Cohomology and products of real weight filtrations

Thierry Limoges, Fabien Priziac

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Abstract

We associate to each algebraic variety defined over $\mathbb{R}$ a filtered cochain complex, which computes the cohomology with compact supports and $\mathbb{Z}_2$-coefficients of the set of its real points. This filtered complex is additive for closed inclusions and acyclic for resolution of singularities, and is unique up to filtered quasi-isomorphism. It is represented by the dual filtration of the geometric filtration on semialgebraic chains with closed supports defined by McCrory and Parusiński, and leads to a spectral sequence which computes the weight filtration on cohomology with compact supports. This spectral sequence is a natural invariant which contains the additive virtual Betti numbers.

We then show that the cross product of two varieties allows us to compare the product of their respective weight complexes and spectral sequences with those of their product, and prove that the cup and cap products in cohomology and homology are filtered with respect to the real weight filtrations.

1 Introduction

In [3], Deligne showed the existence of a so-called weight filtration on the rational cohomology with compact supports of complex algebraic varieties, using mixed Hodge structures. In the real case, where there is no such structure, an analog of the weight filtration on the cohomology with compact supports and $\mathbb{Z}_2$-coefficients of real algebraic varieties was proposed by Totaro in [17], and in [14], McCrory and Parusiński developped a homological analog on the Borel-Moore homology with $\mathbb{Z}_2$-coefficients of the set of real points of real algebraic varieties. These real weight filtrations can be defined using the work of Guillén and Navarro-Aznar on cubical hyperresolutions ([6] and [7]) : for a compact variety, the weight filtration is induced by the spectral sequence (from its level two) associated to a cubical hyperresolution. Furthermore, unlike the complex case with coefficients in $\mathbb{Q}$, in the real case, where we are dealing with coefficients in $\mathbb{Z}_2$ in order to have a ($\mathbb{Z}_2$-)orientation, the associated spectral sequence does not degenerate at level two in general.

In [14], McCrory and Parusiński showed furthermore that the spectral sequence inducing the weight filtration is itself a natural invariant of a real algebraic variety in the following sense. There is a functor that assigns to each real algebraic variety a filtered chain complex,

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the so-called weight complex, which is unique up to filtered quasi-isomorphism, and functorial for proper regular morphisms, inducing the weight filtration on Borel-Moore homology. The spectral sequence induced by the weight complex, called the weight spectral sequence, coincides from level one with the spectral sequence associated to a cubical hyperresolution in the compact case, and we can extract from its level one the virtual Betti numbers ([12]), which are the unique additive invariants of real algebraic varieties coinciding with the usual Betti numbers on compact nonsingular varieties. Moreover, the weight complex can be realized at the geometric chain level by a filtration defined on semialgebraic chains with closed supports, using resolution of singularities. Extending it to the wider category of $\mathcal{AS}$-sets ([9], [10]) using Nash-constructible functions ([11], [13]), McCrory and Parusiński showed that their geometric filtration is also functorial with respect to semialgebraic maps with $\mathcal{AS}$-graph.

In this paper, we first achieve the cohomological counterpart of McCrory and Parusiński's work. Using the extension criterion of Guillén and Navarro-Aznar ([7]), we associate to any real algebraic variety a filtered cochain complex, which induces Totaro's weight filtration on the cohomology with compact supports and $\mathbb{Z}_2$-coefficients of the set of real points of real algebraic varieties (theorem 3.4, proposition 3.7 and corollary 3.8). This functor, that we call the cohomological weight complex, is unique (with properties of extension, acyclicity and additivity) and well-defined from the level one of the spectral sequence it induces. The virtual Betti numbers can be recovered from the rows of the level two of the reindexed cohomological weight spectral sequence (proposition 3.10). In paragraph 3.4, we prove that, in the compact case, the reindexed cohomological weight spectral sequence is isomorphic to the spectral sequence associated to a cubical hyperresolution from the level two.

In section 4, we construct a filtration on the semialgebraic cochain complex (which we define in section 2.3 to be the dual of the semialgebraic chain complex with closed supports of [14], showing in proposition 2.2 that it does compute the cohomology with compact supports) that realizes at the cochain level the cohomological weight complex (theorem 4.3). This filtration is a dualization of the geometric filtration of [14]. It verifies on short exact sequences the dual properties of additivity for a closed inclusion and acyclicity for a resolution of singularities (lemma 4.2) and is filtered quasi-isomorphic to the canonical filtration on nonsingular projective real algebraic varieties. Moreover, since the spectral sequence induced by the dualized filtration is naturally isomorphic (from level zero) to the dual spectral sequence of the original filtration (remark 4.1), we deduce that the cohomological weight spectral sequence is dual to the homological one and that the cohomological and homological weight filtrations are dual too (corollary 4.4).

The second part of this paper is devoted to the question of the compatibility of the real weight filtrations with products. First, if X and Y are two real algebraic varieties, we define the product of two respective semialgebraic chains of X and Y in a natural way (definition 5.1). We then look at its compatibility with the geometric filtration (proposition 5.6). Finally, we show that there is a filtered quasi-isomorphism between the tensor product of the geometric filtrations of X and Y and the geometric filtration of the cross product $X \times Y$ (theorem 5.15). In particular, the weight complex of the product is isomorphic to the product of the weight complexes and the Künneth isomorphism is filtered with respect to the weight filtration. The induced relations on weight spectral sequences can also be used to prove the multiplicativity of the virtual Poincaré polynomial.

These results have their cohomological counterparts (paragraph 5.3) and we use them with $Y = X$ to define a cup product on the dual geometric filtration of X considered in the category of filtered cochain complexes localized with respect to filtered quasi-isomorphisms (paragraph 5.4). We obtain an induced cup product on the cohomological weight spectral sequence of X and furthermore the usual cup product on the cohomology with compact supports of (the set of real points of) X is filtered with respect to the cohomological weight filtration (proposition 5.22). We define also a cap product, inducing the properties of the usual cap product on cohomology and homology from the ones on the (co)chain level (section 5.5). In the last subsection 5.6, we study the cap product with the fundamental class of a compact variety X at the weight spectral sequences level. This morphism coincides with Poincaré duality isomorphism for X nonsingular. We show that, when X is singular, the kernel contains the non-pure cohomology classes (non-minimal weight) and the image is included in the pure homology classes (minimal weight). However these inclusions are not equalities in general (remark 5.28). In particular, this brings us some obstructions for a compact real algebraic variety to satisfy Poincaré duality.

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2 Framework

In this section, we set the context in which we work in this paper. We first fix precisely the source categories (categories of schemes over $\mathbb{R}$) and target categories (categories of filtered chain and cochain complexes) of the functors we are going to deal with. We then describe the geometry of real algebraic varieties we study.

2.1 Filtered cochain complexes

In this paper, we will work with cochain complexes equipped with a decreasing filtration. We will use the following notations :

- $\mathfrak{C}$ will denote the category of bounded cochain complexes of $\mathbb{Z}_2$-vector spaces with bounded decreasing filtration.

  To each filtered complex $(K^*, F^\bullet)$ of $\mathfrak{C}$ is associated a second quadrant spectral sequence E with
$$E_0^{p,q} = \frac{F^p K^{p+q}}{F^{p+1} K^{p+q}} \text{ and } E_1^{p,q} = H^{p+q}\left(\frac{F^p K^*}{F^{p+1} K^*}\right),$$
  the differential d of the spectral sequence being induced by the coboundary operator of K^*.

  The filtration $F^\bullet$ on K^* induces naturally a filtration on the cohomology of K^* by setting
$$F^p H^q(K) := \operatorname{im}\left[H^q(F^p K^*) \longrightarrow H^q(K^*)\right],$$

and the spectral sequence E converges to the cohomology $H^*(K^*)$ of K^*, that is

$$E_\infty^{p,q} = \frac{F^p H^{p+q}(K^*)}{F^{p+1} H^{p+q}(K^*)}.$$

A morphism of filtered complexes which induces an isomorphism on E_1 (and therefore on all E_r from $r \geqslant 1$) will be called a *quasi-isomorphism of* $\mathfrak{C}$, or simply a *filtered quasi-isomorphism*.

- $Ho\,\mathfrak{C}$ denotes the localization of $\mathfrak{C}$ with respect to quasi-isomorphisms of $\mathfrak{C}$ (we keep the notation of [7], 1.5.1).
- $\mathfrak{D}$ will denote the category of complexes of $\mathbb{Z}_2$-vector spaces, and a morphism of cochain complexes which induces an isomorphism on their cohomology will be called a *quasi-isomorphism of* $\mathfrak{D}$, or simply a *quasi-isomorphism*.
- $Ho\,\mathfrak{D}$ denotes the localisation of $\mathfrak{D}$ with respect to quasi-isomorphisms of $\mathfrak{D}$.

Remark 2.1. Let $\phi : \mathfrak{C} \longrightarrow \mathfrak{D}$ be the forgetful functor forgetting the filtration. It can be localized into a functor $Ho\,\mathfrak{C} \longrightarrow Ho\,\mathfrak{D}$, which we denote again by ϕ, because a filtered quasi-isomorphism is in particular a quasi-isomorphism.

2.2 Real algebraic varieties

We are interested in the study of the geometry of the set of real points of real algebraic varieties. In this paper, a real algebraic variety will be a reduced scheme of finite type defined over $\mathbb{R}$. We denote by :

- $\mathbf{Sch}_c(\mathbb{R})$ the category of real algebraic varieties and proper regular morphisms.
- $\mathbf{Reg}_{comp}(\mathbb{R})$ the full subcategory of $\mathbf{Sch}_c(\mathbb{R})$ whose objects are compact nonsingular varieties, that is proper regular schemes.
- $\mathbf{V}(\mathbb{R})$ the full subcategory of $\mathbf{Sch}_c(\mathbb{R})$ whose objects are nonsingular projective varieties, that is regular projective schemes.

For X a real algebraic variety of $\mathbf{Sch}_c(\mathbb{R})$, we denote by $X_\mathbb{R}$ the set of its real points. Equipped with its sheaf of regular functions, the set $X_\mathbb{R}$ is a real algebraic variety in the sense of [1], which can be locally embedded in an affine space $\mathbb{R}^n$. We equip it with the strong topology of $\mathbb{R}^n$, and then $X_\mathbb{R}$ is a Haussdorff space, locally compact.

2.3 Semialgebraic chain and cochain complexes

Let X be a real algebraic variety. We will consider complexes of semialgebraic chains defined using semialgebraic subsets of $X_\mathbb{R}$. In this paper, we will always work with $\mathbb{Z}_2$-coefficients, so that real algebraic varieties and arc-symmetric sets ([1], [10]) always have a ($\mathbb{Z}_2$-)orientation and a fundamental class (recall that real algebraic varieties may not be $\mathbb{Z}$-oriented, as $\mathbb{P}^2(\mathbb{R})$).

We will consider the two following dual complexes :

- the chain complex $(C_*(X), \partial_*)$ of semialgebraic chains of $X_{\mathbb{R}}$ with closed supports, whose homology is the Borel-Moore homology $H_*(X) := H_*^{BM}(X_{\mathbb{R}}, \mathbb{Z}_2)$ of $X_{\mathbb{R}}$ with coefficients in $\mathbb{Z}_2$ (see Appendix, paragraph 6 of [14]),
- the cochain complex $(C^*(X), \delta^*)$ which will be by definition the dual of $(C_*(X), \partial_*)$ and whose cohomology $(H_*(X))^\vee$ is, by proposition 2.2 below, isomorphic to the cohomology with compact supports $H_c^*(X_{\mathbb{R}}, \mathbb{Z}_2)$ of $X_{\mathbb{R}}$ with coefficients in $\mathbb{Z}_2$.

Proposition 2.2. *With the notations above, the Borel-Moore homology $H_*(X) := H_*^{BM}(X_{\mathbb{R}}, \mathbb{Z}_2)$ and the cohomology with compact supports $H^*(X) := H_c^*(X_{\mathbb{R}}, \mathbb{Z}_2)$ of $X(\mathbb{R})$ are dual :*

$$H^*(X) = (H_*(X))^\vee$$

Proof. Borel-Moore homology and cohomology with compact supports of $X_{\mathbb{R}}$ can be defined using relative homology and cohomology of pairs. We have

$$H_*^{BM}(X_{\mathbb{R}}) = H_*(\overline{X}_{\mathbb{R}}, \overline{X}_{\mathbb{R}} \setminus X_{\mathbb{R}}) := H_*(C_*(\overline{X}_{\mathbb{R}}, \overline{X}_{\mathbb{R}} \setminus X_{\mathbb{R}}))$$

and

$$H_c^*(X_{\mathbb{R}}) = H^*(\overline{X}_{\mathbb{R}}, \overline{X}_{\mathbb{R}} \setminus X_{\mathbb{R}}) := H^*(C^*(\overline{X}_{\mathbb{R}}, \overline{X}_{\mathbb{R}} \setminus X_{\mathbb{R}})),$$

with $C_*(\overline{X}_{\mathbb{R}}, \overline{X}_{\mathbb{R}} \setminus X_{\mathbb{R}}) := \frac{C_*(\overline{X})}{C_*(\overline{X} \setminus X)}$ and $C^*(\overline{X}_{\mathbb{R}}, \overline{X}_{\mathbb{R}} \setminus X_{\mathbb{R}}) := \frac{C^*(\overline{X})}{C^*(\overline{X} \setminus X)}$, where $X \hookrightarrow \overline{X}$ is an open compactification, that is an embedding of X in a compact variety $\overline{X}$, whose image is open in $\overline{X}$ (notice that we have $\overline{X}_{\mathbb{R}} \setminus X_{\mathbb{R}} = (\overline{X} \setminus X)_{\mathbb{R}}$).

The closed inclusion $\overline{X}_{\mathbb{R}} \setminus X_{\mathbb{R}} \hookrightarrow \overline{X}_{\mathbb{R}}$ induces the following long exact sequences of homology and cohomology of the pair $(\overline{X}_{\mathbb{R}}, \overline{X}_{\mathbb{R}} \setminus X_{\mathbb{R}})$:

$$\cdots \longrightarrow H_n^{BM}(\overline{X}_{\mathbb{R}} \setminus X_{\mathbb{R}}) \longrightarrow H_n^{BM}(\overline{X}_{\mathbb{R}}) \longrightarrow H_n^{BM}(X_{\mathbb{R}}) \longrightarrow H_{n-1}^{BM}(\overline{X}_{\mathbb{R}} \setminus X_{\mathbb{R}}) \longrightarrow \cdots$$

$$\cdots \longleftarrow H_c^n(\overline{X}_{\mathbb{R}} \setminus X_{\mathbb{R}}) \longleftarrow H_c^n(\overline{X}_{\mathbb{R}}) \longleftarrow H_c^n(X_{\mathbb{R}}) \longrightarrow H_c^{n-1}(\overline{X}_{\mathbb{R}} \setminus X_{\mathbb{R}}) \longleftarrow \cdots$$

The dual of the first sequence is then isomorphic to the second one by the five lemma, because the sets $\overline{X}_{\mathbb{R}}$ and $\overline{X}_{\mathbb{R}} \setminus X_{\mathbb{R}}$ are compact and consequently their Borel-Moore homology and cohomology with compact supports are respectively isomorphic to their singular homology and cohomology which are dual to each other. □

Remark 2.3. The cohomology with compact supports is normally computed from the complex of cochains with compact supports. However, this complex does not have good additivity properties, in contrast with the complex C^*.

3 Cohomological weight complex

We prove the existence and uniqueness of the cohomological weight complex in a way similar to the method of [14], using Theorem 2.2.2 of Guillén and Navarro-Aznar in [7]. The cohomological weight complex is the unique extension to all real algebraic varieties and proper regular morphisms, satisfying conditions of additivity for closed inclusions and acyclicity for generalized blow-ups, of the functor that assigns to a nonsingular projective variety its complex of semialgebraic cochains with the canonical filtration.

The Theorem 2.2.2 of [7] is a criterion of extension for functors defined on nonsingular projective varieties. Precisely, suppose that G is a functor defined on nonsingular projective varieties. Theorem 2.2.2 of [7] ensures the existence of an extension G' of G defined for all (possibly singular or non-compact) varieties, as soon as some relation between the values of G on a nonsingular projective variety X, a smooth closed subvariety Y, the blowup $\widetilde{X}$ of X along Y and the exceptional divisor $\widetilde{Y}$ (these four varieties form a so-called elementary acyclic square) is verified (proof of theorem 3.4, condition (F2)). The extended functor G' then satisfies a generalization (Theorem 3.4, condition (Ac)) of such a blowup formula for any morphism $f : \widetilde{X} \longrightarrow X$ of varieties that is an isomorphism over the complement of a subvariety Y of X (this constitutes a so-called acyclic square).

The structure of the target category of the functor G is important in this theory. The prototype is the derived category of chain complexes of an abelian category, where the set of morphisms between two complexes is expanded to include the inverses of quasi-isomorphisms (morphisms that induce isomorphisms on homology). Guillén and Navarro-Aznar introduced in [7] a generalization of the category of chain complexes with the notion of category of cohomological descent, such a category possessing a class of morphisms analogous to quasi-isomorphims and a functor $\mathbf{s}$ from diagrams to objects that is analogous to the operation associating to a diagram of chain complexes its total complex.

The category $\mathfrak{D}$ that we defined above in 2.1 is an abelian category and its derived category $Ho\,\mathfrak{D}$ is a triangulated category. However $\mathfrak{C}$ is not an abelian category, nevertheless it is a category of cohomological descent (Proposition **(1.7.5)** of [7]). In order to replace the notions of exact sequences and distinguished triangles in an abelian category, Guillén and Navarro-Aznar introduced the notions of acyclicity of diagrams and acyclicity of objects in a category of cohomological descent. In our context, an object in $\mathfrak{C}$ is acyclic if by definition $E_1 = 0$ for the associated spectral sequence.

In this paper we consider varieties over $\mathbb{R}$. The target category will be $Ho\,\mathfrak{C}$, which is the localization of the category $\mathfrak{C}$ of filtered cochain complexes of $\mathbb{Z}_2$-vector spaces, with respect to the class of filtered quasi-isomorphisms (2.1). The diagrams we will consider will be cubical diagrams, on which is defined a functor $\mathbf{s}$ that associates to each cubical diagram its simple filtered complex : see definition 3.1 below.

The initial functor will be the functor which assigns to a real nonsingular projective variety its complex of semialgebraic cochains (2.3) equipped with the canonical filtration defined below (definition 3.2). The blowup formula will follow from a short exact sequence for the cohomology of a blowing-up (remark 3.5), showing the existence of an acyclic and additive extension called the cohomological weight complex (paragraph 3.1).

In paragraph 3.2, we show that the spectral sequence induced by the cohomological weight complex (well-defined only from level one) converges to the cohomogical weight filtration on the cohomology with compact supports and, in paragraph 3.3, that one can recover, as in [14], section 1, the virtual Betti numbers ([12]) from its level one terms. More precisely, the virtual Betti numbers coincide with the Euler characteristics of the rows of the reindexed cohomological weight spectral sequence.

Finally, in paragraph 3.4, we give to the cohomological weight spectral sequence of a compact real algebraic variety the following viewpoint : it can be regarded as the spectral sequence naturally induced from a cubical hyperresolution of the variety. Precisely, the spectral sequence associated to a cubical hyperresolution of a compact variety is isomorphic from level two to its cohomological weight spectral sequence, by the Deligne shift.

Definition 3.1. Keeping the notations from [14] and [7], for $n \geqslant 0$, let $\square_n^+$ be the set of subsets of $\{0, 1, \ldots, n\}$, partially ordered by inclusion. A *cubical diagram* of type $\square_n^+$ in a category $\mathcal{X}$ is by definition a contravariant functor from $\square_n^+$ to $\mathcal{X}$. If $\mathcal{K}$ is a cubical diagram of type $\square_n^+$ in $\mathfrak{C}$, let $K^{*,S}$ be the complex labelled by the subset $S \subset \{0, 1, \ldots n\}$ and $|S|$ denote the number of elements of S. The simple complex $\mathbf{s}\mathcal{K}$ is defined by

$$\mathbf{s}\mathcal{K}^k := \bigoplus_{i+|S|-1=k} \mathcal{K}^{i,S}$$

with differentials $\delta : \mathbf{s}\mathcal{K}^k \longrightarrow \mathbf{s}\mathcal{K}^{k+1}$ defined as follows. For each S, let $\delta' : K^{i,S} \longrightarrow K^{i+1,S}$ be the differential of $K^{*,S}$. If $T \subset S$ and $|T| = |S| - 1$, let $\delta_{S,T} : K^{*,S} \longrightarrow K^{*,T}$ be the cochain map corresponding to the inclusion of T in S. If $a \in K^{i,S}$, let

$$\delta''(a) := \sum \delta_{S,T}(a)$$

where the sum is taken over all $T \subset S$ such that $|T| = |S| - 1$, and

$$\delta(a) := \delta'(a) + \delta''(a). \tag{3.1}$$

The induced filtration on $\mathbf{s}\mathcal{K}$ is given by $F^p\mathbf{s}\mathcal{K} := \mathbf{s}F^p\mathcal{K}$

$$(F^p\mathbf{s}\mathcal{K})^k = \bigoplus_{i+|S|-1=k} F^p(\mathcal{K}^{i,S}) \tag{3.2}$$

Definition 3.2. Let (K^*, δ) be a cochain complex. We define the canonical filtration $F_{can}^{\bullet}$ by

$$F_{can}^p K^q := \begin{cases} K^q & \text{if } q < -p \\ \ker \delta_q & \text{if } q = -p \\ 0 & \text{if } q > -p \end{cases}$$

Such a filtered complex defines a spectral sequence that converges to the cohomology of K^* at level one :

Lemma 3.3. *The associated spectral sequence of the filtered complex $F_{can}^{\bullet}K^*$ is a second quadrant spectral sequence satisfying*

$$E_\infty^{p,q} = E_1^{p,q} = \begin{cases} \dfrac{\ker \delta_{-p}}{\mathrm{im}\delta_{-p-1}} = H^{p+q}(K^*) & \text{if } p + q = -p, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We compute the first terms of the spectral sequence. We have

$$E_0^{p,q} = \begin{cases} \ker \delta_{-p} & \text{if } p+q=-p, \\ \dfrac{K^{-p-1}}{\ker \delta_{-p-1}} & \text{if } p+q=-p-1, \\ 0 & \text{otherwise,} \end{cases}$$

and consequently

$$E_1^{p,q} = \begin{cases} \dfrac{\ker \delta_{-p}}{\mathrm{im}\delta_{-p-1}} = H^{p+q}(K^*) & \text{if } p+q=-p \\ 0 & \text{otherwise.} \end{cases}$$

□

3.1 The construction of the cohomological weight complex

We define a functor $\mathcal{W}C^* : \mathbf{Sch}_c(\mathbb{R}) \longrightarrow Ho\,\mathfrak{C}$ such that, for X an objet of $\mathbf{Sch}_c(\mathbb{R})$, the homology of the complex $\phi(\mathcal{W}C^*(X))$ is $H^*(X)$ (recall that ϕ denotes the forgetful functor). The spectral sequence E_r, $r=1,2,\dots$ associated to $\mathcal{W}C^*(X)$, converges to $H^*(X)$. In particular, it induces a filtration on the cohomology with compact supports of $X_{\mathbb{R}}$.

Theorem **(2.2.2)** of [7] allows us to prove the existence and uniqueness of the functor $\mathcal{W}C^*$ with properties of extension, additivity and acyclicity. We keep the notations from [7] and [14].

Theorem 3.4. *The contravariant functor*

$$F_{can}C^* \ : \ \mathbf{V}(\mathbb{R}) \longrightarrow Ho\,\mathfrak{C}$$

which assigns to a nonsingular projective variety M the semialgebraic cochain complex with closed supports $C^*(M)$ equipped with the canonical filtration extends to a contravariant functor

$$\mathcal{W}C^* \ : \ \mathbf{Sch}_c(\mathbb{R}) \longrightarrow Ho\,\mathfrak{C}$$

satisfying :

(Ac) For an acyclic square

$$\begin{array}{ccc} \widetilde{Y} & \overset{j}{\hookrightarrow} & \widetilde{X} \\ \downarrow_{\pi} & & \downarrow_{\pi} \\ Y & \overset{i}{\hookrightarrow} & X \end{array}$$

the simple filtered complex of the diagram

$$\begin{array}{ccc} \mathcal{W}C^*(\widetilde{Y}) & \overset{j^*}{\longleftarrow} & \mathcal{W}C^*(\widetilde{X}) \\ \uparrow_{\pi^*} & & \uparrow_{\pi^*} \\ \mathcal{W}C^*(Y) & \overset{i^*}{\longleftarrow} & \mathcal{W}C^*(X) \end{array}$$

is acyclic.

(Ad) For a closed inclusion $Y \overset{i}{\hookrightarrow} X$*, the simple filtered complex of the diagram*

$$\mathcal{W}C^*(Y) \longleftarrow \mathcal{W}C^*(X)$$

is quasi-isomorphic in $\mathfrak{C}$ *to* $\mathcal{W}C^*(X \setminus Y)$*.*

Such a functor $\mathcal{W}C^*$ *is unique up to a unique filtered quasi-isomorphism.*

Remark 3.5. The proof uses ingredients analogous to the ones in homological weight complex existence and uniqueness' proof in [14], in particular the fact that, for an elementary acyclic square

$$\begin{array}{ccc} \widetilde{Y} & \overset{j}{\hookrightarrow} & \widetilde{X} \\ \downarrow_{\pi} & & \downarrow_{\pi} \\ Y & \overset{i}{\hookrightarrow} & X \end{array}$$

the sequences

$$0 \longrightarrow H^q(X) \longrightarrow H^q(\widetilde{X}) \oplus H^q(Y) \longrightarrow H^q(\widetilde{Y}) \longrightarrow 0$$

are exact for all $q \in \mathbb{N}$ (this uses Poincaré duality : see proof of Proposition 2.1 of [12]).

Proof. (of Theorem 3.4) Since the functor $F_{can}C^* \; : \; \mathbf{V}(\mathbb{R}) \longrightarrow Ho\,\mathfrak{C}$ can be factorized through $\mathfrak{C}$, it is Φ-rectified. It remains to check the hypotheses (F1) and (F2) of theorem **(2.2.2)** of [7].

(F1) The inclusions $X \overset{i_X}{\hookrightarrow} X \sqcup Y$ and $Y \overset{i_Y}{\hookrightarrow} X \sqcup Y$ glue into an isomorphism $C^*(X \sqcup Y) \overset{i_X^* \oplus i_Y^*}{\longrightarrow} C^*(X) \oplus C^*(Y)$. As a consequence, $F_{can}C^*(X \sqcup Y) \cong F_{can}C^*(X) \oplus F_{can}C^*(Y)$.
(F2) For an elementary acyclic square

$$\begin{array}{ccc} \widetilde{Y} & \overset{j}{\hookrightarrow} & \widetilde{X} \\ \downarrow_{\pi} & & \downarrow_{\pi} \\ Y & \overset{i}{\hookrightarrow} & X \end{array}$$

we check that the following diagram, denoted by $\mathcal{K}$,

$$\begin{array}{ccc} F_{can}C^*(\widetilde{Y}) & \overset{j^*}{\longleftarrow} & F_{can}C^*(\widetilde{X}) \\ \uparrow_{\pi^*} & & \uparrow_{\pi^*} \\ F_{can}C^*(Y) & \overset{i^*}{\longleftarrow} & F_{can}C^*(X) \end{array}$$

is acyclic. In other words, we check that the spectral sequence associated to its simple filtered diagram satisfies $E_1(\mathbf{s}\mathcal{K}) = 0$.

Let $p \in \mathbb{Z}$. We compute the p-th column $(E_0^{p,*}(\mathbf{s}\mathcal{K}), d^{p,*})$ of $E_0(\mathbf{s}\mathcal{K})$ and we check that its homology is 0. The terms $E_0^{p,*}(\mathbf{s}\mathcal{K})$ are given by :

$$\frac{F^p(\mathbf{s}\mathcal{K})}{F^{p+1}(\mathbf{s}\mathcal{K})} = \frac{\mathbf{s}(F^p\mathcal{K})}{\mathbf{s}(F^{p+1}\mathcal{K})}$$

However $\dfrac{(F^p\mathbf{s}\mathcal{K})^k}{(F^{p+1}\mathbf{s}\mathcal{K})^k} \neq 0$ only if $-p \leqslant k \leqslant -p+3$, and we have

$$E_0^{p,k}(\mathbf{s}\mathcal{K}) = \begin{cases} 0 & \text{for } k \leqslant -p-1, \\ \dfrac{C^{-p-1}(X)}{\ker \delta_{-p-1,X}} & \text{for } k = -p, \\ \ker \delta_{-p,X} \oplus \dfrac{C^{-p-1}(\widetilde{X})}{\ker \delta_{-p-1,\widetilde{X}}} \oplus \dfrac{C^{-p-1}(Y)}{\ker \delta_{-p-1,Y}} & \text{for } k = -p+1, \\ 0 \oplus \ker \delta_{-p,\widetilde{X}} \oplus \ker \delta_{-p,Y} \oplus \dfrac{C^{-p-1}(\widetilde{Y})}{\ker \delta_{-p-1,\widetilde{Y}}} & \text{for } k = -p+2, \\ 0 \oplus 0 \oplus 0 \oplus \ker \delta_{-p,\widetilde{Y}} & \text{for } k = -p+3, \\ 0 & \text{for } k \geqslant -p+4. \end{cases}$$

The differentials are sums of morphisms (3.1), induced by the functoriality and the differentials of the semialgebraic cochain complex C^*, and the homology of $(E_0^{p,*}, d^{p,*})$ is given by

$$H^k(E_0^{p,*}) = \begin{cases} 0 & \text{for } k = -p, \\ \ker\left[H^{-p}(X) \longrightarrow H^{-p}(\widetilde{X}) \oplus H^{-p}(Y)\right] & \text{for } k = -p+1, \\ \dfrac{\ker\left[H^{-p}(\widetilde{X}) \oplus H^{-p}(Y) \longrightarrow H^{-p}(\widetilde{Y})\right]}{\operatorname{im}\left[H^{-p}(X) \longrightarrow H^{-p}(\widetilde{X}) \oplus H^{-p}(Y)\right]} & \text{for } k = -p+2, \\ \dfrac{H^{-p}(\widetilde{Y})}{\operatorname{im}\left[H^{-p}(\widetilde{X}) \oplus H^{-p}(Y) \longrightarrow H^{-p}(\widetilde{Y})\right]} & \text{for } k = -p+3, \\ 0 & \text{otherwise.} \end{cases}$$

These spaces are all 0 (see previous remark 3.5) and therefore $E_1(\mathbf{s}\mathcal{K}) = 0$. □

Remark 3.6.

- The extension theorem 2.2.2 of [7] gives us also the fact that the functor $\mathcal{W}C^*$ is Φ-rectified.
- Let $(E_r)_{r\geqslant 0}$ be the spectral sequence associated to the filtered complex $\mathcal{W}C^*(X)$ provided by theorem 3.4. By definition of the category $Ho\,\mathfrak{C}$, the terms E_r for $r = 1, 2, \ldots$ are well-defined and do not depend on the construction of $\mathcal{W}C^*(X)$. On the other hand, E_0 depends on this construction, that is on the chosen cubical hyperresolution of X (see paragraph 3.4 below).

3.2 Cohomological weight filtration

For X a real algebraic variety, we call the filtered complex $\mathcal{W}C^*(X)$ the *cohomological weight complex* of X.

To show that the cohomological weight complex computes the cohomology with compact supports of the set of real points of real algebraic varieties, we prove that the functor $C^*(\cdot)$ satisfy the additivity and acyclicity properties of Theorem **(2.2.2)** of [7]. Therefore, since $\phi \circ \mathcal{W}C^*$ and $C^*(\cdot)$ both satisfy these additivity and acyclicity properties and because they are equal on objects of $\mathbf{V}(\mathbb{R})$, thanks to the unicity provided by the extension theorem, these functors $\mathbf{Sch}_c(\mathbb{R}) \longrightarrow H \circ \mathfrak{D}$ are isomorphic. In particular, the semialgebraic cochain complex with closed supports and the cohomological weight complex compute the same homology.

Proposition 3.7. *For X an object of $\mathbf{Sch}_c(\mathbb{R})$, the homology of the complex $\phi(\mathcal{W}C^*(X))$ is $H^*(X)$.*

Proof. The functor $C^*(\cdot)$ is additive : consider, for a closed inclusion $Y \overset{i}{\hookrightarrow} X$, the sequence

$$0 \longrightarrow C^*(X \setminus Y) \xrightarrow{r^\vee} C^*(X) \xrightarrow{i^*} C^*(Y) \longrightarrow 0 \tag{3.3}$$

which is the dual sequence of the sequence (Proposition 1.5 of [14])

$$0 \longrightarrow C_*(Y) \xrightarrow{i} C_*(X) \xrightarrow{r} C_*(X \setminus Y) \longrightarrow 0$$

(where r is the restriction of chains of X to $X \setminus Y$ given by $\begin{array}{ccc} C_*(X) & \longrightarrow & C_*(X \setminus Y) \\ [A] & \longmapsto & [A \cap (X \setminus Y)] \end{array}$). Since the latter is exact, the former is exact as well.

Considering now an acyclic square $\begin{array}{ccc} \widetilde{Y} & \overset{j}{\hookrightarrow} & \widetilde{X} \\ \downarrow_\pi & & \downarrow_\pi \\ Y & \overset{i}{\hookrightarrow} & X \end{array}$. Writing the additivity of the two closed inclusions $Y \overset{i}{\hookrightarrow} X$ and $\widetilde{Y} \overset{j}{\hookrightarrow} \widetilde{X}$, remarking that $C^*(X \setminus Y) \cong C^*(\widetilde{X} \setminus \widetilde{Y})$ and chasing in the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*(X \setminus Y) & \overset{\alpha}{\longrightarrow} & C^*(X) & \overset{i^*}{\longrightarrow} & C^*(Y) & \longrightarrow & 0 \\ & & \downarrow^{\cong}_{\pi^*} & & \downarrow_{\pi^*} & & \downarrow_{\pi^*} & & \\ 0 & \longrightarrow & C^*(\widetilde{X} \setminus \widetilde{Y}) & \overset{\beta}{\longrightarrow} & C^*(\widetilde{X}) & \overset{j^*}{\longrightarrow} & C^*(\widetilde{Y}) & \longrightarrow & 0 \end{array}$$

we obtain the sequence

$$0 \longrightarrow C^*(X) \xrightarrow{\pi^* \oplus i^*} C^*(\widetilde{X}) \oplus C^*(Y) \xrightarrow{\pi^* + j^*} C^*(\widetilde{Y}) \longrightarrow 0$$

and its exactness, that is the acyclicity of $C^*(\cdot)$. Notice the latter short exact sequence is the dual of the corresponding sequence of Proposition 1.5 in [14]. □

Corollary 3.8. *By theorem 3.4 and previous property 3.7, we obtain a filtration on the cohomology with compact supports of the set of real points of real algebraic varieties :*

$$H^k(X) = \mathcal{W}^{-k} H^k(X) \supset \mathcal{W}^{-k+1} H^k(X) \supset \cdots \supset \mathcal{W}^0 H^k(X) \supset \mathcal{W}^{+1} H^k(X) = \{0\},$$

called the cohomological weight filtration.

We say that the cohomological weight filtration of a variety X is pure *if for all $k \in \mathbb{Z}$ the space $\mathcal{W}^{-k+1} H^k(X)$ is 0.*

It will be shown in section 4.1 that the cohomological weight filtration is dual to the weight filtration on Borel-Moore homology of [14]. In particular, the cohomological weight filtration of a real algebraic variety X is pure if and only if its homological weight filtration is pure.

3.3 Cohomological weight spectral sequence and virtual Betti numbers

Analogously to [14], we recover the virtual Betti numbers from the *cohomological weight spectral sequence*, which is by definition the spectral sequence E_r associated to the cohomological weight complex $\mathcal{W}C^*(X)$ of X (it is well-defined for $r \geqslant 1$: see remark 3.6). We reindex it by setting $\widetilde{E}_r^{p,q} = E_{r-1}^{-q,p+2q}$. Notice that the column $(-p)$ of E_1 is sent to the row p of $\widetilde{E}_2$, the line $p+q=-p$ is sent to the vertical line $p=0$ and the lines $p+q=$ constant are globally preserved.

Lemma 3.9. *The vector spaces appearing in E_1 (or equivalently $\widetilde{E}_2$) have finite dimension.*

Proof. It will be shown in paragraph 3.4 that, for compact varieties, the spectral sequence $\widetilde{E}_r$ is isomorphic to the spectral sequence $\widehat{E}_r$ of the double complex associated to a cubical hyperresolution (from $r \geqslant 2$). Since the latter is computed from homologies of real algebraic varieties, its terms are finite-dimensional. We next use the additivity property of the cohomological weight complex to prove the non-compact case. □

Proposition 3.10. *The q-th virtual Betti number can be read on the q-th row of $\widetilde{E}_2$:*

$$\beta_q(X) = \sum_{p=0}^{\dim X} (-1)^p \dim_{\mathbb{Z}_2} \widetilde{E}_2^{p,q}(X)$$

Proof. We show that the right-hand side of the formula equals $b_q(X) := \dim_{\mathbb{Z}_2} H^q(X)$ for X compact nonsingular, and is additive for a closed inclusion $Y \overset{i}{\hookrightarrow} X$: this gives us the result by the unicity with such properties of the virtual Betti numbers, see [12].

If X is compact nonsingular, then, according to proposition 3.11 below, $\mathcal{W}C^*(X)$ is filtered quasi-isomorphic to $C^*(X)$ equipped with the canonical filtration and, taking into account the reindexing, we have

$$\widetilde{E}_2^{p,q} = \begin{cases} H^{p+q}(X) = H^q(X) & \text{if } p=0, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently,

$$\sum_{p=0}^{\dim X} (-1)^p \dim_{\mathbb{Z}_2} \widetilde{E}_2^{p,q}(X) = \dim \widetilde{E}_2^{0,q}(X) = b_q(X).$$

Now, if $Y \overset{i}{\hookrightarrow} X$ is a closed inclusion, the property *(Ad)* of Theorem 3.4 tells us that $\mathcal{W}C^*(X \setminus Y)$ is filtered quasi-isomorphic to $\mathbf{s}\,[\mathcal{W}C^*(Y) \longleftarrow \mathcal{W}C^*(X)]$. This means, if we denote the q-th line of $\widetilde{E}_1(X)$ by $C^*(X,q)$, with differential $d^{*,q}$ (and so as for Y and $X \setminus Y$), that the simple filtered complex $\mathbf{s}\,[C^*(X \setminus Y, q) \longrightarrow C^*(X,q) \longrightarrow C^*(Y,q)]$ is acyclic (i.e. filtered quasi-isomorphic to the zero complex), and consequently, we have a long exact sequence :

$$\cdots \longrightarrow \widetilde{E}_2^{p,q}(X \setminus Y) \longrightarrow \widetilde{E}_2^{p,q}(X) \longrightarrow \widetilde{E}_2^{p,q}(Y) \longrightarrow \widetilde{E}_2^{p+1,q}(X \setminus Y) \longrightarrow \cdots$$

In particular,

$$\sum_{p=0}^{\dim X} (-1)^p \dim_{\mathbb{Z}_2} \widetilde{E}_2^{p,q}(X) = \sum_{p=0}^{\dim Y} (-1)^p \dim_{\mathbb{Z}_2} \widetilde{E}_2^{p,q}(Y) + \sum_{p=0}^{\dim X\setminus Y} (-1)^p \dim_{\mathbb{Z}_2} \widetilde{E}_2^{p,q}(X \setminus Y).$$

□

Proposition 3.11. *For X compact nonsingular, the cohomology of the complex $\dfrac{\mathcal{W}^p C^*(X)}{\mathcal{W}^{p+1} C^*(X)}$ satisfies*

$$H_k\left(\frac{\mathcal{W}^p C^*(X)}{\mathcal{W}^{p+1} C^*(X)}\right) = \begin{cases} H^p(X) & \text{if } k = -p, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the cohomological weight filtration of a compact nonsingular real algebraic variety is pure.

Proof. The proof is similar to the proof of Proposition 1.8. in [14]. Since the inclusion $\mathbf{V}(\mathbb{R}) \to \mathbf{Reg}_{comp}(\mathbb{R})$ has the extension property of [7] (2.1.10), the functor $F^*_{can} C^* : \mathbf{V}(\mathbb{R}) \to Ho\,\mathfrak{C}$ extends to a functor $\mathbf{Reg}_{comp}(\mathbb{R}) \to Ho\,\mathfrak{C}$, unique up to filtered quasi-isomorphism with extension, additivity and acyclicity properties. Since $F^*_{can} C^* : \mathbf{Reg}_{comp}(\mathbb{R}) \to Ho\,\mathfrak{C}$ and $\mathcal{W}C^* : \mathbf{Reg}_{comp}(\mathbb{R}) \to Ho\,\mathfrak{C}$ both are such extensions (see the remark 3.5 and the proof of theorem 3.4 : $F_{can} C^*$ is acyclic on acyclic squares in $\mathbf{Reg}_{comp}(\mathbb{R})$), they are quasi-isomorphic in $\mathfrak{C}$ and we obtain the result by lemma 3.3. □

3.4 Cohomological weight complex, cubical hyperresolutions and the Deligne shift

This subsection gives the following viewpoint for the weight complex : the spectral sequence given by Theorem 3.4 is quasi-isomorphic to the spectral sequence of the double complex associated to a cubical hyperresolution. For an introduction to cubical hyperresolutions of algebraic varieties, see [15], ch. 5.

Here, we keep the notations from [7]. For any compact real algebraic variety X, there exists a so-called *cubical hyperresolution* of X, which is a special case of cubical diagram (see definition 3.1) denoted by

$$X_\bullet = \left[X_\bullet^+ \longrightarrow X\right],$$

where $X_\bullet^+$ is the so-called *augmented cubical diagram.*

A cubical hyperresolution $X_\bullet$ of X is composed of varieties X_S, $S \in \mathcal{P}[\![1, n]\!]$, associated to the vertices of a n-dimensional cube, with $X_\emptyset = X$ and X_S compact nonsingular for $S \neq \emptyset$, and of morphisms $\pi_{S,T} : X_S \longrightarrow X_T$ for $T \subset S$, such that $\pi_{R,T} = \pi_{R,S} \circ \pi_{S,T}$ if $T \subset S \subset R$.

According to [7], proof of Theorem 2.1.5, we can compute the cohomological weight complex of X from the functor $\mathcal{W}C^*$ applied to a cubical hyperresolution of X. Guillén and Navarro-Aznar use such a property to extend functors (a cubical hyperresolution is a particular case of diagram of *cohomological descent* : see [15], Definition 3.6).

Let $X_\bullet$ be a cubical hyperresolution of a compact real algebraic variety X. We associate to $X_\bullet$ a double complex $C^{i,j}$, defined by

$$X^{(i)} := \bigsqcup_{S \subset [\![1,n]\!], |S|=i+1} X_S$$

and

$$C^{i,j} := C^j(X^{(i)}) \cong \bigoplus_{|S|=i+1} C^j(X_S)$$

for $i, j \geqslant 0$, and equipped with the differentials

$$\delta'_i \ : \ C^j(X^{(i)}) \longrightarrow C^j(X^{(i+1)}),$$

induced by the morphisms $X^{(i+1)} \longrightarrow X^{(i)}$ (that is by the morphisms $X_S \to X_T$ with $T \subset S$ and $|T| = |S| - 1$), and

$$\delta''_j \ : \ C^j(X^{(i)}) \longrightarrow C^{j+1}(X^{(i)})$$

which are the coboundary operators of the $X^{(i)}$'s (recall that we do not need to care about signs because we work with $\mathbb{Z}_2$ coefficients).

The double complex $C^{i,j}$ leads to a filtered complex $(C^*, \hat{F})$: it is the associated total complex equipped with the *naive* filtration coming from the cubical diagram structure. Precisely, we set

$$C^k := \bigoplus_{i+j=k} C^j(X^{(i)})$$

and

$$\hat{F}^p C^k := \bigoplus_{i \geqslant p} C^{k-i}(X^{(i)}),$$

the differential being $\delta := \delta' + \delta''$.

In the following, we show that the spectral sequence $\widehat{E}$ induced by the filtered complex $(C^*, \hat{F})$ associated to the cubical hyperresolution $X_\bullet$ of X is isomorphic to the (reindexed) cohomological weight spectral sequence of X (from level two).

First, the construction of cubical hyperresolutions implies that the functor $\mathcal{W}C^*$ is acyclic for cubical hyperresolutions (see again [7], proof of Theorem 2.1.5) : the cubical diagram $\mathcal{W}C^*(X_\bullet)$ is acyclic, that is $\mathbf{s}\,[\mathcal{W}C^*(X_\bullet)]$ is isomorphic to the zero complex in $Ho\,\mathfrak{C}$. In other words, the cohomological weight complex $\mathcal{W}C^*(X)$ of X is filtered quasi-isomorphic to the total complex associated to the double complex given by $\mathcal{W}C^{i,j} := \mathcal{W}C^j(X^{(i)})$, equipped with the filtration induced as in 3.1. Since the varieties X_S are compact nonsingular for $S \neq \emptyset$, the latter filtered cochain complex is filtered quasi-isomorphic to the cochain complex C^* equipped with (the filtration induced by) the canonical filtration denoted by F_{can}.

Now, using the so-called Deligne shift ([2] Paragraph 1.3), we give, in lemma 3.12 below, a relation between the filtrations $\hat{F}$ and F^{can} of C^*, the latter computing the weight filtration of X.

If (K, F) is a filtered cochain complex, the *Deligne shift* $\mathrm{Dec}(F)$ of the filtration F is a new filtration on K^*, given by

$$\mathrm{Dec}(F)^p K^n := \ker \left[\delta \ : \ F^{p+n} K^n \longrightarrow \frac{K^{n+1}}{F^{p+n+1} K^{n+1}} \right],$$

such that ([2])

$$E_r^{p,n-p}(K, \mathrm{Dec}(F)) = E_{r+1}^{p+n,-p}(K, F) \tag{3.4}$$

In our situation, this gives :

Lemma 3.12. *The Deligne shift of the naive filtration $\hat{F}$ on C^* is the canonical filtration*

$$\mathrm{Dec}(\hat{F})^{p-k} C^k = F_{can}^{p-k} C^k$$

Proof. We compute the Deligne shift of the filtration $\hat{F}$ on C^* :

$$\begin{aligned}
\mathrm{Dec}(\hat{F})^{p-k} C^k \quad &= \ker \left[\delta \ : \ \hat{F}^p C^k \longrightarrow \frac{C^{k+1}}{\hat{F}^{p+1} C^{k+1}} \right] \text{ (by definition of the Deligne shift)} \\
&= \ker \left[\delta \ : \bigoplus_{i \geqslant p} C^{k-i}(X^{(i)}) \longrightarrow \bigoplus_{i \leqslant p} C^{k+1-i}(X^{(i)}) \right] \\
&= \ker \left[\delta''_{k-p} \ : C^{k-p}(X^{(p)}) \longrightarrow C^{k-p+1}(X^{(p)}) \right] \oplus \bigoplus_{i > p} C^{k-i}(X^{(i)}) \\
&= F_{can}^{p-k} C^k
\end{aligned}$$

□

This finally allows us to prove the isomorphism between the cohomological weight spectral sequence of X and the spectral sequence induced by the filtration $\hat{F}$ on C^* :

Proposition 3.13. *For $r \geqslant 2$*

$$\widetilde{E}_r^{a,b} = E_r^{a,b}(C^*, \hat{F})$$

Proof. We have the following relations, for $r \geqslant 1$:

$$\begin{aligned}
\widetilde{E}_{r+1}^{p+n,-p}(X) \quad &= E_r^{p,n-p}(X) && \text{by definition} \\
&= E_r^{p,n-p}(C^*, F_{can}) && \text{by the theory of cubical hyperresolutions} \\
&= E_r^{p,n-p}(C^*, \mathrm{Dec}(\hat{F})) && \text{by Lemma 3.12} \\
&= E_{r+1}^{p+n,-p}(C^*, \hat{F}) && \text{by the relation 3.4.}
\end{aligned}$$

□

Remark 3.14. If U is a non-compact real algebraic variety, take a compactification X of U and consider the complement $X \setminus U$ of U in X. Then, using the additivity property of the weight complex and the proposition 3.13 above, we can compute the cohomological weight spectral sequence of X from the spectral sequences induced by cubical hyperresolutions of X and $X \setminus U$.

4 The dual geometric filtration

In [14], McCrory and Parusiński built a functor $\mathcal{G}_\bullet C_* \ : \ \mathbf{Sch}_c(\mathbb{R}) \longrightarrow \mathcal{C}$ (where $\mathcal{C}$ is the category of bounded filtered chain complexes, see [14]) representing the weight complex functor $\mathcal{W}C_*$ defined in $Ho\,\mathcal{C}$ (up to filtered quasi-isomorphisms only). Dualizing the geometric filtration $\mathcal{G}_\bullet$, we obtain a functor representing the cohomological weight complex $\mathcal{W}C^*$ at the cochain level. Therefore, our cohomological weight complex functor

$$\mathcal{W}C^* \ : \ \mathbf{Sch}_c(\mathbb{R}) \longrightarrow Ho\,\mathfrak{C}$$

can be factorized into a functor

$$\mathcal{G}^\bullet C^* \ : \ \mathbf{Sch}_c(\mathbb{R}) \longrightarrow \mathfrak{C}$$

through the canonical localization $\mathfrak{C} \longrightarrow Ho\,\mathfrak{C}$.

Analogously we can dualize the Nash filtration $\mathcal{N}_\bullet$ that extends the geometric filtration on the wider category $\chi_{\mathcal{AS}}$ of $\mathcal{AS}$-sets (if X is a real algebraic variety, we have $\mathcal{N}_\bullet C_*(X) = \mathcal{G}_\bullet C_*(X)$, see section 3 of [14]), and then extend the functor $\mathcal{G}^\bullet C^*$ to the category $\chi_{\mathcal{AS}}$, showing in particular that the semialgebraic chain complex equipped with the dualized geometric filtration is functorial with respect to semialgebraic morphisms with $\mathcal{AS}$ graph.

We remark also that the cohomological weight spectral sequence E_r is dual to the homological weight spectral sequence for $r \geqslant 0$ and deduce that the cohomological weight filtration can be obtained by dualizing the homological weight filtration.

4.1 Definition

Let X be a real algebraic variety. We dualize the geometric filtration on the semialgebraic chain complex of (the set of real points of) X in the following way. We set

$$\mathcal{G}^p C^q(X) := \{\varphi \in C^q(X) \mid \varphi \equiv 0 \text{ on } \mathcal{G}_{p-1}C_q(X)\}$$

i.e. $\mathcal{G}^p C^q(X)$ consists of the linear forms defined on $C_q(X)$ and vanishing on $\mathcal{G}_{p-1}C_q(X)$.

We get a decreasing filtration on $C^*(X)$:

$$C^k(X) = \mathcal{G}^{-k}C^k(X) \supset \mathcal{G}^{-k+1}C^k(X) \supset \cdots \supset \mathcal{G}^0 C^k(X) \supset \mathcal{G}^1 C^k(X) = 0,$$

that we call the cohomological geometric filtration of X. We show in Proposition 4.3 that the induced spectral sequence E_r (well-defined for $r = 0, 1, \ldots$ and functorial in X) coincides with the spectral sequence of the weight complex from $r \geqslant 1$.

The cohomological geometric filtration satisfies the properties of short exact sequences of additivity and acyclicity (Lemma 4.2), dual to the ones of Theorem **(2.7)** and **(3.6)** of [14]. These properties are stronger than the additivity and acyclicity properties (Ad) and (Ac) of $\mathcal{W}C^*$, which can be recovered by the snake lemma.

Remark 4.1. We have $\mathcal{G}^p C^q(X) \cong \left(\dfrac{C_q(X)}{\mathcal{G}_{p-1}C_q(X)}\right)^\vee$, since we can factorize a linear form on $C_q(X)$ which kernel contains $\mathcal{G}_{p-1}C_q(X)$ through $C_q(X) \to \frac{C_q(X)}{\mathcal{G}_{p-1}C_q(X)}$. Furthermore, if we consider the restriction of morphisms of $\mathcal{G}^pC^q(X)$ to $\mathcal{G}_pC_q(X)$, the quotients on chains and cochains are related by :

$$\frac{\mathcal{G}^pC^q(X)}{\mathcal{G}^{p+1}C^q(X)} = \left(\frac{\mathcal{G}_pC_q(X)}{\mathcal{G}_{p-1}C_q(X)}\right)^\vee .$$

Since the isomorphisms are compatible with the differentials of the complexes which are dual to one another, they induce a duality between the spectral sequence associated the cochain complex $\mathcal{G}^\bullet C^*(X)$ and the spectral sequence associated to the chain complex $\mathcal{G}_\bullet C_*(X)$:

$$E_r^{p,q} = (E^r_{p,q})^\vee$$

(from $r \geqslant 0$).

Notice that the construction of the dual geometric filtration can be generalized to any filtered chain complex of $\mathcal{C}$, providing the same duality on the induced spectral sequences : if $(K_*, F_\bullet) \in \mathcal{C}$, we define the *dual filtration* of $F_\bullet$ by

$$F^p_\vee K^q_\vee := \left\{\varphi \in (K_q)^\vee \mid \varphi \equiv 0 \text{ on } F_{p-1}K_q(X)\right\}$$

and then

$$E_r^{p,q}(K_\vee, F_\vee) = (E^r_{p,q}(K,F))^\vee$$

from $r \geqslant 0$.

Lemma 4.2. *For any closed inclusion $Y \overset{i}{\hookrightarrow} X$ and any $p, q \in \mathbb{Z}$, the sequence*

$$0 \longrightarrow \mathcal{G}^pC^q(X \setminus Y) \overset{r^*}{\longrightarrow} \mathcal{G}^pC^q(X) \overset{i^*}{\longrightarrow} \mathcal{G}^pC^q(Y) \longrightarrow 0$$

is exact.

For an acyclic square $\begin{array}{ccc} \widetilde{Y} & \overset{j}{\hookrightarrow} & \widetilde{X} \\ \downarrow_\pi & & \downarrow_\pi \\ Y & \overset{i}{\hookrightarrow} & X \end{array}$ *and any $p, q \in \mathbb{Z}$, the sequence*

$$0 \longrightarrow \mathcal{G}^pC^q(X) \overset{\pi^* \oplus i^*}{\longrightarrow} \mathcal{G}^pC^q(\widetilde{X}) \oplus \mathcal{G}^pC^q(Y) \overset{\pi^* - j^*}{\longrightarrow} \mathcal{G}^pC^q(\widetilde{Y}) \longrightarrow 0$$

is exact.

Proof. We prove the exactness of the short sequence of additivity. The exactness of the short sequence of acyclicity comes from similar arguments.

Let $Y \overset{i}{\hookrightarrow} X$ be a closed inclusion. The short sequences on chains with closed supports

$$0 \longrightarrow \mathcal{G}_{p-1}C_q(Y) \overset{i_*}{\longrightarrow} \mathcal{G}_{p-1}C_q(X) \overset{r_*}{\longrightarrow} \mathcal{G}_{p-1}C_q(X \setminus Y) \longrightarrow 0$$

are exact for $p, q \in \mathbb{Z}$. Consequently, we have the following exact sequences of quotients :

$$0 \longrightarrow \frac{C_q(Y)}{\mathcal{G}_{p-1}C_q(Y)} \overset{i_*}{\longrightarrow} \frac{C_q(X)}{\mathcal{G}_{p-1}C_q(X)} \overset{r_*}{\longrightarrow} \frac{C_q(X \setminus Y)}{\mathcal{G}_{p-1}C_q(X \setminus Y)} \longrightarrow 0$$

that we dualize to obtain the exactness of the short sequences for the cohomological geometric filtration (remark 4.1). ☐

4.2 Realization of the cohomological weight complex

We now use the previous lemma 4.2 to show that the cohomological geometric filtration realizes the cohomological weight filtration :

Proposition 4.3. *The dual geometric filtration* $\mathcal{G}^\bullet C^* : \mathbf{Sch}_c(\mathbb{R}) \longrightarrow \mathcal{C}$ *induces the functor* $\mathcal{W}C^* : \mathbf{Sch}_c(\mathbb{R}) \longrightarrow Ho\,\mathcal{C}$.

Proof. The functor $\mathcal{G}^\bullet C^*$, composed with the canonical localization $\mathfrak{C} \to Ho\,\mathfrak{C}$ verifies the properties (Ac) and (Ad) of Theorem 3.4 by Lemma 4.2 (use the snake lemma). If it verifies also the extension property, by unicity of the cohomological weight complex, the two functors $\mathcal{G}^\bullet C^*,\ \mathcal{W}C^* : \mathbf{Sch}_c(\mathbb{R}) \longrightarrow Ho\,\mathfrak{C}$ will be isomorphic.

Let X be a nonsingular projective real algebraic variety. We show that $\mathcal{G}^\bullet C^*(X)$ is filtered quasi-isomorphic to $F^\bullet_{can} C^*(X)$. According to [14] Theorem 2.8., the complexes $\mathcal{G}_\bullet C_*(X)$ and $F^{can}_\bullet C_*(X)$ are filtered quasi-isomorphic (through the inclusion morphism). By remark 4.1, we deduce that, on the cohomological spectral sequences level,

$$E^1\left(\mathcal{G}^\bullet C^*\right) = \left(E_1(\mathcal{G}_\bullet C_*)\right)^\vee \cong \left(E_1(F^{can}_\bullet C_*(X))\right)^\vee = E^1\left((F^{can})^\bullet_\vee C^*(X)\right),$$

where the dual canonical filtration $(F^{can})^\bullet_\vee C^*(X)$ is given by

$$(F^{can})^p_\vee C^q(X) = \left\{\varphi \in C^q(X) \mid \varphi \equiv 0 \text{ on } F^{can}_{p-1} C_q(X)\right\} = \begin{cases} 0 & \text{if } q > -(p-1), \\ \operatorname{im} \delta_{q-1} & \text{if } q = -(p-1), \\ C^q(X) & \text{if } q < -(p-1), \end{cases}$$

(notice that a linear form on $C_q(X)$ which vanishes on $\ker \partial_q$ can be factorised into a linear form on $C^{q-1}(X)$ through ∂_q and then belongs to im δ_{q-1}).

We observe that there is an inclusion of the canonical filtration given by

$$F^p_{can} C^q(X) = \begin{cases} 0 & \text{if } q > -p, \\ \ker \delta_q & \text{if } q = -p, \\ C^q(X) & \text{if } q < -p, \end{cases}$$

in the (bounded decreasing) filtration $(F^{can})^\bullet_\vee C^*(X)$, that induces a quasi-isomorphism in $\mathfrak{C}$. Indeed

$$E_1^{p,q}((F^{can})^\bullet_\vee C^*(X)) = (E_1^{p,q}(F^{can}_\bullet C_*(X)))^\vee = \begin{cases} H^{p+q}(X) & \text{if } p+q = -p, \\ 0 & \text{otherwise.} \end{cases} = E_1^{p,q}\left(F^\bullet_{can} C^*(X)\right)$$

(see Paragraph 1.1 of [14] and lemma 3.2).

As a consequence, $E^1\left(\mathcal{G}^\bullet C^*\right) = E^1\left((F^{can})^\bullet_\vee C^*(X)\right) = E_1\left(F^\bullet_{can} C^*(X)\right)$ and $\mathcal{G}^\bullet C^*(X)$ and $F^\bullet_{can} C^*(X)$ are isomorphic in $Ho\,\mathfrak{C}$. □

Corollary 4.4. *We have isomorphisms*

$$\frac{\mathcal{W}^p H^q(X)}{\mathcal{W}^{p+1} H^q(X)} = \left(\frac{\mathcal{W}_p H_q(X)}{\mathcal{W}_{p-1} H_q(X)}\right)^\vee ,$$

and the weight filtrations on Borel-Moore homology in [14] and cohomology with compact supports in 3.8 are related by

$$\mathcal{W}^p H^q(X) = \{\varphi \in H^q(X) \mid \varphi \equiv 0 \text{ on } \mathcal{W}_{p-1} H_q(X)\}$$

Proof. We use the facts that the homological and cohomological geometric filtrations realize respectively the homological and cohomological weight spectral sequences, and that these spectral sequences are dual to one another (remark 4.1) : the first assertion is just the isomorphism

$$E_\infty = (E^\infty)^\vee .$$

We deduce that

$$\mathcal{W}^p H^q(X) = \left(\frac{H_q(X)}{\mathcal{W}_{p-1} H_q(X)}\right)^\vee ,$$

i.e.

$$\mathcal{W}^p H^q(X) = \{\varphi \in H^q(X) \mid \varphi \equiv 0 \text{ on } \mathcal{W}_{p-1} H_q(X)\} .$$

□

In a similar way, we can define a filtration $\mathcal{N}^\bullet$ on the semialgebraic cochain complex $C^*(X)$ of an $\mathcal{AS}$-set, dual to the Nash filtration $\mathcal{N}_\bullet$ of [14], section 3. This dual filtration defines a functor

$$\mathcal{N}^\bullet C^* \ : \ \chi_{\mathcal{AS}} \longrightarrow \mathcal{C}$$

from the category of $\mathcal{AS}$-sets, which extends $\mathcal{G}^\bullet C^* \ : \ \mathbf{Sch}_c(\mathbb{R}) \longrightarrow \mathcal{C}$. In particular we obtain :

Proposition 4.5. *The dual geometric filtration and its spectral sequence are functorial with respect to semialgebraic morphisms with $\mathcal{AS}$-graph.*

5 Weight filtrations and products

In this section, we define the cross product of semialgebraic chains with closed supports, which induces a filtered quasi-isomorphism with respect to the geometric filtration, relating the weight complex of the cross product of real algebraic varieties with the tensor product of the weight complexes by an isomorphism of $Ho\,\mathcal{C}$. The isomorphism of $Ho\,\mathfrak{C}$ on the cohomological counterpart allows us to define cup and cap products on the localized chain and cochain level. This shows in particular that the induced cup and cap products on the (co)homology level are filtered with respect to the weight filtrations. Finally, we give obstructions for a compact real algebraic variety to satisfy Poincaré duality, relating to its weight filtrations.

If X and Y are two real algebraic varieties and c and c' two respective chains, we define the chain $c \times c'$ of $X \times Y$ in a natural way (definition 5.1). We first check that it is well-defined and give its behaviour under the boundary operator (lemmas 5.2 and 5.4). We then look at its behaviour with respect to the geometric filtration (proposition 5.6) : if c is a q-dimensional chain of X of filtration index p and c' a q'-dimensional chain of Y of index p', the product $c \times c'$ is a $(q+q')$-dimensional chain of $X \times Y$ of index $(p+p')$. The product of chains induces then a well-defined morphism u from the tensor product of the geometric filtrations of X and Y to the geometric filtration of the cross product $X \times Y$ (theorem 5.15). Using the naturality property of the extension criterion of Guillén and Navarro-Aznar in [7], we show that it is a filtered quasi-isomorphism. Consequently, the tensor product of the weight complexes is isomorphic in $Ho\mathcal{C}$ to the weight complex of the product, the induced relations between the weight spectral sequences terms implying in particular the multiplicativity of the virtual Poincaré polynomial (without the use of the weak factorization theorem) and the fact that the Künneth isomorphism is filtered with respect to the weight filtration. In paragraph 5.3, dualizing the quasi-isomorphism u, we show the cohomological counterparts of these results : the tensor product of the dual geometric filtrations of X and Y and the dual geometric filtration of their product are related by the two filtered quasi-isomorphisms in opposite directions $u^\vee$ and w (proposition 5.19).

Composing the isomorphism $(u^\vee)^{-1} \circ w$ of $H \circ \mathfrak{C}$ with the morphism induced by the diagonal map, we define a cup product on the dual geometric filtration of a real algebraic variety X at the localized cochain level (subsection 5.4). It induces the usual cup product on the cohomology of its real points, showing that the latter is filtered with respect to the cohomological weight filtration. We then use this cup product in $Ho\,\mathfrak{C}$ to define also a cap product on the cochain and chain level (subsection 5.5). Finally, in paragraph 5.6, we focus on the cap product with the fundamental class $[X]$ of a compact real algebraic variety X, showing that the image of any cohomology class by Poincaré duality map in (co)homology is pure with respect to the weight filtration, the non-pure classes of the cohomological weight filtration being sent to zero. In particular, a compact variety with non-pure weight filtration do not satisfy Poincaré duality.

5.1 Product of semialgebraic chains

Let X and Y two real algebraic varieties. We define a product operation between the chains of X and the chains of Y, checking in lemma 5.2 that this operation is well-defined.

Definition 5.1. For any chains $c = [A] \in C_q(X)$ and $c' = [B] \in C_{q'}(Y)$, we define

$$c \times c' := [A \times B] \in C_{q+q'}(X \times Y).$$

Lemma 5.2. *Let A and A', respectively B and B', two closed semialgebraic subsets of (the set of real points) of X, respectively Y, such that $[A] = [A']$ in $C_n(X)$ and $[B] = [B']$ in $C_m(Y)$ for some nonnegative integers n and m. Then*

$$[A \times B] = [A' \times B']$$

in $C_{n+m}(X \times Y)$.

Proof. We check that $[A\times B]+[A'\times B']=0$ in $C_{n+m}(X\times Y)$. By the definition of semialgebraic chains with closed supports in the Appendix of [14], we have

$$[A\times B]+[A'\times B'] = [cl_{X\times Y}\left(A\times B \div A'\times B'\right)].$$

Since $A\times B\cup A'\times B' \subset (A\cup A')\times(B\cup B')$ and $A\times B\cap A'\times B' = (A\cap A')\times(B\cap B')$, we have

$$A\times B\div A'\times B' \subset (A\cup A')\times(B\cup B')\setminus(A\cap A')\times(B\cap B') = \left((A\div A')\times(B\cup B')\right)\cup\left((A\cup A')\times(B\div B')\right).$$

But $\dim(A\div A')<n$ and $\dim(B\div B')<m$, therefore

$$\dim cl_{X\times Y}\left(A\times B\div A'\times B'\right) = \dim A\times B\div A'\times B' < n+m$$

and $[A\times B]+[A'\times B']=0$. □

We then verify that the product of chains is distributive over the sum :

Lemma 5.3. *If $c_1,\ c_2$ are two chains of X and c' is a chain of Y,*

$$(c_1+c_2)\times c' = c_1\times c' + c_2\times c',$$

and if c is a chain of X and $c'_1,\ c'_2$ are two chains of Y,

$$c\times(c'_1+c'_2) = c\times c'_1 + c\times c'_2.$$

Proof. We write $c_1=[A_1],\ c_2=[A_2]$ and $c'=[B]$. We then have

$$\begin{aligned}
c_1\times c' + c_2\times c' &= [A_1\times B]+[A_2\times B]\\
&= [cl_{X\times Y}\left((A_1\times B)\div(A_2\times B)\right)]\\
&= [cl_{X\times Y}\left((A_1\div A_2)\times B\right)]\\
&= [cl_{X\times Y}\left(A_1\div A_2\right)\times B]\\
&= [cl_{X\times Y}\left(A_1\div A_2\right)]\times[B]\\
&= (c_1+c_2)\times c'
\end{aligned}$$

The equality $(c'_1+c'_2)=c\times c'_1+c\times c'_2$ comes from a symmetric computation.

□

The next lemma describes the behaviour of the semialgebraic boundary operator with respect to the product on semialgebraic chains we defined above.

Lemma 5.4. *The boundary of the product of two chains $c\in C_q(X)$ and $c'\in C_{q'}(Y)$ verifies, in $C_{q+q'-1}(X\times Y)$,*

$$\partial(c\times c') = \partial c\times c' + c\times\partial c'.$$

Proof. Let $A \subset X$ and $B \subset Y$ be closed semialgebraic sets representing respectively c and c'. Then, by definition, the closed semialgebraic set $A \times B \subset X \times Y$ represents the chain $c \times c'$ and $\partial(c \times c') = [\partial(A \times B)]$ (see Appendix of [14]).

We show that

$$\partial(A \times B) = \partial A \times B \cup A \times \partial B,$$

and then $\partial(c \times c') = [\partial A \times B] + [A \times \partial B] = \partial c \times c' + c \times \partial c'$ (notice that $\partial A \times B \cap A \times \partial B = \partial A \times \partial B$ and $\dim \partial A \times \partial B \leqslant q + q' - 2$).

First recall that, for a semialgebraic set S, $\partial S = \{x \in S \mid \chi(\mathrm{lk}(x, S)) \equiv 1 \mod 2\}$, where $\mathrm{lk}(x, S) := S(x, \epsilon) \cap S$ for ϵ small enough. In the lemma 5.5 below, we prove that, for a fixed point $(a, b) \in A \times B$, the link $\mathrm{lk}((a, b), A \times B)$ of (a, b) in $A \times B$ is semialgebraically homeomorphic to the set

$$\{\lambda(a, \beta) + (1 - \lambda)(\alpha, b) \mid \lambda \in [0, 1],\ \beta \in \mathrm{lk}(b, B),\ \alpha \in \mathrm{lk}(a, A)\}.$$

By additivity of the Euler characteristic χ, we deduce

$$\chi(\mathrm{lk}((a, b), A \times B)) = \chi(\{a\} \times \mathrm{lk}(b, B)) + \chi(\mathrm{lk}(a, A) \times \{b\}) + \chi(C'),$$

where $C' := \{\lambda(a, \beta) + (1 - \lambda)(\alpha, b) \mid \lambda \in]0, 1[,\ \beta \in \mathrm{lk}(b, B),\ \alpha \in \mathrm{lk}(a, A)\}$.

Notice that two segments $](a, \beta), (\alpha, b)[$ and $](a, \beta'), (\alpha', b)[$ of C', with $(\alpha, \beta) \neq (\alpha', \beta')$, do not intersect, providing us a semialgebraic homeomorphism between C' and the product $(\{a\} \times \mathrm{lk}(b, B)) \times]0, 1[\times (\mathrm{lk}(a, A) \times \{b\})$.

As a consequence we have

$$\chi(\mathrm{lk}((a, b), A \times B)) = \chi(\mathrm{lk}(a, A)) + \chi(\mathrm{lk}(b, B)) - \chi(\mathrm{lk}(a, A))\chi(\mathrm{lk}(b, B)).$$

Therefore, if $(a, b) \in \partial(A \times B)$, that is by definition $\chi(\mathrm{lk}((a, b), A \times B)) \equiv 1 \mod 2$, we deduce from the above equality that $\chi(\mathrm{lk}(a, A)) \equiv 1 \mod 2$ or $\chi(\mathrm{lk}(b, B)) \equiv 1 \mod 2$ i.e. $a \in \partial A$ or $b \in \partial B$.

Conversely, if $a \in \partial A$, i.e. $\chi(\mathrm{lk}(a, A)) \equiv 1 \mod 2$, then necessarily, $\chi(\mathrm{lk}((a, b), A \times B)) \equiv 1 \mod 2$. Consequently, $\partial A \times B \subset \partial(A \times B)$ and symmetrically $A \times \partial B \subset \partial(A \times B)$.

We proved

$$\partial(A \times B) = \partial A \times B \cup A \times \partial B.$$

□

Lemma 5.5. *Let $(a, b) \in A \times B$, then the link $\mathrm{lk}((a, b), A \times B)$ of (a, b) in $A \times B$ is semialgebraically homeomorphic to the set*

$$\{\lambda(a, \beta) + (1 - \lambda)(\alpha, b) \mid \lambda \in [0, 1],\ \beta \in \mathrm{lk}(b, B),\ \alpha \in \mathrm{lk}(a, A)\}.$$

Proof. Suppose $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ for some $n,\ m \geqslant 0$.

Consider the continuous semialgebraic function $p_1 : \mathbb{R}^n \times \{b\} \to \mathbb{R}\ ;\ (x,b) \mapsto \|x-a\|^2$. By Hardt's theorem, for ϵ small enough, there exists a semialgebraic trivialization of p_1 over $]0,\epsilon^2]$, compatible with $A \times \{b\}$, given by :

$$\overline{B}_n(a,\epsilon) \times \{b\} \to]0,\epsilon^2] \times S_n(a,\epsilon) \times \{b\}\ ;\ (x,b) \mapsto (\|x-a\|^2, \widetilde{h}_1(x), b)$$

with $\widetilde{h}_1$ continuous and semialgebraic such that $\widetilde{h}_{1\,|S_n(a,\epsilon)} = Id$.

Symmetrically, there exists a semialgebraic trivialization of the function $p_2 : \{a\} \times \mathbb{R}^m \to \mathbb{R}\ ;\ (a,y) \mapsto \|y-b\|^2$ over $]0,\epsilon^2]$, compatible with $\{a\} \times B$ given by

$$\{a\} \times \overline{B}_m(b,\epsilon) \to]0,\epsilon^2] \times \{a\} \times S_m(b,\epsilon)\ ;\ (a,y) \mapsto (\|y-b\|^2, a, \widetilde{h}_2(y)$$

with $\widetilde{h}_2$ continuous and semialgebraic such that $\widetilde{h}_{2\,|S_m(b,\epsilon)} = Id$.

We then define

$$f : S_{m+n}((a,b),\epsilon) \to C\ ;\ (x,y) \mapsto \frac{\|x-a\|^2}{\epsilon^2}(\widetilde{h}_1(x), b) + \frac{\|y-b\|^2}{\epsilon^2}(a, \widetilde{h}_2(y)),$$

where $C := \{\lambda(a,\beta) + (1-\lambda)(\alpha,b) \mid \lambda \in [0,1],\ \beta \in S_m(b,\epsilon),\ \alpha \in S_n(a,\epsilon)\}$, which is a semialgebraic homeomorphism. Since the trivializations of p_1 and p_2 over $]0,\epsilon^2]$ are compatible with $A \times \{b\}$ and $\{a\} \times B$ respectively, we have

$$f(S_{n+m}((a,b),\epsilon) \cap A \times B) = \{\lambda(a,\beta) + (1-\lambda)(\alpha,b) \mid \lambda \in [0,1],\ \beta \in S_m(b,\epsilon) \cap B,\ \alpha \in S_n(a,\epsilon) \cap A\}.$$

□

5.2 Product and geometric filtration

Now we study the behaviour of the product of chains with respect to the geometric filtration :

Proposition 5.6.
(1) If $c \in \mathcal{G}_p C_q(X)$ and $c' \in \mathcal{G}_{p'} C_{q'}(Y)$, then

$$c \times c' \in \mathcal{G}_{p+p'} C_{q+q'}(X \times Y)$$

(2) If $c \in C_q(X)$, $c' \in C_{q'}(Y)$ and $c \times c' \in \mathcal{G}_s C_{q+q'}(X \times Y)$, then there exists p, p' with $p+p' = s$, such that

$$c \in \mathcal{G}_p C_q(X) \text{ and } c' \in \mathcal{G}_{p'} C_{q'}(Y)$$

Remark 5.7. Because the filtration $\mathcal{G}_\bullet$ is increasing, the proposition shows in particular that the index $p+p'$ of the product $c \times c'$ in the filtration is minimal if and only if the indices p of c and p' of c' are minimal. In other words, if $c \in \mathcal{G}_p C_q(X) \setminus \mathcal{G}_{p-1} C_q(X)$ and $c' \in \mathcal{G}_{p'} C_{q'}(Y) \setminus \mathcal{G}_{p'-1} C_{q'}(Y)$, then

$$c \times c' \in \mathcal{G}_{p+p'} C_{q+q'}(X \times Y) \setminus \mathcal{G}_{p+p'-1} C_{q+q'}(X \times Y),$$

and if $c \in \mathcal{G}_p C_q(X)$ and $c' \in \mathcal{G}_{p'} C_{q'}(Y)$ with $c \times c' \notin \mathcal{G}_{p+p'-1} C_{q+q'}(X \times Y)$ then

$$c \notin \mathcal{G}_{p-1} C_q(X) \text{ and } c' \notin \mathcal{G}_{p'-1} C_{q'}(Y).$$

For the proof of Proposition 5.6, we use the notion of adapted resolutions (see [14], section 2). Adapted resolutions allow us to work with chains lying in a nonsingular ambient space, with boundary belonging to a normal crossing divisor.

Lemma 5.8. *Suppose X is compact and consider a chain $c = [A]$ of X. We can assume that the dimension of A is maximal, equal to the dimension of X, by considering the Zariski closure $A \subset \overline{A}^Z$ of A, since the filtration is only depending on the support of c (Theorem* **2.1** *(1) of [14]) : we have*

$$c \in \mathcal{G}_p C_k(X) \Longleftrightarrow c \in \mathcal{G}_p C_k(\overline{A}^Z).$$

With this assumption, there exists a resolution of singularities $\pi : \widetilde{X} \longrightarrow X$ of X such that $\operatorname{supp}(\partial(\pi^{-1}c)) \subset D$*, where D is a normal crossing divisor of $\widetilde{X}$.*

Such a resolution is called a *resolution of X adapted to the chain c*. Notice that the pullback $\pi^{-1}c$ of c is defined because π is a resolution of singularities (the pullback operation on chains is more generally defined for any acyclic square of real algebraic varieties : see [14] Appendix).

Proof. First consider a resolution $\pi' : X' \longrightarrow X$ of X to make the ambient space nonsingular, then consider a resolution $\widetilde{X} \longrightarrow X'$ of the embedded variety $\operatorname{supp}\left(\partial(\pi'^{-1}c)\right)$ (which is a hypersurface of X'), so that it is in a normal crossing divisor. □

Proof. (*of Proposition 5.6)* The first point can be proved using the description of geometric filtration using Nash-constructible fonctions. We keep the notations from [14]. There exist generically Nash-constructible functions $\varphi : X \longrightarrow 2^{q+p}\mathbb{Z}$ and $\psi : Y \longrightarrow 2^{q'+p'}\mathbb{Z}$ such that

$$c = \left[\left\{x \in X \mid \varphi(x) \notin 2^{q+p+1}\mathbb{Z}\right\}\right] \text{ and } c' = \left[\left\{y \in Y \mid \psi(y) \notin 2^{q'+p'+1}\mathbb{Z}\right\}\right].$$

Denote by $\pi_X : X \times Y \longrightarrow X$ and $\pi_Y : X \times Y \longrightarrow Y$ the projections. We define the function

$$\eta := \pi_X^*(\varphi) \cdot \pi_Y^*(\psi) \ : \ \begin{array}{ccc} X \times Y & \longrightarrow & 2^{q+q'p+p'}\mathbb{Z} \\ (x,y) & \longmapsto & \varphi(x) \cdot \psi(y) \end{array}$$

which is Nash-constructible because the pullback of a Nash-constructible function and the product of Nash-constructible functions are Nash-constructible. Since,

$$c \times c' = \left[\left\{(x,y) \in X \times Y \mid \eta(x,y) \notin 2^{q+q+p+p'+1}\mathbb{Z}\right\}\right],$$

the chain $c \times c'$ is in $\mathcal{G}_{p+p'}C_{q+q'}(X \times Y)$.

To prove the second point of the proposition, we use the very definition of the geometric filtration (see [14] Theorem 2.1. and Proposition 2.6.). We first assume X and Y to be compact and we proceed by induction on the dimension of $X \times Y$. Suppose $c \times c' \in \mathcal{G}_s C_{q+q'}(X \times Y)$.

Let $\pi : \widetilde{X} \longrightarrow X$ be an adapted resolution of c (it exists by lemma 5.8 above). Then, if we set $\widetilde{c} := \pi^{-1}(c)$, the support $\operatorname{supp}(\partial\widetilde{c})$ is included in a normal crossing divisor D of $\widetilde{X}$, and by definition of the geometric filtration, we have

$$c \in \mathcal{G}_p C_q(X) \Longleftrightarrow \partial\widetilde{c} \in \mathcal{G}_p C_{q-1}(D).$$

In the same way, consider $\pi' : \widetilde{Y} \longrightarrow Y$ an adapted resolution of c'. We have $\mathrm{supp}(\partial \widetilde{c'}) \subset D'$, with $\widetilde{c'} := \pi'^{-1}(c')$ and D' a normal crossing divisor of $\widetilde{Y}$, and

$$c' \in \mathcal{G}_{p'} C_{q'}(Y) \Longleftrightarrow \partial \widetilde{c'} \in \mathcal{G}_{p'} C_{q'-1}(D').$$

Now $\pi \times \pi' : \widetilde{X} \times \widetilde{Y} \longrightarrow X \times Y$; $(x, y) \mapsto \pi(x) \times \pi'(y)$ is an adapted resolution of $c \times c'$. Indeed, by Lemma 5.4,

$$\partial(\widetilde{c} \times \widetilde{c'}) = \partial\widetilde{c} \times \widetilde{c'} + \widetilde{c} \times \partial\widetilde{c'}, \tag{5.1}$$

in particular,

$$\mathrm{supp}(\partial(\widetilde{c} \times \widetilde{c'})) \subset (D \times \widetilde{Y}) \cup (\widetilde{X} \times D') :$$

the subvarieties $D \times \widetilde{Y}$ and $\widetilde{X} \times D'$ are normal crossing divisors of $\widetilde{X} \times \widetilde{Y}$ because $\widetilde{X}$ and $\widetilde{Y}$ are nonsingular, and their union that we denote by $\widetilde{D}$ is again a normal crossing divisor of $\widetilde{X} \times \widetilde{Y}$ since they have no common irreducible component (their intersection is $D \times D'$ which has strictly smaller dimension).

Therefore $c \times c' \in \mathcal{G}_s C_{q+q'}(X \times Y) \Leftrightarrow \partial(\widetilde{c} \times \widetilde{c'}) \in \mathcal{G}_s C_{q+q'-1}(\widetilde{D})$. We then use the lemma 5.9 below to deduce that $\partial(\widetilde{c}) \times \widetilde{c'} \in \mathcal{G}_s C_{q+q'-1}(D \times \widetilde{Y})$ (and $\widetilde{c} \times \partial(\widetilde{c'}) \in \mathcal{G}_s C_{q+q'-1}(\widetilde{X} \times D')$).

By induction on dimension, there exists p and p' in $\mathbb{Z}$ such that $p+p' = s$, $\partial(\widetilde{c}) \in \mathcal{G}_p C_{q-1}(D)$ and $\widetilde{c'} \in \mathcal{G}_{p'} C_{q'}(\widetilde{Y})$, that is $c \in \mathcal{G}_p C_q(X)$ and $c' \in \mathcal{G}_{p'} C_{q'}(Y)$ (since $\widetilde{Y}$ is nonsingular and $\widetilde{c'} \in D'$, we have $\widetilde{c'} \in \mathcal{G}_{p'} C_{q'}(\widetilde{Y}) \Leftrightarrow \partial(\widetilde{c'}) \in \mathcal{G}'_p C_{q'-1}(D')$).

Finally, to prove the result in the general case, consider real algebraic compactifications $\overline{X}$ and $\overline{Y}$ of X and Y respectively. Then $\overline{X} \times \overline{Y}$ is a compactification of $X \times Y$ and, by definition of the geometric filtration,

$$c \times c' \in \mathcal{G}_s C_{q+q'}(X \times Y) \Leftrightarrow \overline{c \times c'} \in \mathcal{G}_s C_{q+q'}(\overline{X} \times \overline{Y}).$$

The closure $\overline{c \times c'}$ of the chain $c \times c'$ is equal to $\overline{c} \times \overline{c'}$ and, by what we proved above in the compact case, there exist p and p' in $\mathbb{Z}$ such that $p + p' = s$ and

$$\overline{c} \in \mathcal{G}_p C_q(\overline{X}) \text{ and } \overline{c'} \in \mathcal{G}_{p'} C_{q'}(\overline{Y})$$

which is equivalent to

$$c \in \mathcal{G}_p C_q(X) \text{ and } c' \in \mathcal{G}_{p'} C_{q'}(Y).$$

□

Lemma 5.9. *With the above notations, we have*

$$\partial(\widetilde{c} \times \widetilde{c'}) \in \mathcal{G}_s C_{q+q'-1}(\widetilde{D}) \Longleftrightarrow \begin{cases} \partial(\widetilde{c}) \times \widetilde{c'} \in \mathcal{G}_s C_{q+q'-1}(D \times \widetilde{Y}) \\ \text{and} \\ \widetilde{c} \times \partial(\widetilde{c'}) \in \mathcal{G}_s C_{q+q'-1}(\widetilde{X} \times D') \end{cases}$$

Proof. The implication from right to left follows from the definition of the geometric filtration ($D \times \widetilde{Y}$ and $\widetilde{X} \times D'$ are two subvarieties of $\widetilde{D}$) and the formula 5.1.

We prove the implication from left to right using the description of the geometric filtration via Nash-constructible functions ([14], section 3).

First denote $A := \operatorname{supp}(\partial(\widetilde{c}) \times \widetilde{c'})$, $B := \operatorname{supp}(\widetilde{c} \times \partial(\widetilde{c'}))$, then the closed semialgebraic set $A \cup B$ represents the chain $\partial(\widetilde{c} \times \widetilde{c'})$ in $C_{q+q'-1}(\widetilde{D})$ (because $A \cap B \subset D \times D'$ is of strictly smaller dimension). Furthermore, because it belongs to $\mathcal{G}_s C_{q+q'-1}(\widetilde{D})$, the chain $\partial(\widetilde{c} \times \widetilde{c'})$ is represented by a Nash-constructible function $\varphi : \widetilde{D} \to 2^{q+q'-1+s}\mathbb{Z}$ and we have

$$A \cup B = \{(x,y) \in \widetilde{D} \mid \varphi(x,y) \notin 2^{q+q'+s}\mathbb{Z}\},$$

up to a set of dimension $< q + q' - 1$.

Consider now the characteristic functions ψ_A and ψ_B on $\widetilde{D}$ of the Zariski closures of A and B respectively. The Nash-constructible function $\varphi \cdot \psi_A : \widetilde{D} \to 2^{q+q'-1+s}\mathbb{Z}$ represents the chain $\left[(A \cup B) \cap \overline{A}^Z\right]$ in $C_{q+q'-1}(\widetilde{D})$. But, since the intersection of the Zariski closures of A and B is of dimension $< q+q'-1$ (because it is a subvariety of $D \times D'$), we have $\left[(A \cup B) \cap \overline{A}^Z\right] = [A]$ and the Nash-constructible function $\varphi \cdot \psi_A : \widetilde{D} \to 2^{q+q'-1+s}\mathbb{Z}$ represents the chain $[A] = \partial(\widetilde{c}) \times \widetilde{c'}$. Consequently, $\partial(\widetilde{c}) \times \widetilde{c'} \in \mathcal{G}_s C_{q+q'-1}(\widetilde{D})$ and, since $\operatorname{supp}(\partial(\widetilde{c}) \times \widetilde{c'}) \subset D \times \widetilde{Y}$, we have

$$\partial(\widetilde{c}) \times \widetilde{c'} \in \mathcal{G}_s C_{q+q'-1}(D \times \widetilde{Y}).$$

Symmetrically, the Nash-constructible function $\varphi \cdot \psi_B : \widetilde{D} \to 2^{q+q'-1+s}\mathbb{Z}$ represents $c \times \partial(\widetilde{c'})$ and

$$\widetilde{c} \times \partial(\widetilde{c'}) \in \mathcal{G}_s C_{q+q'-1}(\widetilde{X} \times D').$$

□

Next, we want to find a relation between the geometric filtration of the product variety $X \times Y$ and the product of the geometric filtrations of X and Y. First, we have to make precise what we mean by a product of filtered complexes :

Definition 5.10. Let (K_*, F) et (M_*, J) be two filtered complexes in the category $\mathcal{C}$. We define $((K \otimes_{\mathbb{Z}_2} M)_*, F \otimes J)$ to be the complex given by

$$(K \otimes M)_n := \bigoplus_{i+j=n} K_i \otimes_{\mathbb{Z}_2} M_j$$

equipped with the differential

$$d(x \otimes y) := dx \otimes y + x \otimes dy$$

and the bounded increasing filtration given by

$$(F \otimes J)_p (K \otimes M)_n := \bigoplus_{i+j=n} \sum_{a+b=p} F_a K_i \otimes_{\mathbb{Z}_2} J_b M_j$$

Remark 5.11. Notice that there is no sign in the definition of the differential because we are working with $\mathbb{Z}_2$ coefficients.

Therefore, a product of filtered complexes of $\mathcal{C}$ induces a spectral sequence converging to its homology. In lemma 5.13 below, we give the relation between this spectral sequence and some product of the spectral sequences induced by each filtered complex. This result will follow from the Künneth isomorphism (see for instance [5], Theorem 29.10) that we recall here. This theorem allows one to compare the homology a product of complexes with the tensor product of their homologies. Because $\mathbb{Z}_2$ is a field, there is no torsion.

Theorem 5.12. *For any K_* and M_* two chain complexes over $\mathbb{Z}_2$, we have a so-called Künneth isomorphism*

$$H_*(K) \otimes H_*(M) \longrightarrow H_*(K \otimes M) \ ;$$

in particular, for all $n \in \mathbb{Z}$,

$$H_n(K \otimes M) \cong \bigoplus_{i+j=n} H_i(K) \otimes_{\mathbb{Z}_2} H_j(M).$$

Lemma 5.13. *Let (K_*, F) and (M_*, J) be two filtered chain complexes. The spectral sequence of their product verifies, for $r \geqslant 0$,*

$$E^r_{a,b}(K \otimes M) \cong \bigoplus_{p+s=a, q+t=b} E^r_{p,q}(K) \otimes E^r_{s,t}(M). \tag{5.2}$$

Proof. We prove this lemma by induction on r. First we have, for $a,\ b \in \mathbb{Z}$,

$$\begin{aligned} E^0_{a,b}(K \otimes M) &= \frac{(F \otimes J)_a (K \otimes M)_{a+b}}{(F \otimes J)_{a-1} (K \otimes M)_{a+b}} \\ &= \bigoplus_{i+j=a+b} \frac{\sum_{\alpha+\beta=a} F_\alpha K_i \otimes J_\beta M_j}{\sum_{\alpha+\beta=a-1} F_\alpha K_i \otimes J_\beta M_j} \\ &= \bigoplus_{i+j=a+b} \bigoplus_{\alpha+\beta=a} \frac{F_\alpha K_i}{F_{\alpha-1} K_i} \otimes \frac{J_\beta M_j}{J_{\beta-1} M_j} \\ &= \bigoplus_{p+s=a, q+t=b} E^0_{p,q}(K) \otimes E^0_{s,t}(M) \end{aligned}$$

We prove the equality $\bigoplus_{i+j=a+b} \frac{\sum_{\alpha+\beta=a} F_\alpha K_i \otimes J_\beta M_j}{\sum_{\alpha+\beta=a-1} F_\alpha K_i \otimes J_\beta M_j} = \bigoplus_{i+j=a+b} \bigoplus_{\alpha+\beta=a} \frac{F_\alpha K_i}{F_{\alpha-1} K_i} \otimes \frac{J_\beta M_j}{J_{\beta-1} M_j}$ in the lemma 5.14 below.

Then suppose the property is true for a fixed $r \geqslant 0$. The term $E^r(K \otimes M)$ of the spectral sequence induced by the filtered tensor product of K_* and M_* is composed of chain complexes $(E^r_{*,*}, d^r_{*,*})$ whose homology computes the term $E^{r+1}(K \otimes M)$. Applying homology and Künneth isomorphism to the formula 5.2 given at level r by induction, we obtain the property at level $r + 1$.

□

Lemma 5.14. *Let a and b be in $\mathbb{Z}$. Keeping the notations from Lemma 5.13 above, we have*

$$\bigoplus_{i+j=a+b} \frac{\sum_{\alpha+\beta=a} F_\alpha K_i \otimes J_\beta M_j}{\sum_{\alpha+\beta=a-1} F_\alpha K_i \otimes J_\beta M_j} = \bigoplus_{i+j=a+b} \bigoplus_{\alpha+\beta=a} \frac{F_\alpha K_i}{F_{\alpha-1} K_i} \otimes \frac{J_\beta M_j}{J_{\beta-1} M_j}.$$

Proof. Let $i, j \in \mathbb{Z}$ such that $i + j = a + b$, we prove that

$$\frac{\sum_{\alpha+\beta=a} F_\alpha K_i \otimes J_\beta M_j}{\sum_{\alpha+\beta=a-1} F_\alpha K_i \otimes J_\beta M_j} \cong \bigoplus_{\alpha+\beta=a} \frac{F_\alpha K_i}{F_{\alpha-1} K_i} \otimes \frac{J_\beta M_j}{J_{\beta-1} M_j}.$$

Denoting simply $F_\alpha K_i$ by F_α and $J_\beta M_j$ by J_β, let ψ be the $\mathbb{Z}_2$-linear map

$$\sum_{\alpha+\beta=a} F_\alpha \otimes J_\beta \to \bigoplus_{\alpha+\beta=a} \frac{F_\alpha}{F_{\alpha-1}} \otimes \frac{J_\beta}{J_{\beta-1}}$$

(well-)defined by, if $x \otimes y \in F_\alpha \otimes J_\beta$, $\psi(x \otimes y) := \overline{x} \otimes \widehat{y} \in \frac{F_\alpha}{F_{\alpha-1}} \otimes \frac{J_\beta}{J_{\beta-1}}$.

The map ψ is surjective and $\sum_{r+s=a-1} F_r \otimes J_s \subset \ker \psi$. Now let $\gamma \in \ker \psi$. Then, $\gamma = \sum_{\alpha+\beta=a} \sum_{i \in I_{\alpha,\beta}} x_i^\alpha \otimes y_i^\beta$ with $x_i^\alpha \in F_\alpha$ and $y_i^\beta \in J_\beta$ for all α, β, i. We have

$$0 = \psi(\gamma) = \sum_{\alpha,\beta} \sum_i \overline{x_i^\alpha} \otimes \widehat{y_i^\beta}$$

that is, for all α, β, $\sum_i \overline{x_i^\alpha} \otimes \widehat{y_i^\beta} = 0$ and thus, for all i, there exist $z_i^{\alpha-1} \in F_{\alpha-1}$ and $w_i^{\beta-1} \in J_{\beta-1}$ such that $\sum_i (x_i^\alpha + z_i^{\alpha-1}) \otimes (y_i^\beta + w_i^{\beta-1}) = 0$ i.e.

$$\sum_i x_i^\alpha \otimes y_i^\beta = \sum_i x_i^\alpha \otimes w_i^{\beta-1} + \sum_i z_i^{\alpha-1} \otimes y_i^\beta + \sum_i z_i^{\alpha-1} \otimes w_i^{\beta-1} \in F_\alpha \otimes J_{\beta-1} + F_{\alpha-1} \otimes J_\beta.$$

As a consequence, $\gamma \in \sum_{r+s=a-1} F_r \otimes J_s$ and we get the result by universal quotient property. □

This lemma 5.13 will allow us to relate the weight spectral sequence of a product of real algebraic varieties and the product of the weight spectral sequences, as an interpretation of the following result. We relate the product of geometric filtrations with the geometric filtration of the product : the map that associates to a tensor product of chains their cross product is a filtered quasi-isomorphism with respect to the geometric filtration.

Theorem 5.15. *We have a filtered quasi-isomorphism*

$$u : \mathcal{G}_\bullet C_*(X) \otimes \mathcal{G}_\bullet C_*(Y) \longrightarrow \mathcal{G}_\bullet C_*(X \times Y)$$

given by

$$c_X \otimes c_Y \longmapsto c_X \times c_Y$$

if $c_X \in \mathcal{G}_p C_q(X)$ *and* $c_Y \in \mathcal{G}_{p'} C_{q'}(Y)$ *with* $p + p' = s$ *and* $q + q' = n$.

The morphism u is filtered by proposition 5.6 (recall also the definition 5.1 of the product of filtered complexes).

Corollary 5.16. *The filtered complexes $\mathcal{W}C_*(X)\otimes_{\mathbb{Z}_2}\mathcal{W}C_*(Y)$ and $\mathcal{W}C_*(X\times Y)$ are isomorphic in $Ho\,\mathcal{C}$ and the above map u induces a filtered isomorphism*

$$u_\infty : \mathcal{W}_\bullet H_*(X) \otimes \mathcal{W}_\bullet H_*(Y) \longrightarrow \mathcal{W}_\bullet H_*(X \times Y).$$

Above theorem 5.15 has an interpretation from the viewpoint of spectral sequences : u is a morphism of filtered complexes which induces an isomorphism on spectral sequences from level one

$$u_r \ : \ E^r_{a,b}(\mathcal{G}_\bullet C_*(X) \otimes \mathcal{G}_\bullet C_*(Y)) \stackrel{\sim}{\longrightarrow} E^r_{a,b}(X \times Y)$$

for $r \geqslant 1$. Using Lemma 5.13, we get an isomorphism

$$u'_r \ : \ \bigoplus_{p+s=a,\ q+t=b} E^r_{p,q}(X) \otimes_{\mathbb{Z}_2} E^r_{s,t}(Y) \stackrel{\sim}{\longrightarrow} E^r_{a,b}(X \times Y) \qquad (5.3)$$

(for $r \geqslant 1$). In particular, using this isomorphism, we show the multiplicativity property of the virtual Poincaré polynomial

$$\beta(\,\cdot\,)(u) := \sum_{q\geqslant 0} \beta_q(\,\cdot\,)u^q,$$

without using the weak factorization theorem as in [12] and [4]. Precisely, taking the alternating sum of the E^1 terms in the equation 5.3 as in proposition 3.10, we obtain the following relation on the virtual Betti numbers :

$$\beta_n(X \times Y) = \sum_{p+q=n} \beta_p(X)\beta_q(Y).$$

Consequently,

$$\beta(X \times Y)(u) = \beta(X)(u)\beta(Y)(u).$$

Furthermore, the Künneth isomorphism in homology is filtered through the isomorphisms

$$u'_\infty \ : \ \bigoplus_{p+s=a,q+t=b} E^\infty_{p,q}(X) \otimes_{\mathbb{Z}_2} E^\infty_{s,t}(Y) \stackrel{\sim}{\longrightarrow} E^\infty_{a,b}(X \times Y)$$

Remark 5.17. To prove theorem 5.15, we use the naturality property of the extension theorem of [7] (Proposition 1.4 of [14]). We first show that u is a filtered quasi-isomorphism for nonsingular projective real algebraic varieties and then use this naturality to prove that u is a filtered quasi-isomorphism for all real algebraic varieties. We do not know whether the theorem follows directly from the geometric filtration of [14] and Proposition 5.6, without the theory of cubical hyperresolutions of [7].

Proof. (of Theorem 5.15) When X and Y are nonsingular projective varieties, so is the product variety $X \times Y$ and the three induced weight spectral sequences verify

$$E^\infty_{p,q} = E^1_{p,q} = \begin{cases} H_{p+q} & \text{if } p+q=-p \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, by lemma 5.13, the morphism $u : \mathcal{G}_\bullet C_*(X) \otimes \mathcal{G}_\bullet C_*(Y) \longrightarrow \mathcal{G}_\bullet C_*(X \times Y)$ induces on $E^1 = E^\infty$ the morphism $H_*(X) \otimes_{\mathbb{Z}_2} H_*(Y) \longrightarrow H_*(X \times Y)$, which is the classical Künneth isomorphism in singular homology. Thus, u is a filtered quasi-isomorphism when X and Y are projective nonsingular.

Let Y be now a fixed nonsingular projective variety and consider the two functors

$$\phi_1 : \mathbf{Sch}_c(\mathbb{R}) \to \mathcal{C} \ ; \ X \mapsto \mathcal{G}C(X) \otimes \mathcal{G}C(Y)$$

and

$$\phi_2 : \mathbf{Sch}_c(\mathbb{R}) \to \mathcal{C} \ ; \ X \mapsto \mathcal{G}C(X \times Y)$$

(in this part of the proof, we drop the subscripts of filtrations and complexes for readability). We proved above that these two functors are quasi-isomorphic in $\mathcal{C}$ on $\mathbf{V}(\mathbb{R})$ (we denote by φ_1 and φ_2 their respective restrictions to $\mathbf{V}(\mathbb{R})$). Furthermore, they both verify the additivity and acyclicity conditions of Theorem (2.2.2) of [7]. Indeed, if $Z \hookrightarrow X$ is a closed inclusion, the additivity of the geometric filtration (see Theorem 2.7. of [14]) induces the exactness of the sequences

$$0 \to \mathcal{G}C(Z) \otimes \mathcal{G}C(Y) \to \mathcal{G}C(X) \otimes \mathcal{G}C(Y) \to \mathcal{G}C(X \setminus Z) \otimes \mathcal{G}C(Y) \to 0$$

and

$$0 \to \mathcal{G}C(Z \times Y) \to \mathcal{G}C(X \times Y) \to \mathcal{G}C((X \setminus Z) \times Y) \to 0$$

(the induced morphism $Z \times Y \hookrightarrow X \times Y$ is also a closed inclusion). Now, if the diagram

$$\begin{array}{ccc} \widetilde{Z} & \to & \widetilde{X} \\ \downarrow & & \downarrow \\ Z & \to & X \end{array}$$

is an acyclic square, we check that the simple filtered complexes associated to the induced diagrams

$$\begin{array}{ccc} \mathcal{G}C(\widetilde{Z}) \otimes \mathcal{G}C(Y) & \longrightarrow & \mathcal{G}C(\widetilde{X}) \otimes \mathcal{G}C(Y) \\ \downarrow & & \downarrow \\ \mathcal{G}C(Z) \otimes \mathcal{G}C(Y) & \longrightarrow & \mathcal{G}C(X) \otimes \mathcal{G}C(Y) \end{array}$$

denoted by $\mathcal{K}_1(Y)$, and

$$\begin{array}{ccc} \mathcal{G}C(\widetilde{Z} \times Y) & \longrightarrow & \mathcal{G}C(\widetilde{X} \times Y) \\ \downarrow & & \downarrow \\ \mathcal{G}C(Z \times Y) & \longrightarrow & \mathcal{G}C(X \times Y) \end{array}$$

denoted by $\mathcal{K}_2(Y)$, are acyclic. The simple filtered complex $\mathbf{s}\mathcal{K}_2(Y)$ is acyclic because the geometric filtration verifies the acyclicity condition for an acyclic square (see Theorem 2.7. of [14]) and the diagram

$$\begin{array}{ccc} \widetilde{Z} \times Y & \to & \widetilde{X} \times Y \\ \downarrow & & \downarrow \\ Z \times Y & \to & X \times Y \end{array}$$

is acyclic. The spectral sequence induced by $\mathbf{s}\mathcal{K}_1(Y)$ verifies $E^1 = 0$ because $\mathbf{s}\mathcal{K}_1(Y)$ is nothing more than the tensor product of filtered complexes $\mathbf{s}\mathcal{K}_0 \otimes \mathcal{G}C(Y)$, where $\mathcal{K}_0$ is the diagram

$$\begin{array}{ccc} \mathcal{G}C(\widetilde{Z}) & \longrightarrow & \mathcal{G}C(\widetilde{X}) \\ \downarrow & & \downarrow \\ \mathcal{G}C(Z) & \longrightarrow & \mathcal{G}C(X) \end{array}$$

and therefore, by lemma 5.13, $E^1_{a,b}(\mathcal{K}_1(Y)) = \bigoplus_{p+s=a, q+t=b} E^1_{p,q}(\mathbf{s}\mathcal{K}_0) \otimes E^1_{s,t}(\mathcal{G}C(Y)) = 0$, for all $a, b \in \mathbb{Z}$, again because of the acyclicity of the geometric filtration (notice that in both cases, we did not use the fact that Y was projective nonsingular).

Consequently, the localizations $\phi_1', \phi_2' : \mathbf{Sch}_c(\mathbb{R}) \to Ho\,\mathcal{C}$ of ϕ_1 and ϕ_2 respectively are the unique extensions of their respective restrictions $\varphi_1', \varphi_2' : \mathbf{V}(\mathbb{R}) \to Ho\,\mathcal{C}$ given by the Theorem 2.2.2. of [7] (notice that the above arguments prove also that the functors φ_1 and φ_2 satisfy the disjoint additivity condition (F_1) and the elementary acyclicity condition (F_2)). By the naturality of this extension (see Proposition 1.4 of [14]), the localization of the filtered quasi-isomorphism $u(Y) : \varphi_1 \to \varphi_2$ extends uniquely into a morphism $\phi_1' \to \phi_2'$, and this morphism is an isomorphism of $Ho\,\mathcal{C}$. Since the localization of

$$u(Y) : \phi_1 \to \phi_2 \ ; \ X \mapsto (\mathcal{G}_\bullet C_*(X) \otimes \mathcal{G}_\bullet C_*(Y) \longrightarrow \mathcal{G}_\bullet C_*(X \times Y))$$

is such an extension, the latter is a quasi-isomorphism of $\mathcal{C}$, that is the morphism $u : \mathcal{G}_\bullet C_*(X) \otimes \mathcal{G}_\bullet C_*(Y) \longrightarrow \mathcal{G}_\bullet C_*(X \times Y)$ is a filtered quasi-isomorphism for any real algebraic variety X and Y a nonsingular projective variety.

Now fix X to be any real algebraic variety and consider the morphism of functors

$$u(X) : Y \mapsto (\mathcal{G}_\bullet C_*(X) \otimes \mathcal{G}_\bullet C_*(Y) \longrightarrow \mathcal{G}_\bullet C_*(X \times Y)).$$

We prove in the same way as above that the localization of the restriction of $u(X)$ to $\mathbf{V}(\mathbb{R})$ extends uniquely into a morphism of functors on $\mathbf{Sch}_c(\mathbb{R})$, which is an isomorphism of $Ho\,\mathcal{C}$ because so is the restriction to $\mathbf{V}(\mathbb{R})$. Because of the uniqueness of the extension, we finally obtain that

$$u : \mathcal{G}_\bullet C_*(X) \otimes \mathcal{G}_\bullet C_*(Y) \longrightarrow \mathcal{G}_\bullet C_*(X \times Y)$$

is a filtered quasi-isomorphism for any real algebraic varieties X and Y. □

Remark 5.18. A morphism between the filtered complexes $\mathcal{W}C_*(X) \otimes \mathcal{W}C_*(Y)$ and $\mathcal{W}C_*(X \times Y)$ for any varieties X and Y can also be obtained without using the geometric filtration. Indeed, using a method similar to the one in the previous proof, we can extend to all real algebraic varieties the morphism of filtered complexes

$$F^{can}_\bullet C_*(X) \otimes F^{can}_\bullet C_*(Y) \longrightarrow F^{can}_\bullet C_*(X \times Y)$$

(given by the product in Definition 5.1) restricted to nonsingular projective varieties.

5.3 Product and cohomological weight complex

As for the homological weight complex, we show that we can relate the cohomological weight complex of a product with the tensor product of the cohomological weight complexes, so that these two filtered complexes are isomorphic in the localized category $Ho\,\mathfrak{C}$. More precisely, this isomorphism of $Ho\,\mathfrak{C}$ is induced by two opposite-directional filtered quasi-isomorphisms of cochain complexes, one of them being the dualization of the quasi-isomorphism in $\mathcal{C}$ in Theorem 5.15.

Proposition 5.19. *The filtered complexes $\mathcal{G}^\bullet C^*(X) \otimes \mathcal{G}^\bullet C^*(Y)$ and $\mathcal{G}^\bullet C^*(X \times Y)$ are isomorphic in $Ho\,\mathfrak{C}$.*

Corollary 5.20. *The filtered complexes $\mathcal{W}C^*(X) \otimes \mathcal{W}C^*(Y)$ and $\mathcal{W}C^*(X \times Y)$ are isomorphic in $Ho\,\mathfrak{C}$ and the Künneth isomorphism in cohomology*

$$\mathcal{W}^\bullet H^*(X) \otimes \mathcal{W}^\bullet H^*(Y) \longrightarrow \mathcal{W}^\bullet H^*(X \times Y)$$

is a filtered isomorphism with respect to the cohomological weight filtration.

Proof. Consider the filtered quasi-isomorphism

$$u : \mathcal{G}_\bullet C_*(X) \otimes \mathcal{G}_\bullet C_*(Y) \longrightarrow \mathcal{G}_\bullet C_*(X \times Y)$$

defined in Theorem 5.15. Its dual

$$u^\vee \;:\; \begin{array}{ccc} (C_*(X \times Y))^\vee & \longrightarrow & (C_*(X) \otimes C_*(Y))^\vee \\ \eta & \longmapsto & \left[\sum_i c_{X,i} \otimes c_{Y,i} \longmapsto \sum_i \eta(c_{X,i} \times c_{Y,i})\right] \end{array}$$

is also a filtered quasi-isomorphism if we equip the dualized complexes with the corresponding dual filtrations (remark 4.1).

On the other hand, the map

$$w \;:\; \begin{array}{ccc} (C_*(X))^\vee \otimes (C_*(Y))^\vee & \longrightarrow & (C_*(X) \otimes C_*(Y))^\vee \\ \varphi \otimes \psi & \longmapsto & \left[\sum_i c_{X,i} \otimes c_{Y,i} \longmapsto \sum_i \varphi(c_{X,i}) \cdot \psi(c_{Y,i})\right] \end{array}$$

where the right-hand side complex is equipped with the same filtration as above (induced by the geometric filtrations of X and Y) and the left-hand side complex is the filtered tensor product of the dual geometric filtrations of X and Y, is also a filtered quasi-isomorphism. Indeed, for $a, b \in \mathbb{Z}$, we have

$$E_1^{a,b}\left((C_*(X))^\vee \otimes_{\mathbb{Z}_2} (C_*(Y))^\vee\right) = \bigoplus_{p+s=a, q+t=b} \left(E^1_{p,q}(X)\right)^\vee \otimes_{\mathbb{Z}_2} \left(E^1_{s,t}(Y)\right)^\vee,$$

and

$$E_1^{a,b}\left((C_*(X)\otimes_{\mathbb{Z}_2} C_*(Y))^\vee\right) = \left(E^1_{a,b}\left(C_*(X)\otimes_{\mathbb{Z}_2} C_*(Y)\right)\right)^\vee = \bigoplus_{p+s=a,q+t=b}\left(E^1_{p,q}(X)\otimes_{\mathbb{Z}_2} E^1_{s,t}(Y)\right)^\vee,$$

by lemma 5.13 (or its cohomological version), and the morphism w induces on the E_1-level the morphisms

$$\left(E^1_{p,q}(X)\right)^\vee \otimes_{\mathbb{Z}_2} \left(E^1_{s,t}(Y)\right)^\vee \longrightarrow \left(E^1_{p,q}(X)\otimes_{\mathbb{Z}_2} E^1_{s,t}(Y)\right)^\vee,$$

given by $\overline{\varphi}\otimes\widehat{\psi}\mapsto [\sum_i \overline{c_{X,i}}\otimes\widehat{c_{Y,i}}\mapsto \sum_i \varphi(c_{X,i})\cdot\psi(c_{Y,i})]$, which are isomorphisms (the terms of the weight spectral sequences of X and Y are finite-dimensional).

Therefore, we have the following diagram in $\mathfrak{C}$

$$(C_*(X\times Y))^\vee \xrightarrow{u^\vee} (C_*(X)\otimes_{\mathbb{Z}_2} C_*(Y))^\vee \xleftarrow{w} (C_*(X))^\vee\otimes_{\mathbb{Z}_2}(C_*(Y))^\vee,$$

where the morphisms $u^\vee$ and w are filtered quasi-isomorphisms. Consequently, in the localization $Ho\,\mathfrak{C}$ of $\mathfrak{C}$ with respect to filtered quasi-isomorphisms, the filtered complexes $\mathcal{G}^\bullet C^*(X)\otimes\mathcal{G}^\bullet C^*(Y)$ and $\mathcal{G}^\bullet C^*(X\times Y)$ are isomorphic.

□

Remark 5.21. As for the homological case, a morphism between the filtered complexes $\mathcal{W}C^*(X)\otimes\mathcal{W}C^*(Y)$ and $\mathcal{W}C^*(X\times Y)$ can be obtained without using the geometric filtration. Indeed, in the previous proof, consider the canonical filtration in place of the geometric filtration : one can show in the same way that there is an isomorphism of $Ho\,\mathfrak{C}$ between $(F^{can})^\bullet_\vee C^*(X)\otimes(F^{can})^\bullet_\vee C^*(Y)$ and $(F^{can})^\bullet_\vee C^*(X\times Y)$. Since the dual canonical filtration and the cohomological canonical filtration are filtered quasi-isomorphic (see the proof of theorem 4.3), we deduce an isomorphism of $Ho\,\mathfrak{C}$ between $F^\bullet_{can}C^*(X)\otimes F^\bullet_{can}C^*(Y)$ and $F^\bullet_{can}C^*(X\times Y)$. Restricting this isomorphism to projective nonsingular varieties, we extend it to all real algebraic varieties in the same way as in proof of theorem 5.15 (see also remark 5.18) to obtain an isomorphism between $\mathcal{W}C^*(X)\otimes\mathcal{W}C^*(Y)$ and $\mathcal{W}C^*(X\times Y)$.

5.4 Cup product

Let X be a real algebraic variety.

We show below that the cup product on the cohomology with compact supports $H^*(X)$ of the set of real points of X is filtered with respect to the cohomological weight filtration. Precisely, we define a cup product on the cochain level in the derived category $Ho\,\mathfrak{C}$, on the cohomological geometric filtration, using the filtered quasi-isomorphisms w and $u^\vee$ defined above in the proof of 5.19, that induces a cup product on the cohomological weight spectral sequence of X and the usual cup product on the cohomology of X.

Let Δ denote the diagonal map

$$\Delta\ :\ \begin{array}{ccc} X & \longrightarrow & X\times X \\ x & \longmapsto & (x,x)\end{array}$$

Now consider the cohomological geometric filtration $\mathcal{G}^\bullet C^*(X)$ of X as an object of the derived category $Ho\,\mathfrak{C}$. We can apply the composition $\Delta^* \circ (u^\vee)^{-1} \circ w$ to the tensor product $\mathcal{G}^\bullet C^*(X) \otimes \mathcal{G}^\bullet C^*(X)$ ($u^\vee$ is an isomorphism of $Ho\,\mathfrak{C}$, see the proof of proposition 5.19) :

$$\Delta^* \circ (u^\vee)^{-1} \circ w : \mathcal{G}^\bullet C^*(X) \otimes \mathcal{G}^\bullet C^*(X) \xrightarrow{(u^\vee)^{-1} \circ w} \mathcal{G}^\bullet C^*(X \times X) \xrightarrow{\Delta^*} \mathcal{G}^\bullet C^*(X)$$

We denote this morphism of $Ho\,\mathfrak{C}$ by $\smile$.

Proposition 5.22.
The cup product

$$\smile : \mathcal{G}^\bullet C^*(X) \otimes \mathcal{G}^\bullet C^*(X) \longrightarrow \mathcal{G}^\bullet C^*(X)$$

in $Ho\,\mathfrak{C}$ induces a morphism of spectral sequences

$$\smile'_r : \bigoplus_{p+s=a,\ q+t=b} E_r^{p,q}(X) \otimes_{\mathbb{Z}_2} E_r^{s,t}(X) \longrightarrow E_r^{a,b}(X)$$

and the usual cup product

$$\begin{array}{ccc} H^*(X) \otimes_{\mathbb{Z}_2} H^*(X) & \overset{\smile}{\longrightarrow} & H^*(X) \\ \varphi \otimes \psi & \longmapsto & \varphi \smile \psi = [\Delta^*(\varphi \times \psi)] \end{array}.$$

In particular, the cup product in cohomology is a filtered map with respect to the cohomological weight filtration.

Proof. The first fact follows from the cohomological version of lemma 5.13, and the cup product in cohomology is the composition of Δ^* and the Künneth isomorphism in cohomology, which is itself induced by $(u^\vee)^{-1} \circ w$ (see proposition 5.19 and corollary 5.20). □

5.5 Cap product

In this section, we define a cap product on the homological and cohomological geometric filtrations considered in the corresponding derived categories $Ho\,\mathcal{C}$ and $Ho\,\mathfrak{C}$. This cap product on chain level induces a cap product on the homological and cohomological weight spectral sequences, showing that the cap product on homology and cohomology is a filtered morphism with respect to the homological and cohomological weight filtrations.

First, we give a filtered chain complex structure to the tensor product of a filtered cochain complex and a filtered chain complex :

Definition 5.23. Let (K^*, F) and (M_*, J) be respectively a filtered cochain complex of $\mathfrak{C}$ and a filtered chain complex of $\mathcal{C}$. We define $((K^* \otimes M_*)_*, F \otimes J)$ to be the chain complex given by

$$(K \otimes M)_n := \bigoplus_{j-i=n} K^i \otimes_{\mathbb{Z}_2} M_j,$$

equipped with the differential

$$\partial(x \otimes y) := dx \otimes y + x \otimes \partial y$$

and the bounded increasing filtration given by

$$(F \otimes J)_p (K \otimes M)_n = \bigoplus_{j-i=n} \sum_{b-a=p} F^a K^i \otimes J_b M_j$$

Considering the semialgebraic chain and cochain complexes $C_*(X)$ and $C^*(X)$ of X, implicitely equipped with the homological and cohomological geometric filtrations (for sake of readability), as objects of the respective derived categories $Ho\,\mathcal{C}$ and $Ho\,\mathfrak{C}$, we are going to define a cap product $C^*(X) \otimes C_*(X) \to C_*(X)$ in $Ho\,\mathcal{C}$.

First, let ω denote the morphism $C^*(X) = (C_*(X))^\vee \to (C^*(X) \otimes C_*(X))^\vee$ of $Ho\,\mathfrak{C}$ given by

$$\begin{array}{rcl} (C_{l-m}(X))^\vee & \longrightarrow & (C^m(X) \otimes C_l(X))^\vee \\ \psi & \longmapsto & [\varphi \otimes c \longmapsto (\psi \smile \varphi)(c)] \end{array}$$

The cap product that we define below will be obtained from the dual of this filtered morphism, in order to have a formula

$$\psi(\varphi \frown c) = (\psi \smile \varphi)(c) \tag{5.4}$$

on the chain level. We make precise what we mean by the dual filtered chain complex of a filtered cochain complex :

Definition 5.24. If $F^\bullet K^*$ is a filtered cochain complex of $\mathfrak{C}$, we define its dual filtered chain complex $F^\vee_\bullet K^\vee_*$ of $\mathcal{C}$ by

$$F^\vee_p K^\vee_q := \left\{ \eta \in K^\vee_q \mid \eta \equiv 0 \text{ on } F^{p+1} K^q \right\}.$$

Notice that, as in remark 4.1, we have the natural isomorphism of spectral sequences given by $E^r_{a,b}(F^\vee K^\vee) = \left(E_r^{a,b}(FK) \right)^\vee$.

Consider the dual filtered chain complexes $(C_*(X))^{\vee\vee}$ and $(C^*(X) \otimes C_*(X))^{\vee\vee}$ of $(C_*(X))^\vee$ and $(C^*(X) \otimes C_*(X))^\vee$ respectively. We have natural filtered morphisms $\nu : C_*(X) \to (C_*(X))^{\vee\vee}$ and $\mu : C^*(X) \otimes C_*(X) \to (C^*(X) \otimes C_*(X))^{\vee\vee}$, inducing the natural morphims $E^r_{a,b} \to \left(E^r_{a,b} \right)^{\vee\vee}$ on the spectral sequence level, which are isomorphisms from $r \geqslant 1$ (the terms of the spectral sequence are finite-dimensional from level one).

Therefore, the morphims ν and μ are quasi-isomorphisms of $\mathcal{C}$ and we can define the morphism

$$\nu^{-1} \circ \omega^\vee \circ \mu : C^*(X) \otimes C_*(X) \to C_*(X)$$

of $Ho\,\mathcal{C}$ given by

$$\begin{array}{rcl} C^m(X) \otimes C_l(X) & \longrightarrow & C_{l-m}(X) \\ \varphi \otimes c & \longmapsto & \varphi \frown c := \nu^{-1} \circ \omega^\vee \circ \mu(\varphi \otimes c) \end{array}$$

We denote it also by $\frown$ and we have :

Proposition 5.25. *The cap product on the geometric filtrations of X induces a cap product*

$$E_r^{p,q}(X) \otimes E^r_{s,t}(X) \longrightarrow E^r_{s-p,\ t-q}(X)$$

on the weight spectral sequences of X, and the usual cap product

$$H^*(X) \otimes H_*(X) \stackrel{\frown}{\longrightarrow} H_*(X)$$

on the homology and cohomology of X. In particular, the latter is a filtered morphism with respect to the weight filtrations (the filtration on the tensor product of cohomology and homology is defined in a way similar to definition 5.23).

Proof. Similarly to lemma 5.13, the term of level r and indices a, b of the spectral sequence induced by $C^*(X) \otimes C_*(X)$ is given by $\bigoplus_{s-p=a,\ t-q=b} E_r^{p,q}(X) \otimes E^r_{s,t}(X)$. Then the cap product on chains and cochains induces morphisms

$$\frown'_r\colon \bigoplus_{s-p=a,\ t-q=b} E_r^{p,q}(X) \otimes E^r_{s,t}(X) \longrightarrow E_r^{a,b}(X).$$

Now, notice that the formula 5.4 on the chain level induce that, if $\varphi \in E_r^{p,q}(X)$, $c \in E^r_{s,t}(X)$ and $\psi \in E_r^{p-s,q-t}(X)$ (then $\psi \smile \varphi \in E_r^{p,q}(X)$, which is isomorphic to $\left(E^r_{p,q}(X)\right)^\vee$), we have

$$\psi(\varphi \frown'_r c) = (\psi \smile'_r \varphi)(c).$$

Since the cup product on the cohomological weight spectral sequence induces the cup product on cohomology and because the cap product on cohomology and homology

$$H^m(X) \times H_l(X) \stackrel{\frown}{\longrightarrow} H_{l-m}(X)$$

is characterized by the formula

$$\psi(\varphi \frown c) = (\psi \smile \varphi)(c)$$

(if $\varphi \in H^m(X)$ and $c \in H_l(X)$, $\varphi \frown c$ is the unique element of $H_{l-m}(X)$ verifying this formula for all $\psi \in H^{l-m}(X)$), the cap product on the cohomological and homological weight spectral sequences induce the cap product on cohomology and homology. ☐

Remark 5.26.

1. If $\varphi \in E_r^{p,q}(X)$ and $c \in E^r_{s,t}(X)$ then $\varphi \frown'_r c$ is the unique element of $E^r_{s-p,\ t-q}(X)$ verifying
$$\psi(\varphi \frown'_r c) = (\psi \smile'_r \varphi)(c)$$
for all $\psi \in E_r^{p-s,q-t}(X)$.

2. Another possible definition for the cap product on the chain level is the following one (see [16]). Consider the morphism
$$h: \begin{array}{ccc} C^*(X) \otimes (C_*(X) \otimes C_*(X)) & \longrightarrow & C_*(X) \\ \varphi \otimes (a \otimes b) & \longmapsto & \varphi(a) \cdot b \end{array}.$$
Then we can also define the cap product on the cohomological and homological geometric filtrations of X (regarded as objects of $Ho\,\mathfrak{C}$ and $Ho\,\mathcal{C}$) by setting
$$\varphi \frown c := h(\varphi \otimes u^{-1}(\Delta_*(c))).$$
Notice that this definition would be valid with integer coefficients as well.

5.6 Weight filtrations and Poincaré duality map

Let X be a compact real algebraic variety of dimension n.

The semialgebraic chain $[X]$ is pure, that is $[X] \in \mathcal{G}_{-n}C_n(X)$. For $r \geqslant 1$, it induces homology classes in the weight spectral sequence terms $E^r_{-n,2n}(X)$.

By taking the cap product with $[X]$, we obtain a map D on the cohomological weight spectral sequence of X, given by :

$$D_r^{s,t} := \cdot \frown [X] \ : \ \begin{array}{ccc} E_r^{s,t}(X) & \longrightarrow & E^r_{-n-s,2n-t}(X) \\ \varphi & \longmapsto & \varphi \frown [X] \end{array} \tag{5.5}$$

Recall that the non-zero terms of the weight spectral sequences lie in the triangle given by the inequalities $t \geqslant -2s$, $s \leqslant 0$ and $t \leqslant -s+n$, the terms induced by the pure chains lying in the line $t = -2s$. Then if, for any $r \geqslant 1$, we consider the cap product of non-pure classes by $[X]$, it is identically zero. Indeed, for $t > -2s$, the term $E^r_{-n-s,2n-t}(X)$, where lie the values of $D_r^{s,t}$, is zero since $2n - t < -2(-n-s)$.

The map D on the cohomological weight spectral sequence induces, on the E^∞ and E_∞ level, the classical Poincaré duality map on the cohomology of X (that we denote again by D) given by

$$\begin{array}{ccc} H^k(X) & \longrightarrow & H_{n-k}(X) \\ \varphi & \longmapsto & \varphi \frown [X] \end{array}$$

($[X]$ corresponds here to the fundamental homology class of X) and :

Proposition 5.27. *For all p and k in $\mathbb{Z}$, the image of $\mathcal{W}^p H^k(X)$ by Poincaré duality map is in $\mathcal{W}_{-p-n}H_{n-k}(X)$:*

$$D(\mathcal{W}^p H^k(X)) \subset \mathcal{W}_{-p-n}H_{n-k}(X).$$

In particular, for all $k \in \mathbb{Z}$, $D(H^k(X)) \subset \mathcal{W}_{k-n}H_{n-k}(X)$ and, if $p > -k$, $D(\mathcal{W}^p H^k(X)) = 0$. In other words, all the non-pure cohomology classes are in the kernel of Poincaré duality map and the pure cohomology classes are the only classes which may be sent to a nonzero pure homology class by Poincaré duality map. Therefore, if its weight filtrations are not pure, a real algebraic variety does not satisfy Poincaré duality.

Remark 5.28. On the other hand, there exist varieties having pure weight filtration but not satisfying Poincaré duality.

For example, let X denote the pinched torus, obtained from a torus T by identifying a circle which generates it as a revolution surface to a point x_0. To compute its weight spectral sequence, we consider the cubical hyperresolution of X given by the blowing-up at x_0 :

$$\begin{array}{ccc} S^1 & \hookrightarrow & T \\ \downarrow & & \downarrow \\ \bullet & \hookrightarrow & X \end{array}$$

We obtain a pure weight filtration given by the term $\widetilde{E}^2 = \widetilde{E}^\infty$:

$$\widetilde{E}^2 = \left| \begin{array}{ccc} \mathbb{Z}_2 \cdot [X] & & \\ \mathbb{Z}_2 \cdot \overline{[a]} & 0 & \\ \mathbb{Z}_2 & 0 & 0 \end{array} \right.$$

(if $H_1(T) = \mathbb{Z}_2\overline{[a]} \oplus \mathbb{Z}_2\overline{[b]}$ with $b = S^1$ the exceptional divisor of the blowing-up). However, the variety X does not satisfy Poincaré duality since $\overline{[a]}^\vee \frown [X] = 0$.

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