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EXPONENTIAL DECAY FOR THE SCHRÖDINGER EQUATION ON A DISSIPATIVE WAVEGUIDE

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Abstract. We prove exponential decay for the solution of the Schrödinger equation on a dissipative waveguide. The absorption is effective everywhere on the boundary but the geometric control condition is not satisfied. The proof relies on separation of variables and the Riesz basis property for the eigenfunctions of the transverse operator. The case where the absorption index takes negative values is also discussed.

1. Introduction

Let $l > 0$ and $d \geqslant 2$. Let Ω denote the straight waveguide $\mathbb{R}^{d-1}\times]0,l[\subset \mathbb{R}^d$. We consider on Ω the Schrödinger equation with dissipative boundary condition

$$\begin{cases} i\partial_t u + \Delta u = 0 & \text{on }]0,+\infty[\times\Omega, \\ \partial_\nu u = iau & \text{on }]0,+\infty[\times\partial\Omega, \\ u(0,\cdot) = u_0 & \text{on } \Omega. \end{cases} \tag{1.1}$$

Here $u_0 \in L^2(\Omega)$ and ∂_ν denotes the outward normal derivative. The absorption index a belongs to $W^{1,\infty}(\partial\Omega)$. In the main result of the paper, a takes positive values, but we also discuss the case where a takes (small) negative values.

We will first prove well-posedness for this problem. Then it is standard computation to check that when $a \geqslant 0$ the norm of $u(t)$ in $L^2(\Omega)$ is non-increasing, and that the decay is due to the boundary condition:

$$\frac{d}{dt}\left\|u(t)\right\|^2_{L^2(\Omega)} = -2\int_{\partial\Omega} a\left|u(t)\right|^2 \leqslant 0.$$

Whether this norm goes to 0 for large times, and then the rate of decay, are questions which have been extensively studied in different contexts. For the Schrödinger equation as in the present paper, or for the (damped) wave equation which is a closely related problem.

Many papers deal with the wave equation on compact manifolds, with dissipation in the interior of the domain or at the boundary.

We know from [Har85, Leb96] that a weak assumption on the absorption index a (for instance the dissipation is effective on any open subset of the domain) is enough to ensure that the energy goes to 0 for any initial datum.

Uniform exponential decay has been obtained in [RT74, BLR92] under the now usual geometric control condition. Roughly speaking, the assumption is that any (generalized) bicharacteristic (or classical trajectory, or ray of geometric optics) meets the damping region (in the interior of the domain or at the boundary). For the free wave equation on a subset of $\mathbb{R}^d$, the spatial projections of these bicharacteristics are straight lines, reflected at the boundary according to classical laws of geometric optics. This condition is essentially necessary and sufficient (we do not discuss here the subtilities due to the trajectories which meet the boundary tangentially).

Then the question was to understand what happens when this damping condition fails to hold. In [Leb96, LR97] it is proved that we have at least a logarithmic decay of the energy if the initial datum belongs to the domain of the infinitesimal generator of the problem. This can be optimal, in particular when non controlled trajectories are stable. Intermediate rates of decay have been obtained for several examples where the flow is unstable near these trajectories (see for instance [LR05, BH07, Chr10, Sch11, AL]).

The same questions have been investigated for the Schrödinger or wave equation on the Euclidean space. Even in the self-adjoint case, where the norm for the solution of the Schrödinger equation or the energy for the solution of the wave equation are conserved, it is interesting to study the local energy decay, which measures the fact that the energy escapes at infinity. For the free case and with compactly supported initial conditions, explicit computations show that the energy on a compact vanishes after finite time for the wave in odd dimension $d \geqslant 3$, while the decay is of size t^{-d} in even dimension. For the Schrödinger equation, the norm of the solution decays like $t^{-\frac{d}{2}}$.

Many authors have proved similar estimates for perturbed problems. For instance the wave or Schrödinger equations can be stated outside a compact obstacle of $\mathbb{R}^d$ (we can also consider a perturbation of the Laplace operator). For the wave equation on an exterior domain, we have uniform (exponential) decay for the local energy decay if and only if the non-trapping assumption holds. This means that there is no trapped bicharacteristic. See [LP67, Ral69, Mel79]. See [Bur98] for logarithmic decay without the non-trapping assumption, and [NZ09] for an intermediate situation.

For the dissipative equations, this non-trapping assumption can be replaced by the same damping condition (geometric control condition) on bounded trajectories as in the compact case (where all the trajectories were bounded). This means that the energy of the wave (or at least the contribution of high frequencies) escapes at infinity or is dissipated by the medium. This has been used in [AK07] for a dissipation in the interior of the domain and in [Alo02, AK10] for dissipation at the boundary, for the free equations on an exterior domain in both cases. See also [Roy10b] for the corresponding resolvent estimates, and [BR] for the damped wave equation with a Laplace-Beltrami operator corresponding to a metric which is a long-range perturbation of the flat one.

In this paper we consider a domain which is neither bounded nor the complement of a bounded obstacle. In particular, compared to the situations mentioned above, the boundary of the waveguide is not compact.

More precisely, we are going to use the fact that Ω is a Euclidean space in some directions and compact with respect to the last coordinate, so that properties of both compact and Euclidean domains will appear in our analysis.

The main result of this paper is the following:

Theorem 1.1. *Assume that there exist two constants a_0, a_1 such that on $\partial\Omega$ we have*

$$0 < a_0 \leqslant a \leqslant a_1.$$

Then there exist $\gamma > 0$ and $C \geqslant 0$ such that for all $u_0 \in L^2(\Omega)$ the solution u of the problem (1.1) satisfies

$$\forall t \geqslant 0, \quad \|u(t)\|_{L^2(\Omega)} \leqslant Ce^{-\gamma t} \|u_0\|_{L^2(\Omega)}.$$

In this theorem the absorption is effective everywhere on the boundary, so the damping condition is clearly satisfied on bounded trajectories. However, it is important to note that we prove exponential decay for the total energy and not only for local energy. This means that all the energy is dissipated by the medium, including the energy going at infinity. This suggests that all the classical trajectories, and not only the bounded ones, should be controlled by the dissipation. This is clearly not the case here.

Thus this theorem provides a new example for the already mentioned general idea that given a result for which the non-trapping condition (or the damping condition for dissipative problems) is necessary, we get a close result if there are only a few trajectories which contradict the assumption. Here the initial conditions in $T^*\Omega$ whose corresponding trajectory avoid the boundary form a submanifold of codimension 1 (the frequency vector has no tranversal component). Moreover the flow is linearly unstable. Compared to the related results on compact or Euclidean domains, our analysis will be based on simpler arguments. The key argument is that our problem will inherit the decay property of the transverse problem (that is the Schrödinger equation on $]0, l[$ with the same dissipative boundary condition), for which the geometric control assumption holds.

In order to prove time-dependent estimates for an evolution equation as in Theorem 1.1, we often use spectral properties for the generator of the problem (spectral gap for the eigenvalues

on a compact manifold, absence of resonances close to a positive energy on a perturbation of the Euclidean space, etc.). Here, the problem is governed by the operator

$$H_a\varphi = -\Delta\varphi \tag{1.2}$$

defined on the domain

$$\mathcal{D}(H_a) = \left\{\varphi \in H^2(\Omega) \,:\, \partial_\nu\varphi = ia\varphi \text{ on } \partial\Omega\right\} \subset L^2(\Omega). \tag{1.3}$$

In particular, when $a \geqslant 0$ the solution u of the problem (1.1) is given by the semi-group e^{-itH_a} generated by this operator. When $u_0 \in \mathcal{D}(H_a)$ this solution belongs to $C^0(\mathbb{R}_+, \mathcal{D}(H_a)) \cap C^1(\mathbb{R}_+, L^2(\Omega))$.

The main part of this paper is devoted to the proof of resolvent estimates for the operator H_a. More precisely, in order to prove Theorem 1.1 we will use the following result:

Theorem 1.2. *Let a be as in Theorem 1.1. Then there exist $\gamma > 0$ and $C \geqslant 0$ such that any $z \in \mathbb{C}$ with $\mathrm{Im}(z) \geqslant -3\gamma$ is in the resolvent set of H_a and moreover*

$$\left\|(H_a - z)^{-1}\right\|_{L^2(\Omega)} \leqslant C.$$

When a is constant, our analysis will provide a good description of the spectrum of H_a, and in particular there is a spectral gap as stated in Theorem 1.2. However, for a non-selfadjoint operator the norm of the resolvent can be large far from the spectrum, so the uniform bound of the resolvent on a strip around the real axis is not a consequence of the spectral gap and has to be proved directly.

In the papers we have mentioned above concerning the damped Schrödinger equation [Alo08b, Alo08a, AKV13, AKR], it is proved that this equation satisfies the Kato smoothing effect, that is an estimate of the form

$$\int_0^\infty \left\|\langle x\rangle^{-\delta}(1-\Delta)^{\frac{1}{4}}u(t)\right\|_{L^2}^2 dt \lesssim \|u_0\|_{L^2}^2 .$$

In these papers, the absorption is in the interior of the domain and, more important, it is of the form $a(x)(1-\Delta)^{\frac{1}{2}}a(x)$. This is stronger than multiplication by $a(x)^2$ for high-frequencies. It is not the case here, which is why we have no smoothing effect for our problem. However, we still have the usual smoothing property in the unbounded directions:

Theorem 1.3. *Let a be as in Theorem 1.1. Let $\delta > \frac{1}{2}$. Then there exists $C \geqslant 0$ such that for all $u_0 \in L^2(\Omega)$ the solution u of (1.1) satisfies*

$$\int_0^\infty \left\|\langle x\rangle^{-\delta}\left(1-\Delta_x\right)^{\frac{1}{4}}u(t)\right\|_{L^2(\Omega)}^2 dt \leqslant C\,\|u_0\|_{L^2(\Omega)}^2 ,$$

where $\Delta_x = \sum_{n=1}^{d-1}\partial_{x_n}^2$ is the partial Laplacian acting on the unbounded directions.

In the results above, we can relax the assumption that the absorption index is positive everywhere. For instance, the proofs hold true without modification in the case where a is a positive constant one one side of the boundary, say $\mathbb{R}^{d-1}\times\{l\}$, and vanishes on the other side $\mathbb{R}^{d-1}\times\{0\}$. What happens if the absorption index is a negative constant on one side is not so clear.

Let $a_l, a_0 \in \mathbb{R}$ be such that

$$a_l + a_0 > 0 \tag{1.4}$$

and consider the problem

$$\begin{cases} i\partial_t u + \Delta u = 0 & \text{on }]0,+\infty[\times\Omega, \\ \partial_\nu u = ia_l u & \text{on }]0,+\infty[\times\mathbb{R}^{d-1}\times\{l\}, \\ \partial_\nu u = ia_0 u & \text{on }]0,+\infty[\times\mathbb{R}^{d-1}\times\{0\}, \\ u(0,\cdot) = u_0 & \text{on } \Omega. \end{cases} \tag{1.5}$$

When a_l and a_0 have different signs but satisfy (1.4), we will say that the boundary condition is weakly dissipative. Estimates like those of Theorems 1.1 and 1.2 are not likely to hold without this assumption. For instance, when $a_0 = -a_l$ we get a $\mathcal{PT}$-symmetric waveguide. Such a boundary condition has been studied in [KBZ06, BK08]. See also [KT08]. In particular it

is known that in this case the spectrum is real, so we cannot expect any generalization of our results. The case $a_l + a_0 < 0$ is dual to the case $a_l + a_0 > 0$. We will see that in this case the norm of the solution grows exponentially (see Remark 7.5).

In the weakly dissipative case we have the following theorem:

Theorem 1.4. *Let $a_l, a_0 \in \mathbb{R}$ satisfy Assumption (1.4). If $u_0 \in \mathcal{D}(H_a)$ then the problem (1.5) has a unique solution $u \in C^1([0,+\infty[, L^2(\Omega)) \cap C^0([0,+\infty[, \mathcal{D}(H_a))$. Moreover, if $|a_l|$ and $|a_0|$ are small enough there exist $\gamma > 0$ and $C \geqslant 0$ such that for all u_0 we have*

$$\forall t \geqslant 0, \quad \|u(t)\|_{L^2(\Omega)} \leqslant Ce^{-\gamma t} \|u_0\|_{L^2(\Omega)}.$$

Let us discuss the assumption that the absorption has to be small. In [Roy] it was proved in another context that for high frequencies the properties of a dissipative problem remain valid when the absorption index is positive on average along the corresponding classical flow. This is the case here under Assumption (1.4). The restriction comes from low frequencies. When $a \neq 0$, the boundary condition can be rewritten

$$u = \frac{1}{a}\partial_\nu u \quad \text{on } \partial\Omega.$$

When a is large compared to the frequency, this is close to a Dirichlet condition. That for large dissipation we recover a self-adjoint problem is usually called the overdamping phenomenon. This suggests that the problem is now governed by the quantity $1/a$. And when $ab < 0$, Assumption (1.4) can be rewritten as

$$\frac{1}{a_l} + \frac{1}{a_0} < 0.$$

It turns out that for a_l, a_0 large enough with $a_l a_0 < 0$ and (1.4), the contribution of low frequencies is indeed exponentially increasing (see Proposition 7.4).

In this paper we prove all these results on the model case of a straight waveguide with a one-dimensional section. The purpose is on one hand to observe all the non-trivial phenomenons mentioned above on a quite simple example. On the other hand our analysis is the first step toward the understanding of similar properties for the wave equation on a more general domain.

On this model case, and with the additional assumption that a is constant, we can rewrite H_a as the sum of the usual Laplacian on the Euclidean space $\mathbb{R}^{d-1}$ and a Laplace operator on the compact section. Since this section is of dimension 1, we can give quite explicitely many spectral properties for this operator. In particular, we will see that its eigenfunctions form a Riesz basis, which will give a good description of the spectrum of H_a, first when a is constant and then in the general case. The time dependent estimate will follow.

In Section 2 we prove that H_a is maximal dissipative, which gives in particular well-posedness for the problem (1.1). In Section 3 we study the transverse operator, that is the Laplace operator on the section $]0, l[$. Spectral properties of H_a are obtained for a constant absorption index in Section 4, and then Theorems 1.2 and 1.3 are proved in Section 5. Once the spectral properties of H_a are well-understood, the proof of Theorem 1.1 is given in Section 6. Finally, the problem (1.5) where a can take negative values is discussed in Section 7.

All along the paper, a general point in Ω will be written $(x, y) \in \Omega \simeq \mathbb{R}^{d-1} \times]0, l[$, with $x \in \mathbb{R}^{d-1}$ and $y \in]0, l[$. As in Theorem 1.3, we denote by Δ_x the usual laplacian on $\mathbb{R}^{d-1}$. For Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is the set of bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. For $\gamma > 0$ we finally set

$$\mathbb{C}_\gamma = \{z \in \mathbb{C} : \operatorname{Im} z > -\gamma\}, \quad \text{and then} \quad \mathbb{C}_+ := \mathbb{C}_0.$$

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2. Operator associated to the dissipative waveguide

In this section we consider a more general waveguide Ω of the form $\mathbb{R}^p \times \omega \subset \mathbb{R}^d$ where $p \in \{1, \dots, d-1\}$ and ω is a smooth open bounded subset of $\mathbb{R}^{d-p}$. In particular, Ω is open in $\mathbb{R}^d$.

Let $a \in W^{1,\infty}(\partial\Omega)$. Until Proposition 2.5, we make no assumption on the sign of a. We consider on $L^2(\Omega)$ the operator H_a defined by (1.2) with domain (1.3). For all $\varphi \in \mathcal{D}(H_a)$ we have

$$\langle H_a\varphi, \varphi\rangle_{L^2(\Omega)} = -\langle \Delta\varphi, \varphi\rangle_{L^2(\Omega)} = \langle \nabla\varphi, \nabla\varphi\rangle_{L^2(\Omega)} - i\langle a\varphi, \varphi\rangle_{L^2(\partial\Omega)}. \tag{2.1}$$

On the other hand, we consider the quadratic form defined for $\varphi \in \mathcal{D}(q_a) = H^1(\Omega)$ by

$$q_a(\varphi) = \int_\Omega |\nabla\varphi|^2 - i\int_{\partial\Omega} a\,|\varphi|^2.$$

We also denote by q_a the corresponding sesquilinear form on $\mathcal{D}(q_a)^2$. That this quadratic form is sectorial and closed follows from the following lemma and traces theorems:

Lemma 2.1. *Let q_R be a non-negative, densely defined, closed form on a Hilbert space $\mathcal{H}$. Let q_I be a symmetric form relatively bounded with respect to q_R. Then the form $q_R - iq_I$ is sectorial and closed.*

It is important to note that there is no smallness assumption on the relative bound of q_I with respect to q_R. In particular, for q_a we do not need any assumption on the size of a in $L^\infty(\partial\Omega)$.

Proof. There exists $C > 0$ such that for all $\varphi \in \mathcal{D}(q_R)$ we have

$$|q_I(\varphi)| \leqslant C\left(q_R(\varphi) + \|\varphi\|^2_{\mathcal{H}}\right).$$

Let $\varepsilon_0 = \frac{1}{2C}$. If we already know that $(q_R - i\lambda q_I)$ is sectorial and closed for some $\lambda \geqslant 0$, then $(q_R - i(\lambda+\varepsilon)q_I)$ is sectorial and closed for all $\varepsilon \in [0, \varepsilon_0]$ according to Theorem VI.3.4 in [Kat80]. Now since q_R is sectorial and closed, we can prove by induction on $n \in \mathbb{N}$ that $(q_R - i\lambda q_I)$ is sectorial and closed for all $\lambda \in [0, n\varepsilon]$, and hence for all $\lambda \geqslant 0$. This is in particular the case when $\lambda = 1$. □

We recall the definitions of accreative and dissipative operators (note that the conventions may be different for other authors):

Definition 2.2. We say that an operator T on the Hilbert space $\mathcal{H}$ is accreative (respectively dissipative) if

$$\forall\varphi \in \mathcal{D}(T), \quad \mathrm{Re}\,\langle T\varphi, \varphi\rangle_{\mathcal{H}} \geqslant 0, \quad \big(\text{respectively} \quad \mathrm{Im}\,\langle T\varphi, \varphi\rangle_{\mathcal{H}} \leqslant 0\big).$$

Moreover T is said to be maximal accreative (maximal dissipative) if it has no other accreative (dissipative) extension on $\mathcal{H}$ than itself. In particuliar T is (maximal) dissipative if and only if iT is (maximal) accreative.

Let us recall that an accreative operator T is maximal accreative if and only if $(T - z)$ has a bounded inverse on $\mathcal{H}$ for some (and hence any) $z \in \mathbb{C}$ with $\mathrm{Re}(z) < 0$. In this case we know from the Hille-Yosida Theorem that $-T$ generates a contractions semi-group $t \mapsto e^{-tT}$. Then for all $u_0 \in \mathcal{D}(T)$ the map $u : t \mapsto e^{-tT}u_0$ belongs to $C^1(\mathbb{R}_+, \mathcal{H}) \cap C^0(\mathbb{R}_+, \mathcal{D}(T))$ and solves the problem

$$\begin{cases} u'(t) + Tu(t) = 0, & \forall t > 0, \\ u(0) = u_0. \end{cases}$$

Let us come back to our context. According to Lemma 2.1 and the Representation Theorem VI.2.1 in [Kat80], there exists a unique maximal accreative operator $\hat{H}_a$ on $L^2(\Omega)$ such that $\mathcal{D}(\hat{H}_a) \subset \mathcal{D}(q_a)$ and

$$\forall\varphi \in \mathcal{D}(\hat{H}_a), \forall\psi \in \mathcal{D}(q_a), \quad \left\langle \hat{H}_a\varphi, \psi\right\rangle_{L^2(\Omega)} = q_a(\varphi, \psi).$$

Moreover we have

$$\mathcal{D}(\hat{H}_a) = \left\{u \in \mathcal{D}(q_a) \,:\, \exists f \in L^2(\Omega), \forall v \in \mathcal{D}(q_a), q_a(u, v) = \langle f, v\rangle\right\},$$

and for $u \in \mathcal{D}(\hat{H}_a)$ the corresponding f is unique and given by $\hat{H}_a u = f$.

Proposition 2.3. *We have $\hat{H}_a = H_a$. In particular H_a is maximal accreative.*

For a one-dimensional section we can essentially follow the proof of Lemma 3.2 in [BK08]. This would be enough for our purpose but, for further use, we prove this result in the general setting.

Proof. It is easy to check that $\mathcal{D}(H_a) \subset \mathcal{D}(\hat{H}_a)$ and $H_a = \hat{H}_a$ on $\mathcal{D}(H_a)$. Now let $u \in \mathcal{D}(\hat{H}_a)$. By definition there exists $f \in L^2(\Omega)$ such that

$$\forall v \in H^1(\Omega), \quad \int_\Omega \nabla u \cdot \nabla \overline{v} - i \int_{\partial\Omega} au\overline{v} = \int_\Omega f\overline{v}.$$

Considering $v \in C_0^\infty(\Omega)$ we see that $-\Delta u = f$ in the sense of distributions and hence in $L^2(\Omega)$. This proves that $u \in H^2_{\mathrm{loc}}(\Omega)$. It remains to prove that $u \in H^2(\Omega)$ and that the boundary condition $\partial_\nu u = iau$ holds on $\partial\Omega$. Let $j \in \{1, \dots, p\}$ and let e_j be the j-th vector in the canonical basis of $\mathbb{R}^p$. Let $\delta \in \mathbb{R}^*$ and $u_\delta : (x, y) \mapsto \frac{1}{\delta}(u(x + \delta e_j, y) - u(x, y)) \in H^1(\Omega)$. See for instance [Eva98, §5.8.2] for the properties of the difference quotients. For all $v \in H^1(\Omega)$ we have

$$q(u_\delta, v) = -\int_\Omega f(x, y)\overline{v_{-\delta}}(x, y)\, dx\, d\sigma(y) - i \int_{\partial\Omega} u(x, y)a_{-\delta}(x, y)\overline{v}(x - \delta e_j, y)\, dx\, d\sigma(y),$$

where σ is the Lebesgue measure on ω. Since $a \in W^{1,\infty}(\partial\Omega)$ there exists $C \geqslant 0$ such that for all $v \in H^1(\Omega)$ and $\delta > 0$ we have

$$|q_a(u_\delta, v)| \leqslant C \|v\|_{H^1(\Omega)}.$$

Applied with $v = u_\delta$ this gives

$$\|u_\delta\|^2_{H^1(\Omega)} = \operatorname{Re} q_a(u_\delta, u_\delta) \leqslant C \|u_\delta\|_{H^1(\Omega)}.$$

Since we already know that $u_\delta \in L^2(\Omega)$ uniformly in $\delta > 0$, we have

$$\|u_\delta\|^2_{H^1(\Omega)} \lesssim 1 + \|u_\delta\|_{H^1(\Omega)},$$

which implies that u_δ is uniformly in $H^1(\Omega)$. This means that $\partial_{x_j} u \in H^1(\Omega)$. Since this holds for any $j \in \{1, \dots, p\}$, this proves that all derivatives of order 2 with at least one derivative in the first p directions belong to $L^2(\Omega)$. Then we get

$$-\Delta_y u = f + \Delta_x u \in L^2(\Omega).$$

According to the Green Formula we have for all $v \in H^1(\Omega)$

$$\int_\Omega -\Delta_y u\, \overline{v}\, dx\, dy = \int_\Omega \nabla_y u \cdot \nabla_y \overline{v}\, dx\, dy - \langle \partial_n u, v\rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}$$

(see for instance [Gri85, eq. (1.5.3.10)]). By density of the trace map, we obtain that

$$\partial_\nu u = iau \quad \text{on } \partial\Omega. \tag{2.2}$$

In particular $\partial_\nu u \in H^{1/2}(\partial\Omega)$. Then there exists $v \in H^2(\Omega)$ such that $\partial_\nu v = \partial_\nu u$ (see [Gri85, Th. 1.5.1.1] for a fonction on $\mathbb{R}^d_+$; for a function on Ω we follow the same idea as for a fonction on a bounded domain, except that we only use a partition of unity for $\partial\omega$, which allows to cover $\partial\Omega$ by a finite number of strips, each of which is diffeomorphic to a strip on $\mathbb{R}^p$). Let $w = u - v$. We have $w \in H^1(\Omega)$, $\Delta_y w \in L^2(\Omega)$ and

$$\begin{cases} -\Delta_y w + w = f + \Delta_x u - \Delta_y v + w & \text{on } \Omega, \\ \partial_\nu w = 0 & \text{on } \partial\Omega. \end{cases}$$

Then for almost all $x \in \mathbb{R}^p$ we have

$$\begin{cases} -\Delta_y w(x) + w(x) = f(x) + \Delta_x u(x) - \Delta_y v(x) + w(x) & \text{on } \omega, \\ \partial_\nu w(x) = 0 & \text{on } \partial\omega. \end{cases}$$

By elliptic regularity for the Neumann problem (see for instance Theorem 9.26 in [Bre11]) we obtain that $w(x) \in H^2(\omega)$ with

$$\|w(x)\|_{H^2(\omega)} \lesssim \|f(x) + \Delta_x u(x) - \Delta_y v(x) + w(x)\|_{L^2(\omega)}.$$

After integration over $x \in \mathbb{R}^p$, this gives

$$\|u\|_{H^2_y(\Omega)} \lesssim \|f + \Delta_x u - \Delta_y v + w\|_{L^2(\Omega)} + \|v\|_{H^2(\Omega)}.$$

Since we already know that second derivatives of u involving a derivation in x are in $L^2(\Omega)$, this proves that $u \in H^2(\Omega)$ and concludes the proof. □

Remark 2.4. We have $H_a^* = H_{-a}$.

Now assume that a takes non-negative values. According to (2.1), a is a dissipative operator. Since it is maximal accreative, it is easy to see that it is in fact maximal dissipative:

Proposition 2.5. *The maximal accreative operator H_a is also maximal dissipative.*

Proof. We already know that H_a is dissipative. Since it is maximal accreative, any $z \in \mathbb{C}$ with $\operatorname{Re} z < 0$ is in its resolvent set. Then it is easy to find z in this resolvent set of H_a with $\operatorname{Im} z > 0$. □

Proposition 2.6. *If $a > 0$ in an open subset of $\partial\Omega$ then H_a has no real eigenvalue.*

Proof. Let $u \in \mathcal{D}(H_a)$, $\lambda \in \mathbb{R}$, and assume that $H_a u = \lambda u$. Taking the imaginary part of the equality $q_a(u,u) = \lambda \|u\|^2$ gives

$$\int_{\partial\Omega} a\,|u|^2 = 0.$$

This implies that $u = 0$ where $a \neq 0$ and $\partial_\nu u = iau = 0$ everywhere on $\partial\Omega$. By unique continuation, this implies that $u = 0$ on Ω. □

3. The Transverse Operator

Let us come back to the case of a one-dimensional cross-section $\omega =]0,l[$. Under the additional assumption that the absorption index a is constant on $\partial\Omega$ the operator H_a can be written as

$$H_a = -\Delta_x \otimes \mathrm{Id}_{L^2(0,l)} + \mathrm{Id}_{L^2(\mathbb{R}^{d-1})} \otimes T_a, \tag{3.1}$$

where $-\Delta_x$ is as before the usual flat Laplacian on $\mathbb{R}^{d-1}$ and T_a is the transverse Laplacian on $]0,l[$. More precisely, we consider on $L^2(0,l)$ the operator $T_a = -\frac{d^2}{dy^2}$ with domain

$$\mathcal{D}(T_a) = \left\{u \in H^2(0,l) \,:\, u'(0) = -iau(0), u'(l) = iau(l)\right\}.$$

This is the maximal accreative and dissipative operator corresponding to the form

$$q : u \in H^1(0,l) \mapsto \int_0^l |u'(x)|^2\,dx - ia\,|u(l)|^2 - ia\,|u(0)|^2.$$

In this section we give the spectral properties of T_a which we need to study the full operator H_a. This operator has compact resolvent, and hence its spectrum is given by a sequence of isolated eigenvalues. When $a = 0$, which corresponds to the Neumann problem, we know that the eigenvalues of T_0 are the real numbers $n^2\nu^2$ for $n \in \mathbb{N}$, where we have set

$$\nu = \frac{\pi}{l}.$$

These eigenvalues are algebraically simple.

Proposition 3.1. *There exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of continuous functions on $\mathbb{R}$ such that $\lambda_n(0) = n\nu$ and for all $a \in \mathbb{R}$ the set of eigenvalues of T_a is $\left\{\lambda_n(a)^2, n \in \mathbb{N}\right\}$. Moreover:*

(i) For $(n,a) \in (\mathbb{N}\times\mathbb{R})\setminus\{(0,0)\}$ the eigenvalue $\lambda_n(a)$ is algebraically simple and a corresponding eigenvector is given by

$$\varphi_n(a) : x \mapsto A_n(a)\left(e^{i\lambda_n(a)x} + \frac{\lambda_n(a)+a}{\lambda_n(a)-a}e^{-i\lambda_n(a)x}\right), \tag{3.2}$$

where we can choose $A_n(a) \in \mathbb{R}_+^*$ in such a way that $\|\varphi_n(a)\|_{L^2(0,l)} = 1$ (when $a = 0$ then 0 is a simple eigenvalue and corresponding eigenvectors are non-zero constant functions).

(ii) For $n \in \mathbb{N}$ and $a \in \mathbb{R}$ we have $\lambda_n(-a) = \overline{\lambda_n(a)}$.

(iii) Let $n \in \mathbb{N}$. For all $a \in \mathbb{R}^*$ we have $\operatorname{Re}(\lambda_n(a)) \in]n\nu, (n+1)\nu[$ (when $n = 0$, we have chosen the square root of $\lambda_0^2(a)$ which has a positive real part).

(iv) For all $n \in \mathbb{N}$ there exists $C_n > 0$ such that for $a > 0$ we have $-C_n < \operatorname{Im}(\lambda_n(a)) < 0$.

(v) Let $a > 0$ be fixed. We have

$$\lambda_n(a) = n\nu - \frac{2ia}{n\nu l} + \underset{n\to+\infty}{O}\left(n^{-2}\right)$$

and hence

$$\lambda_n(a)^2 = (n\nu)^2 - \frac{4ia}{l} + \underset{n\to+\infty}{O}\left(n^{-1}\right).$$

Proof. • It is straightforward computations to check that 0 is an eigenvalue of T_a if and only if $a = 0$ and, if $\lambda \in \mathbb{C}^*$, λ^2 is an eigenvalue if and only if

$$(a-\lambda)^2 e^{2i\lambda l} = (a+\lambda)^2. \tag{3.3}$$

If λ^2 is an eigenvalue then the corresponding eigenfunction is of the form $\varphi : x \mapsto Ae^{i\lambda x} + Be^{-i\lambda x}$ with

$$A = \frac{\lambda - a}{\lambda + a} B = \frac{\lambda + a}{\lambda - a} e^{-2i\lambda l} B. \tag{3.4}$$

Moreover, all these eigenvalues have geometric multiplicity 1. Indeed, given $n \in \mathbb{N}$, the space of eigenvectors corresponding to the eigenvalue $\lambda_n(a)^2$ is stricly included in the space of H^2 functions which are solutions of $-u'' - \lambda_n(a)^2 u = 0$, and this space is of dimension 2. The fact that the eigenvalues of H_{-a} are conjugated to the eigenvalues of H_a is a consequence of Remark 2.4.

• Let $a > 0$ and $\lambda \in \mathbb{C}^*$ be such that λ^2 is an eigenvalue of T_a. Assume that $\operatorname{Re}\lambda \in \nu\mathbb{N}$. Then

$$\left(\frac{a+\lambda}{a-\lambda}\right)^2 = e^{2i\lambda l} \in \mathbb{R}_+$$

(note that λ cannot be equal to a in (3.3)) and hence

$$r := \frac{a+\lambda}{a-\lambda} \in \mathbb{R}.$$

If $r = -1$ then $a = 0$. Otherwise $\lambda = \frac{a(1-r)}{1+r} \in \mathbb{R}$. In both cases we obtain a contradiction (see Proposition 2.6), and hence $\operatorname{Re}\lambda \notin \nu\mathbb{N}$. This proves that for $a > 0$ the operator T_a has no eigenvalue with real part in $\nu\mathbb{N}$.

• Now let $R > 0$. We prove that if $C_R \geqslant 0$ is chosen large enough and if $a \in \mathbb{R}$ and $\lambda \in \mathbb{C}^*$ are such that λ^2 is an eigenvalue of T_a, then

$$|\operatorname{Re}\lambda| \leqslant R \quad \Longrightarrow \quad |\operatorname{Im}\lambda| \leqslant C_R. \tag{3.5}$$

Assume by contradiction that this is not the case. Then for all $m \in \mathbb{N}$ we can find $x_m \in [-R, R]$ and $y_m \in \mathbb{R}$ with $|y_m| \geqslant m$ such that $(x_m + iy_m)^2$ is an eigenvalue of T_{a_m} for some $a_m \in \mathbb{R}$. We have

$$e^{-2y_m l} = \left|\frac{a_m + x_m + iy_m}{a_m - x_m - iy_m}\right|^2 = \frac{(a_m+x_m)^2 + y_m^2}{(a_m - x_m)^2 + y_m^2} \xrightarrow[m\to\infty]{} 1,$$

which gives a contradiction.

• The family of operators T_a for $a \in \mathbb{R}$ is an analytic family of operators of type B in the sense of Kato [Kat80, §VII.4]. We already know that the spectrum of T_0 is given by $\{(n\nu)^2, n \in \mathbb{N}\}$. Then for all $n \in \mathbb{N}$ there exists an analytic function λ_n^2 such that, at least for small a, $\lambda_n^2(a)$ is in the spectrum of T_a (and then we define λ_n as the square root of λ_n^2 with positive imaginary part). See Theorem VII.1.7 in [Kat80].

• Let $n \in \mathbb{N}^*$. We write $\lambda_n(a) = n\nu + \beta a + \gamma a^2 + O_{a\to 0}(a^3)$. We have

$$e^{2i\lambda_n(a)l} = 1 + 2il\beta a + 2il\gamma a^2 - 2l^2\beta^2 a^2 + O(a^3),$$

and on the other hand:

$$\left(\frac{\lambda_n(a)+a}{\lambda_n(a)-a}\right)^2 = 1 + \frac{4a}{n\nu} - \frac{4(\beta-2)a^2}{n^2\nu^2} + O(a^3).$$

Since $\lambda_n(a)$ solves (3.3) for any $a > 0$ we obtain

$$\beta = \frac{2}{iln\nu} = -\frac{2i}{\pi n}$$

and

$$\operatorname{Re}(\gamma) = \frac{4l}{n^3\pi^3}.$$

Since $\mathrm{Re}(\beta) = 0$ and $\mathrm{Re}(\gamma) > 0$ we have $\mathrm{Re}\big(\lambda_n(a)\big) \in]n\nu, (n+1)\nu[$ for $a > 0$ small enough. The functions $a \mapsto \mathrm{Re}\big(\lambda_n(a)\big)$ are continuous and cannot reach $\nu\mathbb{N}$ unless $a = 0$, so this remains true for any $a > 0$ such that $\lambda_n(a)$ is defined. Similarly $\mathrm{Re}\big(\lambda_0(a)\big) \in]0, \nu[$ for all $a > 0$. In particular the curves $a \mapsto \lambda_n(a)$ for $n \in \mathbb{N}$ never meet. Moreover we know from (3.5) that $\lambda_n(a)$ remains in a bounded set of $\mathbb{C}$, so the curves $a \mapsto \lambda_n(a)$ are defined for all $a \in \mathbb{R}$ and for all $a \in \mathbb{R}$ the eigenvalues of T_a are exactly given by $\lambda_n(a)^2$ for $n \in \mathbb{N}$.

- It remains to prove that the asymptotic expansion of $\lambda_n(a)$ for n fixed and a small is also valid for a fixed and n large. Let $a > 0$ be fixed. Derivating (3.3) and using the fact that $|\lambda_n(s)| \geqslant n\nu$ for all $s \in \mathbb{R}$ we obtain that

$$\sup_{s\in[0,a]} |\lambda_n'(s)| = O\big(n^{-1}\big).$$

This means that $\lambda_n(a) = n\nu + O(n^{-1})$. Then we obtain the asymptotic expension of $\lambda_n(a)$ for large n as before, using again (3.3). This gives the last statement of the proposition and concludes the proof. □

Now that we have proved what we need concerning the spectrum of the operator T_a, we study the corresponding sequence of eigenfunctions. In the self-adjoint case $a = 0$, we know that the eigenfunctions $\varphi_n(0)$ form an orthonormal basis. Of course this is no longer the case for the non-selfadjoint operator T_a with $a \neq 0$. However we can prove that in this case we have a Riesz basis.

We recall that the sequence $(\varphi_n)_{n\in\mathbb{N}}$ of vectors in the Hilbert space $\mathcal{H}$ is said to be a Riesz basis if there exists a bounded operator $\Theta \in \mathcal{L}(\mathcal{H})$ with bounded inverse and an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$ such that $\varphi_n = \Theta e_n$ for all $n \in \mathbb{N}$. In this case any $f \in \mathcal{H}$ can be written as $\sum_{n\in\mathbb{N}} f_n\varphi_n$ with $(f_n)_{n\in\mathbb{N}} \in l^2(\mathbb{C})$, and there exists $C \geqslant 1$ such that for all $f = \sum_{n\in\mathbb{N}} f_n\varphi_n \in \mathcal{H}$ we have

$$C^{-1} \sum_{n\in\mathbb{N}} |f_n|^2 \leqslant \left\| \sum_{n\in\mathbb{N}} f_n\varphi_n \right\|_{\mathcal{H}}^2 \leqslant C \sum_{n\in\mathbb{N}} |f_n|^2 .$$

In these estimates we can take $C = \max\Big(\|\Theta\|^2, \big\|\Theta^{-1}\big\|^2\Big)$.

Let $(\varphi_n)_{n\in\mathbb{N}}$ be a Riesz basis of $\mathcal{H}$ with Θ and $(e_n)_{n\in\mathbb{N}}$ as above. If we set $\varphi_n^* = (\Theta^{-1})^* e_n$ for all $n \in \mathbb{N}$ then $(\varphi_n^*)_{n\in\mathbb{N}}$ is also a Riesz basis, called the dual basis of $(\varphi_n)_{n\in\mathbb{N}}$. In particular for all $j, k \in \mathbb{N}$ we have

$$\langle \varphi_j, \varphi_k^* \rangle_{\mathcal{H}} = \delta_{j,k}. \tag{3.6}$$

We refer to [Chr03] for more details about Riesz bases. Now we want to prove that for any $a \in \mathbb{R}$ the sequence of eigenfunctions for the operator T_a is a Riesz basis of $L^2(0, l)$.

Proposition 3.2. *For all $a \in \mathbb{R}$ the sequence $(\varphi_n(a))_{n\in\mathbb{N}}$ defined in (3.2) is a Riesz basis of $L^2(0, l)$. Moreover for all $R > 0$ there exists $C \geqslant 0$ such that for $a \in [-R, R]$ and $(c_n)_{n\in\mathbb{N}} \in l^2(\mathbb{C})$ we have*

$$C^{-1} \sum_{n=0}^{\infty} |c_n|^2 \leqslant \left\| \sum_{n=0}^{\infty} c_n\varphi_n(a) \right\|^2 \leqslant C \sum_{n=0}^{\infty} |c_n|^2 .$$

If C was chosen large enough and if $(c_n)_{n\in\mathbb{N}} \in l^2(\mathbb{C})$ is such that $\sum_{n=0}^{\infty} |\lambda_n(a)c_n|^2 < \infty$ we also have

$$C^{-1} \sum_{n=0}^{\infty} |\lambda_n(a)c_n|^2 \leqslant \left\| \sum_{n=0}^{\infty} c_n\varphi_n'(a) \right\|^2 \leqslant C \sum_{n=0}^{\infty} |\lambda_n(a)c_n|^2 .$$

For similar results we refer to [Mik62] (see also Lemma XIX.3.10 in [DS71]). For the proof we need the following lemma:

Lemma 3.3. *Let $R > 0$. Then there exists $C \geqslant 0$ such that for $a \in [-R, R]$ and $j, k \in \mathbb{N}$ with $j < k$ we have*

$$\langle \varphi_j(a), \varphi_k(a) \rangle_{L^2(0,l)} \leqslant \frac{C}{\langle j \rangle (k - j)} \quad \text{and} \quad \big\langle \varphi_j'(a), \varphi_k'(a) \big\rangle_{L^2(0,l)} \leqslant C\frac{k}{k - j}.$$

Proof. Let $e_n(a,x) = e^{i\lambda_n(a)x}$ and $\tilde{e}_n(a,x) = e^{-i\lambda_n(a)x}$. According to Proposition 3.1 we have $\lambda_n(a) = n\nu + O(n^{-1})$ (here and below all the estimates are uniform in $a \in [-R,R]$), so

$$\|e_n(a)\|^2_{L^2(0,l)} = \frac{e^{-2l\,\mathrm{Im}(\lambda_n(a))} - 1}{-2\,\mathrm{Im}(\lambda_n(a))} = l + O(n^{-1}).$$

Similarly $\|\tilde{e}_n(a)\|^2_{L^2(0,l)} = l + O(n^{-1})$, and $\langle e_n(a), \tilde{e}_n(a)\rangle = O(n^{-1})$. Moreover, with (3.2) we see that

$$A_n(a) = \frac{1}{\sqrt{2l}} + O(n^{-1}).$$

Now let $j,k \in \mathbb{N}$ with $j < k$. We omit the dependence in a for φ_j, e_j, $\tilde{e}_j$ and λ_j. We have

$$\langle e_j, e_k\rangle = \frac{e^{i(\lambda_j - \overline{\lambda_k})l} - 1}{\lambda_j - \overline{\lambda_k}} \quad \text{and} \quad \langle \tilde{e}_j, \tilde{e}_k\rangle = \frac{e^{-i(\lambda_j - \overline{\lambda_k})l} - 1}{-(\lambda_j - \overline{\lambda_k})}.$$

Since

$$\lambda_j - \overline{\lambda_k} = (j-k)\nu + O(1/j)$$

we have

$$|\langle e_j, e_k\rangle| + |\langle \tilde{e}_j, \tilde{e}_k\rangle| \lesssim \frac{1}{k-j} \quad \text{and} \quad |\langle e_j, e_k\rangle + \langle \tilde{e}_j, \tilde{e}_k\rangle| \lesssim \frac{1}{\langle j\rangle\,(k-j)}.$$

Similarly

$$|\langle \tilde{e}_j, e_k\rangle| + |\langle e_j, \tilde{e}_k\rangle| \lesssim \frac{1}{k+j} \quad \text{and} \quad |\langle \tilde{e}_j, e_k\rangle + \langle e_j, \tilde{e}_k\rangle| \lesssim \frac{1}{\langle j\rangle\,(k+j)}.$$

And finally

$$\begin{aligned}
|\langle \varphi_j, \varphi_k\rangle| &= \left| A_j \overline{A_k} \left\langle e_j + \frac{\lambda_j - a}{\lambda_j + a}\tilde{e}_j, e_k + \frac{\lambda_k - a}{\lambda_k + a}\tilde{e}_k \right\rangle \right| \\
&\lesssim |\langle e_j, e_k\rangle + \langle e_j, \tilde{e}_k\rangle + \langle \tilde{e}_j, e_k\rangle + \langle \tilde{e}_j, \tilde{e}_k\rangle| \\
&\quad + \langle j\rangle^{-1} \left(|\langle e_j, e_k\rangle| + |\langle e_j, \tilde{e}_k\rangle| + |\langle \tilde{e}_j, e_k\rangle| + |\langle \tilde{e}_j, \tilde{e}_k\rangle|\right) \\
&\lesssim \frac{1}{\langle j\rangle\,(k-j)}.
\end{aligned}$$

The second estimate is proved similarly, using again that $|\lambda_n(a)| = n\nu + O(n^{-1})$ for large j. □

Proof of Proposition 3.2. • Let $a \in \mathbb{R}$ and $(c_n)_{n\in\mathbb{N}} \in l^2(\mathbb{C})$. Let $C \geqslant 0$ be given by Lemma 3.3. For $N,p \in \mathbb{N}$ we have

$$\left\| \sum_{n=N}^{N+p} c_n \varphi_j(a) \right\|^2 - \sum_{n=N}^{N+p} |c_n|^2 = \sum_{j=N}^{N+p} \sum_{k=j+1}^{N+p} 2\,\mathrm{Re}\left(c_j \overline{c_k}\,\langle \varphi_j, \varphi_k\rangle\right) \leqslant 2C \sum_{j=0}^{\infty} \frac{|c_j|}{\langle j\rangle} \sum_{k=1}^{\infty} \frac{|c_{k+j}|}{k} \lesssim \|c\|^2_{l^2(\mathbb{C})}.$$

This proves that the series $\sum_{n=0}^{\infty} c_n\varphi_n(a)$ converges in $L^2(0,l)$ and

$$\left\| \sum_{n=0}^{\infty} c_n \varphi_n(a) \right\|^2_{L^2(0,l)} \lesssim \sum_{n=0}^{\infty} |c_n|^2.$$

• Let $n,m \in \mathbb{N}$ be such that $n \neq m$. In $L^2(0,l)$ we have

$$\begin{aligned}
\lambda_n(a)^2\,\langle \varphi_n(a), \varphi_m(-a)\rangle &= \langle H_a\varphi_n(a), \varphi_m(-a)\rangle = \langle \varphi_n(a), H_a^*\varphi_m(-a)\rangle \\
&= \lambda_m(a)^2\,\langle \varphi_n(a), \varphi_m(-a)\rangle,
\end{aligned}$$

and hence $\langle \varphi_n(a), \varphi_m(-a)\rangle = 0$. On the other hand, with (3.2) we can check that $\langle \varphi_n(a), \varphi_n(-a)\rangle \neq 0$ at least for n large enough. Now let $(c_n)_{n\in\mathbb{N}} \in l^2(\mathbb{C})$ be such that $\sum_{n=0}^{\infty} c_n\varphi_n(a) = 0$. Taking the inner product with $\varphi_m(-a)$ we see that $c_m = 0$ for $m > N$ if N is chosen large enough. Then for all $k \in \{0,\dots,N\}$ we have

$$0 = H_a^k \sum_{n=0}^{N} c_n\varphi_n(a) = \sum_{n=0}^{N} \lambda_n(a)^{2k} c_n \varphi_n(a).$$

Since the eigenvalues $\lambda_n(a)^2$ for $n \in \{0,\dots,N\}$ are pairwise disjoint, this proves that for all $n \in \{0,\dots,N\}$ we have $c_n\varphi_n(a) = 0$, and hence $c_n = 0$. Finally the map $(c_n) \in l^2 \mapsto \sum c_n\varphi_n(a) \in L^2(0,l)$ is one-to-one.

- As above we can check that for $N \in \mathbb{N}$ we have

$$\left\| \sum_{n=N}^{\infty} c_n \varphi_n(a) \right\|^2 - \sum_{n=N}^{\infty} |c_n|^2 \lesssim \sum_{n=N}^{\infty} |c_n|^2 \sqrt{\sum_{n=N}^{\infty} \frac{1}{\langle n \rangle^2}}.$$

This is less that $\frac{1}{2}\sum_{n=N}^{\infty} |c_n|^2$ is N is chosen large enough. Let such an integer N be fixed. Now assume by contradiction that the sequences $(a_m)_{m\in\mathbb{N}} \in [-R,R]^{\mathbb{N}}$ and $(c^m)_{m\in\mathbb{N}}$ in $l^2(\mathbb{C})^{\mathbb{N}}$ are such that $\|c^m\|_{l^2} = 1$ for all $m \in \mathbb{N}$ and

$$\left\| \sum_{n=0}^{\infty} c_n^m \varphi_n(a_m) \right\|^2 \xrightarrow[m\to\infty]{} 0.$$

For $m \in \mathbb{N}$ we set $f_m = \sum_{n=0}^{N-1} c_n^m \varphi_n(a_m)$ and $g_m = \sum_{n=N}^{\infty} c_n^m \varphi_n(a_m)$. After extracting a subsequence if necessary we can assume that a_m converges to some $a \in [-R,R]$ and f_m converges to some $f \in L^2(0,l)$. Let P_m (respectively P) denote the orthogonal projection on $\mathrm{span}(\varphi_j(a_m))_{j\geqslant N}$ (respectively on $\mathrm{span}(\varphi_n(a))_{n\geqslant N}$). We have

$$\|f_m + g_m\|^2 = \|f_m\|^2 + 2\,\mathrm{Re}\,\langle P_m f_m, g_m\rangle + \|g_m\|^2 \geqslant \|f_m\|^2 - \|P_m f_m\|^2 \xrightarrow[m\to\infty]{} \|f\|^2 - \|Pf\|^2.$$

This gives $f = Pf$, and hence $f = 0$. This gives a contradiction with

$$0 = \lim_{m\to\infty} \|g_m\|^2 \geqslant \lim_{m\to\infty} \frac{1}{2} \sum_{n=N}^{\infty} |c_n^m|^2 = \frac{1}{2}$$

and proves the first inequality of the proposition.

- It remains to prove that the sequence $(\varphi_n(a))$ is complete. We know that the family $\big(\varphi_n(0)\big)_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(0,l)$. Since $\varphi_n(a) = \varphi_n(0) + O\big(n^{-1}\big)$ in $L^2(0,l)$, this follows from a perturbation argument (see Theorem V.2.20 in [Kat80]). This concludes the proof. □

For $a \in \mathbb{R}$ we denote by $(\varphi_n^*(a))_{n\in\mathbb{N}}$ the dual basis of $(\varphi_n(a))_{n\in\mathbb{N}}$. We have $\varphi_n^*(a) = \varphi_n(-a) = \overline{\varphi_n(a)}$.

4. Separation of variables - Spectrum of the model operator

In this section we use the results on the transversal operator T_a to prove spectral properties for the full operator H_a when a is constant on $\partial\Omega$. Most of the results of this section are inspired by the $\mathcal{PT}$-symmetric analogs (see [BK08]). Let $a > 0$ be fixed. We set

$$\mathfrak{S}_a = \bigcup_{n\in\mathbb{N}} \{\lambda_n(a)^2\} + \mathbb{R}_+ = \big\{\lambda_n(a)^2 + r, n \in \mathbb{N}, r \in \mathbb{R}_+\big\} \subset \mathbb{C}.$$

Proposition 4.1. *We have $\mathfrak{S}_a \subset \sigma(H_a)$.*

Proof. Let $n \in \mathbb{N}$, $r \geqslant 0$ and $z = \lambda_n(a)^2 + r \in \mathfrak{S}_a$. Let $(f_m)_{m\in\mathbb{N}}$ be a sequence in $H^2(\mathbb{R}^{d-1})$ such that $\|f_m\|_{L^2(\mathbb{R}^{d-1})} = 1$ for all $m \in \mathbb{N}$ and $\|(-\Delta_x - r)f_m\|_{L^2(\mathbb{R}^{d-1})} \to 0$ as $m \to \infty$. For $m \in \mathbb{N}$ and $(x,y) \in \mathbb{R}^{d-1}\times]0,l[$ we set $u_m(x,y) = f_m(x)\varphi_n(a;y)$. Then $u_m \in \mathcal{D}(H_a)$ and $\|u_m\|_{L^2(\Omega)} = 1$ for all $m \in \mathbb{N}$. Moreover, according to (3.1) we have

$$\|(H_a - z)u_m\|_{L^2(\Omega)} = \|(-\Delta_x - r)f_m\|_{L^2(\mathbb{R}^{d-1})} \xrightarrow[m\to\infty]{} 0.$$

This implies that $z \in \sigma(H_a)$. □

For $u \in L^2(\Omega)$, $n \in \mathbb{N}$ and $x \in \mathbb{R}^{d-1}$ we set $u_n(x) = \big\langle u(x,\cdot), \varphi_j^*(a)\big\rangle_{L^2(0,l)}$. This gives a sequence of functions u_n defined almost everywhere on $\mathbb{R}^{d-1}$.

Proposition 4.2. *Let $u \in L^2(\Omega)$. Then $u_n \in L^2(\mathbb{R}^{d-1})$ for all $n \in \mathbb{N}$ and on $L^2(\Omega)$ we have*

$$u = \sum_{n\in\mathbb{N}} u_n \otimes \varphi_n(a).$$

Proof. For $N \in \mathbb{N}$ we set $v_N = \sum_{n=0}^{N} u_n \otimes \varphi_n(a)$. For almost all $x \in \mathbb{R}^{d-1}$, $v_N(x)$ defines a function in $L^2(0,l)$ which goes to $u(x)$ as $N \to \infty$. According to Proposition 3.2 we have

$$\|v_N(x)\|^2_{L^2(0,l)} \lesssim \sum_{n=0}^{N} \|u_n(x)\|^2_{L^2(0,l)} \lesssim \sum_{n=0}^{\infty} \|u_n(x)\|^2_{L^2(0,l)} \lesssim \|u(x)\|^2_{L^2(0,l)},$$

where all the estimates are uniform in N. Since the map $x \mapsto \|u(x)\|^2_{L^2(0,l)}$ belongs to $L^2(\mathbb{R}^{d-1})$, we can apply the dominated convergence theorem to conclude that $u - v_N \to 0$ in $L^2(\Omega)$. □

Proposition 4.3. *(i) Let $u \in \mathcal{D}(H_a)$. Then $u_n \in H^2(\mathbb{R}^{d-1})$ for all $n \in \mathbb{N}$ and in $L^2(\Omega)$ we have*

$$H_a u = \sum_{n\in\mathbb{N}} \big(-\Delta_x + \lambda_n(a)^2 \big) u_n \otimes \varphi_n(a).$$

(ii) Let $z \in \mathbb{C} \setminus \mathfrak{S}_a$. Then z belongs to the resolvent set of H_a and for all $u \in L^2(\Omega)$ we have

$$(H_a - z)^{-1} u = \sum_{n\in\mathbb{N}} (-\Delta_x + \lambda_n(a)^2 - z)^{-1} u_n \otimes \varphi_n(a).$$

In particular there exists $C \geqslant 0$ such that for all $z \in \mathbb{C} \setminus \mathfrak{S}$ and $u \in L^2(\Omega)$ we have

$$\big\| (H_a - z)^{-1} u \big\|_{L^2(\Omega)} \leqslant \frac{C}{d(z, \mathfrak{S}_a)} \|u\|_{L^2(\Omega)}.$$

Proof. • Let $z \in \mathbb{C} \setminus \mathfrak{S}_a$ and $u \in L^2(\Omega)$. Let $u_n \in L^2(\mathbb{R}^{d-1})$, $n \in \mathbb{N}$, be as above. For $n \in \mathbb{N}$ we set $\tilde{R}_n(u) = (-\Delta_x + \lambda_n(a)^2 - z)^{-1} u_n \in H^2(\mathbb{R}^{d-1})$ and $R_n(u) = \tilde{R}_n(u) \otimes \varphi_n(a) \in \mathcal{D}(H_a)$. Using the standard spectral properties of the self-adjoint operator $-\Delta_x$, we see that on $L^2(\mathbb{R}^{d-1})$ we have

$$\Big\| \tilde{R}_n(u) \Big\| \lesssim \frac{\|u_n\|}{d(z, \lambda_n(a)^2 + \mathbb{R}_+)} \lesssim_z \frac{\|u_n\|}{\langle n \rangle^2} \quad \text{and} \quad \Big\| \partial_x \tilde{R}_n(u) \Big\| \lesssim_z \frac{\|u_n\|}{\langle n \rangle}. \tag{4.1}$$

The first estimate is uniform in z but not the others. With Proposition 3.2 we obtain for $N, p \in \mathbb{N}$

$$\left\| \sum_{n=N}^{N+p} R_n(u) \right\|^2_{H^1(\Omega)} \lesssim_z \sum_{n=N}^{N+p} \left(\Big\| \tilde{R}_n(u) \Big\|^2_{H^1(\mathbb{R}^{d-1})} + \langle n \rangle^2 \Big\| \tilde{R}_n(u) \Big\|^2_{L^2(\mathbb{R}^{d-1})} \right) \xrightarrow[N\to\infty]{} 0.$$

This proves that the series $\sum R_n(u)$ converges to some $R(u) \in H^1(\Omega)$. Moreover, with the first inequality of (4.1) and Proposition 3.2 again, we see that

$$\|R(u)\|_{L^2(\Omega)} \lesssim \frac{\|u\|_{L^2(\Omega)}}{d(z, \mathfrak{S}_a)}, \tag{4.2}$$

uniformly in z. It remains to see that

$$\forall v \in H^1(\Omega), \quad q_a(R(u), v) - z \langle R(u), v \rangle = \langle u, v \rangle,$$

which comes from the fact that this is true with $R(u)$ and u are replaced by $R_n(u)$ and $u_n \otimes \varphi_n(a)$ respectively. This proves that $R(u) \in \mathcal{D}(H_a)$ and $(H_a - z)R(u) = u$. If $\mathrm{Re}(z) < 0$, we already know that $(H_a - z)$ has a bounded inverse on $L^2(\Omega)$, and hence we have $(H_a - z)^{-1} = R$. This proves the second statement of the proposition when $\mathrm{Re}(z) < 0$.

• Let $u \in \mathcal{D}(H_a)$ and $v = (H_a + 1)u \in L^2(\Omega)$. For $n \in \mathbb{N}$ and almost all $x \in \mathbb{R}^{d-1}$ we set $v_n(x) = \langle v(x, \cdot), \varphi_n^*(a) \rangle_{L^2(0,l)}$. According to (ii) applied with $z = -1$ we have

$$u = (H_a + 1)^{-1} v = \sum_{n\in\mathbb{N}} \big(-\Delta_x + \lambda_n(a)^2 + 1 \big)^{-1} v_n \otimes \varphi_n(a).$$

By uniqueness for the decomposition of $u(x, \cdot)$ with respect to the Riesz basis $(\varphi_n(a))_{n\in\mathbb{N}}$, we have for all $n \in \mathbb{N}$

$$u_n = \big(-\Delta_x + \lambda_n(a)^2 + 1 \big)^{-1} v_n.$$

This proves that $u_n \in H^2(\mathbb{R}^{d-1})$. Then

$$\begin{aligned} H_a u = v - u &= \sum_{n\in\mathbb{N}} \Big(1 - \big(-\Delta_x + \lambda_n(a)^2 + 1 \big)^{-1} \Big) v_n \otimes \varphi_n(a) \\ &= \sum_{n\in\mathbb{N}} \big(-\Delta_x + \lambda_n(a)^2 \big) u_n \otimes \varphi_n(a). \end{aligned}$$

This proves the first statement of the proposition.

• It remains to finish the proof of (ii). Let $z \in \mathbb{C} \setminus \mathfrak{S}$ and $w \in \mathcal{D}(H_a)$. With (i) we see that $R\big((H_a - z)w\big) = w$. Since we already know that $(H_a - z)R(u) = u$ for all $u \in L^2(\Omega)$, this proves that R is a bounded inverse for $(H_a - z)^{-1}$ on $L^2(\Omega)$. The estimate on $(H_a - z)^{-1}$ follows from (4.2), and the proposition is proved. □

As a first application of this proposition, we can check that the operator H_a has no eigenvalue, as is the case for $-\Delta_x$:

Corollary 4.4. *The operator H_a has no eigenvalue.*

Proof. Let $z \in \mathbb{C}$ and $u \in \mathcal{D}(H_a)$ be such that $(H_a - z)u = 0$. Then according to the first item of Proposition 4.3 we have

$$\sum_{n\in\mathbb{N}} \big(-\Delta + \lambda_n(a)^2 - z\big)\, u_n \otimes \varphi_n(a) = 0.$$

This implies that for all $n \in \mathbb{N}$ we have

$$\big(-\Delta + \lambda_n(a)^2 - z\big)\, u_n = 0$$

in $L^2(\mathbb{R}^{d-1})$, and hence $u_n = 0$ since the operator $-\Delta_x$ has no eigenvalue. Finally $u = 0$, and the proposition is proved. □

However, the main point in Proposition 4.3 is the second:

Corollary 4.5. *Theorem 1.2 holds when $a > 0$ is constant.*

Proof. According to the last statement of Proposition 3.1 and the fact that $\mathrm{Im}(\lambda_n(a)^2) < 0$ for all $n \in \mathbb{N}$, there exists $\gamma > 0$ such that for all $z \in \mathfrak{S}_a$ we have $\mathrm{Im}(z) \leqslant -3\gamma$. With Proposition 4.3, the conclusion follows. □

5. Non-constant absorption index

In this section we prove Theorems 1.2 and 1.3 for a non-constant absorption index a. For $b \in W^{1,\infty}(\partial\Omega)$ we denote by $\Theta_b \in \mathcal{L}(H^1(\Omega), H^{-1}(\Omega))$ the operator such that for all $\varphi, \psi \in H^1(\Omega)$

$$\langle \Theta_b \varphi, \psi \rangle_{H^{-1}(\Omega), H^1(\Omega)} = \int_{\partial\Omega} b\varphi\overline{\psi}.$$

We denote by θ_b the corresponding quadratic form on $H^1(\Omega)$. We also denote by $\tilde{H}_a$ the operator in $\mathcal{L}\big(H^1(\Omega), H^{-1}(\Omega)\big)$ such that $\left\langle \tilde{H}_a\varphi, \psi \right\rangle_{H^{-1},H^1} = q_a(\varphi, \psi)$ for all $\varphi, \psi \in H^1(\Omega)$. Let $z \in \mathbb{C}_+$. According to the Lax-Milgram Theorem, $(1 + i)(\tilde{H}_a - z)$ is an isomorphism from $H^1(\Omega)$ to $H^{-1}(\Omega)$. Moreover, for $f \in L^2(\Omega) \subset H^{-1}(\Omega)$ we have

$$(\tilde{H}_a - z)^{-1} f = (H_a - z)^{-1} f.$$

The following proposition relies on a suitable version of the so-called quadratic estimates:

Proposition 5.1. *Let $a_0 > 0$ be as in the statement of Theorem 1.1. Let $B \in \mathcal{L}\big(H^1(\Omega), L^2(\Omega)\big)$. Then there exists $C \geqslant 0$ such that for all $z \in \mathbb{C}_+$ we have*

$$\left\| B(\tilde{H}_a - z)^{-1} B^* \right\|_{\mathcal{L}(L^2(\Omega))} \leqslant C \left\| B(\tilde{H}_{a_0} - z)^{-1} B^* \right\|_{\mathcal{L}(L^2(\Omega))}.$$

Proof. For $z \in \mathbb{C}_+$ the resolvent identity applied to $H_a = H_{a_0} + \Theta_{a-a_0}$ gives

$$B(\tilde{H}_a - z)^{-1} B^* = B(\tilde{H}_{a_0} - z)^{-1} B^* - B(\tilde{H}_{a_0} - z)^{-1} \Theta_{a-a_0} (\tilde{H}_a - z)^{-1} B^*. \tag{5.1}$$

Let $\varphi, \psi \in L^2(\Omega)$. Since Θ_{a-a_0} is associated to a non-negative quadratic form on $H^1(\Omega)$, the Cauchy-Schwarz inequality gives

$$\begin{aligned}
&\left\langle B(\tilde{H}_{a_0} - z)^{-1} \Theta_{a-a_0} (\tilde{H}_a - z)^{-1} B^* \varphi, \psi \right\rangle_{L^2} \\
&\quad = \left\langle \Theta_{a-a_0} (\tilde{H}_a - z)^{-1} B^* \varphi, (\tilde{H}_{a_0}^* - \overline{z})^{-1} B^* \psi \right\rangle_{H^{-1},H^1} \\
&\quad \leqslant \theta_{a-a_0}\big((\tilde{H}_a - z)^{-1} B^* \varphi\big)^{\frac{1}{2}} \times \theta_{a-a_0}\big((\tilde{H}_{a_0}^* - \overline{z})^{-1} B^* \psi\big)^{\frac{1}{2}}.
\end{aligned}$$

The first factor is estimated as follows:

$$\begin{aligned}\theta_{a-a_0}\big((\tilde{H}_a-z)^{-1}B^*\varphi\big) &\leqslant \theta_a\big((\tilde{H}_a-z)^{-1}B^*\varphi\big)\\ &\leqslant \frac{1}{2i}\Big\langle 2i(\Theta_a+\operatorname{Im} z)(\tilde{H}_a-z)^{-1}B^*\varphi,(\tilde{H}_a-z)^{-1}B^*\varphi\Big\rangle_{H^{-1},H^1}\\ &\leqslant \frac{1}{2i}\Big\langle B\big((\tilde{H}_a-z)^{-1}-(\tilde{H}_a^*-\overline{z})^{-1}\big)B^*\varphi,\varphi\Big\rangle_{L^2}\\ &\leqslant \Big\|B(\tilde{H}_a-z)^{-1}B^*\Big\|_{\mathcal{L}(L^2)}\|\varphi\|_{L^2}^2\,.\end{aligned}$$

We can proceed similarly for the other factor, using the fact that $a-a_0\leqslant \alpha a_0$ for some $\alpha\geqslant 0$. Thus there exists $C\geqslant 0$ such that

$$\begin{aligned}\Big\|B(\tilde{H}_{a_0}-z)^{-1}\Theta_{a-a_0}(\tilde{H}_a-z)^{-1}B^*\Big\| &\leqslant C\Big\|B(\tilde{H}_{a_0}-z)^{-1}B^*\Big\|^{\frac12}\Big\|B(\tilde{H}_a-z)^{-1}B^*\Big\|^{\frac12}\\ &\leqslant \frac{C^2}{2}\Big\|B(\tilde{H}_{a_0}-z)^{-1}B^*\Big\|+\frac12\Big\|B(\tilde{H}_a-z)^{-1}B^*\Big\|\,.\end{aligned}$$

With (5.1), the conclusion follows. □

Now we can finish the proof of Theorem 1.2:

Proof of Theorem 1.2. According to Corollary 4.5 and Proposition 5.1 applied with $B=\mathrm{Id}_{L^2(\Omega)}$ (note that we can simply replace $\tilde{H}_a$ by H_a when $B\in\mathcal{L}(L^2(\Omega))$), there exists $C>0$ such that $\big\|(H_a-z)^{-1}\big\|_{\mathcal{L}(L^2(\Omega))}\leqslant C$ for all $z\in\mathbb{C}_+$. Then it only remains to choose $\gamma\in\left]0,\frac{1}{3C}\right[$ to conclude.

□

Let us now turn to the proof of Theorem 1.3. We first prove other resolvent estimates. Compared to the estimate of Theorem 1.2, the first estimate of Proposition 5.2 requires a weight but decays for $\operatorname{Re}(z)\gg 1$. This allows to add a power of $-\Delta_x$ in the second estimate, which gives the smoothing effect.

Proposition 5.2. *Let $\delta>\frac12$. Then there exists $C\geqslant 0$ such that for all $z\in\mathbb{C}_+$ we have*

$$\Big\|\langle x\rangle^{-\delta}(H_a-z)^{-1}\langle x\rangle^{-\delta}\Big\|_{\mathcal{L}(L^2(\Omega))}\leqslant C\langle z\rangle^{-\frac12}$$

and

$$\Big\|\langle x\rangle^{-\delta}(1-\Delta_x)^{\frac14}(H_a-z)^{-1}(1-\Delta_x)^{\frac14}\langle x\rangle^{-\delta}\Big\|_{\mathcal{L}(L^2(\Omega))}\leqslant C.$$

Proof. This result is well-known if we replace $(H_a-z)^{-1}$ by $(-\Delta_x-z)^{-1}$ and $\mathcal{L}(L^2(\Omega))$ by $\mathcal{L}(L^2(\mathbb{R}^{d-1}))$. When a is a constant, we obtain the result using the separation of variables as in Section 4. Then we conclude with Proposition 5.1 with $B=\langle x\rangle^{-\delta}(1-\Delta_x)^{\frac14}$. In fact, we first obtain an estimate on the resolvent $(\tilde{H}_a-z)^{-1}$, but this proves that the operator $\langle x\rangle^{-\delta}(1-\Delta_x)^{\frac14}(H_a-z)^{-1}(1-\Delta_x)^{\frac14}\langle x\rangle^{-\delta}$ extends to a bounded operator on $L^2(\Omega)$, and then the same estimate holds for the corresponding closure. □

Proof of Theorem 1.3. With the second estimate of Proposition 5.2, we can apply the theory of relatively smooth operators (see §XIII.7 in [RS79]). However, since H_a is not self-adjoint but only maximal dissipative, we have to use a self-adjoint dilation (see [NF10]) of H_a, as is done in the proof of [Roy10b, Proposition 5.6] (see also Proposition 2.24 in [Roy10a]). □

6. Time decay for the Schrödinger equation

In this section we prove Theorem 1.1. Let $u_0\in\mathcal{D}(H_a)$ and let u be the solution of the problem (1.1). We know that $\|u(t)\|_{L^2(\Omega)}\leqslant\|u_0\|_{L^2(\Omega)}$ for all $t\geqslant 0$, so the result only concerns large times. Let $\gamma>0$ be given by Theorem 1.2. Let $\chi\in C^\infty(\mathbb{R})$ be equal to 0 on $]-\infty,1]$ and equal to 1 on $[2,+\infty[$. For $t\in\mathbb{R}$ we set

$$u_\chi(t)=\chi(t)u(t),$$

and for $z\in\mathbb{C}_+$:

$$v(z)=\int_{\mathbb{R}}e^{itz}u_\chi(t)\,dt.$$

The map $t \mapsto e^{-\gamma t} u_\chi(t)$ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap C^1(\mathbb{R})$ and its derivative is in $L^1(\mathbb{R})$ so $\tau \mapsto v(\tau + i\gamma)$ is bounded and decays at least like $\langle \tau \rangle^{-1}$. In particular it is in $L^2(\mathbb{R})$. For $R > 0$ we set

$$u_R(t) = \frac{1}{2\pi} \int_{-R}^{R} e^{-it(\tau + i\gamma)} v(\tau + i\gamma)\, d\tau.$$

Then

$$\left\| e^{-t\gamma}(u_\chi - u_R) \right\|_{L^2(\mathbb{R}_t, L^2(\Omega))} \xrightarrow[R \to +\infty]{} 0.$$

Since u_χ is continuous, Theorem 1.1 will be proved if we can show that there exists $C \geqslant 0$ which does not depend on u_0 and such that for all $t \geqslant 0$ we have

$$\limsup_{R \to \infty} \|u_R(t)\|_{L^2(\Omega)} \leqslant C e^{-\gamma t} \|u_0\|_{L^2(\Omega)}. \tag{6.1}$$

For $z \in \mathbb{C}$ we set

$$\theta(z) = -i \int_{\mathbb{R}} e^{itz} \chi'(t) u(t)\, dt = -i \int_1^2 e^{itz} \chi'(t) u(t)\, dt.$$

Let $z \in \mathbb{C}_+$. We multiply (1.1) by $\chi(t) e^{itz}$ and integrate over $t \in \mathbb{R}$. After partial integration we obtain

$$v(z) = (H_a - z)^{-1} \theta(z).$$

Then v extends to a holomorphic function on $\mathbb{C}_{3\gamma}$, taking this equality as a definition. According to the Cauchy Theorem we have in $L^2(\mathbb{R}_t)$

$$\lim_{R \to \infty} u_R(t) = \frac{1}{2\pi} \lim_{R \to \infty} \int_{-R}^{R} e^{-it(\tau - 2i\gamma)} v(\tau - 2i\gamma)\, d\tau = e^{-2\gamma t} \lim_{R \to \infty} \widetilde{u_R}(t), \tag{6.2}$$

where for $t \in \mathbb{R}$ we have set

$$\widetilde{u_R}(t) = \int_{-R}^{R} e^{-it\tau} v(\tau - 2i\gamma)\, d\tau.$$

According to Plancherel's equality and Theorem 1.2 we have uniformly in $R > 0$:

$$\begin{aligned}
\int_{\mathbb{R}} \|\widetilde{u_R}(t)\|^2_{L^2(\Omega)}\, dt &= \int_{-R}^{R} \left\| \big(H_a - (\tau - 2i\gamma)\big)^{-1} \theta(\tau - 2i\gamma) \right\|^2_{L^2(\Omega)} d\tau \\
&\lesssim \int_{\mathbb{R}} \|\theta(\tau - 2i\gamma)\|^2_{L^2(\Omega)}\, d\tau \\
&\lesssim \int_{\mathbb{R}} e^{2\gamma t} |\chi'(t)| \, \|u(t)\|^2_{L^2(\Omega)}\, dt \\
&\lesssim \|u_0\|^2_{L^2(\Omega)}.
\end{aligned}$$

In particular there exists $C \geqslant 0$ such that for $u_0 \in \mathcal{D}(H_a)$ and $R > 0$ we can find $T(u_0, R) \in [0, 1]$ which satisfies

$$\|\widetilde{u_R}(T(u_0, R))\|_{L^2(\Omega)} \leqslant C \|u_0\|_{L^2(\Omega)}.$$

Let $R > 0$. Then $\widetilde{u_R} \in C^1(\mathbb{R})$ and for $t \geqslant 1$ we have

$$\widetilde{u_R}(t) = e^{-i(t - T(u_0,R))H_a} \widetilde{u_R}(T(u_0, R)) + \int_{T(u_0,R)}^{t} \frac{\partial}{\partial s} \left(e^{-i(t-s)H_a} \widetilde{u_R}(s) \right) ds,$$

where

$$\begin{aligned}
\frac{\partial}{\partial s} \left(e^{-i(t-s)H_a} \widetilde{u_R}(s) \right) &= \frac{\partial}{\partial s} \int_{-R}^{R} e^{-is\tau} e^{-i(t-s)H_a} \big(H_a - (\tau - 2i\gamma)\big)^{-1} \theta(\tau - 2i\gamma)\, d\tau \\
&= i \int_{-R}^{R} e^{-i(t-s)H_a} e^{-is\tau} (H_a - \tau) \big(H_a - (\tau - 2i\gamma)\big)^{-1} \theta(\tau - 2i\gamma)\, d\tau \\
&= 2\gamma e^{-i(t-s)H_a} \widetilde{u_R}(s) + i e^{-i(t-s)H_a} \int_{-R}^{R} e^{-is\tau} \theta(\tau - 2i\gamma)\, d\tau.
\end{aligned}$$

This proves that the map $s \mapsto \frac{\partial}{\partial s} \left(e^{-i(t-s)H_a} \widetilde{u_R}(s) \right)$ belongs to $L^2([0, t], L^2(\Omega))$ uniformly in t and $R > 0$, and its $L^2([0, t], L^2(\Omega))$ norm is controlled by the norm of u_0 in $L^2(\Omega)$. We finally obtain $C \geqslant 0$ such that for all $t \in \mathbb{R}$ and $R > 0$ we have

$$\|\widetilde{u_R}(t)\|_{L^2(\Omega)} \leqslant C \langle t \rangle^{\frac{1}{2}} \|u_0\|_{L^2(\Omega)}.$$

With (6.2) this proves (6.1) and concludes the proof of Theorem 1.1.

7. The case of a weakly dissipative boundary condition

In this section we prove Theorem 1.4 about the problem (1.5). The absorption index a now takes the value a_l on $\mathbb{R}^{d-1}\times\{l\}$ and a_0 on $\mathbb{R}^{d-1}\times\{0\}$.

The proof follows the same lines as in the dissipative case, except that well-posedness of the problem is not an easy consequence of the general dissipative theory. We will use the separation of variables as in Section 4 instead. Once we have a decomposition as in Proposition 4.2 for the initial datum, we can propagate each term by means of the unitary group generated by $-\Delta_x$ and define the solution of (1.5) as a series of solutions on $\mathbb{R}^{d-1}$.

Let us first look at the transverse problem. The transverse operator on $L^2(0,l)$ corresponding to the problem (1.5) is now given by $T_{a_l,a_0}=-\frac{d^2}{dy^2}$ with domain

$$\mathcal{D}(T_{a_l,a_0})=\left\{u\in H^2(0,l)\,:\,u'(0)=-ia_0u(0),u'(l)=ia_lu(l)\right\}.$$

If $a_l>0$ and $a_0>0$ (or if one of them vanishes) then we can reproduce exactly the same analysis as in Section 3. In particular, there is no restriction on the sizes of these coefficients. The results we give here to handle the weakly dissipative case are also valid in this situation.

The strategy will be the same as in Section 3, so we will only emphasize the differences. We first remark that 0 is an eigenvalue of T_{a_l,a_0} if and only if $a_l=a_0=0$. Otherwise, $\lambda^2\in\mathbb{C}^*$ is an eigenvalue of T_{a_l,a_0} if and only if

$$(\lambda-a_l)(\lambda-a_0)e^{2i\lambda l}=(\lambda+a_l)(\lambda+a_0). \tag{7.1}$$

We recover (3.3) when $a_l=a_0$.

Lemma 7.1. *Let $a_l,a_0\in\mathbb{R}$ and $\lambda\in\mathbb{C}^*$ be such that $a_l+a_0\neq 0$ and λ^2 is an eigenvalue of T_{a_l,a_0}. Then $\mathrm{Re}(\lambda)\notin\nu\mathbb{N}$.*

We recall from [KBZ06] that if $a_l+a_0=0$ ($\mathcal{PT}$-symmectric case) then $n^2\nu^2\in\sigma(T_{a_l,a_0})$ for all $n\in\mathbb{N}^*$ (see also Figure 1 for $a_l+a_0>0$ small).

Proof. • We assume by contradiction that $\mathrm{Re}(\lambda)\in\nu\mathbb{N}$. According to (7.1) we have

$$\frac{(\lambda+a_l)(\lambda+a_0)}{(\lambda-a_l)(\lambda-a_0)}=e^{2il\lambda}=e^{-2l\,\mathrm{Im}(\lambda)}\in\mathbb{R}_+^*.$$

After multiplication by $|\lambda-a_l|^2|\lambda-a_0|^2\in\mathbb{R}_+^*$ we obtain

$$\left(|\lambda|^2-2ia_l\,\mathrm{Im}(\lambda)-a_l^2\right)\left(|\lambda|^2-2ia_0\,\mathrm{Im}(\lambda)-a_0^2\right)\in\mathbb{R}_+^*.$$

Taking the real and imaginary parts gives

$$|\lambda|^4-(a_l^2+a_0^2)|\lambda|^2+a_l^2a_0^2-4a_la_0\,\mathrm{Im}(\lambda)^2>0 \tag{7.2}$$

and

$$2\,\mathrm{Im}(\lambda)(a_l+a_0)\left(|\lambda|^2-a_la_0\right)=0. \tag{7.3}$$

• Assume that $a_la_0\geqslant 0$. In this case $\mathrm{Im}(\lambda)\neq 0$ (for the same reason as in the proof of Proposition 2.6), so (7.3) implies $|\lambda|^2=a_la_0$. Then (7.2) reads

$$-a_la_0(a_l-a_0)^2-4a_la_0\,\mathrm{Im}(\lambda)^2>0,$$

which gives a contradiction.

• Now assume that $a_la_0<0$. Then (7.3) implies $\mathrm{Im}(\lambda)=0$ and hence $e^{2il\lambda}=1$. From (7.1) we now obtain

$$(\lambda-a_l)(\lambda-a_0)=(\lambda+a_l)(\lambda+a_0),$$

which is impossible since $\lambda(a_l+a_0)\neq 0$. This concludes the proof. □

Proposition 7.2. *There exists $\rho>0$ such that if $|a_l|+|a_0|\leqslant\rho$ and $a_l+a_0>0$ then the spectrum of T_{a_l,a_0} is given by a sequence $(\lambda_n(a_l,a_0)^2)_{n\in\mathbb{N}}$ of algebraically simple eigenvalues such that*

$$\sup_{n\in\mathbb{N}}\mathrm{Im}\left(\lambda_n(a_l,a_0)^2\right)<0.$$

Moreover, any sequence of normalized eigenfunctions corresponding to these eigenvalues forms a Riesz basis.

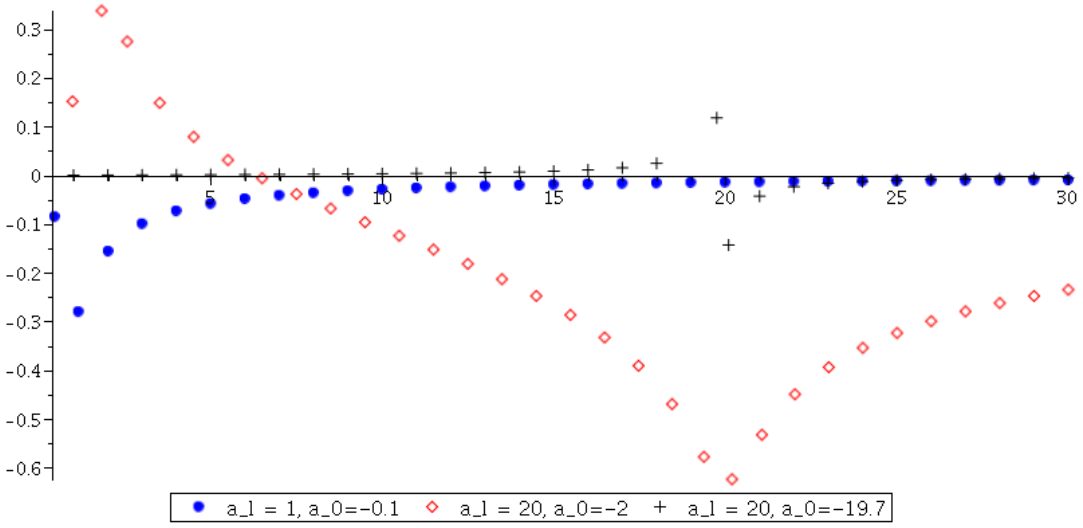

FIGURE 1. $\lambda_n(a_l, a_0)$ for $n \in \{0, \dots, 30\}$ and $l = \pi$.

Proof. As in the proof of Proposition 3.1 we can see that for any $R > 0$ there exists $C_R \geqslant 0$ such that if $\lambda^2 \in \mathbb{C}^*$ is an eigenvalue of T_{a_l,a_0} we have

$$|\mathrm{Re}\,\lambda| \leqslant R \quad \Longrightarrow \quad |\mathrm{Im}\,\lambda| \leqslant C_R. \tag{7.4}$$

The operator T_{a_l,a_0} depends analytically on the parameters a_l and a_0, and we know that when $a_l = a_0 = 0$ the eigenvalues $n^2\nu^2$ for $n \in \mathbb{N}$ are algebraically simple. With the restrictions given by (7.4) and Lemma 7.1, we obtain as in Section 3 a sequence of maps $(a_l, a_0) \mapsto \lambda_n(a_l, a_0)$ such that the eigenvalues of T_{a_l,a_0} are $\lambda_n(a_l, a_0)^2$ for $n \in \mathbb{N}$. Let $n \in \mathbb{N}^*$. We have

$$\lambda_n(a_l, a_0) = n\nu - \frac{i}{n\pi}(a_l + a_0) + \gamma(a_l + a_0)^2 + O\left(|a_l|^3, |a_0|^3\right), \tag{7.5}$$

with $\mathrm{Re}(\gamma) = l/(n\pi)^3 > 0$. As in the dissipative case, we obtain that for any a_l, a_0 with $a_l + a_0 > 0$ these eigenvalues $\lambda_n(a_l, a_0)$ are simple. If moreover a_l and a_0 are small enough, the eigenvalue $\lambda_n(a_l, a_0)^2$ is close to $(n\nu)^2$ and away from the real axis uniformly in $n \in \mathbb{N}^*$ (the first two terms in (7.5) are also the first two terms of the asymptotic expansion for large n and fixed a_l and a_0). It remains to check that we also have $\mathrm{Im}(\lambda_0(a_l, a_0)^2) < 0$. For small a_l, a_0 we denote by $\varphi_{a_l,a_0}(0)$ a normalized eigenvector corresponding to the eigenvalue $\lambda_0(a_l, a_0)^2$ and depending analytically on a_l and a_0. For all $\psi \in H^1(0, l)$ we have

$$\left\langle \varphi'_{a_l,a_0}, \psi' \right\rangle_{L^2(0,l)} - ia_l\varphi_{a_l,a_0}(l)\overline{\psi(l)} - ia_0\varphi_{a_l,a_0}(0)\overline{\psi(0)} = \lambda_0(a_l, a_0)^2 \left\langle \varphi_{a_l,a_0}, \psi \right\rangle_{L^2(0,l)}$$

We apply this with $\psi = \varphi_{a_l,a_0}$, take the derivatives with respect to a_l and a_0 at point $(a_l, a_0) = (0, 0)$, and use the facts that $\varphi_{0,0}$ is constant and $\lambda_0(0, 0) = 0$. We obtain

$$\nabla_{a_l,a_0}\left(\lambda_0^2\right) = -\frac{i}{l}(1, 1).$$

This proves that $\mathrm{Im}\left(\lambda_0(a_l, a_0)^2\right) < 0$ if a_l and a_0 are small enough with $a_l + a_0 > 0$. The Riesz basis property relies as before on the fact that

$$|\lambda_n(a_l, a_0) - n\nu| = O(n^{-1}).$$

For this point we can follow what is done in Section 3 for the dissipative case. □

For $n \in \mathbb{N}$ and $a_l, a_0 \in \mathbb{R}$ with $a_l + a_0 > 0$ we consider a normalized eigenvector $\varphi_n(a_l, a_0) \in L^2(0, l)$ corresponding to the eigenvalue $\lambda_n(a_l, a_0)^2$ of T_{a_l,a_0}. We denote by $(\varphi_n^*(a_l, a_0))_{n\in\mathbb{N}}$ the dual basis.

Proposition 7.3. *Let $u_0 \in \mathcal{D}(H_a)$. Then the problem (1.5) has a unique solution $u \in C^1(\mathbb{R}, L^2(\Omega)) \cap C^0(\mathbb{R}, \mathcal{D}(H_a))$. Moreover if we write*

$$u_0 = \sum_{n\in\mathbb{N}} u_{0,n} \otimes \varphi_n(a_l, a_0)$$

where $u_{0,n} \in L^2(\mathbb{R}^{d-1})$, then u is given by

$$u(t) = \sum_{n\in\mathbb{N}} \left(e^{-it\left(-\Delta_x + \lambda_n(a_l,a_0)^2\right)} u_{0,n} \right) \otimes \varphi_n(a_l, a_0).$$

Proof. • Assume that $u \in C^0(\mathbb{R}_+, \mathcal{D}(H_a)) \cap C^1(\mathbb{R}_+^*, L^2(\Omega))$ is a solution of (1.5). Let $t \in \mathbb{R}_+^*$. For all $n \in \mathbb{N}$ and almost all $x \in \mathbb{R}^{d-1}$ we can define $u_n(t,x) = \langle u(t;x,\cdot), \varphi_n^*(a_l,a_0)\rangle_{L^2(0,l)}$, so that in $L^2(\Omega)$

$$u(t) = \sum_{n\in\mathbb{N}} u_n(t) \otimes \varphi_n(a_l, a_0).$$

According to Proposition 4.3 (which can be proved exactly the same way in this context) we have $u_n(t) \in H^2(\mathbb{R}^{d-1})$ for all $t \in \mathbb{R}_+^*$ and $n \in \mathbb{N}$, and for $s \in \mathbb{R}^*$ we have

$$i\frac{u(t+s)-u(t)}{s} - H_a u(t) = \sum_{n\in\mathbb{N}} \left(i\frac{u_n(t+s)-u_n(t)}{s} - \left(-\Delta_x + \lambda_n(a_l,a_0)^2\right) u_n(t) \right) \otimes \varphi_n(a_l,a_0).$$

Let $n \in \mathbb{N}$. According to Proposition 3.2 we have

$$\left\| i\frac{u_n(t+s)-u_n(t)}{s} - \left(-\Delta_x + \lambda_n(a_l,a_0)^2\right) u(t) \right\|_{L^2(\mathbb{R}^{d-1})} \lesssim \left\| i\frac{u(t+s)-u(t)}{s} - H_a u(t) \right\|_{L^2(\Omega)} \xrightarrow[s\to 0]{} 0.$$

This proves that u_n is differentiable and for all $t > 0$

$$iu_n'(t) = \left(-\Delta_x + \lambda_n(a_l,a_0)^2\right) u_n(t).$$

Then for all $t > 0$

$$u_n(t) = e^{-it\left(-\Delta_x + \lambda_n(a_l,a_0)^2\right)} u_n(0) = e^{-it\left(-\Delta_x + \lambda_n(a_l,a_0)^2\right)} u_{0,n}.$$

• Conversely, let us prove that the function u defined by the statement of the proposition is indeed a solution of (1.5). Let $t \in \mathbb{R}$. According to Proposition 4.3, $u_{0,n}$ and hence $e^{-it\left(-\Delta_x + \lambda_n(a_l,a_0)^2\right)} u_{0,n}$ belong to $H^2(\mathbb{R}^{d-1})$ for all $n \in \mathbb{N}$. Therefore $u(t) \in \mathcal{D}(H_a)$. Then for all $s \in \mathbb{R}^*$ we have

$$\begin{aligned}
&\sum_{n\in\mathbb{N}} \left\| \left(i\frac{e^{-is\left(-\Delta_x + \lambda_n(a_l,a_0)^2\right)} - 1}{s} - \left(-\Delta_x + \lambda_n(a_l,a_0)^2\right) \right) e^{-it\left(-\Delta_x + \lambda_n(a_l,a_0)^2\right)} u_{0,n} \right\|_{L^(\mathbb{R}^{d-1})}^2 \\
&\quad \lesssim_t \sum_{n\in\mathbb{N}} \left\| \frac{1}{s} \int_0^s \left(e^{-is\left(-\Delta_x + \lambda_n(a_l,a_0)^2\right)} - 1 \right) \left(-\Delta_x + \lambda_n(a_l,a_0)^2\right) u_{0,n}\, ds \right\|_{L^(\mathbb{R}^{d-1})}^2 \\
&\quad \lesssim_t \sum_{n\in\mathbb{N}} \left\| \left(-\Delta_x + \lambda_n(a_l,a_0)^2\right) u_{0,n}\, ds \right\|_{L^(\mathbb{R}^{d-1})}^2 \\
&\quad \lesssim_t \|H_a u_0\|_{L^2(\Omega)}^2
\end{aligned}$$

This series of functions converges uniformly in s so we can take the limit, which proves that for any $t \in \mathbb{R}$

$$\left\| i\frac{u(t+s)-u(t)}{s} - H_a u(t) \right\|^2 \xrightarrow[s\to 0]{} 0.$$

This proves that u is differentiable and $iu'(t) + H_a u(t) = 0$, so u is indeed a solution of (1.5). □

Now we can prove Theorem 1.4:

Proof of Theorem 1.4. According to Proposition 7.3 we have existence and uniqueness for the solution u of the problem 1.5. Then with Proposition 3.2 we have

$$\begin{aligned}
\|u(t)\|_{L^2(\Omega)}^2 &\lesssim \sum_{n\in\mathbb{N}} \left\| e^{-it\left(-\Delta_x + \lambda_n(a_l,a_0)^2\right)} u_{0,n} \right\|_{L^2(\mathbb{R}^{d-1})}^2 \\
&\lesssim \sum_{n\in\mathbb{N}} e^{t\,\mathrm{Im}\left(\lambda_n(a_l,a_0)^2\right)} \|u_{0,n}\|_{L^2(\mathbb{R}^{d-1})}^2 .
\end{aligned}$$

Proposition 7.2 gives $\gamma_{a_l,a_0} > 0$ such that

$$\|u(t)\|^2_{L^2(\Omega)} \lesssim e^{-t\gamma_{a_l,a_0}} \sum_{n\in\mathbb{N}} \|u_{0,n}\|^2_{L^2(\mathbb{R}^{d-1})} \lesssim e^{-t\gamma_{a_l,a_0}} \|u_0\|^2_{L^2(\Omega)},$$

which concludes the proof. □

In the end of this section we show that the smallness assumption on $|a_l| + |a_0|$ is necessary in Theorem 1.4. More precisely, if $|a_l|$ and $|a_0|$ are too large, then the transverse operator T_{a_l,a_0} has eigenvalues with positive imaginary parts and hence the solution of the Schrödinger equation grows exponentially.

Proposition 7.4. *Let $a_l, a_0 \in \mathbb{R}$ with $a_l + a_0 > 0$ and $a_l a_0 < 0$. Let $n \in \mathbb{N}$. If $s > 0$ is large enough, we have* $\operatorname{Im}\left(\lambda_n(sa_l, sa_0)^2\right) > 0$.

Proof. We know that the curves $s \mapsto \lambda_n(sa_l, sa_0)$ for $n \in \mathbb{N}$ are defined for all $s \in \mathbb{R}$ and remain bounded. Moreover we have chosen the square root $\lambda_n(sa_l, sa_0)$ of $\lambda_n(sa_l, sa_0)^2$ which has a non-negative real part, so the imaginary parts of $\lambda_n(sa_l, sa_0)$ and $\lambda_n(sa_l, sa_0)^2$ have the same signs. Assume that $a_l a_0 < 0$, and let $n \in \mathbb{N}^*$ be fixed. We have

$$\frac{(\lambda_n(sa_l, sa_0) + sa_l)(\lambda_n(sa_l, sa_0) + sa_0)}{(\lambda_n(sa_l, sa_0) - sa_l)(\lambda_n(sa_l, sa_0) - sa_0)} = 1 + \frac{2\lambda_n(a_l, a_0)}{s}\left(\frac{1}{a_l} + \frac{1}{a_0}\right) + \underset{s\to+\infty}{O}\left(s^{-2}\right). \tag{7.6}$$

Since $\operatorname{Re}(\lambda_n(sa_l, sa_0)) > n\nu$ and $\operatorname{Im}(\lambda_n(sa_l, sa_0))$ is bounded, this quantity is of norm less than 1 when $s > 0$ is large enough, so

$$e^{-2l\operatorname{Im}(\lambda_n(sa_l,sa_0))} = \left|e^{2il\lambda_n(sa_l,sa_0)}\right| < 1,$$

and hence $\operatorname{Im}(\lambda_n(sa_l, sa_0)) > 0$. When $n = 0$, the same holds if we can prove that $\lambda_0(sa_l, sa_0)$ does not go to 0 for large s. Indeed, in this case the only possibility to have

$$e^{2il\lambda_0(sa_l,sa_0)} \to 1$$

is that $\lambda_0(sa_l, sa_0)$ goes to ν, and then $\operatorname{Re}(\lambda_0(sa_l, sa_0))$ is bounded by below by a positive constant, and we can conclude as before. So assume by contradiction that $\lambda_0(sa_l, sa_0)$ goes to 0 as s goes to $+\infty$. Then we have

$$e^{2il\lambda_0(sa_l,sa_0)} = 1 + 2il\lambda_0(sa_l, sa_0) + \underset{s\to+\infty}{O}\left(|\lambda_0(sa_l, sa_0))|^2\right),$$

which gives a contradiction with (7.6), where the rest $O(s^{-2})$ has to be replaced by $O\left(\lambda_0^2 s^{-2}\right)$. This concludes the proof. □

Remark 7.5. We remark from (7.5) (see also Figure 1) that given $a_l, a_0 \in \mathbb{R}$ such that $a_l + a_0 > 0$ we always have $\operatorname{Im}(\lambda_n(a_l, a_0)) < 0$ if n is large enough. By duality, this means that there is always an eigenvalue with positive imaginary part when $a_l + a_0 < 0$, and hence the norm of the solution of (1.5) is always exponentially increasing in this case.

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