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# Two dimensional deterministic model of a thin body with randomly distributed high conductivity fibers

G erard Michaille <sup>\*</sup>, Azdine Nait-ali<sup>†</sup> and St ephane Pagano<sup>‡</sup>

**Abstract** By using ergodic theory of subadditive processes and variational convergence, we study the macroscopic behavior of a thin 3-dimensional composite made up of high conductivity fibers which are randomly distributed according to a stochastic point process in a bounded open set of  $\mathbb{R}^3$ . The thickness of the body, the conductivity and the size of the cross sections of the fibers depend on a small parameter  $\varepsilon$ . The variational limit functional energy obtained when  $\varepsilon$  tends to 0 is deterministic and depends on two variables: one is the solution of a variational problem posed in a 2-dimensional bounded open set and describes the behavior of the medium, the other captures the limit behavior of suitably rescaled solutions in the fibers when the thickness and the size section become increasingly thin and the conductivity of the fibers becomes increasingly large.

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# 1 Introduction

By stochastic homogenization together with a reduction dimension variational process, we propose a two dimensional deterministic model of a randomly fibered composite occupying an open cylinder  $\mathcal{O}_{h(\varepsilon)} = \widehat{\mathcal{O}} \times (0, h(\varepsilon))$  of  $\mathbb{R}^3$ , whose basis is a domain  $\widehat{\mathcal{O}}$  of  $\mathbb{R}^2$  and thickness  $h(\varepsilon)$  goes to zero with  $\varepsilon$ . The random structure may be described as follows: consider the union of cylinders  $T_\varepsilon(\omega) := \varepsilon D(\omega) \times \mathbb{R}$  where  $D(\omega) := \bigcup_{i \in \mathbb{N}} D(\omega_i)$  and  $D(\omega_i)$  are disks distributed at random in  $\mathbb{R}^2$  following a stochastic point process  $\omega = (\omega_i)_{i \in \mathbb{N}}$  of  $\mathbb{R}^2$  associated with a suitable probability space  $(\Omega, \mathcal{A}, \mathbf{P})$ , then the random fibered structure  $\mathcal{O}_{h(\varepsilon)}$  is the union of the matrix  $\mathcal{O}_{h(\varepsilon)} \setminus T_\varepsilon(\omega)$  and the fibers  $\mathcal{O}_{h(\varepsilon)} \cap T_\varepsilon(\omega)$  (Figure 1). For short we sometimes drop  $\omega$  and for instance, write  $T_\varepsilon$  and  $D$  instead of  $T_\varepsilon(\omega)$  and  $D(\omega)$ .

Our objective is to provide a simplified model of the slices of a composite made up of thin fibers with large conductivity, randomly distributed into a matrix with conductivity of order one. This model concerns various steady-states situations like heat diffusion or electrostatic problems. A similar situation has been treated in [4, 5] for a composite with fixed thickness and in the case of a periodic or more general (but deterministic) distribution of very thin fibers. The limit problem obtained in these papers is non local and involves variational capacity theory. By contrast our limit problem is local with a zero-gradient density. Moreover, under some statistical hypothesis on the distribution of fibers, the density is deterministic.

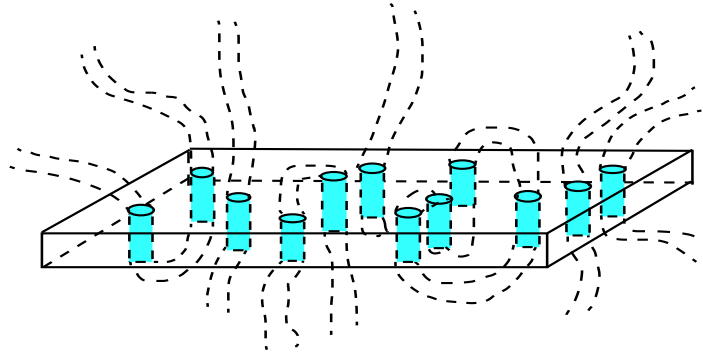


Figure 1: A slice of randomly fibered body of thickness  $h(\varepsilon)$

From the mathematical point of view, we consider the random variational problem  $(\mathcal{P}_\varepsilon(\omega))$

$$\inf_{u \in W_\varepsilon^{1,p}(\mathcal{O}_{h(\varepsilon)})} \left\{ \int_{\mathcal{O}_{h(\varepsilon)} \setminus T_\varepsilon} f(\nabla u) dx + \frac{1}{\varepsilon^\alpha} \int_{\mathcal{O}_{h(\varepsilon)} \cap T_\varepsilon} g(\nabla u) dx - \int_{\mathcal{O}_{h(\varepsilon)}} \mathcal{L}_\varepsilon \cdot u dx - \int_{\widehat{\mathcal{O}} \cap \varepsilon D \times \{h(\varepsilon)\}} l_\varepsilon \cdot u d\mathcal{H}^2 \right\}$$

where  $\mathcal{H}^2$  denotes the two-dimensional Hausdorff measure,  $f, g$  are two convex functions,

$$W_\varepsilon^{1,p}(\mathcal{O}_{h(\varepsilon)}) := \left\{ u \in W^{1,p}(\mathcal{O}_{h(\varepsilon)}) : u = 0 \text{ on } \widehat{\mathcal{O}} \cap \varepsilon D(\omega) \times \{0\} \right\},$$

and  $u$  in the last integral is written for the trace of  $u$  on  $\widehat{\mathcal{O}} \cap \varepsilon D \times \{h(\varepsilon)\}$ , and  $u = 0$  on  $(\widehat{\mathcal{O}} \cap \varepsilon D(\omega)) \times \{0\}$  may be understood in the sense of the traces on  $(\widehat{\mathcal{O}} \cap \varepsilon D(\omega)) \times \{0\}$ . The coefficient  $\frac{1}{\varepsilon^\alpha}$  stands for the high conductivity of the fibers.

We consider the case where  $h(\varepsilon) = \varepsilon^p$  and a scaling where the energy per thickness becomes infinite with rate  $\varepsilon^{-p}$ . It is worth noticing that typically, for dimension reduction problems ones consider a scaling where the energy per thickness is of order one or vanishes. This leads to a problem which is totally different from the one studied in the present paper. More precisely, denoting by  $\bar{u}_\varepsilon(\omega, \cdot) : \mathcal{O}_{\varepsilon^p} \rightarrow \mathbb{R}$  the random minimizer of  $(\mathcal{P}_\varepsilon(\omega))$  which is subjected to a body source  $\mathcal{L}_\varepsilon$ , is null on the lower sections of the fibers, and subjected to a surface source  $l_\varepsilon$  on the upper sections of the fibers, we intend to study the behavior of  $\bar{u}_\varepsilon(\omega, \cdot)$  and  $\mathbb{1}_{\mathcal{O} \cap T_\varepsilon} \bar{u}_\varepsilon(\omega, \cdot)$  where  $\bar{u}_\varepsilon(\omega, \cdot)$  is defined in  $\mathcal{O} := \widehat{\mathcal{O}} \times (0, 1)$  by  $\bar{u}_\varepsilon(\omega, x) = \bar{u}_\varepsilon(\omega, \hat{x}, \varepsilon^p x_3)$ . We will see thereafter that the condition  $u(x) = 0$  on  $(\widehat{\mathcal{O}} \cap \varepsilon D(\omega)) \times \{0\}$  can be generalized by  $u(x) = u_0$  on  $(\widehat{\mathcal{O}} \cap \varepsilon D(\omega)) \times \{0\}$  with  $u_0 \in W^{1,p}(\widehat{\mathcal{O}})$

(see Corollary 1.3). Moreover all our results hold with these two boundary conditions on the lower and upper sections  $\widehat{\mathcal{O}} \times \{0\}$  and  $\widehat{\mathcal{O}} \times \{\varepsilon^p\}$ .

We assume that the sources  $\mathcal{L}_\varepsilon$  and  $l_\varepsilon$  satisfy the following behavior: there exist  $L$  in  $L^q(\mathcal{O})$ ,  $l$  in  $L^q(\widehat{\mathcal{O}})$ ,  $q = \frac{p}{p-1}$ , and  $b$  in  $\mathbb{R}$  such that

$$\begin{cases} \mathcal{L}_\varepsilon(x) = \varepsilon^{-p} L(\hat{x}, \varepsilon^{-p} x_3) \text{ for } x \text{ in } \mathcal{O}_{\varepsilon^p}, \\ l_\varepsilon(\hat{x}) = \varepsilon^{-b} l(\hat{x}) \text{ for } \hat{x} \text{ in } \widehat{\mathcal{O}} \cap \varepsilon D. \end{cases} \quad (1.1)$$

Note that, according to the choice of  $\mathcal{L}_\varepsilon$  in (1.1), the surface limit  $\widehat{\mathcal{O}}$  of the layer  $\mathcal{O}_{\varepsilon^p}$  is submitted to a finite source with density  $\int_0^1 L(\hat{x}, t) dt$ . In what follows we assume  $a > 0$ ,  $p > 1$ ,  $b \leq p - 1 + \frac{a}{p}$ . For carrying out this analysis, we will determine the variational limit of the rescaled energy  $E_\varepsilon$  defined in  $L^p(\mathcal{O})$  by

$$E_\varepsilon(\omega, u) = H_\varepsilon(\omega, u) - \int_{\mathcal{O}} L.u \, dx - \varepsilon^{-b} \int_{\widehat{\mathcal{O}} \cap \varepsilon D \times \{1\}} l.u \, d\mathcal{H}^2.$$

where

$$H_\varepsilon(\omega, u) = \begin{cases} \varepsilon^p \int_{\mathcal{O} \setminus T_\varepsilon} f(\widehat{\nabla} u, \frac{1}{\varepsilon^p} \frac{\partial u}{\partial x_3}) \, dx + \varepsilon^{p-a} \int_{\mathcal{O} \cap T_\varepsilon} g(\widehat{\nabla} u, \frac{1}{\varepsilon^p} \frac{\partial u}{\partial x_3}) \, dx & \text{if } u \in W_\varepsilon^{1,p}(\mathcal{O}) \\ +\infty & \text{otherwise,} \end{cases}$$

and  $W_\varepsilon^{1,p}(\mathcal{O}) := \left\{ u \in W^{1,p}(\mathcal{O}) : u = 0 \text{ on } \widehat{\mathcal{O}} \cap \varepsilon D(\omega) \times \{0\} \right\}$ .

Let us denote by  $\widehat{Y}$  the unit cell of  $\mathbb{R}^2$ , by  $f^{\infty,p}$  the  $p$ -recession function of the function  $f$  and, for all  $\lambda \in \mathbb{R}^2$ , set  $\widehat{f^{\infty,p}}(\lambda) := \inf_{\xi \in \mathbb{R}} f^{\infty,p}(\lambda, \xi)$ . For all  $s \in \mathbb{R}$ , we define  $f_0(s)$  by

$$f_0(s) := \lim_{n \rightarrow +\infty} \inf_{w \in W_0^{1,p}(n\widehat{Y} \setminus D(\omega))} \left\{ \int_{n\widehat{Y}} \widehat{f^{\infty,p}}(\nabla w) \, d\hat{x} : \int_{n\widehat{Y}} w \, d\hat{x} = s \right\}.$$

The existence of this limits holds for  $\mathbf{P}$ -almost every  $\omega$  in  $\Omega$  under certain conditions on the probability space  $(\Omega, \mathcal{A}, P)$  specified further in Section 3 (see Lemma 1.1 below where the properties of  $f_0$  are summarized). Finally, denoting by  $g^{\infty,p}$  the  $p$ -recession of  $g$  we set  $(g^{\infty,p})^\perp(s) = g^{\infty,p}(0, s)$  (the growth conditions fulfilled by  $f$  and  $g$  and the definition of the  $p$ -recession functions are specified in the next section). We define the deterministic functional  $H_0$  defined in  $L^p(\mathcal{O}) \times L^p(\mathcal{O})$  by

$$H_0(u, v) = \begin{cases} \int_{\widehat{\mathcal{O}}} f_0(u) \, d\hat{x} + \theta^{1-p} \int_{\mathcal{O}} (g^{\infty,p})^\perp\left(\frac{\partial v}{\partial x_3}\right) dx & \text{if } (u, v) \in L^p(\mathcal{O}) \times V_0(\mathcal{O}) \\ +\infty & \text{otherwise,} \end{cases}$$

where  $V_0(\mathcal{O}) := \left\{ v \in L^p(\mathcal{O}) : \frac{\partial v}{\partial x_3} \in L^p(\mathcal{O}), v(\hat{x}, 0) = 0 \right\}$ , and the coefficient  $\theta$ , namely the asymptotic volume fraction of the fibers, is introduced in Section 3, Definition 3.1. We consider two cases for which the limit functional  $E_0$  differs following the value of  $b$  compared to  $\gamma := p - 1 + \frac{a}{p}$ :

. **Case (C<sub>1</sub>):**  $b = \gamma$

$$E_0(u, v) = \begin{cases} H_0(u, v) - \int_{\mathcal{O}} L.u \, dx - \int_{\widehat{\mathcal{O}}} l.v \, d\hat{x} & \text{if } (u, v) \in L^p(\mathcal{O}) \times V_0(\mathcal{O}) \\ +\infty & \text{otherwise.} \end{cases}$$

. **Case (C<sub>2</sub>):**  $b < \gamma$

$$E_0(u, v) = \begin{cases} H_0(u, v) - \int_{\mathcal{O}} L.u \, dx & \text{if } (u, v) \in L^p(\mathcal{O}) \times V_0(\mathcal{O}) \\ +\infty & \text{otherwise.} \end{cases}$$

Let us introduce the following convenient notation for any sequence  $(u_\varepsilon)_{\varepsilon>0}$  in  $L^p(\mathcal{O})$  and any  $(u, v)$  in  $L^p(\mathcal{O}) \times L^p(\mathcal{O})$

$$u_\varepsilon \rightharpoonup (u, v) \iff \begin{cases} u_\varepsilon \rightharpoonup u \text{ in } L^p(\mathcal{O}) \\ \varepsilon^{-\gamma} \mathbf{1}_{D(\omega) \cap \hat{\mathcal{O}}} u_\varepsilon \rightharpoonup v \text{ in } L^p(\mathcal{O}), \end{cases}$$

then our main result is

**Theorem 1.1.** *Consider a sequence  $(u_\varepsilon)_{\varepsilon>0}$  in  $L^p(\mathcal{O})$  of bounded energy, i.e., satisfying for  $\mathbf{P}$  a.s.  $\omega \in \Omega$ ,  $\sup_{\varepsilon>0} E_\varepsilon(\omega, u_\varepsilon) < +\infty$ . Then, for  $\mathbf{P}$  a.s.  $\omega \in \Omega$ , there exist a subsequence possibly depending on  $\omega$  and  $(u, v) \in L^p(\mathcal{O}) \times V_0(\mathcal{O})$  possibly depending on  $\omega$  such that :*

$$u_\varepsilon \rightharpoonup u \text{ in } L^p(\mathcal{O}), \text{ and } \frac{\partial u}{\partial x_3} = 0; \quad (1.2)$$

$$\varepsilon^{-\gamma} \mathbf{1}_{D(\omega) \cap \hat{\mathcal{O}}} (\frac{\cdot}{\varepsilon}) u_\varepsilon \rightharpoonup v \text{ in } L^p(\mathcal{O}); \quad (1.3)$$

$$\varepsilon^{-\gamma} \mathbf{1}_{D(\omega) \cap \hat{\mathcal{O}}} (\frac{\cdot}{\varepsilon}) \frac{\partial u_\varepsilon}{\partial x_3} \rightharpoonup \frac{\partial v}{\partial x_3} \text{ in } L^p(\mathcal{O}). \quad (1.4)$$

Furthermore the sequence of functionals  $E_\varepsilon$  almost surely converges to the functional  $E_0$  in the following sense: there exists  $\Omega' \in \mathcal{A}$  with  $\mathbf{P}(\Omega') = 1$  such that for all  $\omega \in \Omega'$  one has

i) for all  $(u, v) \in L^p(\mathcal{O}) \times V_0$  and for all sequences  $(u_\varepsilon)_{\varepsilon>0}$  in  $L^p(\mathcal{O})$  such that  $u_\varepsilon \rightharpoonup (u, v)$ , then  $\liminf_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon) \geq E_0(u, v)$ ;

ii) for all  $(u, v) \in L^p(\mathcal{O}) \times V_0$ , there exists a sequence  $(u_\varepsilon)_{\varepsilon>0}$  in  $L^p(\mathcal{O})$  such that  $u_\varepsilon \rightharpoonup (u, v)$  and  $\limsup_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon) \leq E_0(u, v)$

**Corollary 1.1.** *Let denote by  $\bar{u}_\varepsilon(\omega, \cdot)$  the function  $x \mapsto \bar{u}_\varepsilon(\omega, \hat{x}, \varepsilon^p x_3)$ , where  $\bar{u}_\varepsilon(\omega, \cdot)$  is the solution of  $(\mathcal{P}_\varepsilon(\omega))$ . Then almost surely there exists a subsequence of  $(\bar{u}_\varepsilon(\omega, \cdot))_{\varepsilon>0}$  such that  $\bar{u}_\varepsilon(\omega, \cdot) \rightharpoonup \bar{u}$  in  $L^p(\mathcal{O})$  with for a.e.  $\hat{x} \in \hat{\mathcal{O}}$ ,*

$$\bar{u}(\hat{x}) \in \partial f_0^*(\bar{L})$$

where  $\bar{L}(\hat{x}) = \int_0^1 L_1(\hat{x}, t) dt$ , and  $f_0^*$  is the Fenchel transform of the convex function  $f_0$ . Consequently if  $\partial f_0^*$  is single valued then almost surely all the sequence  $(\bar{u}_\varepsilon(\omega, \cdot))_{\varepsilon>0}$  weakly converges in  $L^p(\mathcal{O})$  to  $\bar{u}$  defined for a.e.  $\hat{x} \in \hat{\mathcal{O}}$  by

$$\bar{u}(\hat{x}) = \partial f_0^*(\bar{L}). \quad (1.5)$$

Assume that  $(g^{\infty, p})^\perp$  is differentiable, then almost surely  $\varepsilon^{-\gamma} \mathbf{1}_{T_\varepsilon \cap \mathcal{O}} \bar{u}_\varepsilon(\omega, \cdot)$  and  $\varepsilon^{-\gamma} \mathbf{1}_{T_\varepsilon \cap \mathcal{O}} \frac{\partial \bar{u}_\varepsilon(\omega, \cdot)}{\partial x_3}$  weakly converge to  $\bar{v}$  and  $\frac{\partial \bar{v}}{\partial x_3}$  in  $L^p(\mathcal{O})$  respectively, where  $\bar{v}$  is the unique solution to

$$\begin{cases} -\frac{\partial}{\partial x_3} \left( \frac{d(g^{\infty, p})^\perp}{ds} \left( \frac{\partial v}{\partial x_3} \right) \right) = 0 \text{ in } \mathcal{O}, \\ v(\hat{x}, 0) = 0 \text{ on } \hat{\mathcal{O}} \times \{0\}, \\ D(g^{\infty, p})^\perp \left( \frac{\partial v}{\partial x_3} \right) = \theta^{p-1} \tilde{l} \text{ on } \hat{\mathcal{O}} \times \{1\}. \end{cases}$$

where  $\tilde{l} = \begin{cases} l \text{ when } b = \gamma \\ 0 \text{ if } b < \gamma. \end{cases}$

By eliminating the function  $v$  regarded as an internal variable, from Theorem 1.1 we easily deduce an almost sure  $\Gamma$ -convergence process when  $L^p(\mathcal{O})$  is equipped with its weak convergence (for the main properties of the  $\Gamma$ -convergence, we refer the reader to [2, 8]). Precisely we have

**Corollary 1.2.** *The sequence of energies  $E_\varepsilon(\omega, \cdot)$  almost surely  $\Gamma$ -converges to the zero-gradient energy functional  $\tilde{E}_0(u) := \inf \{E_0(u, v) : v \in V_0\}$  which is explicitly given in  $L^p(\hat{\mathcal{O}})$  by*

$$\tilde{E}_0(u) = \begin{cases} \int_{\hat{\mathcal{O}}} f_0(u) d\hat{x} - \int_{\hat{\mathcal{O}}} u \cdot \bar{L} d\hat{x} + G_0(\bar{v}) - \int_{\hat{\mathcal{O}}} l \cdot \bar{v} d\hat{x} & \text{if } b = \gamma, \\ \int_{\hat{\mathcal{O}}} f_0(u) d\hat{x} - \int_{\hat{\mathcal{O}}} u \cdot \bar{L} d\hat{x} & \text{if } b < \gamma. \end{cases}$$

More generally, if the conductivity of the fibers is not too high ( $a < p$ ) and if  $b < 0$  or  $l = 0$  we obtain a deterministic energy in the case when  $u(x) = u_0$  on  $(\hat{\mathcal{O}} \cap \varepsilon D) \times \{0\}$  where  $u_0$  is a given function in  $W^{1,p}(\hat{\mathcal{O}})$  (we can also extend this boundary condition to the base  $\hat{\mathcal{O}} \times \{0\}$ ). Indeed let  $\tilde{u} := u - u_0$ , then the energy  $E_\varepsilon$  becomes

$$\begin{aligned} \tilde{E}_\varepsilon(\omega, \tilde{u}) &:= \varepsilon^p \int_{\mathcal{O} \setminus T_\varepsilon} f(\hat{\nabla} \tilde{u} + \hat{\nabla} u_0, \frac{1}{\varepsilon^p} \frac{\partial \tilde{u}}{\partial x_3}) dx + \varepsilon^{p-a} \int_{\mathcal{O} \cap T_\varepsilon} g(\hat{\nabla} \tilde{u} + \hat{\nabla} u_0, \frac{1}{\varepsilon^p} \frac{\partial \tilde{u}}{\partial x_3}) dx \\ &\quad - \int_{\mathcal{O}} L \cdot \tilde{u} dx - \int_{\mathcal{O}} L \cdot u_0 dx - \varepsilon^{-b} \int_{(\hat{\mathcal{O}} \cap \varepsilon D) \times \{1\}} l \cdot \tilde{u} d\mathcal{H}^2 - \varepsilon^{-b} \int_{(\hat{\mathcal{O}} \cap \varepsilon D) \times \{1\}} l \cdot u_0 d\mathcal{H}^2. \end{aligned}$$

Moreover from (2.2) and since  $a < p$  we have

$$\begin{aligned} \tilde{E}_\varepsilon(\omega, \tilde{u}) &\approx \varepsilon^p \int_{\mathcal{O} \setminus T_\varepsilon} f(\hat{\nabla} \tilde{u}, \frac{1}{\varepsilon^p} \frac{\partial \tilde{u}}{\partial x_3}) dx + \varepsilon^{p-a} \int_{\mathcal{O} \cap T_\varepsilon} g(\hat{\nabla} \tilde{u}, \frac{1}{\varepsilon^p} \frac{\partial \tilde{u}}{\partial x_3}) dx \\ &\quad - \int_{\mathcal{O}} L \cdot \tilde{u} dx - \int_{\mathcal{O}} L \cdot u_0 dx - \varepsilon^{-b} \int_{(\hat{\mathcal{O}} \cap \varepsilon D) \times \{1\}} l \cdot \tilde{u} d\mathcal{H}^2 - \varepsilon^{-b} \int_{(\hat{\mathcal{O}} \cap \varepsilon D) \times \{1\}} l \cdot u_0 d\mathcal{H}^2, \end{aligned}$$

where  $\tilde{u} = 0$  on  $\hat{\mathcal{O}} \cap \varepsilon D(\omega) \times \{0\}$ . Then we have

**Corollary 1.3.** *Assume that  $a < p$  and  $b \leq 0$  or  $l = 0$ . The sequence of energy  $\tilde{E}_\varepsilon$  converges almost surely to  $\tilde{E}_0$  in the sense of Theorem 1.1 where  $\tilde{E}_0$  is defined in  $L^p(\mathcal{O}) \times V_0(\mathcal{O})$  by*

$$\tilde{E}_0(u, v) = E_0(u, v) - \int_{\hat{\mathcal{O}}} \bar{L} \cdot u_0 d\hat{x},$$

and  $E_0$  is the limit energy described previously. In this case, the admissible functions are given by  $u + u_0$ .

Let us clarify the limit density energy  $f_0$  by considering a suitable discrete subadditive process on the probabilistic space  $(\Omega, \mathcal{A}, \mathbf{P})$ . (see Section 3 for the definition of  $(\Omega, \mathcal{A}, \mathbf{P})$  and the group  $(\tau_z)_{z \in \mathbf{Z}^2}$  acting on  $(\Omega, \mathcal{A}, \mathbf{P})$ ). Let denote by  $\mathring{\mathcal{I}}$  the set of all open intervals  $(a, b)$  with  $a$  and  $b$  in  $\mathbf{Z}^2$ . For all  $\hat{A} \in \mathring{\mathcal{I}}$  and all  $s$  in  $\mathbb{R}$  set

$$\begin{aligned} \mathbb{S}_{\hat{A}}(\omega, s) &:= \inf \left\{ \int_{\hat{A} \setminus \overline{D(\omega)}} \widehat{f^{\infty,p}}(\nabla w(\hat{x})) d\hat{x} : w \in \text{Adm}_{\hat{A}}(\omega, s) \right\}, \\ \text{Adm}_{\hat{A}}(\omega, s) &:= \{w \in W_0^{1,p}(\hat{A} \setminus \overline{D(\omega)}) : \int_{\hat{A}} w dx = s\}. \end{aligned}$$

The following lemma which summarizes the properties of the subadditive process by which we define  $f_0$ , is crucial for establishing the proof of Theorem 1.1. Its proof and various definitions and results related to subadditive processes are postponed in appendix.

**Lemma 1.1.** *For all fixed  $s$  in  $\mathbb{R}$  the map*

$$\begin{aligned} \mathbb{S}(\cdot, s) &: \quad \mathring{\mathcal{I}} \longrightarrow L^1(\Omega, \mathcal{A}, \mathbf{P}) \\ &\quad \hat{A} \longmapsto \mathbb{S}_{\hat{A}}(\cdot, s) \end{aligned}$$

is a subadditive process with respect to the group  $(\tau_z)_{z \in \mathbf{Z}^2}$ . For  $\delta > 0$  small enough, It satisfies for all  $s \in \mathbb{R}$ , all  $\hat{A} \in \mathring{\mathcal{I}}$  and all  $\omega \in \Omega$

$$\mathbb{S}_{\hat{A}}(\omega, s) \leq C(p) \frac{|s|^p}{\delta^p \left| (\hat{Y} \setminus D(\bar{\omega}))_{2\delta} \right|^p} |\hat{A}| \quad (1.6)$$

where  $C(p)$  is a positive constant that depends on  $p$ , and  $\bar{\omega}$  is the family of centers of the hexagonal close-packing distribution of disks with radius  $\frac{d}{2}$  in  $\mathbb{R}^2$  (cf Remark 3.1).

Therefore for any regular family  $(I_n)_{n \in \mathbb{N}}$  of sets in  $\overset{\circ}{\mathcal{I}}$  the limit  $f_0(s) := \lim_{n \rightarrow \infty} \frac{\mathbb{S}_{I_n}(\omega, s)}{|I_n|}$  exists for  $\mathbf{P}$  almost every  $\omega \in \Omega$  and

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\mathbb{S}_{I_n}(\omega, s)}{|I_n|} &= \inf_{m \in \mathbb{N}^*} \left\{ \mathbf{E} \frac{\mathbb{S}_{[0, m]^2}(\cdot, s)}{m^2} \right\} \\ &= \lim_{n \rightarrow +\infty} \left\{ \mathbf{E} \frac{\mathbb{S}_{[0, n]^2}(\cdot, s)}{n^2} \right\}. \end{aligned}$$

The so defined function  $s \mapsto f_0(s)$  is convex, positively homogeneous of degree  $p$ , satisfies the growth conditions (2.4) with the same constant  $\alpha$ , with a constant  $\beta$  possibly different, and the Lipschitz condition (2.5) with a constant  $\ell$  possibly different.

In the periodic case, one can show that  $f_0$  reduces to a simple expression. The proof of the proposition below is postponed in Section 4.

**Proposition 1.1.** *When the fibers are periodically distributed, for all  $s \in \mathbb{R}$  we have*

$$f_0(s) = \inf \left\{ \int_{\hat{Y}} \widehat{f^{\infty, p}}(\nabla w) \, d\hat{y} : w \in W_{\#}^{1, p}(\hat{Y}), \int_{\hat{Y}} w \, d\hat{y} = s, w = 0 \text{ on } D \right\}$$

where  $W_{\#}^{1, p}(\hat{Y})$  is the set of  $\hat{Y}$ -periodic functions of  $W_{loc}^{1, p}(\mathbb{R}^2)$ .

Taking  $f = g = \frac{1}{2}|\cdot|^2$ , the problem  $(\mathcal{P}_{\varepsilon}(\omega))$  may be formulated in terms of partial differential equations by the following random Dirichlet-Neumann problem which derives from the standard computation of the Euler equation of  $(\mathcal{P}_{\varepsilon, h(\varepsilon)}(\omega))$ :

$$\left\{ \begin{array}{l} -\operatorname{div}(a_{\varepsilon}(\omega, x) \nabla \bar{u}_{\varepsilon}) = \mathcal{L}_{\varepsilon} \text{ in } \mathcal{O}_{h(\varepsilon)}, \\ \bar{u}_{\varepsilon} = 0 \text{ on } \mathcal{O}_{h(\varepsilon)} \cap D_{\varepsilon}(\omega) \times \{0\} \\ \varepsilon^{-a} \frac{\partial}{\partial x_3} g(\nabla \bar{u}_{\varepsilon}) = l_{\varepsilon} \text{ on } \mathcal{O}_{h(\varepsilon)} \cap (D_{\varepsilon}(\omega) \times \{h(\varepsilon)\}), \\ \frac{\partial \bar{u}_{\varepsilon}}{\partial \nu} = 0 \text{ on } \partial \mathcal{O} \setminus \left( \mathcal{O}_{h(\varepsilon)} \cap (D_{\varepsilon}(\omega) \times \{0\}) \right) \cup \left( (\mathcal{O}_{h(\varepsilon)} \cap (D_{\varepsilon}(\omega) \times \{h(\varepsilon)\})) \right). \end{array} \right.$$

where  $a_{\varepsilon}(\omega, x) = 1$  if  $x \in \mathcal{O}_{h(\varepsilon)} \setminus T_{\varepsilon}$ ,  $a_{\varepsilon}(\omega, x) = \frac{1}{\varepsilon^a}$  if  $\mathcal{O}_{h(\varepsilon)} \cap T_{\varepsilon}$  and  $\nu$  is the unit outer normal vector to the boundary  $\partial \mathcal{O}$  of  $\mathcal{O}$ . By an elementary computation one can show that (1.5) reduces to

$$\bar{u}(\hat{x}) = \Lambda \int_0^1 L(\hat{x}, t) \, dt$$

where  $\Lambda$  is defined as follows: consider  $U_n(\omega, \cdot)$  solution to the random Dirichlet problem

$$\left\{ \begin{array}{l} -\Delta U = 1 \text{ in } n\hat{Y} \setminus D(\omega), \\ U \in W_0^{1, 2}(n\hat{Y} \setminus D(\omega)) \end{array} \right.$$

and set  $\Lambda_n(\omega) := \int_{n\hat{Y}} U_n(\omega, \cdot) \, d\hat{x}$ , then one can show that  $\Lambda_n(\omega)$  almost surely converges when  $n$  tends to  $+\infty$  to a deterministic limit that we denote by  $\Lambda$ . The proof is established in Section 5.2, Proposition 5.1. It is interesting to note that  $\bar{u}$  is also the weak limit in  $L^2(\mathcal{O})$  of  $\frac{1}{\varepsilon} u_{\varepsilon}$  where  $u_{\varepsilon}$  is the solution of the random Dirichlet problem

$$\left\{ \begin{array}{l} -\varepsilon^2 \Delta u_{\varepsilon} = \int_0^1 L(\hat{x}, t) \, dt \text{ in } \hat{\mathcal{O}} \setminus \varepsilon D(\omega), \\ u_{\varepsilon} \in W_0^{1, 2}(\hat{\mathcal{O}} \setminus \varepsilon D(\omega)) \end{array} \right.$$

i.e.,  $\bar{u}$  follows a scalar Darcy's law (see [7]). Some numerical experiments are carried out in Section 5.2.

## 2 Functional analysis setting

We are given two strictly convex functions  $f$  and  $g$  defined on  $\mathbb{R}^3$  satisfying the standard growth condition of order  $p > 1$ : there exist two positive constants  $\alpha, \beta$ , such that  $\forall \zeta \in \mathbb{R}^3$

$$\alpha|\zeta|^p \leq f(\zeta) \leq \beta(1 + |\zeta|^p), \quad (2.1)$$

idem for  $g$ . It is well known that  $f$  satisfies automatically the Lipschitz property

$$|f(\zeta) - f(\zeta')| \leq \ell|\zeta - \zeta'|(1 + |\zeta|^{p-1} + |\zeta'|^{p-1}) \quad (2.2)$$

for all  $(\zeta, \zeta') \in \mathbb{R}^3 \times \mathbb{R}^3$  where  $\ell$  is a positive constant; idem for  $g$ . Furthermore, we assume that there exist  $\beta' > 0$ ,  $0 < r < p$  and a  $p$ -positively homogeneous function  $f^{\infty,p}$  (the  $p$ -recession function of  $f$ ) such that for all  $\zeta \in \mathbb{R}^3$

$$|f(\zeta) - f^{\infty,p}(\zeta)| \leq \beta'(1 + |\zeta|^{p-r}). \quad (2.3)$$

From (2.3) we infer  $\lim_{t \rightarrow +\infty} \frac{f(t\zeta)}{t^p} = f^{\infty,p}(\zeta)$  so that from (2.1),  $f^{\infty,p}$  satisfies for all  $\zeta \in \mathbb{R}^3$

$$\alpha|\zeta|^p \leq f^{\infty,p}(\zeta) \leq \beta|\zeta|^p \quad (2.4)$$

and

$$|f^{\infty,p}(\zeta) - f^{\infty,p}(\zeta')| \leq \ell|\zeta - \zeta'|(|\zeta|^{p-1} + |\zeta'|^{p-1}) \quad (2.5)$$

for all  $(\zeta, \zeta') \in \mathbb{R}^3 \times \mathbb{R}^3$ . Finally for all  $\lambda \in \mathbb{R}^2$  we set  $\widehat{f^{\infty,p}}(\lambda) := \inf_{\xi \in \mathbb{R}} f^{\infty,p}(\lambda, \xi)$  which clearly defines a convex function  $\widehat{f^{\infty,p}}$  in  $\mathbb{R}^2$ .

We define the  $p$ -recession function  $g^{\infty,p}$  of  $g$  as in (2.3) and, for all  $s$  in  $\mathbb{R}$ , we assume that

$$\begin{aligned} (g^{\infty,p})^\perp(s) &:= \inf_{\lambda \in \mathbb{R}^2} g^{\infty,p}(\lambda, s) \\ &= g^{\infty,p}(0, s). \end{aligned}$$

Note that  $(g^{\infty,p})^\perp$  is a convex function.

In the proof of Theorem 1.1 we sometimes consider separately the following functionals

$$F_\varepsilon(\omega, u) = \begin{cases} \varepsilon^p \int_{\mathcal{O} \setminus T_\varepsilon} f(\hat{\nabla} u, \frac{1}{\varepsilon^p} \frac{\partial u}{\partial x_3}) dx & \text{if } u \in W_\varepsilon^{1,p}(\mathcal{O}) \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$G_\varepsilon(\omega, u) = \begin{cases} \varepsilon^{p-a} \int_{\mathcal{O} \cap T_\varepsilon} g(\hat{\nabla} u, \frac{1}{\varepsilon^p} \frac{\partial u}{\partial x_3}) dx & \text{if } u \in W_\varepsilon^{1,p}(\mathcal{O}) \\ +\infty & \text{otherwise,} \end{cases}$$

so that  $H_\varepsilon(\omega, \cdot) = F_\varepsilon(\omega, \cdot) + G_\varepsilon(\omega, \cdot)$  in  $L^p(\mathcal{O})$ .

Similarly we will consider the two functionals defined in  $L^p(\mathcal{O})$  by

$$F_0(u) = \int_{\mathcal{O}} f_0(u) d\hat{x}$$

and

$$G_0(u) = \begin{cases} \theta^{1-p} \int_{\mathcal{O}} (g^{\infty,p})^\perp(\frac{\partial v}{\partial x_3}) dx & \text{if } v \in V_0(\mathcal{O}) \\ +\infty & \text{otherwise,} \end{cases}$$

so that  $H_0(u, v) = F_0(u) + G_0(v)$  in  $L^p(\mathcal{O}) \times V_0(\mathcal{O})$ .



### 3 Probabilistic setting

For all  $x = (x_1, x_2, x_3)$  of  $\mathbb{R}^3$ ,  $\hat{x}$  stands for  $(x_1, x_2)$  and we denote by  $\hat{Y}$  the unit cell  $(0, 1)^2$  of  $\mathbb{R}^2$ . For any bounded Borel set  $A$  of  $\mathbb{R}^2$  or  $\mathbb{R}^3$ ,  $|A|$  denotes its Lebesgue measure.

We go back to the probabilistic model suggested in [12]. Let  $d$  be a given number satisfying  $0 < d \leq 1$  and consider the set

$$\Omega = \{(\omega_i)_{i \in \mathbb{N}} : \omega_i \in \mathbb{R}^2, |\omega_i - \omega_j| \geq d \text{ for } i \neq j\}$$

equipped with the trace  $\sigma$ -algebra  $\mathcal{A}$  of the standard product  $\sigma$ -algebra on  $\Omega$ . Let  $\hat{B}_{d/2}(0)$  denote the open ball of  $\mathbb{R}^2$  centered at 0 with radius  $d/2$ , then for every  $\omega = (\omega_i)_{i \in \mathbb{N}}$  we form the disk  $D(\omega_i) := \omega_i + \hat{B}_{d/2}(0)$  and consider  $D(\omega) := \bigcup_{i \in \mathbb{N}} D(\omega_i)$ . Therefore  $\omega \mapsto T(\omega) = D(\omega) \times \mathbb{R}$  is a random set in  $\mathbb{R}^3$ , union of random cylinders, whose basis is the union of the pairwise disjoint disks  $D(\omega_i)$  of  $\mathbb{R}^2$  centered at  $\omega_i$ . We set  $T_\varepsilon(\omega) := \varepsilon D(\omega) \times \mathbb{R}$ . For every  $z \in \mathbf{Z}^2$  we define the operator  $\tau_z : \Omega \rightarrow \Omega$  by  $(\tau_z \omega)_i = \omega_i - z$ . Note that  $D(\tau_z \omega) = D(\omega) - z$ .

We assume that there exists a probability measure on  $(\Omega, \mathcal{A})$  which satisfies the system of three following axioms:

(A<sub>1</sub>) *The sections are non sparsely distributed:*  $\mathbf{P}\left(\left\{\omega \in \Omega : |\hat{Y} \cap D(\omega)| > 0\right\}\right) = 1$ ;

(A<sub>2</sub>) *Stationarity condition:*  $\forall z \in \mathbf{Z}^2, \tau_z \# \mathbf{P} = \mathbf{P}$  where  $\tau_z \# \mathbf{P}$  denotes image measure of  $\mathbf{P}$  by  $\tau_z$ ;

(A<sub>3</sub>) *Asymptotic mixing property:* for all sets  $E$  and  $F$  of  $\mathcal{A}$ ,  $\lim_{|z| \rightarrow +\infty} \mathbf{P}(\tau_z E \cap F) = \mathbf{P}(E)\mathbf{P}(F)$ .

We emphasise the following remarks extracted from [12, Remark 2.1]:

**Remark 3.1.** *i) It would be more natural to consider stationarity condition (A<sub>2</sub>) with respect to the continuous group  $(\tau_t)_{t \in \mathbb{R}^2}$  defined in the same way by  $(\tau_t \omega)_i = \omega_i - t$ . Actually the discrete group  $(\tau_z)_{z \in \mathbf{Z}^2}$  suffices for the mathematical analysis. The size of the cell  $\hat{Y}$  is chosen in such a way to fix the generator of the group  $(\tau_z)_{z \in \mathbf{Z}^2}$ . Condition (A<sub>2</sub>) then says that every random function  $X$  defined on  $\Omega$  is statistically homogeneous. Roughly speaking, moving a window  $\hat{A}$  in  $\mathbb{R}^2$  following translations in  $\mathbb{R}^2$ , the distributions of cross sections in the window are statistically the same.*

*ii) Consider  $\bar{\omega} = (\bar{\omega}_i)_{i \in \mathbb{N}}$  made up of the centers  $\bar{\omega}_i$  of the disks of radius  $d/2$  arranged in an hexagonal lattice, where each disk is surrounded by 8 disks with the same radius  $d/2$ . This configuration is called the hexagonal close-packing of disks in  $\mathbb{R}^2$ . It is known to provide the highest density of disks among all other distributions of centers in  $\Omega$  of disks with the same radius, so that, for all  $\omega$  in  $\Omega$ ,  $|\hat{Y} \cap D(\omega)| \leq |\hat{Y} \cap D(\bar{\omega})|$ .*

We end this section by the following convergence result on some random oscillating sequences, which is a consequence of the multidimensional Birkoff ergodic theorem and whose proof is a straightforward consequence of Theorem 4.2 and Proposition 5.3 in [6].

**Proposition 3.1.** *Let  $n$  be fixed in  $\mathbb{N}^*$ , and  $\psi : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$  be a  $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^2)$ -measurable function satisfying the three conditions:*

*i) for  $\mathbf{P}$ -almost every  $\omega \in \Omega$ ,  $\hat{y} \mapsto \psi(\omega, \hat{y})$  belongs to  $L^1_{loc}(\mathbb{R}^2)$ ;*

*ii) for all bounded Borel set  $\hat{A}$  of  $\mathbb{R}^2$  the map  $\hat{A} \mapsto \int_{\hat{A}} \psi(\omega, \hat{y}) d\hat{y}$  belongs to  $L^1(\Omega, \mathcal{A}, \mathbf{P})$ ;*

*iii) for all  $z \in n\mathbf{Z}^2$ , for all  $\hat{y} \in \mathbb{R}^2$ ,  $\psi(\omega, \hat{y} + z) = \psi(\tau_z \omega, \hat{y})$  for  $\mathbf{P}$ -almost every  $\omega \in \Omega$ .*

*Then almost surely*

$$\psi(\omega, \frac{\cdot}{\varepsilon}) \xrightarrow{*} \mathbf{E} \int_{(0, n)^2} \psi(\cdot, \hat{y}) d\hat{y}$$

*for the  $\sigma(L^1(\mathcal{O}), L^\infty(\mathcal{O}))$  topology.*

Note that the characteristic function of the random set  $T_\varepsilon \cap \mathcal{O}$  may be written  $\mathbf{1}_{D(\omega) \cap \hat{\mathcal{O}}(\frac{\cdot}{\varepsilon})}$  and that  $(\omega, \cdot) \mapsto \mathbf{1}_{D(\omega) \cap \hat{\mathcal{O}}(\cdot)}$  satisfies the condition  $\mathbf{1}_{D(\omega) \cap \hat{\mathcal{O}}(\hat{x} + z)} = \mathbf{1}_{D(\tau_z \omega) \cap \hat{\mathcal{O}}(\hat{x})}$ . Therefore, applying Proposition 3.1 we infer that for  $\mathbf{P}$  a.e.  $\omega$  in  $\Omega$ ,

$$\mathbf{1}_{D \cap \hat{\mathcal{O}}(\frac{\cdot}{\varepsilon})} \xrightarrow{*} \mathbf{E} |\hat{Y} \cap D(\cdot)|. \quad (3.1)$$

**Definition 3.1.** *The limit  $\theta := \mathbf{E} |\hat{Y} \cap D(\cdot)|$  in (3.1) is called the asymptotic volume fraction of the fibers.*

## 4 Proofs of the results

In what follows  $C$  will denote various constants which may depend on  $\omega$  and  $p$  and may vary from line to line.

### 4.1 Proof of compactness properties (1.2), (1.3) and (1.4) of Theorem 1.1

For proving (1.2), (1.3) and (1.4) we will need the following Lemma deduced from the Poincaré-Wirtinger inequality.

**Lemma 4.1.** *For all  $w \in W_\varepsilon^{1,p}(\mathcal{O})$ , we have almost surely*

$$\int_{\mathcal{O}} |w|^p dx \leq C \left[ \frac{h}{|\hat{Y} \cap D(\omega)|} \int_{\mathcal{O} \cap T_\varepsilon} \left| \frac{\partial w}{\partial x_3} \right|^p dx + \int_{\mathcal{O}} |\varepsilon \hat{\nabla} w|^p dx \right], \quad (4.1)$$

where the constant  $C > 0$  depends on  $\omega$  and  $p$ .

*Proof.* We fix  $\omega$  in the subset of  $\Omega$  of full probability for which  $(A_1)$  holds and consider  $w \in W^{1,p}(\mathbb{R}^2)$ . According to Poincaré-Wirtinger's inequality, there exists a constant  $C_{pw}(\omega)$  such that

$$\int_{\hat{Y}} \left| w - \fint_{\hat{Y} \cap D(\omega)} w \, d\hat{y} \right|^p d\hat{x} \leq C_{pw}(\omega) \int_{\hat{Y}} |\nabla w|^p d\hat{x}$$

from which we deduce

$$\int_{\varepsilon \hat{Y}} \left| w - \fint_{\varepsilon \hat{Y} \cap \varepsilon D(\omega)} w \, d\hat{y} \right|^p d\hat{x} \leq C_{pw}(\omega) \int_{\varepsilon \hat{Y}} |\varepsilon \nabla w|^p d\hat{x}$$

and finally

$$\int_{\varepsilon \hat{Y}} |w|^p d\hat{x} \leq C \left[ \varepsilon^2 \fint_{\varepsilon \hat{Y} \cap \varepsilon D(\omega)} |w|^p d\hat{x} + \int_{\varepsilon \hat{Y}} |\varepsilon \nabla w|^p d\hat{x} \right]. \quad (4.2)$$

From (4.2) and the operator  $\tau_{\varepsilon z} w$  defined by  $\tau_{\varepsilon z} w(\hat{x}) := w(\hat{x} + \varepsilon z)$  with  $z \in \mathbf{Z}^2$ , we have

$$\begin{aligned} \int_{\varepsilon(\hat{Y}+z)} |w|^p d\hat{x} &= \int_{\varepsilon \hat{Y}} |\tau_{\varepsilon z} w|^p d\hat{x} \\ &\leq C \left[ \varepsilon^2 \fint_{\varepsilon \hat{Y} \cap \varepsilon D(\omega)} |\tau_{\varepsilon z} w|^p d\hat{x} + \int_{\varepsilon \hat{Y}} |\varepsilon \nabla \tau_{\varepsilon z} w|^p d\hat{x} \right] \\ &= C \left[ \frac{1}{|\hat{Y} \cap D(\omega)|} \int_{\varepsilon(\hat{Y}+z) \cap \varepsilon D(\tau_{-z}\omega)} |w|^p d\hat{x} + \int_{\varepsilon(\hat{Y}+z)} |\varepsilon \nabla w|^p d\hat{x} \right]. \end{aligned} \quad (4.3)$$

Fix now  $w$  in  $W_\varepsilon^{1,p}(\mathcal{O})$ . Noticing that  $\left| \hat{\mathcal{O}} \setminus \bigcup_{z \in I_\varepsilon} \varepsilon(\hat{Y} + z) \right| = 0$  where  $I_\varepsilon$  is a finite subset of  $\mathbf{Z}^2$  and the cells  $(\hat{Y} + z)_{z \in \mathbf{Z}^2}$  are pairwise disjoint, from (4.3) we deduce

$$\begin{aligned} \int_{\mathcal{O}} |w|^p dx &\leq C \left[ \sum_{z \in I_\varepsilon} \frac{1}{|\hat{Y} \cap D(\omega)|} \int_0^h \int_{\varepsilon(\hat{Y}+z) \cap \varepsilon D(\tau_{-z}\omega)} |w|^p dx + \int_{\mathcal{O}} |\varepsilon \hat{\nabla} w|^p dx \right] \\ &\leq C \left[ \frac{1}{|\hat{Y} \cap D(\omega)|} \int_{\mathcal{O} \cap T_\varepsilon} |w|^p dx + \int_{\mathcal{O}} |\varepsilon \hat{\nabla} w|^p dx \right] \\ &\leq C \left[ \frac{h}{|\hat{Y} \cap D(\omega)|} \int_{\mathcal{O} \cap T_\varepsilon} \left| \frac{\partial w}{\partial x_3} \right|^p dx + \int_{\mathcal{O}} |\varepsilon \hat{\nabla} w|^p dx \right], \end{aligned} \quad (4.4)$$

the last inequality is deduced from a Poincaré inequality. Indeed, since  $w = 0$  on  $\hat{\mathcal{O}} \cap (\varepsilon D(\omega) \times \{0\})$ ,

$$\int_{\mathcal{O} \cap T_\varepsilon} |w|^p dx \leq h \int_{\mathcal{O} \cap T_\varepsilon} \left| \frac{\partial w}{\partial x_3} \right|^p dx,$$

which completes the proof.  $\square$

We now turn to the proofs of (1.2), (1.3) and (1.4). Fix  $\omega$  in the subset of  $\Omega$  of full probability for which  $(A_1)$  holds and consider  $(u_\varepsilon)_{\varepsilon>0} \in L^p(\mathcal{O})$  such that  $\sup_{\varepsilon>0} E_\varepsilon(\omega, u_\varepsilon) < +\infty$ . From 4.1 established in the previous lemma, the growth conditions satisfied by  $f$  and  $g$ , and since  $\gamma > 0$  and  $a > 0$ , we have

$$\begin{aligned} \int_{\mathcal{O}} |u_\varepsilon|^p dx &\leq C \int_{\mathcal{O} \cap T_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_3} \right|^p dx + C\varepsilon^p \int_{\mathcal{O} \setminus T_\varepsilon} \left| \hat{\nabla} u_\varepsilon \right|^p dx + C\varepsilon^{p-a} \int_{\mathcal{O} \cap T_\varepsilon} \left| \hat{\nabla} u_\varepsilon \right|^p dx \\ &\leq C \frac{\varepsilon^{p\gamma}}{\alpha} H_\varepsilon(u_\varepsilon) + CH_\varepsilon(u_\varepsilon) \\ &\leq CH_\varepsilon(u_\varepsilon). \end{aligned} \tag{4.5}$$

On the other hand, from  $st \leq \frac{\nu^p}{p} s^p + \frac{1}{q\nu^q} t^q$  with  $s \geq 0$ ,  $t > 0$ , and  $\nu > 0$  suitably chosen later, noticing that

$$\int_{\hat{\mathcal{O}} \cap \varepsilon D} |u_\varepsilon(\hat{x}, 1)|^p d\hat{x} \leq \int_{\mathcal{O} \cap T_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_3} \right|^p dx,$$

and since  $b \geq \gamma$ , we deduce

$$\begin{aligned} H_\varepsilon(\omega, u_\varepsilon) &\leq C + \left| \int_{\mathcal{O}} L u_\varepsilon dx \right| + \left| \int_{\hat{\mathcal{O}} \cap \varepsilon D} \varepsilon^{-b} l u_\varepsilon d\hat{x} \right| \\ &\leq C + \frac{1}{q\nu^q} \int_{\mathcal{O}} |L|^q dx + \frac{\nu^p}{p} \int_{\mathcal{O}} |u_\varepsilon|^p dx + \frac{1}{q\nu^q} \int_{\hat{\mathcal{O}} \cap \varepsilon D} |l|^q d\hat{x} + \frac{\nu^p}{p} \varepsilon^{-pb} \int_{\mathcal{O} \cap T_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_3} \right|^p dx \\ &\leq C + \frac{\nu^p}{p} \int_{\mathcal{O}} |u_\varepsilon|^p dx + \frac{\nu^p}{\alpha p} H_\varepsilon(\omega, u_\varepsilon). \end{aligned} \tag{4.6}$$

Thus

$$\left(1 - \frac{\nu^p}{\alpha p}\right) H_\varepsilon(\omega, u_\varepsilon) \leq C + \frac{\nu^p}{p} \int_{\mathcal{O}} |u_\varepsilon|^p dx. \tag{4.7}$$

Combining (4.7) with (4.5) we infer

$$\int_{\mathcal{O}} |u_\varepsilon|^p dx \leq C + C \frac{\nu^p}{\delta(\nu)} \int_{\mathcal{O} \setminus T_\varepsilon} |u_\varepsilon|^p dx$$

where  $\delta(\nu) := \frac{1}{1 - \frac{\nu^p}{\alpha p}}$ . Choosing  $\nu$  small enough in such a way that  $C \frac{\nu^p}{\delta(\nu)} < \frac{1}{2}$  we obtain

$$\int_{\mathcal{O}} |u_\varepsilon|^p dx \leq C \tag{4.8}$$

so that  $u_\varepsilon$  weakly converges to some  $u$  in  $L^p(\mathcal{O})$ . Moreover (4.7), (4.8) yield  $H_\varepsilon(\omega, u_\varepsilon) \leq C$ . Therefore, according to the coercivity assumption on  $f$  and  $g$ , we infer

$$\begin{aligned} \varepsilon^{-p} \int_{\mathcal{O} \setminus T_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_3} \right|^p dx + \varepsilon^{-p\gamma} \int_{\mathcal{O} \cap T_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_3} \right|^p dx &\leq C; \\ \varepsilon^{-p\gamma} \int_{\mathcal{O} \cap T_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_3} \right|^p dx &\leq C; \\ \varepsilon^{-p\gamma} \int_{\mathcal{O} \cap T_\varepsilon} |u_\varepsilon|^p dx &\leq C, \end{aligned} \tag{4.9}$$

from which we easily deduce (1.2), (1.3) and (1.4).

## 4.2 Proof of the upper bound ii) of Theorem 1.1

This section is devoted to the establishing of the upper bound (ii) in Theorem 1.1.

**Proposition 4.1.** *There exists a set  $\Omega' \in \mathcal{A}$  of full probability such that for all  $(u, v) \in L^p(\mathcal{O}) \times V_0(\mathcal{O})$  and all  $\omega \in \Omega'$  there exists a sequence  $(u_\varepsilon(\omega))_{\varepsilon>0}$  in  $L^p(\mathcal{O})$  satisfying*

$$\begin{aligned} u_\varepsilon(\omega) &\rightharpoonup (u, v) \\ E_0(u, v) &\geq \limsup_{\varepsilon \rightarrow 0} E_\varepsilon(\omega, u_\varepsilon(\omega)). \end{aligned}$$

*Proof.* For any sequence  $(u_\varepsilon(\omega))_{\varepsilon>0}$  satisfying  $u_\varepsilon(\omega) \rightharpoonup (u, v)$ , the limit

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathcal{O}} L.u_\varepsilon(\omega) dx + \varepsilon^{-b} \int_{\hat{\mathcal{O}} \times \{1\}} l.u_\varepsilon(\omega) d\mathcal{H}^2 = \begin{cases} \int_{\mathcal{O}} L.u dx + \int_{\hat{\mathcal{O}}} lv d\hat{x} & \text{when } b = \gamma, \\ \int_{\mathcal{O}} L.u dx & \text{when } b < \gamma. \end{cases}$$

is easy to establish and left to the reader. Therefore we are reduced to prove

$$\begin{aligned} u_\varepsilon(\omega) &\rightharpoonup (u, v) \\ H_0(u, v) &\geq \limsup_{\varepsilon \rightarrow 0} H_\varepsilon(\omega, u_\varepsilon(\omega)), \end{aligned}$$

for a suitable sequence  $(u_\varepsilon(\omega))_{\varepsilon>0}$ . We proceed into three steps.

*Step 1.* We assume  $(u, v) \in \mathcal{C}_c^1(\hat{\mathcal{O}}) \times (\mathcal{C}^1(\mathcal{O}) \cap V_0(\mathcal{O}))$  and we show that there exists a set  $\Omega'$  of full probability and, for all  $\omega \in \Omega'$ , a sequence  $(u_\varepsilon(\omega))_{\varepsilon>0}$  in  $L^p(\mathcal{O})$  such that  $u_\varepsilon(\omega) \rightharpoonup (u, v)$  and

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} F_\varepsilon(\omega, u_\varepsilon(\omega)) &= \int_{\hat{\mathcal{O}}} f_0(u) d\hat{x} \\ \lim_{\varepsilon \rightarrow 0} G_\varepsilon(\omega, u_\varepsilon(\omega)) &= G_0(v). \end{aligned}$$

Let  $\eta \in \mathbf{Q}^+$  intended to go to 0 and let  $(\hat{Q}_{i,\eta})_{i \in I_\eta}$  be a finite family of pairwise disjoint cubes of size  $\eta$  included in  $\hat{\mathcal{O}}$ , such that

$$\left| \hat{\mathcal{O}} \setminus \bigcup_{i \in I_\eta} \hat{Q}_{i,\eta} \right| = 0.$$

Let  $z_\eta := \sum_{i \in I_\eta} u(\hat{x}_{i,\eta}) \mathbf{1}_{\hat{Q}_{i,\eta}}$  where  $x_{i,\eta}$  is arbitrarily chosen in  $\hat{Q}_{i,\eta}$ . Since  $u$  is a Lipschitz function on  $\hat{\mathcal{O}}$ , clearly  $z_\eta \rightarrow u$  in  $L^p(\mathcal{O})$  when  $\eta \rightarrow 0$ .

For every  $i \in I_\eta$ , and for fixed  $n \in \mathbb{N}^*$ , consider  $w_{i,n}(\omega, \cdot) \in \text{Adm}_{n\hat{Y}}(\omega, u(\hat{x}_{i,\eta}))$  and  $\xi_{i,n}(\omega, \cdot) \in \mathcal{C}_c^\infty(n\hat{Y} \setminus D(\omega))$  such that

$$\int_{n\hat{Y} \setminus D(\omega)} f^{\infty,p}(\nabla w_{i,n}(\omega, \hat{x}), \xi_{i,n}(\omega, \hat{x})) d\hat{x} = \inf \left\{ \int_{n\hat{Y}} \widehat{f^{\infty,p}}(\nabla w) dy : w \in \text{Adm}_{n\hat{Y}}(\omega, u(\hat{x}_{i,\eta})) \right\}$$

and extend it on  $\mathbb{R}^2$  as follows:

$$\tilde{w}_{i,n}(\omega, \hat{x}) = w_{i,n}(\tau_z \omega, \hat{x} - z) \text{ if } x \in n\hat{Y} + z, z \in n\mathbf{Z}^2;$$

$$\tilde{\xi}_{i,n}(\omega, \hat{x}) = \xi_{i,n}(\tau_z \omega, \hat{x} - z) \text{ if } x \in n\hat{Y} + z, z \in n\mathbf{Z}^2.$$

It is easy to check that  $\tilde{w}_{i,n}$  and  $\tilde{\xi}_{i,n}$  satisfies:  $\tilde{w}_{i,n}(\omega, \hat{x} + z) = \tilde{w}_{i,n}(\tau_z \omega, \hat{x})$  and  $\tilde{\xi}_{i,n}(\omega, \hat{x} + z) = \tilde{\xi}_{i,n}(\tau_z \omega, \hat{x})$  for all  $z \in n\mathbf{Z}$ . To shorten notation we drop the dependance on  $\eta$  and we still denote by  $w_{i,n}$  and  $\xi_{i,n}$  these two functions. According to Proposition 3.1, we have, almost surely when  $\varepsilon \rightarrow 0$

$$\begin{aligned} f^{\infty,p}(\nabla w_{i,n}(\omega, \frac{\hat{x}}{\varepsilon}), \xi_{i,n}(\omega, \frac{\hat{x}}{\varepsilon})) &\stackrel{*}{=} \mathbf{E} \int_{n\hat{Y}} f^{\infty,p}(\nabla w_{i,n}(\omega, \hat{x}), \xi_{i,n}(\omega, \hat{x})) d\hat{x} \\ &= \mathbf{E} \frac{\mathbf{S}_{(0,n)^2}(\omega, u(\hat{x}_{i,\eta}))}{n^2}, \end{aligned} \tag{4.10}$$

and

$$w_{i,n}(\omega, \frac{\cdot}{\varepsilon}) \rightarrow \mathbf{E} \int_{n\hat{Y}} w_{i,n}(\omega, y) dy = u(\hat{x}_{i,\eta}). \quad (4.11)$$

Let  $(\theta_{i,\delta})_{i \in I_\eta}$  be a partition of unity associated with  $(\hat{Q}_{i,\eta})_{i \in I_\eta}$  with  $\theta_{i,\delta} \rightarrow \mathbb{1}_{\hat{Q}_{i,\eta}}$  when  $\delta \rightarrow 0$  (we omit the dependance on  $\eta$ ), and consider the following function in  $W^{1,p}(\mathcal{O})$ :

$$u_{\delta,n,\varepsilon}(\omega, x) = \frac{1}{\theta} \varepsilon^\gamma v + \sum_{i \in I_\eta} \theta_{i,\delta}(\hat{x}) \left[ w_{i,n}(\omega, \frac{\hat{x}}{\varepsilon}) + \varepsilon^{p-1} x_3 \xi_{i,n}(\omega, \frac{\hat{x}}{\varepsilon}) \right].$$

Clearly  $u_{\delta,n,\varepsilon} = \frac{1}{\theta} \varepsilon^\gamma v$  on  $\mathcal{O} \cap T_\varepsilon(\omega)$ , and

$$\begin{aligned} \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} u_{\delta,n,\varepsilon}(\omega, \cdot) &= z_\eta \text{ weakly in } L^p(\mathcal{O}) \\ \lim_{\varepsilon \rightarrow 0} \varepsilon^{-\gamma} a(\omega, \frac{\cdot}{\varepsilon}) u_{\delta,n,\varepsilon}(\omega, \cdot) &= v \text{ weakly in } L^p(\mathcal{O}). \end{aligned} \quad (4.12)$$

Let  $\Omega_0$  be the set of full probability made up of all  $\omega \in \Omega$  for which  $a(\omega, \frac{\cdot}{\varepsilon}) \rightarrow \theta$  for the  $\sigma(\mathbf{L}^\infty(\mathcal{O}), L^1(\mathcal{O}))$  topology and denote by  $\Omega_{i,\eta,n}$  the set of full probability made up of all  $\omega \in \Omega$  for which (4.10) and (4.11) hold. In what follows we denote the set of full probability  $\bigcap_{n \in \mathbb{N}^*} \bigcap_{\eta \in \mathbf{Q}^+} \bigcap_{i \in I_\eta} \Omega_{i,\eta,n} \cap \Omega_0$  by  $\Omega'$  and we fix  $\omega \in \Omega'$ .

Now we are going to estimate  $F_\varepsilon(\omega, u_{\delta,n,\varepsilon}(\omega, \cdot))$  and  $G_\varepsilon(\omega, u_{n,\delta,\varepsilon}(\omega, \cdot))$ . To simplify the notation we do not indicate the dependance on  $\omega$ . On  $\mathcal{O} \setminus T_\varepsilon$  we have

$$\begin{aligned} \varepsilon \hat{\nabla} u_{\delta,n,\varepsilon}(x) &= \frac{1}{\theta} \varepsilon^{\gamma+1} \hat{\nabla} v(x) + \sum_{i \in I_\eta} \theta_{i,\delta}(\hat{x}) \left[ \hat{\nabla} w_{i,n}(\omega, \frac{\hat{x}}{\varepsilon}) + \varepsilon^{p-1} x_3 \hat{\nabla} \xi_{i,n}(\omega, \frac{\hat{x}}{\varepsilon}) \right] \\ &+ \sum_{i \in I_\eta} \varepsilon \hat{\nabla} \theta_{i,\delta}(\hat{x}) \left[ w_{i,n}(\omega, \frac{\hat{x}}{\varepsilon}) + \varepsilon^{p-1} x_3 \xi_{i,n}(\omega, \frac{\hat{x}}{\varepsilon}) \right] \\ &= O(\varepsilon) + \sum_{i \in I_\eta} \theta_{i,\delta}(\hat{x}) \hat{\nabla} w_{i,n}(\omega, \frac{\hat{x}}{\varepsilon}) \end{aligned}$$

and

$$\begin{aligned} \varepsilon^{1-p} \frac{\partial u_{\delta,n,\varepsilon}}{\partial x_3}(x) &= \frac{1}{\theta} \varepsilon^{\gamma+1-p} \frac{\partial v}{\partial x_3}(x) + \sum_{i \in I_\eta} \theta_{i,\delta}(\hat{x}) \xi_{i,n}(\omega, \frac{\hat{x}}{\varepsilon}) \\ &= O(\varepsilon) + \sum_{i \in I_\eta} \theta_{i,\delta}(\hat{x}) \xi_{i,n}(\omega, \frac{\hat{x}}{\varepsilon}) \end{aligned}$$

where  $O(\varepsilon)$  may depend on  $\eta$ ,  $n$ , and  $\delta$ . Consequently from (2.3), (2.5), (4.10)

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} F_\varepsilon(\omega, u_{\delta,n,\varepsilon}) &= \lim_{\varepsilon \rightarrow 0} \varepsilon^p \int_{\mathcal{O} \setminus T_\varepsilon(\omega)} f(\hat{\nabla} u_{\delta,n,\varepsilon}, \varepsilon^{-p} \frac{\partial u_{\delta,n,\varepsilon}}{\partial x_3}) dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{O} \setminus T_\varepsilon(\omega)} f^{\infty,p}(\varepsilon \hat{\nabla} u_{\delta,n,\varepsilon}, \varepsilon^{1-p} \frac{\partial u_{\delta,n,\varepsilon}}{\partial x_3}) dx \\ &= \lim_{\varepsilon \rightarrow 0} \sum_{i \in I_\eta} \int_{\hat{Q}_{i,\eta}} \theta_{i,\delta}^p f^{\infty,p}(\hat{\nabla} w_{i,n}(\omega, \frac{\hat{x}}{\varepsilon}), \xi_{i,n}(\frac{\hat{x}}{\varepsilon})) d\hat{x} \\ &= \sum_{i \in I_\eta} \int_{\hat{Q}_{i,\eta}} \theta_{i,\delta}^p \mathbf{E} \frac{\mathbf{S}_{(0,n)^2}(\omega, u(\hat{x}_{i,\eta}))}{n^2} d\hat{x}. \end{aligned}$$

Thus, according to Lemma 1.1

$$\begin{aligned} \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} F_\varepsilon(\omega, u_{\delta,n,\varepsilon}) &= \sum_{i \in I_\eta} |\hat{Q}_{i,\eta}| f_0(u(\hat{x}_{i,\eta})) \\ &= \int_{\mathcal{O}} f_0(z_\eta) d\hat{x}. \end{aligned}$$

Finally, letting  $\eta \rightarrow 0$ , we infer

$$\lim_{\eta \rightarrow 0} \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} F_\varepsilon(\omega, u_{\delta, n, \varepsilon}) = \int_{\mathcal{O}} f_0(u) \, dx. \quad (4.13)$$

The same kind of computation gives (recall that  $\gamma = p - 1 + \frac{a}{p}$  and that  $g^{\infty, p}$  is positively homogeneous of degree  $p$ )

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} G_\varepsilon(\omega, u_{\delta, n, \varepsilon}) &= \lim_{\varepsilon \rightarrow 0} \varepsilon^{p-a} \int_{\mathcal{O} \cap T_\varepsilon} g(\varepsilon^\gamma \frac{1}{\theta} \hat{\nabla} v, \varepsilon^{\gamma-p} \frac{1}{\theta} \frac{\partial v}{\partial x_3}) \, dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{O} \cap T_\varepsilon} g^{\infty, p}(\varepsilon^p \frac{1}{\theta} \hat{\nabla} v, \frac{1}{\theta} \frac{\partial v}{\partial x_3}) \, dx \\ &= \theta \int_{\mathcal{O}} g^{\infty, p}(0, \frac{1}{\theta} \frac{\partial v}{\partial x_3}) \, dx = G_0(v). \end{aligned} \quad (4.14)$$

Combining (4.12), (4.13), (4.14) and a standard diagonalization argument<sup>1</sup> furnishes a map  $\varepsilon \mapsto (\eta(\varepsilon), \delta(\varepsilon), n(\varepsilon))$  such that

$$\begin{aligned} u_\varepsilon(\omega, \cdot) &:= u_{\eta(\varepsilon), \delta(\varepsilon), n(\varepsilon), \varepsilon}(\omega, \cdot) \rightharpoonup (u, v) \\ \lim_{\varepsilon \rightarrow 0} H_\varepsilon(\omega, u_\varepsilon(\omega, \cdot)) &= \int_{\hat{\mathcal{O}}} f_0(u(\hat{x})) \, d\hat{x} + G_0(v). \end{aligned}$$

which completes the proof of step 1.

*Step 2.* We fix  $(u, v) \in L^p(\mathcal{O}) \times V_0(\mathcal{O})$  with  $v \in \mathcal{C}^1(\mathcal{O})$  and we show that for all  $\omega \in \Omega'$  there exists  $(u_\varepsilon(\omega))_{\varepsilon > 0}$  in  $L^p(\mathcal{O})$  such that  $u_\varepsilon(\omega) \rightharpoonup (u, v)$  and  $\lim_{\varepsilon \rightarrow 0} H_\varepsilon(\omega, u_\varepsilon(\omega)) = H_0(u, v)$ .

Consider  $u_n \in \mathcal{C}_c^1(\hat{\mathcal{O}})$  weakly converging toward  $u$  in  $L^p(\hat{\mathcal{O}})$  such that

$$\lim_{n \rightarrow +\infty} \int_{\hat{\mathcal{O}}} f_0(u_n) \, d\hat{x} = \int_{\hat{\mathcal{O}}} f_0(u) \, d\hat{x}.$$

Thus according to step 1, there exists  $u_{\varepsilon, n}(\omega, \cdot)$  weakly converging to  $u_n$  when  $\varepsilon \rightarrow 0$  and

$$\lim_{n \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} H_\varepsilon(\omega, u_{\varepsilon, n}(\omega, \cdot)) = \int_{\hat{\mathcal{O}}} f_0(u(\hat{x})) \, d\hat{x} + G_0(v).$$

We conclude by a diagonalization argument.

*Step 3.* For any  $(u, v) \in L^p(\mathcal{O}) \times V_0(\mathcal{O})$  we show that for all  $\omega \in \Omega'$  there exists  $(u_\varepsilon(\omega))_{\varepsilon > 0}$  in  $L^p(\mathcal{O})$  such that  $u_\varepsilon(\omega) \rightharpoonup (u, v)$  and  $\lim_{\varepsilon \rightarrow 0} H_\varepsilon(\omega, u_\varepsilon(\omega)) = H_0(u, v)$ .

Fix  $v \in V_0(\mathcal{O})$ . According to standard relaxation results, there exists a sequence  $(\zeta_n)_{n \in \mathbb{N}}$  in  $\mathcal{C}_c^1(\mathcal{O})$  weakly converging to  $\frac{\partial v}{\partial x_3}$  in  $L^p(\mathcal{O})$  such that

$$\lim_{n \rightarrow +\infty} \int_{\mathcal{O}} (g^{\infty, p})^\perp(\frac{1}{\theta} \zeta_n) = \int_{\mathcal{O}} (g^{\infty, p})^\perp(\frac{1}{\theta} \frac{\partial v}{\partial x_3}) \, dx. \quad (4.15)$$

For all  $x \in \mathcal{O}$  set

$$v_n(x) := \int_0^{x_3} \zeta_n(\hat{x}, s) \, ds.$$

Then  $v_n \in V_0(\mathcal{O}) \cap \mathcal{C}^1(\mathcal{O})$ ,  $v_n \rightharpoonup u$  in  $L^p(\mathcal{O})$  and

$$\lim_{n \rightarrow +\infty} \theta \int_{\mathcal{O}} (g^{\infty, p})^\perp(\frac{1}{\theta} \frac{\partial v_n}{\partial x_3}) = G_0(v).$$

We end the proof by using Step 2 and a diagonalization argument. □

<sup>1</sup>One can easily check that  $u_{\eta, \delta, n, \varepsilon}(\omega, \cdot)$  and  $\varepsilon^{-\gamma} a(\omega, \frac{\cdot}{\varepsilon}) u_{\eta, \delta, n, \varepsilon}$  belongs to a fixed ball  $\mathcal{B}(0, r)$  of  $L^p(\mathcal{O})$ . Since the weak topology of  $L^p(\mathcal{O})$  induces a metric on bounded sets, the diagonalization argument holds.

### 4.3 Proof of the lower bound i) of Theorem 1.1

This section is devoted to the establishing of the lower bound (i) of Theorem 1.1.

**Proposition 4.2.** *For all sequence  $(u_\varepsilon)_{\varepsilon>0}$  such that  $u_\varepsilon \rightharpoonup (u, v)$  one has*

$$E_0(u, v) \leq \liminf_{\varepsilon \rightarrow 0} E_\varepsilon(\omega, u_\varepsilon) \quad (4.16)$$

for  $\mathbf{P}$  a. s.  $\omega \in \Omega$ .

*Proof.* One may assume  $\liminf_{\varepsilon \rightarrow 0} E_\varepsilon(\omega, u_\varepsilon) < +\infty$  otherwise there is nothing to prove. With the notation of Section 2 It suffices to show that for  $\mathbf{P}$  a.s.  $\omega$  in  $\Omega$ ,

$$F_0(v) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(\omega, u_\varepsilon) \quad (4.17)$$

$$G_0(u) \leq \liminf_{\varepsilon \rightarrow 0} G_\varepsilon(\omega, u_\varepsilon). \quad (4.18)$$

Indeed, according to Lemma ??, we easily infer

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathcal{O}} L.u_\varepsilon(\omega) dx + \int_{\widehat{\mathcal{O}} \times \{1\}} l.u_\varepsilon d\mathcal{H}^2 = \begin{cases} \int_{\mathcal{O}} L.u dx + \int_{\widehat{\mathcal{O}}} l.v d\hat{x} & \text{when } b = \gamma, \\ \int_{\mathcal{O}} L.u dx & \text{when } b < \gamma. \end{cases}$$

**Proof of (4.17).** Note that since  $\sup_{\varepsilon>0} E_\varepsilon(\omega, u_\varepsilon) < +\infty$ , from (4.9), we infer that there exists a constant  $C$  such that

$$\varepsilon^p \int_{\mathcal{O} \cap T_\varepsilon} |\hat{\nabla} u_\varepsilon|^p dx < C\varepsilon^a, \quad (4.19)$$

$$\int_{\mathcal{O} \cap T_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_3} \right|^p < C\varepsilon^{p\gamma}. \quad (4.20)$$

On the other hand, according to the compactness lemma, Lemma ??, one has for any subsequence  $\mathbf{1}_{\mathcal{O} \cap T_\varepsilon} u_\varepsilon \rightarrow 0$  in  $L^p(\mathcal{O})$  so that,

$$\mathbf{1}_{\mathcal{O} \setminus T_\varepsilon} u_\varepsilon = u_\varepsilon - \mathbf{1}_{\mathcal{O} \cap T_\varepsilon} u_\varepsilon \rightharpoonup u \text{ in } L^p(\mathcal{O}). \quad (4.21)$$

We will make use of (4.21) in the last step of the proof.

From (2.3), the coercivity condition satisfied by  $f$  and  $g$ , and from (4.19), (4.19), it is easily seen that

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \varepsilon^p \int_{\mathcal{O} \setminus T_\varepsilon} f(\hat{\nabla} u_\varepsilon, \frac{1}{\varepsilon^p} \frac{\partial u_\varepsilon}{\partial x_3}) dx &= \liminf_{\varepsilon \rightarrow 0} \int_{\mathcal{O}} \varepsilon^p f(\hat{\nabla} u_\varepsilon, \frac{1}{\varepsilon^p} \frac{\partial u_\varepsilon}{\partial x_3}) dx \\ &\geq \liminf_{\varepsilon \rightarrow 0} \int_{\mathcal{O}} \widehat{f^{\infty, p}}(\varepsilon \hat{\nabla} u_\varepsilon) dx \end{aligned}$$

Fix  $x_0$  in  $\mathcal{O}$  and set  $Q_\rho(x_0) := S_\rho(\hat{x}_0) \times I_\rho(x_{0,3})$  (to shorten notation we sometimes do not indicate the fixed argument  $x_0$ ). By using a blow up argument, for proving (4.17), it is enough to establish that for a.e.  $x_0$  in  $\mathcal{O}$ ,

$$\lim_{\rho \rightarrow 0} \liminf_{\varepsilon \rightarrow 0} \int_{Q_\rho(x_0)} \widehat{f^{\infty, p}}(\varepsilon \nabla u_\varepsilon) dx \geq f_0(u(x_0)). \quad (4.22)$$

Let  $0 < \delta < 1$  intended to go to 1 and set  $(T_\varepsilon)_\delta = \varepsilon D_\delta(\omega) \times (0, 1)$  where  $D_\delta(\omega) = \bigcup_{i \in \mathbb{N}} (\omega_i + \hat{B}_{\delta \frac{d}{2}}(0))$ . Let denote by  $\hat{A} \mapsto \hat{S}_{\hat{A}}(\omega, s, \delta)$  the subadditive process introduced in Section 5.4 where  $D(\omega_i)$  is replaced by the disk  $D_\delta(\omega_i) := \omega_i + \hat{B}_{\delta \frac{d}{2}}(0)$  and denote by  $\text{Adm}_{\hat{A}}(\omega, s, \delta)$  the associated admissible set. Denoting by  $C_{\varepsilon, \rho}$  the smallest cube in  $\overset{\circ}{\mathcal{I}}$  containing  $\frac{1}{\varepsilon} S_\rho$ , our strategy consists in suitably changing the function  $u_\varepsilon$  in order to obtain

a function  $z_\varepsilon$  whose mean  $\int_{I_\rho} z(\hat{x}, x_3) dx_3$  belongs to  $\text{Adm}_{C_{\varepsilon,\rho}}(\omega, u(x_0), \delta)$  and whose gradient asymptotically decreases the left hand side of (4.22). In the four steps below, to simplify the notation, we do not indicate the dependance on  $\rho$  for the various Sobolev functions.

*First change.* By using a standard truncation argument, we modify  $u_\varepsilon$  into a Sobolev function satisfying  $u_{\varepsilon,\delta} = 0$  in  $(T_\varepsilon)_\delta$  and

$$\int_{Q_\rho} \mathbf{1}_{Q_\rho \setminus T_\varepsilon} \widehat{f^{\infty,p}}(\varepsilon \hat{\nabla} u_\varepsilon) dx \geq \int_{Q_\rho} \widehat{f^{\infty,p}}(\varepsilon \hat{\nabla} u_{\varepsilon,\delta}) dx - \beta \left[ \frac{\varepsilon^{p\gamma}}{(1-\delta)^p} + \varepsilon^a \right] \quad (4.23)$$

Indeed, consider  $\varphi$  in  $C_c^1(S_\rho)$  satisfying  $\varphi = 0$  on  $\varepsilon D_\delta$ ,  $\varphi = 1$  in  $S_\rho \setminus \varepsilon D$  and  $|\nabla \varphi|_\infty \leq \frac{1}{\varepsilon(1-\delta)}$  and set

$$u_{\varepsilon,\delta} = \varphi u_\varepsilon.$$

According to the growth conditions satisfied by  $f^{\infty,p}$  we infer

$$\begin{aligned} \int_{Q_\rho} \widehat{f^{\infty,p}}(\varepsilon \hat{\nabla} u_{\varepsilon,\delta}) d\hat{x} &= \int_{Q_\rho \setminus T_\varepsilon} \widehat{f^{\infty,p}}(\varepsilon \hat{\nabla} u_\varepsilon) d\hat{x} + \int_{(T_\varepsilon \setminus (T_\varepsilon)_\delta) \cap Q_\rho} \widehat{f^{\infty,p}}(\varepsilon \hat{\nabla} u_{\varepsilon,\delta}) d\hat{x} \\ &\leq \int_{Q_\rho} \widehat{f^{\infty,p}}(\varepsilon \hat{\nabla} u_\varepsilon) d\hat{x} + \beta \int_{(T_\varepsilon \setminus (T_\varepsilon)_\delta) \cap Q_\rho} \varepsilon^p |\hat{\nabla} u_\varepsilon|^p d\hat{x} \\ &\quad + \beta \frac{1}{(1-\delta)^p} \int_{(T_\varepsilon \setminus (T_\varepsilon)_\delta) \cap Q_\rho} |u_\varepsilon|^p d\hat{x} \end{aligned}$$

so that, from the Poincaré inequality and (4.19), (4.20), we infer

$$\int_{Q_\rho} \varepsilon^p \widehat{f^{\infty,p}}(\hat{\nabla} u_{\varepsilon,\delta}) d\hat{x} \leq \int_{Q_\rho} \varepsilon^p \widehat{f^{\infty,p}}(\hat{\nabla} u_\varepsilon) d\hat{x} + \beta \left[ \frac{\varepsilon^{p\gamma}}{(1-\delta)^p} + \varepsilon^a \right]$$

which proves (4.23).

*Second change.* By using a standard De Giorgi slicing argument (see for instance [3, proof of Proposition 11.2.3]), there exists  $\eta(\varepsilon) \rightarrow 0^+$ ,  $\eta(\varepsilon) > \varepsilon$ , a  $\eta(\varepsilon)$ -neighborhood  $V\eta(\varepsilon) \subset Q_\rho$  of  $\partial Q_\rho$ , and a Sobolev function  $\tilde{u}_{\varepsilon,\delta}$  vanishing on  $\partial S_\rho \times I_\rho$ , equal to  $u_{\varepsilon,\delta}$  in a  $Q_\rho \setminus V\eta(\varepsilon)$ , satisfying

$$\int_{Q_\rho} f^{\infty,p}(\varepsilon \hat{\nabla} u_{\varepsilon,\delta}) dx \geq \int_{Q_\rho} f^{\infty,p}(\varepsilon \hat{\nabla} \tilde{u}_{\varepsilon,\delta}) dx - \frac{C(\rho)}{\nu} - r_\varepsilon(\rho) \quad (4.24)$$

where  $C(\rho)$  is a positive constant depending only on  $\rho$ ,  $\lim_{\varepsilon \rightarrow 0} r_\varepsilon(\rho) = 0$  and  $\nu \in \mathbb{N}$  is the number of bands slicing  $V\eta(\varepsilon)$  and intended to go to  $+\infty$ . It is worth noticing that  $\tilde{u}_{\varepsilon,\delta}$  remains equal to 0 in  $(T_\varepsilon)_\delta$  since it is of the form  $\varphi_{\eta(\varepsilon)} u_{\varepsilon,\delta}$  for a suitable truncation function  $\varphi_{\eta(\varepsilon)}$ .

*Third change.* We modify  $\tilde{u}_{\varepsilon,\delta}$  into a Sobolev function  $w_{\varepsilon,\delta}$  satisfying

$$w_{\varepsilon,\delta} = 0 \text{ in } (T_\varepsilon)_\delta, \quad w_{\varepsilon,\delta} = 0 \text{ on } \partial Q_\rho, \quad \int_{Q_\rho} w_{\varepsilon,\delta} = u(x_0)$$

and

$$\int_{Q_\rho} f^{\infty,p}(\varepsilon \hat{\nabla} \tilde{u}_{\varepsilon,\delta}) dx \geq \int_{Q_\rho} f^{\infty,p}(\varepsilon \hat{\nabla} w_{\varepsilon,\delta}) dx - C \left| u(x_0) - \int_{Q_\rho} \mathbf{1}_{Q_\rho \setminus (T_\varepsilon)_\delta} \tilde{u}_{\varepsilon,\delta} dy \right|^p. \quad (4.25)$$

Indeed, set

$$w_{\varepsilon,\delta} = \tilde{u}_{\varepsilon,\delta} + \frac{\psi}{\int_{Q_\rho} \psi dx} \left( u(x_0) - \int_{Q_\rho} \mathbf{1}_{Q_\rho \setminus (T_\varepsilon)_\delta} \tilde{u}_{\varepsilon,\delta} dy \right)$$

where  $\psi \in C_c^1(Q_\rho)$  satisfies  $\psi = 0$  in  $T_\varepsilon$ ,  $\psi = 0$  on  $\partial Q_\rho$ ,  $|\nabla \psi|_\infty \leq \frac{C}{\varepsilon}$  and  $|\psi|_\infty \leq C$ .



*Last step.* Collecting (4.23), (4.24) and (4.25) we finally obtain

$$\begin{aligned} \int_{Q_\rho} \widehat{f^{\infty,p}}(\varepsilon \hat{\nabla} u_\varepsilon) dx &\geq \int_{Q_\rho} \widehat{f^{\infty,p}}(\varepsilon \hat{\nabla} w_{\varepsilon,\delta}) dx - \beta \left[ \frac{\varepsilon^{p\gamma}}{(1-\delta)^p} + \varepsilon^a \right] - \frac{C(\rho)}{\nu} - r_\varepsilon(\rho) \\ &\quad - C \left| u(x_0) - \int_{Q_\rho} \mathbf{1}_{Q_\rho \setminus (T_\varepsilon)_\delta} \tilde{u}_{\varepsilon,\delta} dy \right|^p. \end{aligned}$$

Rescaling  $w_{\varepsilon,\delta}$  into  $z_{\varepsilon,\delta}(y) := w_{\varepsilon,\delta}(\varepsilon y)$  then yields

$$\begin{aligned} \int_{Q_\rho} \widehat{f^{\infty,p}}(\varepsilon \hat{\nabla} u_\varepsilon) dx &\geq \int_{\frac{1}{\varepsilon} Q_\rho} \widehat{f^{\infty,p}}(\hat{\nabla} z_{\varepsilon,\delta}) dx - \beta \left[ \frac{\varepsilon^{p\gamma}}{(1-\delta)^p} + \varepsilon^a \right] - \frac{C(\rho)}{\nu} - r_\varepsilon(\rho) \\ &\quad - C \left| u(x_0) - \int_{Q_\rho} \mathbf{1}_{Q_\rho \setminus (T_\varepsilon)_\delta} \tilde{u}_{\varepsilon,\delta} dy \right|^p. \end{aligned}$$

Now we extend  $z_{\varepsilon,\delta}$  by 0 in  $\mathbb{R}^3 \setminus \frac{1}{\varepsilon} S_\rho$ . Then the function  $\tilde{z}_{\varepsilon,\delta}$  defined by

$$\tilde{z}_{\varepsilon,\delta}(\hat{x}) := \frac{|C_{\varepsilon,\rho}|}{|\frac{1}{\varepsilon} S_\rho|} \int_{I_\rho} z_{\varepsilon,\delta}(\hat{x}, x_3) dx_3$$

clearly belongs to  $\text{Adm}_{C_{\varepsilon,\rho}}(\omega, u(x_0), \delta)$ . Therefore, according to Jensen's inequality and from the  $p$ -homogeneity of  $f^{\infty,p}$

$$\begin{aligned} \int_{Q_\rho} \widehat{f^{\infty,p}}(\varepsilon \hat{\nabla} u_\varepsilon) dx &\geq \left( \frac{|C_{\varepsilon,\rho}|}{|\frac{1}{\varepsilon} S_\rho|} \right)^p \frac{\mathbb{S}_{C_{\varepsilon,\rho}}(\omega, u(x_0), \delta)}{|C_{\varepsilon,\rho}|} - \beta \left[ \frac{\varepsilon^{p\gamma}}{(1-\delta)^p} + \varepsilon^a \right] - \frac{C(\rho)}{\nu} - r_\varepsilon(\rho) \\ &\quad - C \left| u(x_0) - \int_{Q_\rho} \mathbf{1}_{Q_\rho \setminus (T_\varepsilon)_\delta} \tilde{u}_{\varepsilon,\delta} dy \right|^p. \end{aligned} \quad (4.26)$$

It is easily seen that from (4.21) and the Lebesgue point Theorem, for a.e.  $x_0$  in  $\mathcal{O}$  one has

$$\lim_{\rho \rightarrow 0} \lim_{\delta \rightarrow 1} \lim_{\varepsilon \rightarrow 0} \left| u(x_0) - \int_{Q_\rho} \mathbf{1}_{Q_\rho \setminus (T_\varepsilon)_\delta} \tilde{u}_{\varepsilon,\delta} dy \right|^p = 0.$$

Moreover  $\lim_{\varepsilon \rightarrow 0} \frac{|C_{\varepsilon,\rho}|}{|\frac{1}{\varepsilon} S_\rho|} = 1$ . On the other hand, according to Lemma 1.1, Theorem 5.2 and Section 6.2 of [11], for  $\mathbf{P}$ -almost every  $\omega \in \Omega$  and for every  $\rho > 0$  one has

$$\lim_{\delta \rightarrow 1} \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{S}_{C_{\varepsilon,\rho}}(\omega, u(x_0), \delta)}{|C_{\varepsilon,\rho}|} = \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{S}_{C_{\varepsilon,\rho}}(\omega, u(x_0))}{|C_{\varepsilon,\rho}|} = f_0(u(x_0)). \quad (4.27)$$

Then, letting successively  $\varepsilon \rightarrow 0$ ,  $\delta \rightarrow 1$ ,  $\nu \rightarrow \infty$  and  $\rho \rightarrow 0$  in (4.26), we obtain for  $\mathbf{P}$ -almost every  $\omega \in \Omega$  and for almost every  $x_0 \in \mathcal{O}$ ,

$$\liminf_{\varepsilon \rightarrow 0} \int_{Q_\rho} f^{\infty,p}(\varepsilon \nabla u_\varepsilon) dx \geq f_0(u(x_0))$$

which ends the proof.

**Proof of (4.18).** Fix  $\omega$  in the set  $\Omega^n$  of full probability given in Proposition 3.1 and assume that

$\liminf_{\varepsilon \rightarrow 0} G_\varepsilon(u_\varepsilon) < +\infty$ . According to the Moreau-Rockafellar duality principle we infer that for all  $\phi$  in  $L^q(\mathcal{O})$ :

$$\begin{aligned}
\liminf_{\varepsilon \rightarrow 0} G_\varepsilon(u_\varepsilon) &= \liminf_{\varepsilon \rightarrow 0} \varepsilon^{p-a} \int_{\mathcal{O}} \mathbf{1}_{D(\omega) \cap \widehat{\mathcal{O}}(\frac{\hat{x}}{\varepsilon})} g^{\infty,p}(\widehat{\nabla} u_\varepsilon, \frac{1}{\varepsilon^p} \frac{\partial u_\varepsilon}{\partial x}) dx \\
&\geq \liminf_{\varepsilon \rightarrow 0} \int_{\mathcal{O}} \mathbf{1}_{D(\omega) \cap \widehat{\mathcal{O}}(\frac{\hat{x}}{\varepsilon})} (g^{\infty,p})^\perp(\varepsilon^{-\gamma} \frac{\partial u_\varepsilon}{\partial x_3}) dx \\
&\geq \liminf_{\varepsilon \rightarrow 0} \left( \int_{\mathcal{O}} \mathbf{1}_{D(\omega) \cap \widehat{\mathcal{O}}(\frac{\hat{x}}{\varepsilon})} \phi \cdot \varepsilon^{-\gamma} \frac{\partial u_\varepsilon}{\partial x_3} dx - \int_{\mathcal{O}} \mathbf{1}_{D(\omega) \cap \widehat{\mathcal{O}}(\frac{\hat{x}}{\varepsilon})} (g^{\infty,p})^{\perp,*}(\phi) dx \right) \\
&= \int_{\mathcal{O}} \phi \cdot \frac{\partial v}{\partial x_3} dx - \theta \int_{\mathcal{O}} (g^{\infty,p})^{\perp,*}(\phi) dx \\
&= \theta \left[ \int_{\mathcal{O}} \frac{1}{\theta} \phi \frac{\partial v}{\partial x_3} dx - \int_{\mathcal{O}} (g^{\infty,p})^{\perp,*}(\phi) dx \right].
\end{aligned}$$

By taking the supremum over all functions  $\phi$  in  $\phi \in L^q(\mathcal{O})$  we finally obtain

$$\begin{aligned}
\liminf_{\varepsilon \rightarrow 0} G_\varepsilon(u_\varepsilon) &\geq \theta \sup_{\phi \in L^q(\mathcal{O})} \left[ \int_{\mathcal{O}} \frac{1}{\theta} \phi \frac{\partial v}{\partial x_3} dx - \int_{\mathcal{O}} (g^{\infty,p})^{\perp,*}(\phi) dx \right] \\
&= \theta \int_{\mathcal{O}} (g^{\infty,p})^\perp \left( \frac{1}{\theta} \frac{\partial v}{\partial x_3} \right) dx = \theta^{1-p} \int_{\mathcal{O}} (g^{\infty,p})^\perp \left( \frac{\partial v}{\partial x_3} \right) dx
\end{aligned}$$

which completes the proof.  $\square$

#### 4.4 Proof of Corollary 1.1

By applying the variational property of the convergence established in Theorem 1.1, and computing the Euler equation associated with the minimization problem  $\min \{E_0(u, v) : (u, v) \in L^p(\mathcal{O}) \times V_0(\mathcal{O})\}$  to obtain:

$$\left\{ \begin{array}{l} \partial f_0(\bar{u}(\hat{x})) \ni \int_0^1 L(\hat{x}, s) ds, \\ -\frac{\partial}{\partial x_3} \left( \frac{dg^{\infty,p}}{ds} \right)^\perp \left( \frac{\partial v}{\partial x_3} \right) = 0 \text{ in } \mathcal{O}, \\ v(\hat{x}, 0) = 0 \text{ on } \widehat{\mathcal{O}}, \\ D(g^{\infty,p})^\perp \left( \frac{\partial v}{\partial x_3} \right) = \theta^{p-1} \tilde{l} \text{ on } \widehat{\mathcal{O}} + \{1\}. \end{array} \right.$$

To end the proof it suffices to apply the subdifferential rule:

$$a^* \in \partial f_0(a) \iff a \in \partial f_0^*(a^*).$$

#### 4.5 Proof of Proposition 1.1

*Proof.* Clearly  $\hat{A} \mapsto \mathbb{S}_{\hat{A}}$  is a deterministic subadditive process and the covariance property becomes  $\mathbb{S}_{\hat{A}+z} = \mathbb{S}_{\hat{A}}$  for all  $z \in \mathbf{Z}^2$ . Therefore, for all  $s \in \mathbb{R}$ ,

$$f_0(s) = \inf_{n \in \mathbb{N}^*} \left( \frac{\mathbb{S}_{(0,n)^2}(s)}{n^2} \right). \quad (4.28)$$

Fix  $s \in \mathbb{R}$ . We first establish the inequality

$$f_0(s) \geq \inf \left\{ \int_{\hat{Y}} \widehat{f^{\infty,p}}(\nabla w) d\hat{y} : w \in W_{\#}^{1,p}(\hat{Y}), \int_{\hat{Y}} w d\hat{y} = s, w = 0 \text{ on } D \right\}. \quad (4.29)$$

Let  $n$  be fixed in  $\mathbb{N}^*$ ,  $\psi$  any element in  $\text{Adm}_{n\hat{Y}}(s)$  and  $w_{\#}$  be a solution of the minimization problem

$$\min \left\{ \int_{\hat{Y}} \widehat{f^{\infty,p}}(\nabla w) d\hat{y} : w \in W_{\#}^{1,p}(\hat{Y}), \int_{\hat{Y}} w d\hat{y} = s, w = 0 \text{ on } D \right\}.$$

Let denote by  $\lambda$  the Lagrange multiplier associated with the constraint  $\int_{\hat{Y}} w \, d\hat{y} = s$ . The Euler equation of the problem (4.29) is

$$\begin{cases} -\operatorname{div}(\widehat{\partial f^{\infty,p}})(\nabla w_{\#}) = \lambda w_{\#} \text{ on } \hat{Y}, \\ w_{\#} \in W_{\#}^{1,p}(\hat{Y}), \\ \int_{\hat{Y}} w_{\#} d\hat{y} = s. \end{cases} \quad (4.30)$$

The periodic expansion of  $w_{\#}$ , still denoted by  $w_{\#}$  then satisfies <sup>2</sup>

$$\begin{cases} -\operatorname{div}(\widehat{\partial f^{\infty,p}})(\nabla w_{\#}) = \lambda w_{\#} \text{ in } n\hat{Y}, \\ w_{\#} \in W_{\#}^{1,p}(n\hat{Y}), \\ \int_{n\hat{Y}} w_{\#} d\hat{y} = s. \end{cases} \quad (4.31)$$

According to the subdifferential inequality, we have

$$\begin{aligned} \int_{n\hat{Y}} \widehat{f^{\infty,p}}(\nabla \psi(x)) dx &\geq \int_{n\hat{Y}} \widehat{f^{\infty,p}}(\nabla w_{\#}(x)) dx + \int_{n\hat{Y}} \widehat{\partial f^{\infty,p}}(\nabla w_{\#}(x)) \cdot (\nabla \psi(x) - \nabla w_{\#}(x)) dx \\ &= \int_{\hat{Y}} \widehat{f^{\infty,p}}(\nabla w_{\#}(x)) dx + \int_{n\hat{Y}} \widehat{\partial f^{\infty,p}}(\nabla w_{\#}(x)) \cdot (\nabla \psi(x) - \nabla w_{\#}(x)) dx. \end{aligned}$$

In order to establish inequality (4.29), it suffices to show that the second term in the right hand side is equal to zero. Integrating by part we obtain

$$\begin{aligned} \int_{n\hat{Y}} \widehat{\partial f^{\infty,p}}(\nabla w_{\#}(x)) \cdot (\nabla \psi(x) - \nabla w_{\#}(x)) dx &= - \int_{n\hat{Y}} \operatorname{div}(\widehat{\partial f^{\infty,p}}(\nabla w_{\#}(x))) \cdot (\psi(x) - w_{\#}(x)) dx \\ &\quad + \int_{\partial(n\hat{Y})} \widehat{\partial f^{\infty,p}}(\nabla w_{\#}(x)) \nu \cdot (\psi(x) - w_{\#}(x)) dx \end{aligned}$$

where  $\nu$  is the unit outer normal to  $n\hat{Y}$ . Since  $\widehat{\partial f^{\infty,p}}(\nabla w_{\#}(x)) \cdot \nu$  is antiperiodic, we have

$$\int_{\partial(n\hat{Y})} \widehat{\partial f^{\infty,p}}(\nabla w_{\#}(x)) \nu \cdot (\psi(x) - w_{\#}(x)) dx = 0.$$

On the other hand from (4.31), the fact that  $\psi \in \operatorname{Adm}_{n\hat{Y}}(s)$ , and from the periodicity of  $w_{\#}$ , we have

$$\begin{aligned} \int_{n\hat{Y}} \operatorname{div}(\widehat{\partial f^{\infty,p}}(\nabla w_{\#}(x))) \cdot (\psi(x) - w_{\#}(x)) dx &= \int_{n\hat{Y}} \lambda \cdot (\psi(x) - w_{\#}(x)) dx \\ &= \int_{n\hat{Y}} \lambda \psi(x) dx - \int_{n\hat{Y}} \lambda w_{\#}(x) dx \\ &= \lambda \cdot s - \lambda \cdot s \end{aligned}$$

then (4.29) is established.

We establish the inequality

$$f_0(s) \leq \inf \left\{ \int_{\hat{Y}} \widehat{f^{\infty,p}}(\nabla w) \, d\hat{y} : w \in W_{\#}^{1,p}(\hat{Y}), \int_{\hat{Y}} w \, d\hat{y} = s, w = 0 \text{ on } D \right\}. \quad (4.32)$$

Let  $w_{\#}$  be a solution of the problem

$$\min \left\{ \int_{\hat{Y}} \widehat{f^{\infty,p}}(\nabla w) \, d\hat{y} : w \in W_{\#}^{1,p}(\hat{Y}), \int_{\hat{Y}} w \, d\hat{y} = s, w = 0 \text{ on } D \right\}. \quad (4.33)$$

---

<sup>2</sup> To simplify the notation, we denote abusively  $\widehat{\partial f^{\infty,p}}(\nabla w_{\#}(x))$  to denote any element of the set  $\widehat{\partial f^{\infty,p}}(\nabla w_{\#}(x))$

Let us expand as previously the function  $w_\#$  by  $\hat{Y}$ -periodicity in  $\mathbb{R}^2$ . We then obtain a function still denoted by  $w_\#$  in  $W_{loc}^{1,p}(\mathbb{R}^2)$ . For all  $n \in \mathbb{N}$ , and all  $\hat{x} \in \mathbb{R}^2$ , set  $w_n(\hat{x}) := w_\#(n\hat{x})$ . We use now the following result of Proposition 4.2: for all  $u \in L^p(\mathcal{O})$ , and all sequence  $(u_\varepsilon)_{\varepsilon>0}$  such that  $u_\varepsilon \rightharpoonup u$  dans  $L^p(\mathcal{O})$ , we have

$$\int_{\hat{\mathcal{O}}} f_0(u) d\hat{x} \leq \liminf_{\varepsilon \rightarrow 0} \int_{\mathcal{O} \setminus T_\varepsilon(\omega)} \widehat{f^{\infty,p}}(\varepsilon \nabla u_\varepsilon) dx.$$

We apply this estimate with

$$\varepsilon = \frac{1}{n}, \quad \mathcal{O} = \hat{Y} \times (0, 1), \quad w_n = u_\varepsilon \text{ and } u = s.$$

Clearly the periodicity of  $w_n$  yields the weak convergence  $w_n \rightharpoonup f_{\hat{Y}} w_\# = s$  in  $L^p(\hat{Y} \times (0, 1))$ . Since  $w_n = 0$  in  $D$ , we then obtain,

$$\begin{aligned} \int_{\hat{Y}} f_0(u) d\hat{x} = f_0(s) &\leq \liminf_{\varepsilon \rightarrow 0} \int_0^1 \int_{\hat{Y}} \widehat{f^{\infty,p}}\left(\frac{1}{n} \nabla w_n(\hat{x})\right) d\hat{x} dx_3 \\ &= \liminf_{\varepsilon \rightarrow 0} \int_{\hat{Y}} \widehat{f^{\infty,p}}(\nabla w_\#(n\hat{x})) d\hat{x}. \end{aligned}$$

The change of variable  $n\hat{x} = y$  then gives

$$\begin{aligned} f_0(s) &\leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{n^2} \int_{n\hat{Y}} \widehat{f^{\infty,p}}(\nabla w_\#(y)) dy \\ &= \liminf_{\varepsilon \rightarrow 0} \int_{\hat{Y}} \widehat{f^{\infty,p}}(\nabla w_\#(y)) dy \\ &= \int_{\hat{Y}} \widehat{f^{\infty,p}}(\nabla w_\#(y)) dy \end{aligned}$$

which completes the proof since  $w_\#$  is a minimizer of (4.33). □

## 5 Numerical results when $f = g = \frac{1}{2} |\cdot|^2$

### 5.1 Computation of $\bar{u}$

In this section we establish the expression of  $\bar{u}$  when  $f = \frac{1}{2} |\cdot|^2$ . With the notation of Corollary 1.1 we have:

**Proposition 5.1.** *Let denote by  $U_n(\omega, \cdot)$  the unique solution of the scalar random Dirichlet problem*

$$\mathcal{P}_1 \begin{cases} -\Delta U = 1 \text{ in } n\hat{Y} \setminus D(\omega), \\ U \in W_0^{1,2}(n\hat{Y} \setminus D(\omega)), \end{cases} \quad (5.1)$$

and set  $\Lambda_n(\omega) := \int_{n\hat{Y}} U_n(\omega, \hat{x}) d\hat{x}$ . Then for  $\mathbf{P}$  a.e.  $\omega \in \Omega$ ,  $\Lambda_n(\omega)$  converges to a deterministic value  $\Lambda > 0$  and  $\bar{u}$  is uniquely determined by the formula

$$\bar{u}(\hat{x}) = \Lambda \int_0^1 L(\hat{x}, t) dt. \quad (5.2)$$

*Proof.* Consider the Lagrange multiplier  $\lambda^{s,n}(\omega) \in \mathbb{R}$  of the optimization problem

$$f_n(\omega, s) := \inf \left\{ \frac{1}{2} \int_{n\hat{Y}} |\nabla w|^2 d\hat{x} : w \in W_0^{1,2}(n\hat{Y} \setminus D(\omega)), \int_{n\hat{Y}} w d\hat{x} = s \right\},$$

whose (random) minimizer  $w^s$  satisfies

$$\begin{cases} -\Delta w^{s,n} = \lambda^{s,n}(\omega) \text{ in } n\hat{Y} \setminus D(\omega), \\ w^{s,n} \in W_0^{1,2}(n\hat{Y} \setminus D(\omega)), \\ \int_{n\hat{Y}} w^{s,n} d\hat{x} = s. \end{cases} \quad (5.3)$$

Applying (5.3) for  $s = 1$  we deduce  $f_n(\omega, 1) = \frac{1}{2}\lambda^{1,n}(\omega)$ . This proves that  $f_n(\omega, \cdot)$  is a quadratic form (note that  $f_n(\omega, \cdot)$  is homogeneous of degree 2), and that  $f_n(\omega, s) = \frac{1}{2}\lambda^{1,n}(\omega)s^2$ .

Let us compute  $\lambda^{1,n}(\omega)$ . Since  $\lambda^{1,n}(\omega) = 2f_n(\omega, 1)$ , one has  $\lambda^{1,n}(\omega) \geq 2\alpha > 0$  and from (5.3), we infer that  $\frac{w^{1,n}(\omega)}{\lambda^{1,n}(\omega)}$  solves the scalar Dirichlet problem

$$\begin{cases} -\Delta U_n = 1 \text{ in } n\hat{Y} \setminus D(\omega), \\ U_n \in W_0^{1,2}(n\hat{Y} \setminus D(\omega)). \end{cases} \quad (5.4)$$

Let us denote by  $U_n(\omega)$  its unique solution, then

$$\int_{n\hat{Y}} \frac{w^{1,n}(\omega)}{\lambda^{1,n}(\omega)} d\hat{x} = \int_{n\hat{Y}} U_n d\hat{x}$$

yields

$$\lambda^{1,n}(\omega) = \frac{\int_{n\hat{Y}} w^{1,n}(\omega) d\hat{x}}{\int_{n\hat{Y}} U_n d\hat{x}}.$$

But from (5.3)

$$\int_{n\hat{Y}} w^{1,n}(\omega) d\hat{x} = 1,$$

so that

$$\lambda^{1,n}(\omega) = \frac{1}{\int_{n\hat{Y}} U_n d\hat{x}}.$$

According to the Akcoglu-Krengel subadditive ergodic theorem we are going to establish that  $\Lambda_n(\omega) := \int_{n\hat{Y}} U_n(\omega) d\hat{x}(\omega)$  almost surely converges towards a constant  $\Lambda > 0$ : indeed from (5.4) the function  $U_n$  satisfies

$$\int_{n\hat{Y}} |\nabla U_n|^2 d\hat{x} = \int_{n\hat{Y}} U_n d\hat{x}$$

thus, since

$$\begin{aligned} \frac{1}{2} \int_{n\hat{Y}} |\nabla U_n|^2 d\hat{x} - \int_{n\hat{Y}} U_n d\hat{x} &= \inf \left\{ \frac{1}{2} \int_{n\hat{Y}} |\nabla U|^2 d\hat{x} - \int_{n\hat{Y}} U d\hat{x} : U \in W_0^{1,2}(n\hat{Y} \setminus D(\omega)) \right\}, \\ \Lambda_n(\omega) &= -2 \frac{\mathbb{S}_{n\hat{Y}}}{n^2} \end{aligned}$$

where, for all interval  $\hat{A}$  generated by  $\hat{Y}$ ,

$$\tilde{\mathbb{S}}_{\hat{A}} := \inf \left\{ \frac{1}{2} \int_{\hat{A}} |\nabla U|^2 d\hat{x} - \int_{\hat{A}} U d\hat{x} : U \in W_0^{1,2}(\hat{A} \setminus D(\omega)) \right\}.$$

Therefore it suffices to notice that  $\hat{A} \mapsto \tilde{\mathbb{S}}_{\hat{A}}$  is a subadditive process and to apply the Akcoglu-Krengel subadditive theorem [1] to infer the almost sure convergence of  $\Lambda_n(\omega)$ . For a detailed proof we refer to [7]. Consequently, for  $\mathbf{P}$ - a.e.  $\omega \in \Omega$  and for all  $s \in \mathbb{R}$ ,

$$\lim_{n \rightarrow +\infty} f_n(\omega, s) = f_0(s) = \frac{1}{2\Lambda} s^2$$

and  $\partial f_0(s) = \frac{1}{\Lambda} s$ . The conclusion follows from  $\partial f_0^*(s) = \Lambda s$ .  $\square$

## 5.2 Numerical computations of $\Lambda$

We compute a numerical approximation  $\Lambda_n(\omega)$  of the constant  $\Lambda$  determined in the preceding section which provides an approximation of  $\bar{u}$  thanks to (5.2). We take  $a = 4$ ,  $p = 2$  and  $b = 3$  (i.e  $b = p - 1 + \frac{a}{p}$ ) and we make use of the cast3M program [13] to solve the scalar random Dirichlet problem (5.1) in three geometrical situations: in the periodic case, in the situation of the random checkerboard-like with  $\Omega = \Omega_0^{\mathbb{Z}^2}$ ,  $\Omega_0$  made up of 9 points, and a general ergodic situation. More precisely, in the random checkerboard-like case, the sections of the fibers are randomly placed following 9 places in each cell (in the 4 corners, the 4 sides and in the center), and we consider a configuration which is neither periodic nor a checkerboard-like's case but satisfies the axioms described in in section 3.

The first step of the computation consists in constructing the triangulation mesh of the random set  $n\hat{Y} \setminus D(\omega)$  (Figure 2, Figure 3, Figure 4).

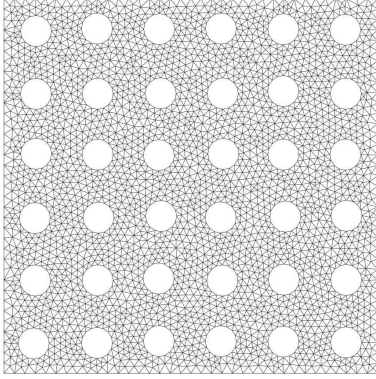


Figure 2: A periodic triangle mesh ( $n=6$ ).

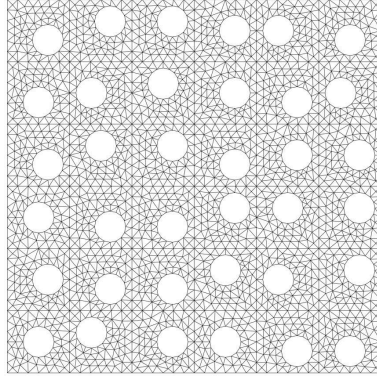


Figure 3: A "checkerboard-like" random triangle mesh ( $n=6$ ).

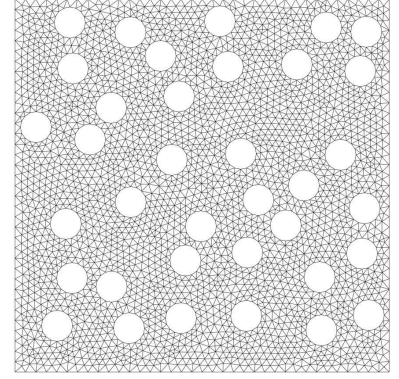


Figure 4: A general ergodic triangle mesh ( $n=6$ ).

The second step concerns the evolution of  $n \mapsto \Lambda_n(\omega)$ . Figure 5 represents the evolution of  $n \mapsto \Lambda_n(\omega)$  for various realizations  $\omega$  with equi-probability presence (for the checkerboard-like's case that corresponds to  $\alpha_k = \frac{1}{9}$ ,  $k = 1, \dots, 9$ ) when  $n$  increases. For each  $\omega$  we can see that the sequence  $(\Lambda_n(\omega))_{n \in \mathbb{N}}$  converges to the same constant  $\Lambda$  (cf Figures 5 and 6). This illustrates the ergodic hypothesis.

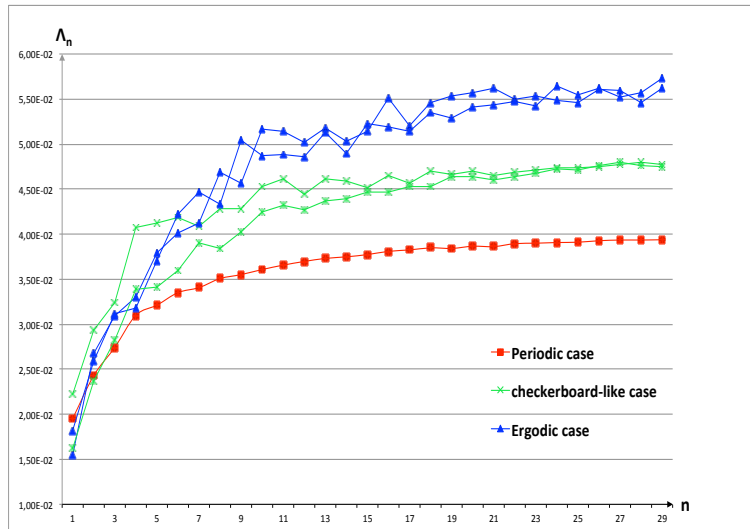


Figure 5: The curves  $n \mapsto \Lambda_n(\omega)$  for various realizations  $\omega$  with equi-probability presence in our three random situations.

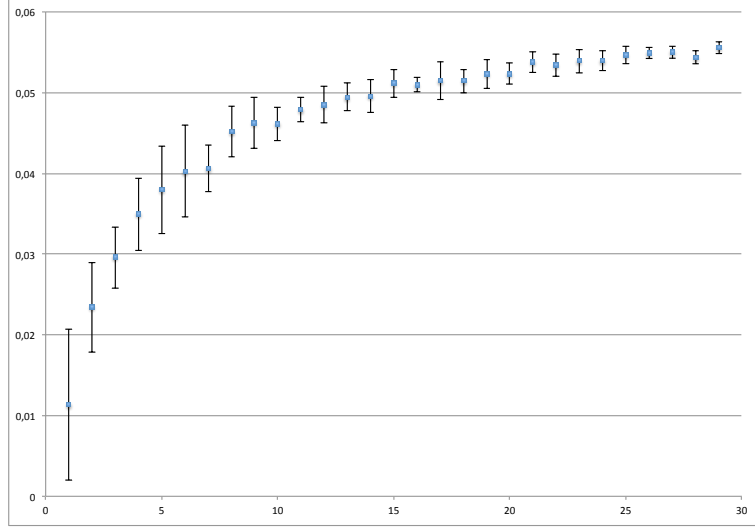


Figure 6: Mean and standard deviation of  $n \mapsto \Lambda_n(\omega)$  for ten realizations  $\omega$  in the general ergodic situation.

### 5.3 Estimate of the error between the solutions of $\mathcal{P}_{\varepsilon, h(\varepsilon)}(\omega)$ and $\mathcal{P}$ in the scalar case

In order to validate our theoretical results in a scalar situation, we simulate the evolution of a suitable error between  $\hat{x} \mapsto \bar{u}_\varepsilon(\omega, \hat{x}, 0)$  and  $\bar{u}$  on the one hand, and  $\bar{v}_\varepsilon := 1_{T_\varepsilon(\omega)} \varepsilon^{-\gamma} \bar{u}_\varepsilon$  and  $\bar{v}$  on the other hand when  $\varepsilon$  decreases to 0. Recall that  $\bar{u}_\varepsilon$ ,  $\bar{v}$  and  $\bar{u}$  are solutions to the problems

INITIAL PROBLEM

$$\left\{ \begin{array}{l} -\operatorname{div}(a_\varepsilon(\omega, x) \nabla \bar{u}_\varepsilon) = \mathcal{L}_\varepsilon \quad \text{in } \mathcal{O}_{h(\varepsilon)}, \\ \bar{u}_\varepsilon \in W^{1,2}(\mathcal{O}_{h(\varepsilon)} \setminus T_\varepsilon(\omega)), \\ \bar{u}_\varepsilon = 0 \quad \text{on } \mathcal{O}_{h(\varepsilon)} \cap (D_\varepsilon(\omega) \times \{0\}) \\ \frac{1}{\varepsilon^4} \frac{\partial \bar{u}_\varepsilon}{\partial x_3} = l_\varepsilon \quad \text{on } \mathcal{O}_{h(\varepsilon)} \cap (D_\varepsilon(\omega) \times \{h(\varepsilon)\}) \end{array} \right.$$

LIMIT PROBLEM

$$\left\{ \begin{array}{l} \bar{u} = \Lambda \bar{L} \quad p.p \text{ in } \mathcal{O}, \\ -\frac{\partial^2 \bar{v}}{\partial x_3^2} = 0 \quad p.p \text{ in } \mathcal{O}, \\ \bar{v} = 0 \quad p.p \text{ in } \hat{\mathcal{O}} \times \{0\}, \\ \frac{\partial \bar{v}}{\partial x_3} = \theta \tilde{l} \quad p.p \text{ on } \hat{\mathcal{O}} \times \{1\}. \end{array} \right.$$

A numerical approximation of  $\theta$  is obtained by averaging  $|\hat{Y} \cap D(\omega)|$  over 10000 drawings. We perform the calculations with  $f = g = \frac{1}{2}|\cdot|^2$ ,  $p = 2$ ,  $a = 4$ ,  $b = \gamma = p - 1 + \frac{a}{p}$ ,  $\mathcal{L}_\varepsilon = \varepsilon^{-p}$ , and  $l_\varepsilon = \varepsilon^{-b}$ .

For any function  $w$  in  $L^2(\mathcal{O})$  we denote by  $\tilde{w}$  its numerical approximation in  $\mathbb{R}^N$ , where  $N$  is the number of nodes given by the software Cast3M [13]. Furthermore  $\|\cdot\|_2$  denotes the euclidian norm in  $\mathbb{R}^N$ . Precisely, we compute the relative error  $\frac{\|\tilde{\bar{u}}_\varepsilon(\hat{x}, 0) - \tilde{\bar{u}}(\hat{x})\|_2}{\|\tilde{\bar{u}}(\hat{x})\|_2}$  and  $\frac{\|\tilde{\bar{v}}_\varepsilon(x) - \tilde{\bar{v}}(x)\|_2}{\|\tilde{\bar{v}}(x)\|_2}$ . We illustrate these two convergences in Figures 7 et 8 respectively by taking  $h(\varepsilon) = \varepsilon^2 = \frac{1}{n^2}$  where  $n$  is the size of the cell  $n\hat{Y}$ , and with an arbitrary realization of  $\omega$ . At the end of the process, the error in the matrix is  $< 8\%$  and in the fibers is  $< 4\%$ .



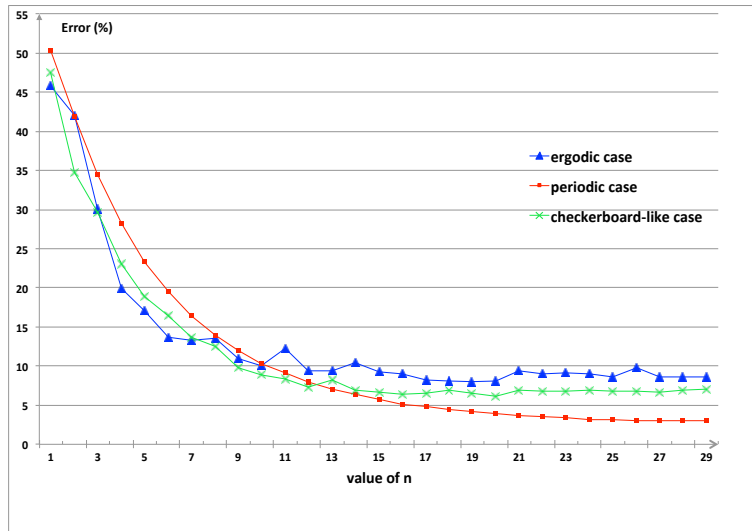


Figure 7: Error convergence in the matrix

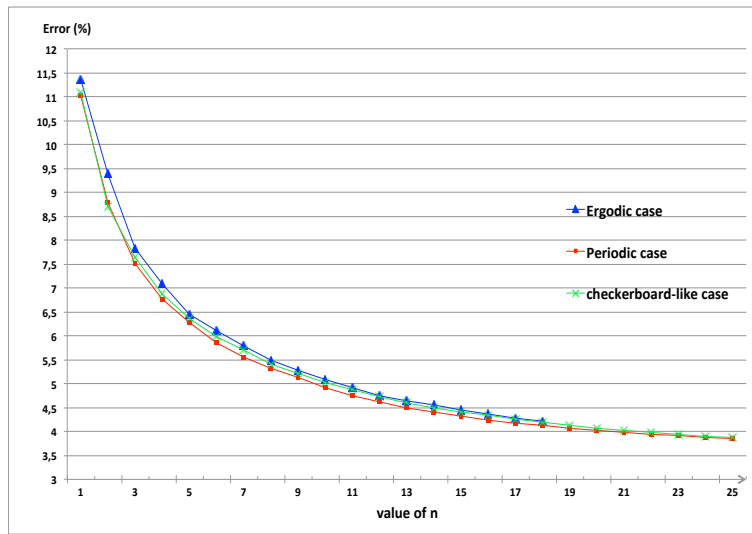


Figure 8: Error convergence in the fiber

## 5.4 Appendix

### 5.5 Discrete subadditive processes

Let us denote by  $\mathcal{I}$  the set of all the half open intervals  $[a, b)$  with  $a$  and  $b$  in  $\mathbf{Z}^N$ . Given a dynamical system  $(\Omega, \mathcal{A}, \mathbf{P}, (T_z)_{z \in \mathbf{Z}^N})$ , i.e., a probability space  $(\Omega, \mathcal{A}, \mathbf{P})$  equipped with a group  $(T_z)_{z \in \mathbf{Z}^N}$  of  $\mathbf{P}$ -preserving transformation on  $\Omega$ , we call discrete subadditive process covariant with respect to  $(T_z)_{z \in \mathbf{Z}^N}$ , a set function  $\mathbb{S} : \mathcal{I} \rightarrow L^1_{\mathbf{P}}(\Omega)$  satisfying

- (i) for every  $I \in \mathcal{I}$  such that there exists a finite family  $(I_j)_{j \in J}$  of disjoint intervals in  $\mathcal{I}$  with  $I = \bigcup_{j \in J} I_j$ ,

$$\mathbb{S}_I(\cdot) \leq \sum_{j \in J} \mathbb{S}_{I_j}(\cdot),$$

- (ii)  $\forall I \in \mathcal{I}, \forall z \in \mathbf{Z}^N, \mathbb{S}_I \circ \tau_z = \mathbb{S}_{z+I}$

- (iii)  $\gamma(\mathbb{S}) := \inf \left\{ \int_{\Sigma} \frac{\mathbb{S}_I}{|I|} d\mathbf{P} : I \in \mathcal{I} \right\} > -\infty$ .

A sequence  $(I_n)_{n \in \mathbb{N}}, I_n \in \mathcal{I}$  is said to be regular if there exists a nondecreasing sequence  $(I'_n)_{n \in \mathbb{N}}$  of sets in  $\mathcal{I}$  and a constant  $C_{reg} > 0$  such that  $I_n \subset I'_n$  and  $\sup_{n \in \mathbb{N}} |I'_n|/|I_n| \leq C_{reg}$ .

Let us denote by  $\mathcal{F}$  the  $\sigma$ -algebra of invariant sets of  $\mathcal{A}$  by the group  $(T_z)_{z \in \mathbf{Z}^N}$ , i.e.,  $E \in \mathcal{F}$  iff  $T_z E = E$  for all  $z \in \mathbf{Z}^N$ , the dynamical system  $(\Omega, \mathcal{A}, \mathbf{P}, (T_z)_{z \in \mathbf{Z}^N})$  is said to be ergodic if  $\mathcal{F}$  is made up of sets  $E$  satisfying  $\mathbf{P}(E) \in \{0, 1\}$ .

The ergodic theorem below is a crucial tool in stochastic homogenization.

**Theorem 5.1.** *Let  $\mathbb{S} : \mathcal{I} \rightarrow L^1_{\mathbf{P}}(\Omega)$  be a discrete subadditive process and  $(I_n)_{n \in \mathbb{N}}$  a regular sequence of  $\mathcal{I}$  satisfying  $\lim_{n \rightarrow +\infty} \rho(I_n) = +\infty$ . Then for  $\mathbf{P}$  almost every  $\omega \in \Omega$*

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{S}_{I_n}(\omega)}{|I_n|} = \inf_{m \in \mathbb{N}^*} \mathbf{E}^{\mathcal{F}} \frac{\mathbb{S}_{[0, m]^N}}{m^N}(\omega).$$

*If moreover the dynamical system  $(\Omega, \mathcal{A}, \mathbf{P}, (T_z)_{z \in \mathbf{Z}^N})$  is ergodic, then*

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{S}_{I_n}(\omega)}{|I_n|} = \inf_{m \in \mathbb{N}^*} \mathbf{E} \frac{\mathbb{S}_{[0, m]^N}}{m^N} = \gamma(\mathbb{S}).$$

For a proof, we refer the reader to [1, 11]. One can also consider the restriction of  $\mathbb{S}$  to the family  $\overset{\circ}{\mathcal{I}}$  of open intervals  $(a, b)$  with  $a$  and  $b$  in  $\mathbf{Z}^N$ . In that case, subadditivity condition (i) can be weakened in the following sense (see [1, 11]):

- (i)' for every  $I \in \mathcal{I}$  such that there exists a finite family  $(I_j)_{j \in J}$  of disjoint intervals in  $\overset{\circ}{\mathcal{I}}$  with  $|I \setminus \bigcup_{j \in J} I_j| = 0$ ,

$$\mathbb{S}_I(\cdot, s) \leq \sum_{j \in J} \mathbb{S}_{I_j}(\cdot, s).$$

The conclusion of Theorem 5.1 remains valid under these conditions.

### 5.6 Proof of Lemma 1.1

We reproduce with minor change the proofs of Theorem 2.2 and Proposition 2.1 in [12]. For any  $\delta > 0$  and any set  $\hat{A}$  of  $\mathbb{R}^2$ , we make use of the following notation:  $\hat{A}_\delta := \left\{ x \in \hat{A} : d(x, \mathbb{R}^2 \setminus \hat{A}) > \delta \right\}$ . For any bounded Borel set  $A$  of  $\mathbb{R}^2$ ,  $\#(A)$  denotes its cardinal when it is finite.

From now on  $\overset{\circ}{\mathcal{I}}$  denotes the set of all open intervals  $(a, b)$  with  $a$  and  $b$  in  $\mathbf{Z}^2$ . We recall the following notation in Section 1: for all  $\hat{A} \in \overset{\circ}{\mathcal{I}}$  and all  $s$  in  $\mathbb{R}$

$$\begin{aligned} \mathbb{S}_{\hat{A}}(\omega, s) &:= \inf \left\{ \int_{\hat{A} \setminus \overline{D(\omega)}} \widehat{f^{\infty, p}}(\nabla w(\hat{x})) \, d\hat{x} : w \in \text{Adm}_{\hat{A}}(\omega, s) \right\}, \\ \text{Adm}_{\hat{A}}(\omega, s) &:= \{w \in W_0^{1, p}(\hat{A} \setminus \overline{D(\omega)}) : \int_{\hat{A}} w \, dx = s\}. \end{aligned}$$

Note that the random set  $D(\omega)$  is not necessarily included in  $\hat{A}$ . It is standard to see that the random functionals defined in the introduction are measurable when  $\Omega \times L^p(\mathcal{O})$  is equipped with the product  $\sigma$ -algebra  $\mathcal{A} \otimes \mathcal{B}$  where  $\mathcal{B}$  is the Borel  $\sigma$ -algebra associated with the normed space  $L^p(\mathcal{O})$ . Consequently, for all fixed  $\hat{A}$  in  $\overset{\circ}{\mathcal{I}}$  and all fixed  $s$  in  $\mathbb{R}$ , the map  $\omega \mapsto \mathbb{S}_{\hat{A}}(\omega, s)$  is measurable.

For each fixed  $s$  in  $\mathbb{R}$ , it is easily seen that  $\mathbb{S}(\cdot, s)$  satisfies the subadditivity condition (i)' in  $L^1_{\mathbf{P}}(\Omega)$ . Moreover  $\mathbb{S}(\cdot, s)$  is clearly covariant with respect to the group  $(\tau_z)_{z \in \mathbf{Z}^2}$ , i.e., for all  $\hat{A} \in \overset{\circ}{\mathcal{I}}$  and all  $z \in \mathbf{Z}^2$ ,

$$\mathbb{S}_{\hat{A}+z}(\cdot, s) = \mathbb{S}_{\hat{A}}(\cdot, s) \circ \tau_z.$$

We are going to show that  $\text{Adm}_{\hat{A}}(\omega, s) \neq \emptyset$  and that  $\mathbb{S}_{\hat{A}} \in L^1(\Omega, \mathcal{A}, \mathbf{P})$  by establishing (1.6), then to apply the Akcoglu-Krengel subadditive ergodic theorem (see [1] and Annex). Fix  $\hat{A}$  in  $\overset{\circ}{\mathcal{I}}$ ,  $0 < \delta$  small enough, and set  $\phi_\delta(\omega, \cdot) = \rho_\delta * \mathbb{1}_{(\hat{A} \setminus D(\omega))_\delta}$  where  $\rho_\delta = \frac{1}{\delta^2} \rho(\frac{\cdot}{\delta})$  is a mollifier kernel defined from the standard mollifier  $\rho$  with support the unit ball of  $\mathbb{R}^2$ . Clearly we have

$$\phi_\delta(\omega, \hat{x}) = \begin{cases} 1 & \text{if } \hat{x} \in (\hat{A} \setminus D(\omega))_{2\delta}, \\ 0 & \text{if } \hat{x} \in \mathbb{R}^2 \setminus (\hat{A} \setminus D(\omega)) \end{cases}$$

and

$$\int_{\hat{A}} \phi_\delta \, d\hat{x} \geq \frac{|(\hat{A} \setminus D(\omega))_{2\delta}|}{|\hat{A}|}.$$

(In order to shorten the notation we omit the variable  $\omega$  for the function  $\phi_\delta$ ). Therefore, involving the family  $\bar{\omega}$  and using Remark 3.1, ii), we infer that for all  $\omega \in \Omega$

$$\begin{aligned} \int_{\hat{A}} \phi_\delta \, d\hat{x} &\geq \frac{|\sum_{z \in \hat{A} \cap \mathbf{Z}^2} (\hat{Y} + z \setminus D(\omega))_{2\delta}|}{|\hat{A}|} \\ &= \frac{|\sum_{z \in \hat{A} \cap \mathbf{Z}^2} (\hat{Y} \setminus D(\tau_z \omega))_{2\delta}|}{|\hat{A}|} \\ &\geq \frac{\#(\hat{A})}{|\hat{A}|} \left| (\hat{Y} \setminus D(\bar{\omega}))_{2\delta} \right| = \left| (\hat{Y} \setminus D(\bar{\omega}))_{2\delta} \right|. \end{aligned} \tag{5.5}$$

The random function defined by

$$w_\delta(\hat{x}, x_3) = s \frac{\phi_\delta(\hat{x})}{\int_{\hat{A}} \phi_\delta \, d\hat{x}}$$

belongs to  $\text{Adm}_{\hat{A}}(\omega, s)$ . From the definition of  $\rho_\delta * \mathbb{1}_{(\hat{A} \setminus D(\omega))_\delta}$  we have  $|\nabla \phi_\delta|_\infty \leq \frac{C}{\delta}$  for some positive constant  $C$  depending on  $\rho$ . Thus, from (5.5) and the growth condition (2.4) satisfied by  $f^{\infty, p}$ , we obtain

$$\begin{aligned} \mathbb{S}_{\hat{A}}(\omega, s) &\leq \int_{\hat{A} \setminus D(\omega)} f^{\infty, p}(\nabla w_\delta) \, d\hat{x} \\ &\leq \frac{C(p)}{\delta^p \left| (\hat{Y} \setminus D(\bar{\omega}))_{2\delta} \right|^p} |s|^p |\hat{A}|. \end{aligned}$$

where  $C(p) = \beta C^p$ .

It is easily seen that  $f_0$  is a positively  $p$ -homogeneous convex function. The upper bound in (2.4) is a straightforward consequence of (1.6), and (2.5) is deduced from (2.4) by a standard argument of convex analysis. It remains to establish that  $f_0(s) \geq \alpha |s|^p$  for all  $s \in \mathbb{R}$ . Let  $w \in \text{Adm}_{n\hat{Y}}(\omega, s)$ . By using Jensen's inequality for the convex function  $s \mapsto |s|^p$  and Poincaré-Wirtinger's inequality with optimal constant 1, we have

$$\begin{aligned} \alpha |s|^p &= \alpha \left| \int_{n\hat{Y}} w \, dx \right|^p \\ &\leq \alpha \int_{n\hat{Y}} |w|^p \, d\hat{x} \\ &\leq \alpha \int_{n\hat{Y}} |\hat{\nabla} w|^p \, d\hat{x} \\ &\leq \int_{n\hat{Y}} \widehat{f^{\infty,p}}(\hat{\nabla} w) \, d\hat{x}. \end{aligned}$$

We end the proof by taking the infimum over all the functions  $w$  in  $\text{Adm}_{n\hat{Y}}(\omega, s)$ .

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