasonable conditions $-\Delta_V$ is a Schrödinger operator $-\Delta_x + U(x)$ with a potential $U(x)$ positive outside some compact and slowly decreasing at the infinity. The threshold spectral properties for this class of selfadjoint elliptic operators can be well analyzed. The open question is to study to which extent the approximation of the KFP operator by the Witten Laplacian is valid in scattering problems. We hope to return to this problem in a forthcoming work. See [8, 9, 11] the use of the Witten Laplacian in eigenvalue problems when the spectrum is discrete near 0.

Appendix A. A family of complex harmonic oscillators

In this appendix, we study some basic spectral properties of a family of non-selfadjoint harmonic oscillators

$$\hat{P}_0(\xi) = -\Delta_v + \frac{v^2}{4} - \frac{n}{2} + iv \cdot \xi, \tag{A.1}$$

where $\xi \in \mathbb{R}^n$ are regarded as parameters. By Fourier transform in x-variables, the free KFP P_0 is a direct integral of the family $\{\hat{P}_0(\xi); \xi \in \mathbb{R}^n\}$. We give here some quantitative results with explicit bounds in ξ. Note that non-selfadjoint harmonic operators with complex frequency $-\Delta_x + \omega x^2$, $\omega \in \mathbb{C}$, are studied by several authors. See for example [2, 3] and references quoted therein.

The operator $\hat{P}_0(\xi)$ can be written as

$$\hat{P}_0(\xi) = -\Delta_v + \frac{1}{4}\sum_{j=1}^{n}(v_j + 2i\xi_j)^2 - \frac{n}{2} + |\xi|^2.$$

$\{\hat{P}_0(\xi), \xi \in \mathbb{R}^n\}$ is a holomorphic family of type (A) in sense of Kato with constant domain $D = D(-\Delta_v + \frac{v^2}{4})$ in $L^2(\mathbb{R}^n_v)$. Let $F_j(s) = (-1)^j e^{\frac{s^2}{2}}\frac{d^j}{ds^j}e^{-\frac{s^2}{2}}, j \in \mathbb{N}$, be the Hermite polynomials and

$$\varphi_j(s) = (j!\sqrt{2\pi})^{-\frac{1}{2}} e^{-\frac{s^2}{4}} F_j(s)$$

the normalized Hermite functions. For $\xi \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}^n$, define

$$\psi_\alpha(v) = \prod_{j=1}^{n} \varphi_{\alpha_j}(v_j) \text{ and } \psi^\xi_\alpha(v) = \psi_\alpha(v + 2i\xi). \tag{A.2}$$

Then $\xi \to \langle \psi^\xi_\alpha, \psi_j^{-\xi}\rangle$ extends to an entire function for $\xi \in \mathbb{C}$ and is constant on $i\mathbb{R}$. Therefore $\langle \psi^\xi_\alpha, \psi_j^{-\xi}\rangle$ is a constant and one has

$$\langle \psi^\xi_\alpha, \psi_\beta^{-\xi}\rangle = \delta_{\alpha\beta} = \begin{cases} 1, & \alpha = \beta, \\ 0, & \alpha \neq \beta. \end{cases}, \quad \forall \alpha, \beta \in \mathbb{N}^n, \xi \in \mathbb{R}^n. \tag{A.3}$$

Using the definition of the Hermite functions, one can check that for $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$

$$\|\psi^\xi_\alpha\|^2 = e^{2\xi^2}\prod_{m=1}^{n}(\sum_{j=1}^{\alpha_m}\frac{C^j_{\alpha_m}}{j!}(2\xi_m)^{2j}). \tag{A.4}$$

In fact, when $n = 1$, $\xi = \xi_1$ and $k \in \mathbb{N}$, one has

$$\begin{aligned}
\|\psi^\xi_k\|^2 &= \int_{\mathbb{R}} \varphi_k(y + 2i\xi)\varphi_k(y - 2i\xi)dy \\
&= (k!\sqrt{2\pi})^{-1}e^{2\xi^2}\int_{\mathbb{R}} F_k(y + 2i\xi)F_k(y - 2i\xi)e^{-\frac{y^2}{2}}dy \\
&= (k!\sqrt{2\pi})^{-1}e^{2\xi^2}\int_{\mathbb{R}}(\sum_{j=0}^{k} C^j_k(2i\xi)^{k-j}F_j(y))\sum_{l=0}^{k} C^l_k(-2i\xi)^{k-l}F_l(y))e^{-\frac{y^2}{2}}dy \\
&= e^{2\xi^2}\sum_{j,l=0}^{k}(k!\sqrt{2\pi})^{-1}C^j_k C^l_k(j!\sqrt{2\pi})^{\frac{1}{2}}(l!\sqrt{2\pi})^{\frac{1}{2}}(2i\xi)^{k-j}(-2i\xi)^{k-l}\int_{\mathbb{R}}\psi_j(y)\psi_l(y)dy \\
&= e^{2\xi^2}\sum_{j=0}^{k}\frac{j!(C^j_k)^2}{k!}(4\xi^2)^{k-j} \\
&= e^{2\xi^2}\sum_{j=0}^{k}\frac{C^j_k}{j!}(4\xi^2)^j.
\end{aligned}$$

The general case $n \geq 1$ follows from the product formula:

$$\|\psi^\xi_\alpha\|^2 = \prod_{j=1}^{n}\|\varphi_{\alpha_j}(\cdot + 2i\xi_j)\|^2.$$

(A.4) shows that if $\xi \neq 0$, $\|\psi_\alpha^\xi\|$ grows exponentially as $|\alpha| \to +\infty$.

Lemma A.1. *The spectrum of $\hat{P}_0(\xi)$ is purely discrete:*

$$\sigma(\hat{P}_0(\xi)) = \{E_l \triangleq l + \xi^2; l \in \mathbb{N}\}. \tag{A.5}$$

Each eigenvalue E_l is semi-simple (i.e., its algebraic multiplicity and geometric multiplicity are equal) with multiplicity $m_l = \#\{\alpha \in \mathbb{N}^n; |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n = l\}$. The Riesz projection associated with the eigenvalue $l + \xi^2$ is given by

$$\Pi_l^\xi \phi = \sum_{\alpha, |\alpha| = l} \langle \psi_\alpha^{-\xi}, \phi \rangle \psi_\alpha^\xi, \quad \phi \in L^2. \tag{A.6}$$

Proof. It is clear that the spectrum of $\hat{P}_0(\xi)$ is purely discrete and $\psi_\alpha^\xi(v)$ is an eigenfunction associated with the eigenvalue E_l. This means that $\sigma(\hat{P}_0(\xi)) \supset \{l + \xi^2; l \in \mathbb{N}\}$. Since $\hat{P}_0(\xi)^* = \hat{P}_0(-\xi)$ and that the linear span of $\{\psi_\alpha^{-\xi}(v); \alpha \in \mathbb{N}^n\}$ is dense in L^2, $\hat{P}_0(-\xi)$ can not have other eigenvalues than E_l, $l = 0, 1, \cdots$. This proves that $\sigma(\hat{P}_0(\xi)) = \{E_l = l + \xi^2; l \in \mathbb{N}\}$.

To show that E_l is semisimple, assume by contradiction that $\exists \varphi \in D$ such that $(\hat{P}_0(\xi) - E_l)\varphi = c\psi_\alpha^\xi$, $|\alpha| = l$ and $c \in \mathbb{C}^*$. Then ψ_α^ξ is in the range of $\hat{P}_0(\xi) - E_l$ and hence is orthogonal to the kernel of $(\hat{P}_0(\xi) - E_l)^* = \hat{P}_0(-\xi) - E_l$. In particular, one has

$$\langle \psi_\alpha^\xi, \psi_\alpha^{-\xi} \rangle = 0.$$

This is impossible due to (A.3). This contradiction shows that the eigenvalue $E_l = l + \xi^2$ is semisimple. Since the pole of the resolvent $(\hat{P}_0(\xi) - z)^{-1}$ is simple at $z = E_l$, the range of the associated Riesz projection defined by

$$\Pi_l^\xi \phi = \frac{i}{2\pi} \int_{|z - E_l| = \frac{1}{2}} (\hat{P}_0(\xi) - z)^{-1} \phi dz$$

is equal to $\ker(\hat{P}_0(\xi)) - E_l)$ and the kernel of Π_l^ξ is equal to the range of $\hat{P}_0(\xi) - E_l$. The latter is the orthogonal complement of $\ker(\hat{P}_0(-\xi)) - E_l)$ which is spanned by $\{\psi_\alpha^{-\xi}; |\alpha| = l\}$. The representation formula (A.6) then follows from (A.3). □

Lemma A.2. *Let $n = 1$. Then one has for $t > 0$*

$$\sum_{k=0}^{\infty} e^{-t(k+\xi^2)} \|\Pi_k^\xi\| = \frac{e^{-\xi^2(t-2)}}{1 - e^{-t}} e^{\frac{4\xi^2}{e^t - 1}}, \quad \xi \in \mathbb{R}. \tag{A.7}$$

Proof. In the case $n = 1$, one has for any $k \in \mathbb{N}$

$$\|\Pi_k^\xi\| = \|\psi_k^\xi\| \|\psi_k^{-\xi}\| = \|\psi_k^\xi\|^2 = e^{2\xi^2} \sum_{j=0}^{k} \frac{C_k^j}{j!} (4\xi^2)^j. \tag{A.8}$$

The right-hand side of (A.7) is norm convergent when $t > 0$. In fact, one can calculate the sum of the series as follows

$$\begin{aligned}
&\sum_{k=0}^{\infty} e^{-t(k+\xi^2)} \|\Pi_k^\xi\| \\
&= \sum_{k=0}^{\infty} e^{-t(k+\xi^2)+2\xi^2} \sum_{j=0}^{k} \frac{C_k^j}{j!} (4\xi^2)^j \\
&= \sum_{k=0}^{\infty} e^{-t(k+\xi^2)+2\xi^2} + \sum_{k=1}^{\infty} e^{-t(k+\xi^2)+2\xi^2} \sum_{j=1}^{k} \frac{C_k^j}{j!} (4\xi^2)^j \\
&= \frac{e^{-\xi^2(t-2)}}{1-e^{-t}} + e^{-\xi^2(t-2)} \sum_{j=1}^{\infty} \frac{(4\xi^2)^j}{j!} \sum_{k=j}^{\infty} C_k^j e^{-tk} \\
&= \frac{e^{-\xi^2(t-2)}}{1-e^{-t}} + e^{-\xi^2(t-2)} \sum_{j=1}^{\infty} \frac{(4\xi^2 e^{-t})^j}{(j!)^2} \sum_{k=j}^{\infty} k(k-1)\cdots(k-j+1) e^{-t(k-j)} \\
&= \frac{e^{-\xi^2(t-2)}}{1-e^{-t}} + e^{-\xi^2(t-2)} \sum_{j=1}^{\infty} \frac{(4\xi^2 e^{-t})^j}{(j!)^2} \sum_{k=0}^{\infty} (k+j)(k+j-1)\cdots(k+1) e^{-tk} \\
&= \frac{e^{-\xi^2(t-2)}}{1-e^{-t}} + e^{-\xi^2(t-2)} \sum_{j=1}^{\infty} \frac{(4\xi^2 e^{-t})^j}{(j!)^2} \{\frac{d^j}{dx^j} \frac{1}{1-x}\} \mid_{x=e^{-t}} \\
&= \frac{e^{-\xi^2(t-2)}}{1-e^{-t}} (1 + \sum_{j=1}^{\infty} \frac{1}{j!} (\frac{4\xi^2}{e^t-1})^j) \\
&= \frac{e^{-\xi^2(t-2)}}{1-e^{-t}} e^{\frac{4\xi^2}{e^t-1}}.
\end{aligned}$$

□

Proposition A.3. *Let $n \geq 1$. For any $\xi \in \mathbb{R}^n$ and $t > 0$, one has the following spectral decomposition:*

$$e^{-t\hat{P}_0(\xi)} = \sum_{l=0}^{\infty} e^{-t(l+\xi^2)} \Pi_l^\xi, \tag{A.9}$$

where Π_l^ξ is the Riesz projection associated with the eigenvalue l of $\hat{P}_0(\xi)$ and the series is norm convergent as operators on $L^2(\mathbb{R}_v^n)$.

Proof. For $n \geq 1$, one has

$$\|\Pi_l^\xi\| \leq \sum_{k=(k_1,\cdots,k_n)\in\mathbb{N}^n; |k|=l} \prod_{j=1}^{n} \|\varphi_{k_j}(\cdot + 2i\xi_{k_j})\|^2.$$

By Lemma A.2, the righthand side of (A.9) is norm convergent for every $t > 0$ and can be evaluated by

$$\begin{aligned}\sum_{l=0}^{\infty} e^{-t(l+\xi^2)}\|\Pi_l^\xi\| &\leq \prod_{j=1}^{n}\left(\sum_{\alpha_j=0}^{\infty} e^{-t(\alpha_j+\xi_j^2)}\|\Pi_{\alpha_j}^{\xi_j}\|\right) \\ &= \frac{e^{-\xi^2(t-2-\frac{4}{e^t-1})}}{(1-e^{-t})^n}.\end{aligned} \tag{A.10}$$

Since the both sides of (A.9) are equal on the dense subspace spanned by $\{\psi_\alpha^\xi; \alpha \in \mathbb{N}\}$, an argument of density shows that they are equal on the whole space L^2. □

As a consequence of the proof, we obtain the following estimate on the semigroup.

Corollary A.4. *The following estimate holds for $t > 0$ and $\xi \in \mathbb{R}^n$*

$$\|e^{-t\hat{P}_0(\xi)}\|_{\mathcal{B}(L^2(\mathbb{R}^n_v))} \leq \frac{e^{-\xi^2(t-2-\frac{4}{e^t-1})}}{(1-e^{-t})^n} \tag{A.11}$$

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