



HAL
open science

Normal class and normal lines of algebraic surfaces

Alfrederic Josse, Françoise Pene

► **To cite this version:**

Alfrederic Josse, Françoise Pene. Normal class and normal lines of algebraic surfaces. 2014. hal-00953669v3

HAL Id: hal-00953669

<https://hal.science/hal-00953669v3>

Preprint submitted on 12 Oct 2014 (v3), last revised 2 Apr 2016 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

NORMAL CLASS AND NORMAL LINES OF ALGEBRAIC SURFACES

ALFREDERIC JOSSE AND FRANÇOISE PÈNE

ABSTRACT. We are interested in the normal class of an algebraic surface \mathcal{S} of the complex projective space \mathbb{P}^3 , that is the number of normal lines to \mathcal{S} passing through a generic point of \mathbb{P}^3 . Thanks to the notion of normal polar, we state a formula for the normal class valid for a general surface \mathcal{S} . We give a generic result and we illustrate our formula with examples. We define the orthogonal incidence variety and compute the Schubert class of the variety of projective normal lines in the Chow ring of $\mathbb{G}(1, 3)$. We complete our work with a generalization of Salmon's formula for the normal class of a Plücker curve to any planar curve with any kind of singularity.

INTRODUCTION

The notion of normal lines to an hypersurface of an euclidean space is extended here to the complex projective space \mathbb{P}^n . The aim of the present work is to study the **normal class** $c_\nu(\mathcal{S})$ of a surface \mathcal{S} of \mathbb{P}^3 , that is the number of $m \in \mathcal{S}$ such that the projective normal line $\mathcal{N}_m(\mathcal{S})$ to \mathcal{S} at m passing through a generic $m_1 \in \mathbb{P}^3$ (see Section 1 for a precise definition of the projective normal lines and of the normal class of an hypersurface of \mathbb{P}^n). Let us consider the **variety** $\mathfrak{N}_{\mathcal{S}}$ of **projective normal lines of \mathcal{S}** by

$$\mathfrak{N}_{\mathcal{S}} := \overline{\{\mathcal{N}_m(\mathcal{S}); m \in \mathcal{S}\}} \subset \mathbb{G}(1, n)$$

and its Schubert class $\mathfrak{n}_{\mathcal{S}} := [\mathfrak{N}_{\mathcal{S}}] \in A^{n-1}(\mathbb{G}(1, n))$. The fact that $PGL(n, \mathbb{C})$ **does not preserve normal lines** complicates our study compared to the study of tangent hyperplanes (see Section 1). We prove namely the following result valid for a wide family of surfaces of \mathbb{P}^3 . We recall that the umbilical curve at infinity \mathcal{C}_∞ is the intersection of the hyperplane at infinity \mathcal{H}^∞ with any sphere (see Section 5.1 for a precise definition). Let $\mathcal{S} = V(F)$ be an irreducible surface of \mathbb{P}^3 . We write $\mathcal{S}_\infty = \mathcal{S} \cap \mathcal{H}^\infty$. Observe that singular points of \mathcal{S}_∞ correspond to points of tangency of \mathcal{S} with \mathcal{H}^∞ . In the statement of our first result, we assume that \mathcal{S} is smooth and that \mathcal{S}_∞ has no worse singularities than ordinary multiple points and ordinary cusps.

Theorem 1 ($n=3$). *Let $\mathcal{S} \subset \mathbb{P}^3$ be a smooth irreducible surface of degree $d_{\mathcal{S}} \geq 2$ such that:*

- (i) *in \mathcal{H}^∞ , the curve \mathcal{S}_∞ has a finite number of points of contact with \mathcal{C}_∞ ,*
- (ii) *any singular point of \mathcal{S}_∞ is either an ordinary multiple point or an ordinary cusp,*
- (iii) *at any (non singular) contact point of \mathcal{S}_∞ with \mathcal{C}_∞ , the contact is ordinary,*
- (iv) *at any singular point of \mathcal{S}_∞ contained in \mathcal{C}_∞ , the tangent line to \mathcal{C}_∞ is not contained in the tangent cone to \mathcal{S}_∞ .*

Then

$$\mathfrak{n}_{\mathcal{S}} = c_\nu(\mathcal{S}).\sigma_2 + d_{\mathcal{S}}(d_{\mathcal{S}} - 1).\sigma_{1,1} \in A^2(\mathbb{G}(1, 3))$$

Date: October 12, 2014.

2000 Mathematics Subject Classification. 14J99,14H50,14E05,14N05,14N10.

Key words and phrases. projective normal, class, Plücker, Grassmannian, Puiseux

Françoise Pène is partially supported by the french ANR project GEODE (ANR-10-JCJC-0108).

and the normal class of \mathcal{S} is

$$c_\nu(\mathcal{S}) = d_{\mathcal{S}}^3 - d_{\mathcal{S}}^2 + d_{\mathcal{S}} - \sum_{k \geq 2} ((k-1)^2 m_\infty^{*(k)} + k(k-1) \tilde{m}_\infty^{(k)}) - 2\kappa_\infty^* - 3\tilde{\kappa}_\infty - c_\infty,$$

where

- $m_\infty^{*(k)}$ (resp. $\tilde{m}_\infty^{(k)}$) is the number of ordinary multiple points of order k of \mathcal{S}_∞ outside (resp. contained in) \mathcal{C}_∞ ,
- κ_∞^* (resp. $\tilde{\kappa}_\infty$) is the number of ordinary cusps of \mathcal{S}_∞ outside (resp. contained in) \mathcal{C}_∞ ,
- c_∞ is the number of ordinary (non singular) contact points of \mathcal{S}_∞ with \mathcal{C}_∞ .

Example 2. The surface $\mathcal{S} = V(xzt - tx^2 - zt^2 - xz^2 + y^3) \subset \mathbb{P}^3$ is smooth, its only contact point with $\mathcal{H}_\infty = V(t)$ is $P[1 : 0 : 0 : 0]$ which is an ordinary cusp of $\mathcal{S}_\infty = V(t, -xz^2 + y^3)$. Moreover \mathcal{S}_∞ has no contact point with \mathcal{C}_∞ . Hence the normal class of \mathcal{S} is $27 - 9 + 3 - 2 = 19$.

Actually we establish a general formula valid for a wider family of surfaces. The notion of normal polar plays an important role in our study. It is a notion analogous to the notion of polar [2]. Given an irreducible surface $\mathcal{S} \subset \mathbb{P}^3$ of degree $d_{\mathcal{S}}$, we extend the definition of the line $\mathcal{N}_m(\mathcal{S})$ to $m \in \mathbb{P}^3$. In our approach, we use the Plücker embedding to identify a line with an element of $\mathbb{G}(1, 3) \subset \mathbb{P}^5$. We then define a regular map $\alpha : \mathbb{P}^3 \setminus \mathcal{B} \rightarrow \mathbb{P}^5$ corresponding to $m \mapsto \mathcal{N}_m(\mathcal{S})$ (where \mathcal{B} is the set of base points of α). We will see that $\mathcal{B}_{\mathcal{S}} := \mathcal{B} \cap \mathcal{S}$ corresponds to the union of the set of singular points of \mathcal{S} , of the set of contact points of \mathcal{S} with \mathcal{H}_∞ and of the set of contact points of \mathcal{S}_∞ with \mathcal{C}_∞ . For any $P \in \mathbb{P}^3$, we will introduce the notion of **normal polar** of \mathcal{S} with respect to P as the set of $m \in \mathbb{P}^3$ such that either $\mathcal{N}_m(\mathcal{S})$ contains P or $m \in \mathcal{B}$. We will see that, if $\dim \mathcal{B} \leq 1$, then the normal polar has dimension 1 and degree $d_{\mathcal{S}}^2 - d_{\mathcal{S}} + 1$ (for a generic $P \in \mathbb{P}^3$) and we also prove the following result.

Theorem 3. Let \mathcal{S} be an irreducible surface of \mathbb{P}^3 with isolated singularities, admitting a finite number of contact points with \mathcal{H}_∞ and such that \mathcal{S}_∞ has a finite number of (non singular) contact points with \mathcal{C}_∞ . Then $\mathbf{n}_{\mathcal{S}} = c_\nu(\mathcal{S}) \cdot \sigma_2 + d_{\mathcal{S}}(d_{\mathcal{S}} - 1) \cdot \sigma_{1,1} \in A^2(\mathbb{G}(1, 3))$ and the normal class $c_\nu(\mathcal{S})$ of \mathcal{S} is equal to $d_{\mathcal{S}}(d_{\mathcal{S}}^2 - d_{\mathcal{S}} + 1)$ minus the intersection multiplicity of \mathcal{S} with its generic normal polars at points of $\mathcal{B}_{\mathcal{S}}$.

Theorem 1 is a consequence of Theorem 3. In a more general setting, we can replace α in $\tilde{\alpha} = \frac{\alpha}{H}$ (for some homogeneous polynomial H of degree d_H) so that the set $\tilde{\mathcal{B}}$ of base points of $\tilde{\alpha}$ has dimension at most 1. In this case, we consider a notion of normal polars associated to $\tilde{\alpha}$ which have generically dimension 1 and degree $\tilde{d}_{\mathcal{S}}^2 - \tilde{d}_{\mathcal{S}} + 1$ (with $\tilde{d}_{\mathcal{S}} = d_{\mathcal{S}} - d_H$).

Theorem 4. Let \mathcal{S} be an irreducible surface of \mathbb{P}^3 . If the set $\tilde{\mathcal{B}} \cap \mathcal{S}$ is finite, then $\mathbf{n}_{\mathcal{S}} = c_\nu(\mathcal{S}) \cdot \sigma_2 + d_{\mathcal{S}}(\tilde{d}_{\mathcal{S}} - 1) \cdot \sigma_{1,1} \in A^2(\mathbb{G}(1, 3))$ and the normal class $c_\nu(\mathcal{S})$ of \mathcal{S} is equal to $d_{\mathcal{S}}(\tilde{d}_{\mathcal{S}}^2 - \tilde{d}_{\mathcal{S}} + 1)$ minus the intersection multiplicity of \mathcal{S} with its generic normal polars at points $m \in \tilde{\mathcal{B}} \cap \mathcal{S}$.

When the surface is a cylinder or a surface of revolution, its normal class is equal to the normal class of its planar base curve. The normal class of any planar curve is given by the simple formula given in Theorem 5 below, that we give for completeness. Let us recall that, when $\mathcal{C} = V(F)$ is an irreducible curve of \mathbb{P}^2 , the evolute of \mathcal{C} is the curve tangent to the family of normal lines to \mathcal{Z} and that the evolute of a line or a circle is reduced to a single point. Hence, except for lines and circles, the normal class of \mathcal{C} is simply the class (with multiplicity) of its evolute. The following result generalizes the result by Salmon [9, p. 137] proved in the case of Plücker curves (planar curves with no worse multiple tangents than ordinary double tangents, no singularities other than ordinary nodes and cusps) to any planar curve (with any type of singularities). We write ℓ_∞ for the line at infinity of \mathbb{P}^2 . We define $I[1 : i : 0]$ and $J[1 : -i : 0]$ in \mathbb{P}^2 . Recall that I and J are the two cyclic points (i.e. the circular points at infinity).

Theorem 5 ($n=2$). *Let $\mathcal{C} = V(F)$ be an irreducible curve of \mathbb{P}^2 of degree $d \geq 2$ with class d^\vee . Then its normal class is*

$$c_\nu(\mathcal{C}) = d + d^\vee - \Omega(\mathcal{C}, \ell_\infty) - \mu_I(\mathcal{C}) - \mu_J(\mathcal{C}),$$

where Ω denotes the sum of the contact numbers between two curves and where $\mu_P(\mathcal{C})$ is the multiplicity of P on \mathcal{C} .

In [4], Fantechi proved that the evolute map is birational from \mathcal{C} to its evolute curve unless if¹ $F_x^2 + F_y^2$ is a square modulo F and that in this latest case the evolute map is $2 : 1$ (if \mathcal{C} is neither a line nor a circle). Therefore, the normal class $c_\nu(\mathcal{C})$ of a planar curve \mathcal{C} corresponds to the class of its evolute unless $F_x^2 + F_y^2$ is a square modulo F and in this last case, the normal class $c_\nu(\mathcal{C})$ of \mathcal{C} corresponds to the class of its evolute times 2 (if \mathcal{C} is neither a line nor a circle).

The notion of focal loci generalizes the notion of evolute to higher dimension [10, 1]. The normal lines of an hypersurface \mathcal{Z} are tangent to the focal loci hypersurface of \mathcal{Z} but of course the normal class of \mathcal{Z} does not correspond anymore (in general) to the class of its focal loci (the normal lines to \mathcal{Z} are contained in but are not equal to the tangent planes of its focal loci).

As usual, we write elements of \mathbb{C}^{n+1} either by (x_1, \dots, x_{n+1}) or by $\begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix}$ (our convention

for line vectors is to write them $(x_1 \cdots x_{n+1})$).

In Section 1, we endow \mathbb{P}^n with an orthogonality structure, we define the projective normal lines and the normal class of an hypersurface \mathcal{Z} of \mathbb{P}^n . Sections 2 to 8 are devoted to the study of the normal classes of surfaces of \mathbb{P}^3 . In Section 2, we study the normal lines to a surface \mathcal{S} of \mathbb{P}^3 , we introduce the regular map α . Section 3 deals with the notion of normal polars. In section 4, we recall some classical facts on the Schubert classes. In Section 5, we interpret geometrically the set $\mathcal{B}_{\mathcal{S}}$ and we prove Theorems 3 and 4. In Section 6, we prove Theorem 1. In Section 7, we apply our results to compute the normal class of every quadric. In Section 8, we use our method to compute of the normal class of a cubic surface with singularity E_6 . In Section 9, we prove Theorem 5. We end this paper with an appendix on the normal class of two particular kinds of surfaces: the cylinders and the surfaces of revolution.

1. ORTHOGONALITY IN \mathbb{P}^n

1.1. Orthogonal subspaces. Let E_n be an euclidean affine n -space of direction the n -vector space \mathbf{E}_n (endowed with some fix basis). Let $\mathbf{V} := (\mathbf{E}_n \oplus \mathbb{R}) \otimes \mathbb{C}$ (endowed with the induced basis $\mathbf{e}_1, \dots, \mathbf{e}_{n+1}$). We consider the complex projective space $\mathbb{P}^n := \mathbb{P}(\mathbf{V})$ with projective coordinates x_1, \dots, x_{n+1} . Let us write $\pi : \mathbf{V} \setminus \{0\} \rightarrow \mathbb{P}^3$ for the canonical projection. We denote by $\mathcal{H}^\infty := V(x_{n+1}) \subset \mathbb{P}^n$ the hyperplane at infinity. We consider the affine space $A^n := \mathbb{P}^n \setminus \mathcal{H}^\infty$ endowed with the vector space $\vec{\mathbf{E}} := \text{Span}(\mathbf{e}_1, \dots, \mathbf{e}_n) \subset \mathbf{V}$ (with the affine structure $m + \vec{\mathbf{v}} = \pi(\mathbf{m} + \vec{\mathbf{v}})$ if $\vec{\mathbf{v}} \in \vec{\mathbf{E}}$ and $m = \pi(\mathbf{m}) \in A^n$ with $\mathbf{m}(x_1, \dots, x_n, 1)$).

Let us consider $\mathcal{W}_1 = \mathbb{P}(\mathbf{W}_1) \subset \mathbb{P}^n$ and $\mathcal{W}_2 = \mathbb{P}(\mathbf{W}_2) \subset \mathbb{P}^n$ where \mathbf{W}_1 and \mathbf{W}_2 are two vector subspaces of \mathbf{V} not contained in $\vec{\mathbf{E}}$ such that $\dim \mathbf{W}_1 + \dim \mathbf{W}_2 = n + 2$. Since \mathcal{W}_i is not contained in \mathcal{H}^∞ , $W_i := \mathcal{W}_i \setminus \mathcal{H}^\infty$ is an affine subspace of A^n with vector space $\vec{\mathbf{W}}_i := \mathbf{W}_i \cap \vec{\mathbf{E}}$, that is to say that there exists m_i such that $W_i = m_i + \vec{\mathbf{W}}_i$ in A^n . Consider the usual bilinear symmetric form $\langle u, v \rangle = \sum_{i=0}^3 u_i v_i$ on \mathbf{V} , the associated orthogonality on \mathbf{V} is written \perp .

¹We write $[x : y : z]$ for the coordinates of $m \in \mathbb{P}^2$ and F_x, F_y, F_z for the partial derivatives of F .

Definition 6. Let us consider $\mathcal{W}_1 = \mathbb{P}(\mathbf{W}_1) \subset \mathbb{P}^n$ and $\mathcal{W}_2 = \mathbb{P}(\mathbf{W}_2) \subset \mathbb{P}^n$ where \mathbf{W}_1 and \mathbf{W}_2 are two vector subspaces of \mathbf{V} not contained in $\vec{\mathbf{E}}$ and such that $\dim \mathbf{W}_1 + \dim \mathbf{W}_2 = n + 2$. With the above notations, we say that \mathcal{W}_1 and \mathcal{W}_2 are orthogonal in \mathbb{P}^3 if $\vec{\mathbf{W}}_1 \perp \vec{\mathbf{W}}_2$. We then write $\mathcal{W}_1 \perp \mathcal{W}_2$.

Observe that if $\mathcal{H} = V(a_1x_1 + \cdots + a_{n+1}x_{n+1})$ is an hyperplane and $\mathcal{L} = \mathbb{P}(\mathbf{L})$ a line of \mathbb{P}^n which are not contained in \mathcal{H}^∞ , then

$$\mathcal{H} \perp \mathcal{L} \Leftrightarrow (a_1, \dots, a_n, 0) \in \mathbf{L}.$$

This leads us to the following generalization of normal lines to an hyperplane.

Definition 7. We say that a projective hyperplane $\mathcal{H} = V(a_1x_1 + \cdots + a_{n+1}x_{n+1}) \subset \mathbb{P}^n$ and a projective line $\mathcal{L} = \mathbb{P}(\mathbf{L}) \subset \mathbb{P}^n$ are orthogonal in \mathbb{P}^3 if $(a_1, \dots, a_n, 0) \in \mathbf{L}$. We then write $\mathcal{L} \perp \mathcal{H}$.

1.2. Orthogonal incidence variety. Let us write as usual $\mathbb{G}(1, n)$ (resp. $\mathbb{G}(n-1, n)$) for the grassmanian of the lines (resp. of the hyperplanes) of \mathbb{P}^n . Let us define $pr_1 : \mathbb{G}(1, n) \times \mathbb{G}(n-1, n) \rightarrow \mathbb{G}(1, n)$ and $pr_2 : \mathbb{G}(1, n) \times \mathbb{G}(n-1, n) \rightarrow \mathbb{G}(n-1, n)$. We define the **orthogonal incidence variety** \mathcal{I}^\perp by

$$\mathcal{I}^\perp := \{(\mathcal{L}_1, \mathcal{H}_1) \in \mathbb{G}(1, n) \times \mathbb{G}(n-1, n) : \mathcal{L}_1 \perp \mathcal{H}_1\}.$$

Let us write $p_1 : \mathcal{I}^\perp \rightarrow \mathbb{G}(1, n)$ and $p_2 : \mathcal{I}^\perp \rightarrow \mathbb{G}(n-1, n)$ for the restrictions of pr_1 and pr_2 . We want to describe in the Chow ring of $\mathbb{G}(1, n)$ and $\mathbb{G}(n-1, n) \cong \mathbb{P}^{n \vee}$ the rational equivalence class of $p_2 p_1^{-1}(\mathcal{L})$ and $p_1 p_2^{-1}(\mathcal{H})$.

Lemma 8. $p_2 \circ p_1^{-1} : \mathbb{G}(1, n) \setminus \{\mathcal{L} \subset \mathcal{H}^\infty\} \rightarrow \mathbb{G}(n-1, n)$ is a line projective bundle and $p_1 \circ p_2^{-1} : \mathbb{G}(n-1, n) \setminus \{\mathcal{H}^\infty\} \rightarrow \mathbb{G}(1, n)$ and is a plane projective bundle.

Proof. Let $\mathcal{H} = V(a_1x_1 + \cdots + a_{n+1}x_{n+1})$ be a projective hyperplane of \mathbb{P}^n , which is not \mathcal{H}^∞ . Then

$$p_1(p_2^{-1}(\mathcal{H})) = \{\mathcal{L} \in \mathbb{G}(1, n), (a_1, \dots, a_n, 0) \in \mathbf{L}\}.$$

Moreover $p_1(p_2^{-1}(\mathcal{H}^\infty)) = \mathbb{G}(1, n)$.

Let $\mathcal{L} \not\subset \mathcal{H}^\infty$ be a line of \mathbb{P}^3 , let $A_0[a_1 : \cdots : a_n : 0]$ be the only point in $\mathcal{L} \cap \mathcal{H}^\infty$, we have

$$p_2(p_1^{-1}(\mathcal{L})) = \{\mathcal{H} \in \mathbb{G}(n-1, n) : \exists [a : b] \in \mathbb{P}^1, \mathcal{H} = V(aa_1x_1 + \cdots + aa_nx_n + bx_{n+1})\}.$$

Finally, if $\mathcal{L} = \mathbb{P}(\mathbf{L}) \subset \mathcal{H}^\infty$ is a projective line, then we have

$$p_2(p_1^{-1}(\mathcal{L})) = \{\mathcal{H} \in \mathbb{G}(n-1, n) : \exists a, b \in \mathbb{C}, \exists (a_1, \dots, a_n, 0) \in \mathbf{L}, \mathcal{H} = V(aa_1x_1 + \cdots + aa_nx_n + bx_{n+1})\}.$$

□

It follows directly from the proof of this lemma that if $\mathcal{H} \in \mathbb{G}(n-1, n) \setminus \{\mathcal{H}^\infty\}$, the class of $p_1(p_2^{-1}(\mathcal{H}))$ in the Chow ring $A^*(\mathbb{G}(1, n))$ is simply the Schubert class σ_{n-1} (see Section 4 for some recalls on the Schubert cycles in the case $n = 3$).

1.3. Projective normal lines. Given $\mathcal{Z} = V(F) \neq \mathcal{H}_\infty$ an irreducible hypersurface of \mathbb{P}^n (with $F \in \mathbb{P}(\text{Sym}(\mathbf{V}^\vee)) \cong \mathbb{C}[x_1, \dots, x_{n+1}]$), we consider the rational map $n_{\mathcal{Z}} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ given by $n_{\mathcal{Z}} = [F_{x_1} : \cdots : F_{x_n} : 0]$.

Definition 9. The projective normal line $\mathcal{N}_m \mathcal{Z}$ to \mathcal{Z} at $m \in \mathcal{Z}$ is the line $(m n_{\mathcal{Z}}(m))$ when $n_{\mathcal{Z}}(m)$ is well defined in \mathbb{P}^n and not equal to m .

If F has real coefficients and if $m \in \mathcal{Z} \setminus \mathcal{H}_\infty$ has real coordinates $[x_1^{(0)} : \cdots : x_n^{(0)} : 1]$, then $\mathcal{N}_m \mathcal{Z}$ corresponds to the affine normal line of the affine hypersurface $V(F(x_1, \dots, x_n, 1)) \subset E_n$ at the point of coordinates $(x_1^{(0)}, \dots, x_n^{(0)})$. The aim of this work is the study of the notion of normal class.

Definition 10. *Let \mathcal{Z} be an irreducible hypersurface of \mathbb{P}^n . The normal class of \mathcal{Z} is the number $c_\nu(\mathcal{Z})$ of projective normal lines of \mathcal{Z} through a generic $m_1 \in \mathbb{P}^n$, i.e. the number of $m \in \mathcal{Z}$ such that $\mathcal{N}_m(\mathcal{Z})$ contains m_1 for a generic $m_1 \in \mathbb{P}^n$.*

We will observe that the projective normal lines are preserved by some changes of coordinates in \mathbb{P}^n , called **projective similitudes** of \mathbb{P}^n and defined as follows.

1.4. Projective similitudes. Recall that, for every field \mathbb{k} ,

$$GO(n, \mathbb{k}) = \{A \in GL(n, \mathbb{k}); \exists \lambda \in \mathbb{k}^*, A \cdot {}^t A = \lambda \cdot I_n\}$$

is the **orthogonal similitude group** (for the standard products) and that $GOAff(n, \mathbb{k}) = \mathbb{k}^n \rtimes GO(n, \mathbb{k})$ is the **orthogonal similitude affine group**. We have a natural monomorphism of groups $\kappa : Aff(n, \mathbb{R}) = \mathbb{R}^n \rtimes GL(n, \mathbb{R}) \longrightarrow GL(n+1, \mathbb{R})$ given by

$$\kappa(b, A) = \begin{pmatrix} a_{11} & \dots & \dots & a_{1n} & b_1 \\ a_{21} & \dots & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & \dots & a_{nn} & b_n \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix} \quad (1)$$

and, by restriction, $\kappa|_{GOAff(n, \mathbb{R})} : GOAff(n, \mathbb{R}) = \mathbb{R}^n \rtimes GO(n, \mathbb{R}) \longrightarrow GL(n+1, \mathbb{R})$. Analogously we have a natural monomorphism of groups $\kappa' := (\kappa \otimes 1)|_{GOAff(n, \mathbb{C})} : GOAff(n, \mathbb{C}) = \mathbb{C}^n \rtimes GO(n, \mathbb{C}) \longrightarrow GL(n+1, \mathbb{C})$. Composing with the canonical projection $\pi : GL(n+1, \mathbb{C}) \longrightarrow \mathbb{P}(GL(n+1, \mathbb{C}))$ we obtain the **projective complex similitude Group**:

$$\widehat{Sim}_{\mathbb{C}}(n) := (\pi \circ \kappa')(GOAff(n, \mathbb{C})).$$

which acts naturally on \mathbb{P}^n .

Definition 11. *An element of $\mathbb{P}(GL(\mathbf{V}))$ corresponding to an element of $\widehat{Sim}_{\mathbb{C}}(n)$ with respect to the basis $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ is called a **projective similitude of \mathbb{P}^n** .*

The set of projective similitudes of \mathbb{P}^n is isomorphic to $\widehat{Sim}_{\mathbb{C}}(n)$.

Lemma 12. *The projective similitude preserves the orthogonality structure. They preserve namely the normal lines and the normal class of surfaces of \mathbb{P}^n .*

This lemma has a straightforward proof that is omitted.

2. PROJECTIVE NORMAL LINES TO A SURFACE OF \mathbb{P}^3

Let \mathbf{V} be a four dimensional complex vector space (endowed with a basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$) and $\mathbb{P}^3 := \mathbb{P}(\mathbf{V})$ with projective coordinates x, y, z, t . In \mathbb{P}^3 , we consider an irreducible surface $\mathcal{S} = V(F)$ with homogeneous $F \in \mathbb{C}[x, y, z, t]$ of degree $d_{\mathcal{S}}$. Recall that we write $\mathcal{H}^\infty = V(t)$ for the plane at infinity. For any nonsingular $m \in \mathcal{S}$ (with coordinates $\mathbf{m} = (x, y, z, t) \in \mathbb{C}^4$), we write $\mathcal{T}_m \mathcal{S}$ for the tangent plane to \mathcal{S} at m . If $\mathcal{T}_m \mathcal{S} \neq \mathcal{H}^\infty$, then

$$n_{\mathcal{S}}(m) = [F_x(\mathbf{m}) : F_y(\mathbf{m}) : F_z(\mathbf{m}) : 0] \quad (2)$$

is well defined in \mathbb{P}^3 . If moreover $n_{\mathcal{S}}(m) \neq m$, we define the projective normal line $\mathcal{N}_m\mathcal{S}$ to \mathcal{S} at m as the line $(mn_{\mathcal{S}}(m))$. We will associate to a generic $m \in \mathcal{S}$ a system of equations of $\mathcal{N}_m\mathcal{S}$. This will be done thanks to the Plücker embedding. Observe that $n_{\mathcal{S}}$ given by (2) defines a rational map $n_{\mathcal{S}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$.

2.1. Plücker embedding. Let $W \equiv \mathbf{V}^{\vee}$ be the set of homogeneous polynomials of degree one in $\mathbb{C}[x, y, z, t]$, endowed with $(e_1^*, e_2^*, e_3^*, e_4^*)$ the dual basis of $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$. We consider the duality $\delta : W \dashrightarrow \mathbb{P}^3$ given by $\delta(ae_1^* + be_2^* + ce_3^* + de_4^*) = [a : b : c : d]$. We define $\delta^{(4)} : W^4 \dashrightarrow (\mathbb{P}^3)^4$ by

$$\delta^{(4)}(l_1, l_2, l_3, l_4) = (\delta(l_1), \delta(l_2), \delta(l_3), \delta(l_4)).$$

We consider now $\Lambda^3 : \mathbf{V} \times \mathbf{V} \times W^4 \rightarrow W^4$ given by $\Lambda^3(\mathbf{u}, \mathbf{v}, \ell) = (\widetilde{M}_1, -\widetilde{M}_2, \widetilde{M}_3, -\widetilde{M}_4)$, where \widetilde{M}_j is the determinant of the minor obtained from $M = (\mathbf{u} \ \mathbf{v} \ \ell)$ by deleting the j -th row of M . Let Δ be the diagonal of $\mathbb{P}^3 \times \mathbb{P}^3$. We define $\mu : (\mathbb{P}^3 \times \mathbb{P}^3) \setminus \Delta \rightarrow (\mathbb{P}^3)^4$ by

$$\mu(\mathbf{u}, \mathbf{v}) = \delta^{(4)}(\Lambda^3(\mathbf{u}, \mathbf{v}, L)) \text{ where } L := \begin{pmatrix} e_1^* \\ e_2^* \\ e_3^* \\ e_4^* \end{pmatrix}.$$

Observe that when $u \neq v$, $\Lambda^3(\mathbf{u}, \mathbf{v}, L)$ gives four equations in $\mathbb{C}[x, y, z, t]$ of the line (uv) , i.e. $(uv) = \cap_{i=1}^4 V(\langle \mu_i(u, v), \cdot \rangle) \subset \mathbb{P}^3$, where $\mu(u, v) = (\mu_1(u, v), \mu_2(u, v), \mu_3(u, v), \mu_4(u, v))$ and where we write $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^4 u_i v_i$ for the complexification of the usual scalar product on \mathbb{R}^4 to \mathbf{V} . We will express ν in terms of the **Plücker embedding** $(\mathbb{P}^3 \times \mathbb{P}^3) \setminus \Delta \xrightarrow{Pl} \mathbb{P}^5 = \mathbb{P}(\Lambda^2 \mathbb{C}^4)$ defined by

$$Pl(u, v) = [\mathbf{u} \wedge \mathbf{v}] = \begin{bmatrix} u_1 v_2 - v_1 u_2 \\ u_1 v_3 - v_1 u_3 \\ u_1 v_4 - v_1 u_4 \\ u_2 v_3 - v_2 u_3 \\ u_2 v_4 - v_2 u_4 \\ u_3 v_4 - v_3 u_4 \end{bmatrix}.$$

The image of the Plücker embedding is the quadric hypersurface (see [3])

$$\mathbb{G}(1, 3) := Pl((\mathbb{P}^3)^2 \setminus \Delta) = V(w_1 w_6 - w_2 w_5 + w_3 w_4) \subset \mathbb{P}^5.$$

We have a first commutative diagram (for $i = 1, \dots, 4$)

$$\begin{array}{ccccc} & & (\mathbb{P}^3)^4 & \xrightarrow{pr_i} & \mathbb{P}^3 \\ & & \uparrow \Psi & \nearrow \psi_i & \\ & \mu \nearrow & & & \\ (\mathbb{P}^3 \times \mathbb{P}^3) \setminus \Delta & \xrightarrow{Pl} & \mathbb{P}^5 = \mathbb{P}(\Lambda^2 \mathbf{V}) & & \end{array} \quad (3)$$

where pr_i is the canonical projection (on the i -th coordinate) and where $\Psi = (\psi_1, \psi_2, \psi_3, \psi_4)$ is given on coordinates by

$$\Psi(w_1, w_2, w_3, w_4, w_5, w_6) = \left(\begin{pmatrix} 0 \\ w_6 \\ -w_5 \\ w_4 \end{pmatrix}, \begin{pmatrix} -w_6 \\ 0 \\ w_3 \\ -w_2 \end{pmatrix}, \begin{pmatrix} w_5 \\ -w_3 \\ 0 \\ w_1 \end{pmatrix}, \begin{pmatrix} -w_4 \\ w_2 \\ -w_1 \\ 0 \end{pmatrix} \right) \in \mathbf{V}^4.$$

Observe that the images of Ψ correspond to antisymmetric matrices (by identifying $(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{w}') \in \mathbf{V}^4$ with the matrix $M_{\psi} := (\mathbf{u} \ \mathbf{v} \ \mathbf{w} \ \mathbf{w}')$).

2.2. Equations of projective normal lines. Now we come back to normal lines. Let $j : \mathbb{P}^3 \setminus V(F_x, F_y, F_z) \rightarrow \mathbb{P}^3 \times \mathbb{P}^3$ be the morphism defined by $j(m) = (m, n_{\mathcal{S}}(m))$. We define $\alpha : \mathbb{P}^3 \setminus \mathcal{B} \rightarrow \mathbb{P}^5$ by $\alpha := Pl \circ j$, with $\mathcal{B} := V(F_x, F_y, F_z) \cup j^{-1}(\Delta)$, i.e. $\mathcal{B} = \{m \in \mathbb{P}^3; \text{rank}(\mathbf{m}, \mathbf{n}_{\mathcal{S}}(\mathbf{m})) < 2\}$, with $\mathbf{n}_{\mathcal{S}} = (F_x, F_y, F_z)$. We define also $\nu := \mu \circ j$ which maps any $m \in \mathcal{S} \setminus \Delta$ to the coordinates of a system of four equations of the normal line $\mathcal{N}_m \mathcal{S}$ to \mathcal{S} at m . We deduce from (3) a second commutative diagram

$$\begin{array}{ccc} & (\mathbb{P}^3)^4 & \xrightarrow{pr_i} \mathbb{P}^3 \\ & \nearrow \nu & \uparrow \Psi \nearrow \psi_i \\ \mathbb{P}^3 \setminus \mathcal{B} & \xrightarrow{\alpha} & \mathbb{P}^5 \end{array} . \quad (4)$$

Observe that $\alpha(x, y, z, t) = \begin{pmatrix} xF_y - yF_x \\ xF_z - zF_x \\ -tF_x \\ yF_z - zF_y \\ -tF_y \\ -tF_z \end{pmatrix}$ and so

$$\nu(x, y, z, t) = \left(\begin{bmatrix} 0 \\ -tF_z \\ tF_y \\ yF_z - zF_y \end{bmatrix}, \begin{bmatrix} tF_z \\ 0 \\ -tF_x \\ zF_x - xF_z \end{bmatrix}, \begin{bmatrix} -tF_y \\ tF_x \\ 0 \\ xF_y - yF_x \end{bmatrix}, \begin{bmatrix} zF_y - yF_z \\ xF_z - zF_x \\ yF_x - xF_y \\ 0 \end{bmatrix} \right).$$

In fact, if $\mathbb{U} = \mathcal{S} \setminus \mathcal{B}$ we have an exact sequence of $\mathcal{O}_{\mathbb{U}}^4$ -modules :

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{O}_{\mathbb{U}}^4 \xrightarrow{f} \mathcal{O}_{\mathbb{U}}^4 \rightarrow \text{Coker}(f) \rightarrow 0$$

which defines for all fibers at $m[\mathbf{m}]$ (with $\mathbf{m}(x_0, x_1, x_2, x_3) \in \mathcal{V}$) of \mathbb{U} the projective line $\mathcal{N}_m(\mathcal{S}) := \mathbb{P}(\mathcal{K}_m) \hookrightarrow \mathbb{P}^3$ where f_m is represented by the antisymmetric matrix associated to $\nu(\mathbf{m})$ (this matrix has rank 2).

3. THE NORMAL POLARS OF $\mathcal{S} \subset \mathbb{P}^3$

Let m_1 be a generic point of \mathbb{P}^3 . By construction, for every $m[x : y : z : t] \in \mathcal{S} \setminus \mathcal{B}$, we have $\mathcal{N}_m \mathcal{S} = \bigcap_{i=1}^4 V(\langle \nu_i(\mathbf{m}), \cdot \rangle)$ where $\nu_i := pr_i \circ \nu$. For every $A \in \mathbf{V}^{\vee}$, we define **the normal polar** $\mathcal{P}_{A, \mathcal{S}}$ of \mathcal{S} with respect to A by

$$\mathcal{P}_{A, \mathcal{S}} := \bigcap_{i=1}^4 V(A \circ \nu_i) = \mathcal{B} \cup \left(\bigcap_{i=1}^4 \nu_i^{-1} \mathcal{H}_A \right),$$

where $\mathcal{H}_A := V(A) \subset \mathbb{P}^3$. For every $m \in \mathbb{P}^3$ and every $A \in \mathbf{V}^{\vee}$, we have

$$m \in \mathcal{P}_{A, \mathcal{S}} \Leftrightarrow m \in \mathcal{B} \text{ or } \delta(A) \in \mathcal{N}_m \mathcal{S},$$

extending the definition of $\mathcal{N}_m \mathcal{S}$ from $m \in \mathcal{S}$ to $m \in \mathbb{P}^3$.

Lemma 13 (The projective similitudes preserve the normal polars). *Let $\mathcal{S} = V(F) \subset \mathbb{P}^3$ be a surface and φ be any projective similitude, then $\varphi(\mathcal{P}_{A, \mathcal{S}}) = \mathcal{P}_{\varphi(A), \varphi(\mathcal{S})}$ (with the classical identification $\mathbf{V}^{\vee} \cong \mathbf{V}$).*

Proof. Due to Lemma 12, $\varphi(\mathcal{N}_m \mathcal{S}) = \mathcal{N}_{\varphi(m)}(\varphi(\mathcal{S}))$ which gives the result. \square

From the geometric point of view, due to (4), we observe that

$$\mathcal{P}_{A,S} = \mathcal{B} \cup \left(\bigcap_{i=1}^4 \alpha^{-1} \mathcal{H}_{A,i} \right),$$

where $\mathcal{H}_{A,i}$ is the hyperplane of \mathbb{P}^5 given by $\mathcal{H}_{A,i} := V(A \circ \psi_i) = \text{Base}(\psi_i) \cup \psi_i^{-1}(\mathcal{H})$, where $\text{Base}(\psi_i)$ is the set of base points of ψ_i .

Lemma 14. *For every $A \in \mathbf{V}^\vee \setminus \{0\}$, the set $\bigcap_{i=1}^4 \mathcal{H}_{A,i}$ is a plane of \mathbb{P}^5 contained in $\mathbb{G}(1,3)$.*

Proof. Let $A = ae_1^* + be_2^* + ce_3^*z + de_4^*$ be an element of \mathbf{V}^\vee and write $B := \delta(A) = [a : b : c : d]$ and $\mathbf{B} := (a, b, c, d) \in \mathbf{V}$. Assume for example $d \neq 0$ (the proof being analogous when $a \neq 0$, $b \neq 0$ or $c \neq 0$, for symmetry reason). Observe that $a \cdot A \circ \psi_1 + b \cdot A \circ \psi_2 + c \cdot A \circ \psi_3 + d \cdot A \circ \psi_4 = ({}^t\mathbf{B}) \cdot M_\psi \cdot \mathbf{B} = 0$ since M_ψ is antisymmetric. Hence $\bigcap_{i=1}^4 \mathcal{H}_{A,i} = \bigcap_{i=1}^3 \mathcal{H}_{A,i}$. Recall that $A \circ \psi_1 = bw_6 - cw_5 + dw_4$, $A \circ \psi_2 = -aw_6 + cw_3 - dw_2$ and $A \circ \psi_3 = aw_5 - bw_3 + dw_1$. Since $d \neq 0$, these three linear equations are linearly independent and so $\bigcap_{i=1}^3 \mathcal{H}_{A,i}$ is a plane of \mathbb{P}^5 . Moreover, we observe that

$$w_3 \cdot (A \circ \psi_1) + w_5 \cdot (A \circ \psi_2) + w_6 \cdot (A \circ \psi_3) = d(w_1w_6 - w_2w_5 + w_3w_4)$$

and so, since $d \neq 0$, it follows that $\bigcap_{i=1}^3 \mathcal{H}_{A,i} \subset V(w_1w_6 - w_2w_5 + w_3w_4) = \mathbb{G}(1,3)$. \square

Due to the proof of the previous lemma that, if $d \neq 0$, we have

$$\mathcal{P}_{A,S} = \bigcap_{i=1}^3 V(A \circ \nu_i). \quad (5)$$

Proposition 15. *Assume $d_S \geq 2$ and $\dim \mathcal{B} \leq 1$. Then, for a generic $A \in \mathbf{V}^\vee$, we have $\dim \mathcal{P}_{A,S} = 1$ and $\deg \mathcal{P}_{A,S} = d_S^2 - d_S + 1$.*

Proof of Proposition 15. Up to a composition by a projective similitude of \mathbb{P}^3 , we can suppose that $F_x F_y F_z \neq 0$ in $\mathbb{C}[x, y, z, t]$. Let us notice that, since $\dim \mathcal{B} \leq 1$, at least one of the following varieties has dimension less than one : $V(tF_x, tF_y, F_z)$, $V(tF_x, tF_z, F_y)$ and $V(tF_y, tF_z, F_x)$. Assume that $\dim V(tF_x, tF_y, F_z) \leq 1$ (the proof being analogous in the other cases). For $A \in \mathbf{V}^\vee$, we write $[a : b : c : d] := \delta(A)$. Assume that $d \neq 0$. Then $\mathcal{P}_{A,S} = \bigcap_{i=1}^3 V(A \circ \nu_i)$ with

$$\begin{aligned} A \circ \nu_1 &= F_z(dy - bt) + F_y(ct - dz), & A \circ \nu_2 &= F_z(at - dx) + F_x(dz - ct), \\ A \circ \nu_3 &= F_y(dx - at) + F_x(tb - dy). \end{aligned}$$

We observe that $V(A \circ \nu_1, A \circ \nu_2) = \mathcal{P}_{A,S} \cup V(F_z, ct - dz)$ and that $\#\mathcal{P}_{A,S} \cap V(F_z, ct - dz) < \infty$ for a generic $A \in \mathbf{V}^\vee$. Hence

$$\deg \mathcal{P}_{A,S} = \deg(V(A \circ \nu_1, A \circ \nu_2)) - \deg(V(F_z, ct - dz)) = d_S^2 - d_S + 1. \quad \square$$

Analogously we have the following.

Proposition 16. *If $\dim \mathcal{B} = 2$, then the two dimensional part of \mathcal{B} is $V(H) \subset \mathbb{P}^3$ for some homogeneous polynomial $H \in \mathbb{C}[x, y, z, t]$ of degree d_H . We write $\alpha = H \cdot \tilde{\alpha}$. Observe that the regular map $\tilde{\alpha} : \mathbb{P}^3 \setminus \tilde{\mathcal{B}} \rightarrow \mathbb{P}^3$ (with $\dim \tilde{\mathcal{B}} \leq 1$) associated to $\tilde{\alpha}$. We then adapt our study by replacing ν by $\tilde{\nu}$ associated to $\tilde{\nu}_i := \frac{\nu_i}{H}$ and by defining the corresponding polar $\tilde{\mathcal{P}}_{A,S}$. Then, we have $\deg \tilde{\mathcal{P}}_{A,S} = (d_S - d_H)^2 - d_S + d_H + 1$.*

Example 17. Observe that the only irreducible quadrics $\mathcal{S} = V(F) \subset \mathbb{P}^3$ such that $\dim \mathcal{B} \geq 2$ are the spheres and cones, i.e. with F of the following form

$$F(x, y, z, t) = (x - x_0t)^2 + (y - y_0t)^2 + (z - z_0t)^2 + a_0t^2,$$

where x_0, y_0, z_0, a_0 are complex numbers (it is a sphere if $a_0 \neq 0$ and it is a cone otherwise).

Hence, due to Proposition 15, the degree of a generic normal polar of any irreducible quadric of \mathbb{P}^3 which is neither a sphere nor a cone is 3.

Moreover, for a sphere or for a cone, applying Proposition 16 with $H = t$, $\tilde{\mathcal{P}}_{A,S}$ is a line for a generic $A \in \mathbf{V}^\vee$.

4. SCHUBERT CLASSES

Given a flag $\mathbf{F} = \{\mathbf{V}_1 \subset \mathbf{V}_2 \subset \mathbf{V}_3 \subset \mathbf{V}_2 = \mathbf{V}\}$ of \mathbf{V} with $\dim_{\mathbb{C}}(V_i) = i$ for all integer i , we consider its associated projective flag \mathcal{F} of \mathbb{P}^3 (image by the canonical projection $\pi : \mathbf{V} \setminus \{\mathbf{0}\} \rightarrow \mathbb{P}^3$)

$$\mathcal{F} = \{p \in \mathcal{D} \subset \mathcal{P} \subset \mathbb{P}^3\}.$$

Let \mathcal{Z}^k denote the set of cycles of codimension k in $\mathbb{G}(1, 3)$. We recall that the **Schubert cycles** of $\mathbb{G}(1, 3)$ associated to \mathcal{F} (or to \mathbf{F}) are given by

$$\left\{ \begin{array}{l} \Sigma_{0,0} := \mathbb{G}(1, 3) \in \mathcal{Z}^0(\mathbb{G}(1, 3)) \\ \Sigma_{1,0} := \{\mathcal{L} \in \mathbb{G}(1, 3); \mathcal{D} \cap \mathcal{L} \neq \emptyset\} \in \mathcal{Z}^1(\mathbb{G}(1, 3)) \\ \Sigma_{2,0} := \{\mathcal{L} \in \mathbb{G}(1, 3); p \in \mathcal{L}\} \in \mathcal{Z}^2(\mathbb{G}(1, 3)) \\ \Sigma_{1,1} := \{\mathcal{L} \in \mathbb{G}(1, 3); \mathcal{L} \subset \mathcal{P}\} \in \mathcal{Z}^2(\mathbb{G}(1, 3)) \\ \Sigma_{2,1} := \Sigma_{2,0} \cap \Sigma_{1,1} \in \mathcal{Z}^3(\mathbb{G}(1, 3)) \\ \Sigma_{2,2} := \{\mathcal{L} \in \mathbb{G}(1, 3); \mathcal{L} = \mathcal{D}\} \in \mathcal{Z}^2(\mathbb{G}(1, 3)) \end{array} \right. \quad (6)$$

We write as usual $A^*(\mathbb{G}(1, 3))$ for the Chow ring of $\mathbb{G}(1, 3)$ and $\sigma_{i,j} := [\Sigma_{i,j}] \in A^{i+j}(\mathbb{G}(1, 3))$ for **Schubert classes**. For commodity we will use the notation $\Sigma_k := \Sigma_{k,0}$ and $\sigma_k := \sigma_{k,0}$. We recall that $A^*(\mathbb{G}(1, 3))$ is freely generated as graded \mathbb{Z} -module by $\{\sigma_{i,j}; 0 \leq j \leq i \leq 2\}$ with the following multiplicative relations

$$(E) \left\{ \begin{array}{l} \sigma_{1,1} = \sigma_1^2 - \sigma_2 \\ \sigma_{1,1} \cdot \sigma_1 = \sigma_1 \cdot \sigma_2 = \sigma_{2,1} \\ \sigma_{2,1} \cdot \sigma_1 = \sigma_{1,1}^2 = \sigma_2^2 = \sigma_{2,2} \\ \sigma_{1,1} \cdot \sigma_2 = 0 \end{array} \right. .$$

Hence, the Chow ring of the grassmannian is

$$A^*(\mathbb{G}(1, 3)) = \frac{\mathbb{Z}[\sigma_1, \sigma_2]}{(2\sigma_1 \cdot \sigma_2 - \sigma_1^2, \sigma_1^2 \cdot \sigma_2 - \sigma_2^2)}.$$

5. PROOFS OF THE GENERAL RESULTS

5.1. Geometric study of \mathcal{B}_S . Recall that $\mathcal{B}_S = \mathcal{B} \cap \mathcal{S}$, that $\mathcal{S}_\infty = \mathcal{S} \cap \mathcal{H}^\infty$ and that the **umbilical curve** \mathcal{C}_∞ is the set of circular points at infinity, i.e.

$$\mathcal{C}_\infty := \{[x : y : z : t] \in \mathbb{P}^3 : t = 0, x^2 + y^2 + z^2 = 0\}.$$

Proposition 18. We have $\mathcal{B}_S = \text{Sing}(\mathcal{S}) \cup \mathcal{K}_\infty(\mathcal{S}) \cup \mathcal{U}_\infty(\mathcal{S})$, where

- $\text{Sing}(\mathcal{S})$ is the set of singular points of \mathcal{S} ,
- $\mathcal{K}_\infty(\mathcal{S})$ is the set of contact at infinity of \mathcal{S} , i.e. the set of points of \mathcal{S} at which the tangent plane is \mathcal{H}_∞ ,

- $\mathcal{U}_\infty(\mathcal{S})$ is the set of umbilical contact of \mathcal{S} , i.e. the set of points of contact of \mathcal{S}_∞ with the umbilical \mathcal{C}_∞ .

Proof. Let $m \in \mathcal{S}$. We have

$$\begin{aligned} m \in \mathcal{B}_\mathcal{S} &\Leftrightarrow \mathbf{m} \wedge \mathbf{n}_\mathcal{S}(\mathbf{m}) = 0 \\ &\Leftrightarrow \mathbf{n}_\mathcal{S}(\mathbf{m}) = \mathbf{0} \text{ or } m = n_\mathcal{S}(m) \\ &\Leftrightarrow m \in V(F_x, F_y, F_z) \text{ or } m = n_\mathcal{S}(m). \end{aligned}$$

Now $m \in V(F_x, F_y, F_z)$ means either that m is a singular point of \mathcal{S} or that $\mathcal{T}_m\mathcal{S} = \mathcal{H}^\infty$.

Let $m = [x : y : z : t] \in \mathcal{S}$ be such that $m = n_\mathcal{S}(m)$. So $[x : y : z : t] = [F_x : F_y : F_z : 0]$. In particular $t = 0$. Due to the Euler identity, we have $0 = xF_x + yF_y + zF_z + tF_t = F_x^2 + F_y^2 + F_z^2$ and so $x^2 + y^2 + z^2 = 0$. Hence m is in the umbilical curve \mathcal{C}_∞ . Observe that the tangent line $\mathcal{T}_m\mathcal{C}_\infty$ to \mathcal{C}_∞ at m has equations $T = 0$ and $\delta^{-1}([x : y : z : 0])$ and that the tangent line $\mathcal{T}_m\mathcal{S}_\infty$ to \mathcal{S}_∞ at m has equations $T = 0$ and $\delta^{-1}(n_\mathcal{S}(m))$. We conclude that $\mathcal{T}_m\mathcal{C}_\infty = \mathcal{T}_m\mathcal{S}_\infty$.

Conversely, if $m = [x : y : z : 0]$ is a nonsingular point of $\mathcal{S}_\infty \cap \mathcal{C}_\infty$ such that $\mathcal{T}_m\mathcal{C}_\infty = \mathcal{T}_m\mathcal{S}_\infty$, then the linear spaces $\text{Span}((x, y, z, t), (0, 0, 0, 1))$ and $\text{Span}((F_x, F_y, F_z, F_t), (0, 0, 0, 1))$ are equal which implies that $[x : y : z : 0] = [F_x : F_y : F_z : 0]$. \square

In other words, $\mathcal{B}_\mathcal{S}$ are the singular points of \mathcal{S} and the common points of \mathcal{S} with its dual surface² at infinity.

Example 19. For the saddle surface $\mathcal{S}_1 = V(xy - zt)$, the set $\mathcal{B}_{\mathcal{S}_1}$ contains a single point $[0 : 0 : 1 : 0]$ which is a point of contact at infinity of \mathcal{S}_1 .

For the ellipsoid $\mathcal{E}_1 := V(x^2 + 2y^2 + 4z^2 - t^2)$, the set $\mathcal{B}_{\mathcal{E}_1}$ is empty.

For the ellipsoid $\mathcal{E}_2 := V(x^2 + 4y^2 + 4z^2 - t^2)$, the set $\mathcal{B}_{\mathcal{E}_2}$ has two elements: $[0 : 1 : \pm i : 0]$ which are umbilical contact points of \mathcal{E}_2 .

5.2. Proofs of Theorems 3 and 4. Recall that we have defined $\mathfrak{N}_\mathcal{S} := \overline{\{\mathcal{N}_m(\mathcal{S}); m \in \mathcal{S}\}} \subset \mathbb{G}(1, 3)$ and $\mathbf{n}_\mathcal{S} := [\mathfrak{N}_\mathcal{S}] \in A^2(\mathbb{G}(1, 3))$.

Proposition 20. Let $\mathcal{S} \subset \mathbb{P}^3$ be an irreducible surface of degree $d \geq 2$ of \mathbb{P}^3 .

- If $\#\mathcal{B}_\mathcal{S} < \infty$, we have

$$\mathbf{n}_\mathcal{S} = c_\nu(\mathcal{S}) \cdot \sigma_2 + d_\mathcal{S}(d_\mathcal{S} - 1) \cdot \sigma_{1,1} \in A^2(\mathbb{G}(1, 3)).$$

- If $\dim \mathcal{B} = 2$ with two dimensional part $V(H)$ and $\#\tilde{\mathcal{B}} \cap \mathcal{S} < \infty$ (with the notations of Proposition 16), then we have

$$\mathbf{n}_\mathcal{S} = c_\nu(\mathcal{S}) \cdot \sigma_2 + d_\mathcal{S}(d_\mathcal{S} - d_H - 1) \cdot \sigma_{1,1} \in A^2(\mathbb{G}(1, 3)).$$

Proof. Since $\mathbf{n}_\mathcal{S} \in A^2(\mathbb{G}(1, 3))$, we have $\mathbf{n}_\mathcal{S} = a \cdot \sigma_2 + b \cdot \sigma_{1,1}$ for some integers a and b . Moreover by Kleiman's transversality theorem (see for example [3, Thm 5.20]), since $\Sigma_{1,1} := \{\mathcal{L} \in \mathbb{G}(1, 3); \mathcal{L} \subset \mathcal{P}\} \in \mathcal{Z}^2(\mathbb{G}(1, 3))$, we have $\mathbf{n}_\mathcal{S} \cdot \sigma_{1,1} = (a\sigma_2 + b\sigma_{1,1}) \cdot \sigma_{1,1}$ and so, using (6), we obtain

$$\mathbf{n}_\mathcal{S} \cdot \sigma_{1,1} = b \cdot \sigma_{1,1}^2 = b \cdot \sigma_{2,2} = b. \quad (7)$$

Analogously, since $\Sigma_2 := \{\mathcal{L} \in \mathbb{G}(1, 3); p \in \mathcal{L}\} \in \mathcal{Z}^2(\mathbb{G}(1, 3))$, due to (6), we have

$$\mathbf{n}_\mathcal{S} \cdot \sigma_2 = (a\sigma_2 + b\sigma_{1,1}) \cdot \sigma_2 = a\sigma_2^2 = a\sigma_{2,2} = a. \quad (8)$$

²We call dual surface of \mathcal{S} its image by the Gauss map, with the identification $(\mathbb{P}^3)^\vee \cong \mathbb{P}^3$ given by δ (see Section 2.1).

Now it remains to compute $n_{\mathcal{S}} \cdot \sigma_2$ and $n_{\mathcal{S}} \cdot \sigma_{1,1}$, i.e. to compute the cardinality of the intersection of $\mathfrak{N}_{\mathcal{S}}$ with $\Sigma_{1,1}$ and with Σ_2 .

Let us start with the computation of $a = n_{\mathcal{S}} \cdot \sigma_2$. If $\#\mathcal{B}_{\mathcal{S}} < \infty$, then, for a generic $P \in \mathbb{P}^3$, we have

$$\mathfrak{N}_{\mathcal{S}} \cap \Sigma_2 = \{\mathcal{L} \in \mathfrak{N}_{\mathcal{S}}; P \in \mathcal{L}\} = \{\mathcal{N}_m \mathcal{S}; m \in \mathcal{S} \setminus \mathcal{B}_{\mathcal{S}}, P \in \mathcal{N}_m \mathcal{S}\}$$

and if $\dim \mathcal{B} = 2$ and $\#\tilde{\mathcal{B}} \cap \mathcal{S} < \infty$, then, for a generic $P \in \mathbb{P}^3$, we have

$$\mathfrak{N}_{\mathcal{S}} \cap \Sigma_2 = \{\mathcal{L} \in \mathfrak{N}_{\mathcal{S}}; P \in \mathcal{L}\} = \{\mathcal{N}_m \mathcal{S}; m \in \mathcal{S} \setminus \tilde{\mathcal{B}}, P \in \mathcal{N}_m \mathcal{S}\}.$$

So, in any case, $a = c_{\nu}(\mathcal{S})$ by definition of the normal class of \mathcal{S} .

Now, for b , since $\#\mathcal{B}_{\mathcal{S}} < \infty$, we observe that, for a generic projective plane $\mathcal{H} \subset \mathbb{P}^3$, we have

$$\mathfrak{N}_{\mathcal{S}} \cap \Sigma_{1,1} = \{\mathcal{L} \in \mathfrak{N}_{\mathcal{S}}; \mathcal{L} \subset \mathcal{H}\} = \{\mathcal{N}_m \mathcal{S}; m \in \mathcal{S} \setminus \mathcal{B}_{\mathcal{S}}, \mathcal{N}_m \mathcal{S} \subset \mathcal{H}\}.$$

We have $\mathcal{H} = V(a_1X + a_2Y + a_3Z + a_4T) \subset \mathbb{P}^3$ for some complex numbers a_1, a_2, a_3 and a_4 . Let $m[x : y : z : t] \in \mathbb{P}^3$. For a generic \mathcal{H} , we have

$$\begin{aligned} m \in \mathcal{S} \setminus \mathcal{B}_{\mathcal{S}}, \mathcal{N}_m \mathcal{S} \subset \mathcal{H} &\Leftrightarrow m \in \mathcal{S} \setminus \mathcal{B}_{\mathcal{S}}, m \in \mathcal{H}, n_{\mathcal{S}}(m) \in \mathcal{H} \\ &\Leftrightarrow \begin{cases} F(x, y, z, t) = 0 \\ a_1F_x + a_2F_y + a_3F_z = 0 \\ a_1x + a_2y + a_3z + a_4t = 0 \end{cases}. \end{aligned}$$

Hence $b = d_{\mathcal{S}}(d_{\mathcal{S}} - 1)$. Assume now that $\dim \mathcal{B} = 2$ with two dimensional part $V(H)$ and $\#\tilde{\mathcal{B}} \cap \mathcal{S} < \infty$. For a generic projective plane $\mathcal{H} = V(A^{\vee}) \subset \mathbb{P}^3$, we have

$$\begin{aligned} \mathfrak{N}_{\mathcal{S}} \cap \Sigma_{1,1} &= \{\mathcal{N}_m \mathcal{S}; m \in \mathcal{S} \setminus \tilde{\mathcal{B}}, \mathcal{N}_m \mathcal{S} \subset \mathcal{H}\} \\ &= \{\mathcal{N}_m \mathcal{S}; m \in \mathcal{S} \setminus \tilde{\mathcal{B}}, m \in \mathcal{H}, n_{\mathcal{S}}(m) \in \mathcal{H}\}. \end{aligned}$$

Now there are two cases:

- If H divides F_x, F_y and F_z and then $n_{\mathcal{S}} = [\frac{F_x}{H} : \frac{F_y}{H} : \frac{F_z}{H}]$ and $b = d_{\mathcal{S}}(d_{\mathcal{S}} - d_H - 1)$.
- Otherwise $H = tH_1$, with H_1 dividing F_x, F_y and F_z and $V(X) \subset V(xF_y - yF_x, xF_z - zF_x, yF_z - zF_y)$. Hence $n_{\mathcal{S}} = [\frac{F_x}{H_1} : \frac{F_y}{H_1} : \frac{F_z}{H_1}]$. We have

$$m \in \mathcal{S} \setminus (\mathcal{H}^{\infty} \cup \tilde{\mathcal{B}}), \mathcal{N}_m \mathcal{S} \subset \mathcal{H} \Leftrightarrow \begin{cases} F(x, y, z, t) = 0, & t \neq 0 \\ a_1 \frac{F_x}{H_1} + a_2 \frac{F_y}{H_1} + a_3 \frac{F_z}{H_1} = 0 \\ a_1x + a_2y + a_3z + a_4t = 0 \end{cases}$$

and

$$m \in \mathcal{S}_{\infty} \setminus \tilde{\mathcal{B}}, \mathcal{N}_m \mathcal{S} \subset \mathcal{H} \Leftrightarrow \begin{cases} F(x, y, z, t) = 0 \\ t = 0 \\ a_1x + a_2y + a_3z = 0 \end{cases},$$

so

$$b = d_{\mathcal{S}}(d_{\mathcal{S}} - d_H) + d_{\mathcal{S}} - \sum_{P \in \mathcal{S} \cap \mathcal{H}^{\infty} \cap \mathcal{H}} i_P \left(\mathcal{S}, V\left(a_1 \frac{F_x}{H_1} + a_2 \frac{F_y}{H_1} + a_3 \frac{F_z}{H_1}\right), \mathcal{H} \right)$$

(due to the Bezout Theorem). Now let $P \in \mathcal{S} \cap \mathcal{H}^\infty \cap \mathcal{H}$, we have $x \neq 0$ or $y \neq 0$ or $z \neq 0$. Assume for example that $x \neq 0$, we have

$$\begin{aligned} & i_P \left(\mathcal{S}, V \left(a_1 \frac{F_x}{H_1} + a_2 \frac{F_y}{H_1} + a_3 \frac{F_z}{H_1} \right), \mathcal{H} \right) = \\ &= i_P \left(\mathcal{S}, V \left(t \left(-a_4 \frac{F_x}{H_1} + a_2 \frac{x F_y - y F_x}{H} + a_3 \frac{x F_z - z F_x}{H} \right), \mathcal{H} \right) \right) \\ &= 1 + i_P \left(\mathcal{S}, V \left(-a_4 \frac{F_x}{H_1} + a_2 \frac{x F_y - y F_x}{H} + a_3 \frac{x F_z - z F_x}{H} \right), \mathcal{H} \right) \\ &= 2 \end{aligned}$$

for a generic \mathcal{H} and so $b = d_{\mathcal{S}}(d_{\mathcal{S}} - d_H + 1)$. □

Proof of Theorem 3. Let \mathcal{S} be an irreducible surface of \mathbb{P}^3 of degree $d_{\mathcal{S}} \geq 2$ such that $\#\mathcal{B}_{\mathcal{S}} < \infty$. It remains to prove that

$$c_\nu(\mathcal{S}) = d_{\mathcal{S}}^3 - d_{\mathcal{S}}^2 + d_{\mathcal{S}} - \sum_{P \in \mathcal{B}_{\mathcal{S}}} i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}), \quad (9)$$

for a generic $A \in \mathbf{V}^\vee$. We keep the notation $\delta(A) = [a : b : c : d]$. Observe that, for a generic $A \in \mathbf{V}^\vee$, since $\overline{\alpha(\mathcal{S})}$ is irreducible of dimension at most 2, we have $\#\bigcap_{i=1}^4 \mathcal{H}_{A,i} \cap \overline{\alpha(\mathcal{S})} < \infty$ and so $\#\mathcal{P}_{A,\mathcal{S}} \cap \mathcal{S} < \infty$ (since $\mathcal{B}_{\mathcal{S}} < \infty$). Since $\dim \mathcal{P}_{A,\mathcal{S}} = 1$ and $\#\mathcal{S} \cap \mathcal{P}_{A,\mathcal{S}} < \infty$ for a generic A in \mathbf{V}^\vee , due to Proposition 15 and to the Bezout formula, we have:

$$\begin{aligned} d_{\mathcal{S}}(d_{\mathcal{S}}^2 - d_{\mathcal{S}} + 1) &= \deg(\mathcal{S} \cap \mathcal{P}_{A,\mathcal{S}}) \\ &= \sum_{P \in \mathcal{B}_{\mathcal{S}}} i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) + \sum_{P \in \mathcal{S} \setminus \mathcal{B}} i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}). \end{aligned}$$

Now let us prove that, for a generic $A \in \mathbf{V}^\vee$,

$$\sum_{P \in \mathcal{S} \setminus \mathcal{B}} i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = \#((\mathcal{S} \cap \mathcal{P}_{A,\mathcal{S}}) \setminus \mathcal{B}). \quad (10)$$

Since α is a rational map, $\overline{\alpha(\mathcal{S})}$ is irreducible and its dimension is at most 2.

Assume first that $\dim \overline{\alpha(\mathcal{S})} < 2$. For a generic $A \in \mathbf{V}^\vee$, the plane $\bigcap_{i=1}^4 \mathcal{H}_{A,i} = \bigcap_{i=1}^4 \psi_i^{-1}(\mathcal{H}_A)$ does not meet $\overline{\alpha(\mathcal{S})}$ and so the left and right hand sides of (10) are both zero. So Formula (9) holds true with $c_\nu(\mathcal{S}) = 0$.

Assume now that $\dim \overline{\alpha(\mathcal{S})} = 2$. Then, for a generic $A \in \mathbf{V}^\vee$, the plane $\bigcap_{i=1}^4 \mathcal{H}_{A,i}$ meets $\overline{\alpha(\mathcal{S})}$ transversally (with intersection number 1 at every intersection point) and does not meet $\overline{\alpha(\mathcal{S})} \setminus \alpha(\mathcal{S})$. This implies that, for a generic $A \in \mathbf{V}^\vee$, we have $i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = 1$ for every $P \in (\mathcal{S} \cap \mathcal{P}_{A,\mathcal{S}}) \setminus \mathcal{B}$ and so (10) follows. Hence, for a generic $A \in \mathbf{V}^\vee$, we have

$$\begin{aligned} d_{\mathcal{S}}(d_{\mathcal{S}}^2 - d_{\mathcal{S}} + 1) &= \sum_{P \in \mathcal{B}_{\mathcal{S}}} i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) + \#\{P \in \mathcal{S} \setminus \mathcal{B} : \delta(A) \in \mathcal{N}_m \mathcal{S}\} \\ &= \sum_{P \in \mathcal{B}_{\mathcal{S}}} i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) + c_\nu(\mathcal{S}), \end{aligned}$$

which gives (9). □

Since $\#\mathcal{B}_{\mathcal{S}} = \emptyset$ for a generic irreducible surface of degree d of \mathbb{P}^3 , we directly obtain the following result.

Corollary 21. *For a generic irreducible surface $\mathcal{S} \subset \mathbb{P}^3$ of degree $d \geq 2$, we have $c_\nu(\mathcal{S}) = d^3 - d^2 + d$ and*

$$n_{\mathcal{S}} = (d^3 - d^2 + d) \cdot \sigma_2 + d(d-1) \cdot \sigma_{1,1} \in A^2(\mathbb{G}(1, 3)).$$

If $\dim \mathcal{B} = 2$, we saw in Proposition 16 that we can adapt our study to compute the degree of the reduced normal polar $\tilde{\mathcal{P}}_{A,\mathcal{S}}$ associated to the rational map $\tilde{\alpha} : \mathbb{P}^3 \setminus \tilde{\mathcal{B}} \rightarrow \mathbb{P}^3$ such that $\alpha = H \cdot \tilde{\alpha}$. Using Proposition 16 and following the proof of Theorem 3, we obtain Theorem 4, which can be rewritten as follows.

Theorem 4. *Assume that $\dim \mathcal{B} = 2$, with two dimensional part $V(H) \subset \mathbb{P}^3$ for some homogeneous polynomial $H \in \mathbb{C}[x, y, z, t]$ of degree d_H . Assume moreover that $\#(\mathcal{S} \cap \tilde{\mathcal{B}}) < \infty$. Then, for a generic $A \in \mathbf{V}^\vee$ we have*

$$c_\nu(\mathcal{S}) = d_{\mathcal{S}} \cdot (d_{\mathcal{S}}^2 - \tilde{d}_{\mathcal{S}} + 1) - \sum_{P \in \tilde{\mathcal{B}} \cap \mathcal{S}} i_P(\mathcal{S}, \tilde{\mathcal{P}}_{A,\mathcal{S}}), \quad \text{with } \tilde{d}_{\mathcal{S}} := d_{\mathcal{S}} - d_H.$$

6. PROOF OF THEOREM 1

We apply Theorem 3. Observe that, since \mathcal{S} is smooth, it has only a finite number of points of contact with \mathcal{H}_∞ (due to Zak's theorem on tangencies [13, corollary 1.8]). Since the surface is smooth, $\mathcal{B}_{\mathcal{S}}$ consists of points of contact of \mathcal{S} with \mathcal{H}_∞ and of points of contact of \mathcal{S}_∞ with \mathcal{C}_∞ . It remains to compute the intersection multiplicity of \mathcal{S} with a generic normal polar at these points. Let us recall that, due to (5), we have

$$i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = \dim_{\mathbb{C}}((\mathbb{C}[x, y, z, t]/I)_P)$$

where I is the ideal (F, A_1, A_2, A_3) of $\mathbb{C}[x, y, z, t]$, with $A_i := A \circ \nu_i$.

To compute these quantities, it may be useful to make an appropriate change of coordinates with the use of a projective similitude of \mathbb{P}^3 . Let us notice that:

- * The umbilical \mathcal{C}_∞ is stable under the action of the group of projective similitudes of \mathbb{P}^3 .
- * For any $P \in \mathcal{H}_\infty \setminus \mathcal{C}_\infty$, there exists a projective similitude ζ of \mathbb{P}^3 mapping $[1 : 0 : 0 : 0]$ to P .³
- * For any $P \in \mathcal{C}_\infty$, there exists a projective similitude ζ of \mathbb{P}^3 mapping $[1 : i : 0 : 0]$ to P .⁴

We recall that a multiple point of order k of a planar curve is ordinary if its tangent cone contains k pairwise distinct lines and that an ordinary cusp of a planar curve is a double point with a single tangent line in the tangent cone, this tangent line being non contained in the cubic cone of the curve at this point.

- Let P be a (non singular) point of contact of \mathcal{S} with \mathcal{H}_∞ .

We prove the following:

³Let $P[x_0 : y_0 : z_0 : 0]$ with $x_0^2 + y_0^2 + z_0^2 = 1$. Assume for example that $z_0^2 \neq 1$ (up to a permutation of the coordinates) and take ζ given by $\kappa'(b, A)$ (for any $b \in \mathbb{C}^n$) with $A = (uvw)$ where $u = (x_0, y_0, z_0)$ and $v = (x_0^2 + y_0^2)^{-\frac{1}{2}}(y_0, -x_0, 0)$ and $w = (x_0^2 + y_0^2)^{-\frac{1}{2}}(x_0 z_0, y_0 z_0, -x_0^2 - y_0^2)$.

⁴Let $P[x_0 : y_0 : z_0 : 0] \in \mathcal{C}_\infty$. Assume for example that $y_0 \neq 0$ and $x_0^2 + y_0^2 \neq 0$ (up to a composition by a permutation matrix). A suitable ζ is given by $\kappa'(b, A)$ (for any $b \in \mathbb{C}^n$) with $A =$

$$\begin{pmatrix} \frac{x_0(y_0^2-1)}{2y_0^2} & -\frac{ix_0(1+y_0^2)}{2y_0^2} & \frac{\sqrt{x_0^2+y_0^2}}{y_0} \\ \frac{1+y_0^2}{2y_0^2} & \frac{i(1-y_0^2)}{y_0} & 0 \\ \frac{i(y_0^4+y_0^2x_0^2-y_0^2-x_0^2)}{y_0^2\sqrt{x_0^2+y_0^2}} & \frac{(1+y_0^2)\sqrt{x_0^2+y_0^2}}{2y_0^2} & \frac{ix_0}{y_0} \end{pmatrix}.$$

- (a) $i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = k^2$ for a generic $A \in \mathbf{V}^\vee$ if P is an ordinary multiple point of order $k + 1$ of $\mathcal{S}_\infty \setminus \mathcal{C}_\infty$.
- (b) $i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = k(k + 1)$ for a generic $A \in \mathbf{V}^\vee$ if P is an ordinary multiple point of order $k + 1$ of \mathcal{S}_∞ , which belongs to \mathcal{C}_∞ and at which the tangent line to \mathcal{C}_∞ is not contain in the tangent cone of \mathcal{S}_∞ .
- (c) $i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = 3$ for a generic $A \in \mathbf{V}^\vee$ if P is an ordinary cusp of \mathcal{S}_∞ , which belongs to \mathcal{C}_∞ and at which the tangent line to \mathcal{C}_∞ is not contain in the tangent cone of \mathcal{S}_∞ .
- (d) $i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = 2$ for a generic $A \in \mathbf{V}^\vee$ if P is an ordinary cusp of $\mathcal{S}_\infty \setminus \mathcal{C}_\infty$.

Due to Lemma 13, we assume that $P[1 : \theta : 0 : 0]$ with $\theta = 0$ (if $P \in \mathcal{H}_\infty \setminus \mathcal{C}_\infty$) or $\theta = i$ (if $P \in \mathcal{C}_\infty$). Since $\mathcal{T}_P\mathcal{S} = \mathcal{H}_\infty$, we suppose that $F_x(P) = F_y(P) = F_z(P) = 0$ and $F_t(P) = 1$ (without any loss of generality). Recall that the Hessian determinant H_F of F satisfies⁵

$$H_F = \frac{(d_{\mathcal{S}} - 1)^2}{x^2} \begin{vmatrix} 0 & F_y & F_z & F_t \\ F_y & F_{yy} & F_{yz} & F_{yt} \\ F_z & F_{yz} & F_{zz} & F_{zt} \\ F_t & F_{yt} & F_{zt} & F_{tt} \end{vmatrix}.$$

Hence $H_F(P) \neq 0 \Leftrightarrow [F_{yy}F_{zz} - F_{yz}^2](P) \neq 0$. Due to (??), for a generic $A \in \mathbf{V}^\vee$, we have

$$\left(\frac{\mathbb{C}[x, y, z, t]}{I} \right)_P \cong \left(\frac{\mathbb{C}[x, y, z, t]}{(F, A_2, A_3)} \right)_P.$$

Recall that $A_2 = atF_z - ctF_x + d(zF_x - xF_z)$ and $A_3 = -atF_y + btF_x + d(xF_y - yF_x)$ (with $[a : b : c : d] = \delta(A)$). Using the Euler identity $xF_x + yF_y + zF_z + tF_t = d_{\mathcal{S}}F$, we obtain that

$$\left(\frac{\mathbb{C}[x, y, z, t]}{I} \right)_P \cong \left(\frac{\mathbb{C}[x, y, z, t]}{(F, A'_2, A'_3)} \right)_P \cong \left(\frac{\mathbb{C}[y, z, t]}{(F_*, A'_{2*}, A'_{3*})} \right)_{(0,0,0)}$$

with

$$A'_2 := atxF_z + ct(yF_y + zF_z + tF_t) - d(z(yF_y + zF_z + tF_t) + x^2F_z),$$

$$A'_3 := -atxF_y - bt(yF_y + zF_z + tF_t) + d(x^2F_y + y(yF_y + zF_z + tF_t))$$

and with $G_*(y, z, t) := G(1, \theta + y, z, t)$ for any homogeneous G . In a neighbourhood of $(0, 0, 0)$, $V(F_*)$ is given by $t = \varphi(y, z)$ with $\varphi(y, z) \in \mathbb{C}[[y, z]]$ and

$$\varphi_y(y, z) = -\frac{F_y(1, \theta + y, z, \varphi(y, z))}{F_t(1, \theta + y, z, \varphi(y, z))} \quad \text{and} \quad \varphi_z(y, z) = -\frac{F_z(1, \theta + y, z, \varphi(y, z))}{F_t(1, \theta + y, z, \varphi(y, z))}. \quad (11)$$

So

$$\left(\frac{\mathbb{C}[x, y, z, t]}{I} \right)_P \cong \frac{\mathbb{C}[[y, z]]}{(A'_{2**}, A'_{3**})},$$

with $G_{**}(y, z) := G(1, \theta + y, z, \varphi(y, z))$. Now due to (11), we have

$$H := A'_{2**} = (F_t)_{**}[-a\varphi\varphi_z + c\varphi(\varphi - (\theta + y)\varphi_y - z\varphi_z) + d(z((\theta + y)\varphi_y + z\varphi_z - \varphi) + \varphi_z)]$$

and

$$K := A'_{3**} = (F_t)_{**}[a\varphi\varphi_y - b\varphi(\varphi - (\theta + y)\varphi_y - z\varphi_z) - d(\varphi_y + (\theta + y)((\theta + y)\varphi_y + z\varphi_z - \varphi))].$$

Hence

$$i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = i_{(0,0)}(\Gamma_H, \Gamma_K),$$

where Γ_H and Γ_K are the analytic planar curves of respective equations H and K of $\mathbb{C}[[y, z]]$.

Observe that $(H_y(0, 0), H_z(0 : 0)) = d(\varphi_{yz}(0, 0), \varphi_{zz}(0, 0))$. Analogously, we obtain $(K_y(0, 0), K_z(0, 0)) = -d(1 + \theta^2)(\varphi_{y,y}(0, 0), \varphi_{yz}(0, 0))$.

⁵see for example [8].

- (a) If $P \notin \mathcal{C}_\infty$ and if P is an ordinary multiple point of order $k + 1$ of \mathcal{S}_∞ , with our change of coordinates we have $P[1 : 0 : 0 : 0]$ (i.e. $\theta = 0$) and $V((\varphi_{k+1})_y)$ and $V((\varphi_{k+1})_z)$ have no common lines.⁶ Then the first homogeneous parts of H and K have order k and are $H_k = d(\varphi_{k+1})_z$ and $K_k = -d(\varphi_{k+1})_y$ respectively. Since Γ_{H_k} and Γ_{K_k} have no common lines, we conclude that $i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = k^2$.
- (b) Assume now that $P \in \mathcal{C}_\infty$ is an ordinary multiple point of \mathcal{S}_∞ , which belongs to \mathcal{C}_∞ and at which the tangent line to \mathcal{C}_∞ is not contain in the tangent cone of \mathcal{S}_∞ . With our changes of coordinates, this means that $P[1 : i : 0 : 0]$ (i.e. $\theta = i$) that y does divide φ_{k+1} (since $V(y)$ is the tangent line to \mathcal{C}_∞ at P) and that $V((\varphi_{k+1})_y)$ and $V((\varphi_{k+1})_z)$ have no common lines.

We observe that the first homogeneous parts of H and K have respective orders k and $k + 1$ and are respectively $H_k = d(\varphi_{k+1})_z$ and

$$\begin{aligned} K_{k+1} &= -di[2y(\varphi_{k+1})_y + z(\varphi_{k+1})_z - \varphi_{k+1}] \\ &= -di \left[\left(2 - \frac{1}{k+1}\right) y(\varphi_{k+1})_y + \left(1 - \frac{1}{k+1}\right) z(\varphi_{k+1})_z \right], \end{aligned}$$

due to the Euler identity applied to φ_{k+1} . Hence $i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = k(k+1)$ if $V((\varphi_{k+1})_z)$ and $V(y(\varphi_{k+1})_y)$ have no common lines, which is true since $V((\varphi_{k+1})_z)$ and $V((\varphi_{k+1})_y)$ have no common lines and since y does not divide $(\varphi_{k+1})_z$.

- (c) Assume that $P \in \mathcal{C}$ is an ordinary cusp (of order 2) of \mathcal{S}_∞ , which belongs to \mathcal{C}_∞ and at which the tangent line to \mathcal{C}_∞ is not contain in the tangent cone of \mathcal{S}_∞ . With our changes of coordinates, this means that $P[1 : i : 0 : 0]$ (i.e. $\theta = i$), that $H_F(P) = 0$ and $F_{zz}(P) \neq 0$ (since $V(y)$ is the tangent line to \mathcal{C}_∞ at P).

We observe that $H = -d(F_{yz}(P)y + F_{zz}(P)z) + \dots$. In a neighbourhood of $(0, 0)$, $H(y, z) = 0 \Leftrightarrow z = h(y)$ with

$$h(y) = -\frac{F_{yz}(P)}{F_{zz}(P)}y - \frac{F_{zzz}(P)F_{yz}^2(P) - 2F_{yzz}(P)F_{yz}(P)F_{zz}(P) + F_{yyz}(P)F_{zz}^2(P)}{F_{zz}^3(P)}y^2 + \dots$$

and we obtain that $\text{val}_y K(y, h(y)) = 3$ if $P \notin V(F_{yyy}F_{zz}^3 - 3F_{yyz}F_{zz}^2F_{yz} + 3F_{yzz}F_{zz}F_{yz}^2 - F_{zzz}F_{yz}^3)$ which means that the line $V(F_{yz}(P)y + F_{zz}(P)z)$ (corresponding to the tangent cone of $V(F(1, y, z, 0))$) is not contained in the cubic cone of $V(F(1, y, z, 0))$. Hence $i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = 3$ if the node P of \mathcal{S}_∞ is ordinary.

- (d) Assume now that P is an ordinary cusp (of order 2) of $\mathcal{S}_\infty \setminus \mathcal{C}_\infty$.

With our change of coordinates, this means that $P[1 : 0 : 0 : 0]$ (i.e. $\theta = 0$), that $H_F(P) = 0$ and $P \notin V(F_{yy}, F_{yz}, F_{zz})$. This implies that $F_{yy}(P) \neq 0$ or $F_{zz}(P) \neq 0$. If $F_{yy}(P) \neq 0$, the tangent line to \mathcal{S}_∞ at P is given by $V(t, F_{yy}(P)y + F_{yz}(P)z)$. If $F_{zz}(P) \neq 0$, the tangent line to \mathcal{S}_∞ at P is given by $V(t, F_{zz}(P)z + F_{yz}(P)y)$. The fact that the cusp is ordinary implies also that the tangent line is not contained in the cubic cone of $V(F(1, y, z, 0))$, i.e. this tangent line is not contained $V(F_{yyy}(P)y^3 + 3F_{yyz}(P)y^2z + 3F_{yzz}(P)yz^2 + F_{zzz}(P)z^3)(P)$.

Hence, we have either

$$F_{yy}[F_{yyy}F_{yz}^3 - 3F_{yyz}F_{yz}^2F_{yy} + 3F_{yzz}F_{yz}F_{yy}^2 - F_{zzz}F_{yy}^3](P) \neq 0$$

or

$$F_{zz}[F_{zzz}F_{yz}^3 - 3F_{yzz}F_{yz}^2F_{zz} + 3F_{yyz}F_{yz}F_{zz}^2 - F_{yyy}F_{zz}^3](P) \neq 0.$$

We can notice that, if $F_{yy}(P)$ and $F_{zz}(P)$ are both non null, these two conditions are equivalent.

⁶Recall that the tangent cone $V(\varphi_{k+1})$ of Γ_φ at $(0, 0)$ (corresponding to the tangent cone of $V(\mathcal{S}_\infty)$ at P) has pairwise distinct tangent lines if and only if $V((\varphi_{k+1})_y)$ and $V((\varphi_{k+1})_z)$ have no common lines.

Assume for example that the first condition holds. In a neighbourhood of $(0, 0)$, $H(y, z) = 0 \Leftrightarrow y = h(z)$ and $K(y, z) = 0 \Leftrightarrow y = k(z)$, with

$$h'(z) = -\frac{H_z(h(z), z)}{H_y(h(z), z)} \quad \text{and} \quad k'(z) = -\frac{K_z(h(z), z)}{K_y(h(z), z)}.$$

Hence we have

$$\left(\frac{\mathbb{C}[x, y, z, t]}{I} \right)_P \cong \frac{\mathbb{C}[[y, z]]}{(y - h(z), y - k(z))} \cong \frac{\mathbb{C}[[z]]}{((h - k)(z))}.$$

We have $h'(0) = k'(0) = -\varphi_{yz}(0, 0)/\varphi_{yy}(0, 0)$ and

$$(h'' - k'')(0) = [\varphi_{yy}^3 \varphi_{zy} (\varphi_{yyy} \varphi_{yz}^3 - 3\varphi_{yyz} \varphi_{yz}^2 \varphi_{yy} + 3\varphi_{yzz} \varphi_{yy}^2 \varphi_{yz} - \varphi_{zzz} \varphi_{yy}^3)](0, 0)$$

Hence $i_P(\mathcal{S}, \mathcal{P}_{A, \mathcal{S}}) = \dim_{\mathbb{C}} \frac{\mathbb{C}[[z]]}{((h-k)(z))} = 2$.

- Let P be a simple (non singular) point of contact of \mathcal{S}_{∞} with \mathcal{C}_{∞} .

Let us prove that $i_P(\mathcal{S}, \mathcal{P}_{A, \mathcal{S}}) = 1$. Due to Lemma 13, we can assume that $P = [1 : i : 0 : 0]$ (i.e. $\theta = i$) and that $F_t(P) = 1$. As previously, we observe that

$$\left(\frac{\mathbb{C}[x, y, z, t]}{I} \right)_P \cong \frac{\mathbb{C}[[y, z]]}{(A_{1**}, A'_{3**})},$$

with

$$A_{1**}(y, z) = (F_t)_{**}[\varphi(b\varphi_z - c\varphi_y) - d((y + i)\varphi_z - z\varphi_y)]$$

and

$$A_{3**} = (F_t)_{**}[a\varphi\varphi_y - b\varphi(\varphi - (\theta + y)\varphi_y - z\varphi_z) - d(\varphi_y + (i + y)((i + y)\varphi_y + z\varphi_z - \varphi))].$$

The fact that P is a simple contact point of \mathcal{S}_{∞} with \mathcal{C}_{∞} implies that $[F_x(P) : F_y(P) : F_z(P)] = [1 : i : 0]$ and that $\varphi_{zz}(0, 0) \neq 1$. Indeed $V(t, F(1, i + y, z, t))$ is given by $t = 0$, $y = g(z)$ with $g(0) = 0$ and $g'(z) = -\varphi_z(g(z), z)/\varphi_y(g(z), z)$ (in particular $g'(0) = 0$), so

$$\frac{\mathbb{C}[[y, z]]}{(y - g(z), 1 + (i + y)^2 + z^2)} \cong \frac{\mathbb{C}[[z]]}{(1 + (i + g(z))^2 + z^2)}$$

and finally

$$i_P(\mathcal{S}_{\infty}, \mathcal{C}_{\infty}) = \text{val}_z(1 + (i + g(z))^2 + z^2) = 1 + \text{val}_z((i + g(z))g'(z) + z)$$

which is equal to 2 if and only if $\varphi_{zz}(0, 0) \neq 1$.

In a neighbourhood of $(i, 0)$, A_{1**} can be rewritten $\varphi - \kappa$ with $\kappa = d((y + i)\varphi_z - z\varphi_y)/(b\varphi_z - c\varphi_y)$. Since $\varphi_{zz}(0) \neq 1$, $\kappa_z(0, 0) \neq 0$ and, in a neighbourhood of 0, $\varphi - \kappa = 0$ corresponds to $y = h(z)$ with $h'(0) \neq 0$ (recall that $\varphi_y(0) = i \neq 0$ and that A is generic) which gives

$$\left(\frac{\mathbb{C}[x, y, z, t]}{I} \right)_P \cong \frac{\mathbb{C}[[y, z]]}{(y - h(z), A'_{3**})}$$

and finally $i_P(\mathcal{S}, \mathcal{P}_{A, \mathcal{S}}) = \text{val}_z A'_{3**}(h(z), z) = 1$.

7. STUDY OF THE NORMAL CLASS OF QUADRICS

The aim of the present section is the study of the normal class of every irreducible quadric. Let $\mathcal{S} = V(F) \subset \mathbb{P}^3$ be an irreducible quadric. We recall that, up to a composition by $\varphi \in \widehat{\text{Sim}}_{\mathbb{C}}(3)$,

one can suppose that F has the one of the following forms:

- (a) $F(x, y, z, t) = x^2 + \alpha y^2 + \beta z^2 + t^2$
- (b) $F(x, y, z, t) = x^2 + \alpha y^2 + \beta z^2$
- (c) $F(x, y, z, t) = x^2 + \alpha y^2 - 2tz$
- (d) $F(x, y, z, t) = x^2 + \alpha y^2 + t^2,$

with α, β two non zero complex numbers. Spheres, ellipsoids and hyperboloids are particular cases of (a), paraboloids (including the saddle surface) are particular cases of (c), (b) correspond to cones and (d) to cylinders.

We will see, in Appendix A, that in the case (d) (cylinders) and in the cases (a) and (b) with $\alpha = \beta = 1$, the normal class of the quadric is naturally related to the normal class of a conic.

Proposition 22. *The normal class of a sphere is 2.*

The normal class of a quadric $V(F)$ with F given by (a) is 6 if $1, \alpha, \beta$ are pairwise distinct.

The normal class of a quadric $V(F)$ with F given by (a) is 4 if $\alpha = 1 \neq \beta$.

The normal class of a quadric $V(F)$ with F given by (b) is 4 if $1, \alpha, \beta$ are pairwise distinct.

The normal class of a quadric $V(F)$ with F given by (b) is 2 if $\alpha = 1 \neq \beta$.

The normal class of a quadric $V(F)$ with F given by (b) is 0 if $\alpha = \beta = 1$.

The normal class of a quadric $V(F)$ with F given by (c) is 5 if $\alpha \neq 1$ and 3 if $\alpha = 1$.

The normal class of a quadric $V(F)$ with F given by (d) is 4 if $\alpha \neq 1$ and 2 if $\alpha = 1$.

Corollary 23. *The normal class of the saddle surface $\mathcal{S}_1 = V(xy - zt)$ is 5.*

The normal class of the ellipsoid $\mathcal{E}_1 = V(x^2 + 2y^2 + 4z^2 - t^2)$ with three different length of axis is 6.

The normal class of the ellipsoid $\mathcal{E}_2 = V(x^2 + 4y^2 + 4z^2 - t^2)$ with two different length of axis is 4.

Proof of Proposition 22. Let $\mathcal{S} = V(F)$ be a quadric with F of the form (a), (b), (c) or (d).

- The easiest cases is (a) with $1, \alpha, \beta$ pairwise distinct since $\mathcal{B}_{\mathcal{S}}$ is empty. In this case, since the generic degree of the normal polar curves is 3 and since \mathcal{E}_1 has degree 2, we simply have $c_{\nu}(\mathcal{E}_1) = 2 \cdot 3 = 6$ (due to Theorem 3).
- The case of a sphere \mathcal{S} is analogous. In this case, $\tilde{\mathcal{B}} \cap \mathcal{S} = \emptyset$ and $\deg \tilde{\mathcal{P}}_{A, \mathcal{S}} = 1$ for a generic $A \in \mathbf{V}^V$ (see Example 17). Hence, we have $c_{\nu}(\mathcal{E}_1) = 2 \cdot 1 = 2$ (due to Theorem 4).
- In case (a) with $\alpha = 1 \neq \beta$, the set $\mathcal{B}_{\mathcal{S}}$ contains two points $[1 : \pm i : 0 : 0]$. We find the parametrization $\psi(y) = [1 : \pm i + y : 0 : 0]$ of $\mathcal{P}_{A, \mathcal{S}}$ at the neighbourhood of $P[1 : \pm i : 0 : 0]$, which gives $i_P(\mathcal{S}, \mathcal{P}_{A, \mathcal{S}}) = \text{val}_z(1 + (\pm i + y)^2) = 1$ and so $c_{\nu}(\mathcal{S}) = 2 \cdot 3 - 1 - 1 = 4$.
- In case (b) with α, β and 1 are pairwise distinct, the set $\mathcal{B}_{\mathcal{S}}$ contains a single point $P[0 : 0 : 0 : 1]$ and a parametrization of $\mathcal{P}_{A, \mathcal{S}}$ in a neighbourhood of P is

$$\psi(x) = \left[x : -\frac{bx}{d(\alpha - 1)x - a} : \frac{cx}{a + d(1 - \beta)x} : 1 \right]. \quad (12)$$

Hence $i_P(\mathcal{S}, \mathcal{P}_{A, \mathcal{S}}) = \text{val}_x(F(\psi(x))) = 2$ and so $c_{\nu}(\mathcal{S}) = 2 \cdot 3 - 2 = 4$.

- In case (b) with $\alpha = 1 \neq \beta$, we have $\mathcal{B}_{\mathcal{S}} = \{P, P'_+, P'_-\}$ with $P[0 : 0 : 0 : 1]$ and $P'_{\pm}[1 : \pm i : 0 : 0]$. A parametrization of $\mathcal{P}_{A, \mathcal{S}}$ in a neighbourhood of P is given by (12)

with $\alpha = 1$ and so $i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = 2$. A parametrization of $\mathcal{P}_{A,\mathcal{S}}$ at a neighbourhood of P'_\pm is $\psi(z) = [1 : \pm i + y : 0 : 0]$ and so $i_{P'_\pm}(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = 1$. Hence $c_\nu(\mathcal{S}) = 2 \cdot 3 - 2 - 1 - 1 = 2$.

- In case (b) with $\alpha = \beta = 1$, for a generic $A \in \mathbf{V}^\vee$, we have $\deg \tilde{\mathcal{P}}_{A,\mathcal{S}} = 1$ (see Example 17) but here $\tilde{\mathcal{B}} \cap \mathcal{S} = \{[0 : 0 : 0 : 1]\}$. We find the parametrization $\psi(x) = [x : (bx/a) : (cx/a) : 1]$ of $\tilde{\mathcal{P}}_{A,\mathcal{S}}$ at the neighbourhood of $P[0 : 0 : 0 : 1]$. Hence $i_P(\mathcal{S}, \tilde{\mathcal{P}}_{A,\mathcal{S}}) = 2$ and so $c_\nu(\mathcal{S}) = 2 \cdot 1 - 2 = 0$.
- In case (c) with $\alpha \neq 1$, the only point of $\mathcal{B}_\mathcal{S}$ is $P_1[0 : 0 : 1 : 0]$ and a parametrization of $\mathcal{P}_{A,\mathcal{S}}$ at the neighbourhood of this point is

$$\psi(t) = \left[\frac{at^2}{d + (d-c)t} : \frac{bt^2}{t(d-c\alpha) + \alpha d} : 1 : t \right], \quad (13)$$

which gives $i_{P_1}(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = 1$. Hence $c_\nu(\mathcal{S}) = 2 \times 3 - 1 = 5$.

- In case (c) with $\alpha = 1$, $\mathcal{B}_\mathcal{S}$ is made of three points: $P_1[0 : 0 : 1 : 0]$, $P_{2,\pm}[1 : \pm i : 0 : 0]$. As in the previous case, a parametrization of $\mathcal{P}_{A,\mathcal{S}}$ at the neighbourhood of P_1 is given by (13) with $\alpha = 1$ and so $i_{P_1}(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = 1$. Now, a parametrization of $\mathcal{P}_{A,\mathcal{S}}$ at the neighbourhood of $P_{2,\pm}$ is $\psi(t) = [1 : \pm i + y : 0 : 0]$ and so $i_{P_{2,\pm}}(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = \text{val}_y(1 + (y \pm i)^2) = 1$.
- For the case (d), due to Proposition 25, $c_\nu(\mathcal{S}) = c_\nu(\mathcal{C})$ with $\mathcal{C} = V(x^2 + \alpha y^2 + z^2) \subset \mathbb{P}^2$ which is a circle if $\alpha = 1$ and an ellipse otherwise. Hence, due to Theorem 5, $c_\nu(\mathcal{C}) = 2 + 2 - 0 - 1 - 1 = 2$ if $\alpha = 1$ and $c_\nu(\mathcal{C}) = 2 + 2 = 4$ otherwise.

□

8. NORMAL CLASS OF A CUBIC SURFACE WITH SINGULARITY E_6

Consider $S = V(F) \subset \mathbb{P}^3$ with $F(x, y, z, t) := x^2z + z^2t + y^3$. \mathcal{S} is a singular cubic surface with E_6 -singularity at $p[0 : 0 : 0 : 1]$. Let a generic $A \in \mathbf{V}^\vee$ with $\delta(A) = [a : b : c : d]$. The ideal of the normal polar $\mathcal{P}_{\mathcal{S},A}$ is given by $I(\mathcal{P}_{\mathcal{S},A}) = \langle H_1, H_2, H_3 \rangle \subset \mathbb{C}[x, y, z, t]$ with $H_1 := (y(x^2 + 2zt) - 3y^2z)d - b(x^2 + 2zt)t + 3y^2ct$, $H_2 := (x(x^2 + 2zt) - 2xz^2)d - a(x^2 + 2zt)t + 2xztc$ and $H_3 := (-2xzy + 3xy^2)d - 3ay^2t + 2xztb$. $\mathcal{B}_\mathcal{S}$ is made of two points: p and $q[0 : 0 : 1 : 0]$. Actually q is the contact point of \mathcal{S} with \mathcal{H}_∞ . This point is an ordinary cusp of \mathcal{S}_∞ .

(1) Study at p .

Near p the ideal of the normal polar, in the chart $t = 1$, $H_3 = 0$ gives $z = g(x, y) := \frac{3y^2(-xd+a)}{2x(-yd+b)}$. Now $V(A_1(x, y, g(x, y), 1))$ corresponds to a quintic with a cusp at the origine (and with tangent cone $V(y^2)$). Its single branch has Puiseux expansion $y^2 = -\frac{b}{3a}x^3 + o(x^3)$, with probranches $y = \varphi_\varepsilon(x)$ with $\varphi_\varepsilon(x) = i\varepsilon\sqrt{\frac{b}{3a}}x^{\frac{3}{2}} + o(x^{\frac{3}{2}})$ for $\varepsilon \in \{\pm 1\}$. Hence, $g(x, \varphi_\varepsilon(x)) = -\frac{x^2}{2} + o(x^2)$. Hence parametrizations of the probranches of $\mathcal{P}_{A,\mathcal{S}}$ at a neighbourhood of p are

$$\Gamma_\varepsilon(x) = [x : \varphi_\varepsilon(x) : g(x, \varphi_\varepsilon(x)) : 1]$$

and $F(\Gamma_\varepsilon(x)) = -\frac{x^4}{4} + o(x^4)$. Therefore $i_p(\mathcal{P}_{\mathcal{S},A}, \mathcal{S}) = 8$.

(2) Study at q .

Assume that $b = 1$. Near $q[0 : 0 : 1 : 0]$, in the chart $z = 1$, $H_3 = 0$ gives $t = h(x, y) := \frac{d(-2+3y)xy}{3ay^2-2x}$ and $V(H_2(x, y, 1, h(x, y)))$ is a quartic with a (tacnode) double point in $(0, 0)$ with vertical tangent and which has Puiseux expansion

$$x = \theta_\varepsilon(y) = \omega_\varepsilon a y^2 + o(y^2),$$

with $\omega_\varepsilon = \frac{3-d}{2} + \frac{\varepsilon}{2}\sqrt{d(d-6)}$ for $\varepsilon \in \{\pm 1\}$ and $h(\theta_\varepsilon(y), y) = -\frac{2d\omega_\varepsilon}{3-2\omega_\varepsilon}y + o(y)$. Hence parametrizations of the probranches of $\mathcal{P}_{\mathcal{S},A}$ in a neighbourhood of q are given by

$$\Gamma_\varepsilon(y) := [\theta_\varepsilon(y) : y : 1 : h(\theta_\varepsilon(y), y)]$$

for $\varepsilon \in \{\pm 1\}$ and $F(\Gamma_\varepsilon(y)) = -\frac{2\omega_\varepsilon d}{3-2\omega_\varepsilon}y + o(y)$. Hence $i_q(\mathcal{P}_{\mathcal{S},A}, \mathcal{S}) = 2$

We can also apply directly Item (c) of Section 6 to prove that $i_q(\mathcal{P}_{\mathcal{S},A}, \mathcal{S}) = 2$.

Therefore, due to Theorem 3, the normal class of $\mathcal{S} = V(x^2z + z^2t + y^3) \subset \mathbb{P}^3(\mathbb{C})$ is

$$c_\nu(\mathcal{S}) = 3 \cdot (3^2 - 3 + 1) - 8 - 2 = 11.$$

9. NORMAL CLASS OF PLANAR CURVES : PROOF OF THEOREM 5

Let \mathbf{V} be a three dimensional complex vector space and set $\mathbb{P}^2 := \mathbb{P}(\mathbf{V})$ with projective coordinates x, y, z . We denote by $\ell_\infty = V(z)$ the line at infinity. We consider the natural map $\delta : \mathbf{V}^\vee \dashrightarrow \mathbb{P}^2$ given by $\delta(A) = [A(1, 0, 0) : A(0, 1, 0) : A(0, 0, 1)]$.

Let $\mathcal{C} = V(F) \subset \mathbb{P}^2$ be an irreducible curve of degree $d \geq 2$. For any nonsingular $m[x : y : z] \in \mathcal{C}$ (with coordinates $\mathbf{m} = (x, y, z) \in \mathbb{C}^3$), we write $\mathcal{T}_m\mathcal{C}$ for the tangent line to \mathcal{C} at m . If $\mathcal{T}_m\mathcal{C} \neq \ell_\infty$, then $n_{\mathcal{C}}(m) = [F_x : F_y : 0]$ is well defined in \mathbb{P}^2 and **the projective normal line** $\mathcal{N}_m\mathcal{C}$ to \mathcal{C} at m is the line $(m n_{\mathcal{C}}(m))$ if $n_{\mathcal{C}}(m) \neq m$. An equation of this normal line is then given by $\delta^{-1}(N_{\mathcal{C}}(m))$ where $N_{\mathcal{C}} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is the rational map defined by

$$\mathbf{N}_{\mathcal{C}}(\mathbf{m}) := \mathbf{m} \wedge \begin{pmatrix} F_x \\ F_y \\ 0 \end{pmatrix} = \begin{pmatrix} -zF_y(m) \\ zF_x(m) \\ xF_y(m) - yF_x(m) \end{pmatrix}. \quad (14)$$

Lemma 24. *The base points of $(N_{\mathcal{C}})_{|\mathcal{C}}$ are the singular points of \mathcal{C} , the contact points with the line at infinity and the points of $\{I, J\} \cap \mathcal{C}$.*

Proof. A point $m \in \mathcal{C}$ is a base point of $N_{\mathcal{C}}$ if and only if $F_x = F_y = 0$ or $z = xF_y - yF_x = 0$. Hence, singular points of \mathcal{C} are base points of $N_{\mathcal{C}}$.

Let $m = [x : y : z]$ be a nonsingular point of \mathcal{C} . First $F_x = F_y = 0$ is equivalent to $\mathcal{T}_m\mathcal{C} = \ell_\infty$. Assume now that $z = xF_y - yF_x = 0$ and $(F_x, F_y) \neq (0, 0)$. Then $m = [x : y : 0] = [F_x : F_y : 0]$ and, due to the Euler formula, we have $0 = -zF_z = xF_x + yF_y$ and so $x^2 + y^2 = 0$, which implies $m = I$ or $m = J$.

Finally observe that if $m \in \{I, J\} \cap \mathcal{C}$, then $m = [-y : x : 0]$ and, due to the Euler formula, $0 = -zF_z = xF_x + yF_y = xF_y - yF_x$. \square

Since the degree of each non zero coordinate of $N_{\mathcal{C}}$ is d , we have

$$c_\nu(\mathcal{C}) = d^2 - \sum_{P \in \text{Base}((N_{\mathcal{C}})_{|\mathcal{C}})} i_P(\mathcal{C}, V(L \circ \mathbf{N}_{\mathcal{C}})), \quad (15)$$

for a generic $L \in \mathbf{V}^\vee$, where we write $\text{Base}((N_{\mathcal{C}})_{|\mathcal{C}})$ for the set of base points of $(N_{\mathcal{C}})_{|\mathcal{C}}$. The set $V(L \circ \mathbf{N}_{\mathcal{C}}) \subset \mathbb{P}^2$ is called **the normal polar** of \mathcal{C} with respect to L . It satisfies

$$m \in V(L \circ \mathbf{N}_{\mathcal{C}}) \iff \mathbf{N}_{\mathcal{C}}(\mathbf{m}) = 0 \text{ or } \delta(L) \in \mathcal{N}_m(\mathcal{C}).$$

Now, to compute the generic intersection numbers, we use the notion of probranches [5, 11, 12]. See section 4 of [6] for details. Let $P \in \mathcal{C}$ be an indeterminacy point of $N_{\mathcal{C}}$ and let us write μ_P for the multiplicity of \mathcal{C} at P . Recall that $\mu_P = 1$ means that P is a nonsingular point of \mathcal{C} . Let $M \in GL(\mathbf{V})$ be such that $M(\mathbf{O}) = \mathbf{P}$ with $\mathbf{O} = (0, 0, 1)$ (we set also $O = [0 : 0 : 1]$) and such that $V(x)$ is not contained in the tangent cone of $V(F \circ M)$ at O . Recall that the equation of

this tangent cone is the homogeneous part of lowest degree in (x, y) of $F(x, y, 1) \in \mathbb{C}[x, y]$ and that this lowest degree is μ_P . Using the combination of the Weierstrass preparation theorem and of the Puiseux expansions,

$$F \circ M(x, y, 1) = U(x, y) \prod_{j=1}^{\mu_P} (y - g_j(x)),$$

for some $U(x, y)$ in the ring of convergent series in x, y with $U(0, 0) \neq 0$ and where $g_j(x) = \sum_{m \geq 1} a_{j,m} x^{\frac{m}{q_j}}$ for some integer $q_j \neq 0$. The $y = g_j(x)$ correspond to the equations of the probranches of \mathcal{C} at P . Since $V(x)$ is not contained in the tangent cone of $V(F \circ M)$ at O , the valuation in x of g_j is strictly larger than or equal to 1 and so the probranch $y = g_j(x)$ is tangent to $V(y - xg'_j(0))$. We write $\mathcal{T}_P^{(i)} := M(V(y - xg'_j(0)))$ the associated (eventually singular) tangent line to \mathcal{C} at P ($\mathcal{T}_P^{(i)}$ is the tangent to the branch of \mathcal{C} at P corresponding to this probranch) and we denote by $i_P^{(j)}$ the tangential intersection number of this probranch:

$$i_P^{(j)} = \text{val}_x(g_j(x) - xg'_j(0)) = \text{val}_x(g_j(x) - xg'_j(x)).$$

We recall that for any homogeneous polynomial $H \in \mathbb{C}[x, y, z]$, we have

$$\begin{aligned} i_P(\mathcal{C}, V(H)) &= i_O(V(F \circ M), V(H \circ M)) \\ &= \sum_{j=1}^{\mu_P} \text{val}_x(H(M(G_j(x)))) \end{aligned}$$

where $G_j(x) := (x, g_j(x), 1)$. With these notations and results, we have

$$\Omega(\mathcal{C}, \ell_\infty) = \sum_{P \in \mathcal{C} \cap \ell_\infty} (i_P(\mathcal{C}, \ell_\infty) - \mu_P(\mathcal{C})) = \sum_{P \in \mathcal{C} \cap \ell_\infty} \sum_{j: \mathcal{T}_P^{(j)} = \ell_\infty} (i_P^{(j)} - 1).$$

For a generic $L \in \mathbf{V}^\vee$, we also have

$$\begin{aligned} i_P(\mathcal{C}, V(L \circ N_{\mathcal{C}})) &= \sum_{j=1}^{\mu_P} \text{val}_x(L(N_{\mathcal{C}}(M(G_j(x)))))) \\ &= \sum_{j=1}^{\mu_P} \min_k \text{val}_x([N_{\mathcal{C}} \circ M]_k(G_j(x))), \end{aligned}$$

where $[\cdot]_k$ denotes the k -th coordinate. Moreover, due to (14), as noticed in Proposition 16 of [7], we have

$$N_{\mathcal{C}} \circ M(\mathbf{m}) = \text{Com}(M) \cdot (\mathbf{m} \wedge [\Delta_{\mathbf{A}} G(\mathbf{m}) \cdot \mathbf{A} + \Delta_{\mathbf{B}} G(\mathbf{m}) \cdot \mathbf{B}]),$$

where $G := F \circ M$, $\mathbf{A} := M^{-1}(1, 0, 0)$, $\mathbf{B} := M^{-1}(0, 1, 0)$ and $\Delta_{(x_1, y_1, z_1)} H = x_1 H_x + y_1 H_y + z_1 H_z$. As observed in Lemma 33 of [6], we have

$$\Delta_{(x_1, y_1, z_1)} G(x, g_j(x), 1) = R_j(x) W_{(x_1, y_1, z_1), j}(x),$$

where $R_j(x) = U(x, g_j(x)) \prod_{j' \neq j} (g_{j'}(x) - g_j(x))$ and $W_{(x_1, y_1, z_1), j}(x) := y_1 - x_1 g'_j(x) + z_1 (x g'_j(x) - g_j(x))$. Therefore, for a generic $L \in \mathbf{V}^\vee$, we have

$$i_P(\mathcal{C}, V(L \circ N_{\mathcal{C}})) = V_P + \sum_{j=1}^{\mu_P} \min_k \text{val}_x([G_j(x) \wedge (W_{\mathbf{A}, j}(x) \cdot \mathbf{A} + W_{\mathbf{B}, j}(x) \cdot \mathbf{B})]_k)$$

where $V_P := \sum_{j=1}^{\mu_P} \sum_{j' \neq j} \text{val}(g_{j'} - g_j)$. Now, we write $h_P^{(j)} := \min_k \text{val}_x([G_j(x) \wedge (W_{\mathbf{A}, j}(x) \cdot \mathbf{A} + W_{\mathbf{B}, j}(x) \cdot \mathbf{B})]_k)$ and $h_P := \sum_{j=1}^{\mu_P} h_P^{(j)}$. Observe that $V(P) = 0$ if P is a nonsingular point of \mathcal{C} .

We recall that, due to Corollary 31 of [6], we have

$$\sum_{P \in \mathcal{C} \cap \text{Base}(N_{\mathcal{C}})} V_P = d(d-1) - d^V$$

and so, due to (15), we obtain

$$c_\nu(\mathcal{C}) = d + d^V - \sum_{P \in \mathcal{C} \cap \text{Base}(N_{\mathcal{C}})} h_P. \quad (16)$$

Now we have to compute the contribution $h_P^{(j)}$ of each probranch of each $P \in \mathcal{C} \cap \text{Base}(N_{\mathcal{C}})$. We have observed, in Proposition 29 of [6], that we can adapt our choice of M to each probranch (or, to be more precise, to each branch corresponding to the probranch). This fact will be useful in the sequel. In particular, for each probranch, we take M such that $g'_j(0) = 0$ so $G_j(x) \wedge (W_{\mathbf{A},j}(x) \cdot \mathbf{A} + W_{\mathbf{B},j}(x) \cdot \mathbf{B})$ can be rewritten:

$$\begin{pmatrix} x \\ g_j(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} x_A y_A - (x_A^2 + x_B^2)g'_j(x) + x_B y_B + (z_A x_A + z_B x_B)(xg'_j(x) - g_j(x)) \\ y_A^2 + y_B^2 - (x_A y_A + x_B y_B)g'_j(x) + (z_A y_A + z_B y_B)(xg'_j(x) - g_j(x)) \\ y_A z_A + y_B z_B - (x_A z_A + x_B z_B)g'_j(x) + (z_A^2 + z_B^2)(xg'_j(x) - g_j(x)) \end{pmatrix}. \quad (17)$$

- Assume first that P is a point of \mathcal{C} outside ℓ_∞ . Then for M as above and such that $z_A = z_B = 0$, we have

$$G_j(0) \wedge (W_{\mathbf{A},j}(0) \cdot \mathbf{A} + W_{\mathbf{B},j}(0) \cdot \mathbf{B}) = \begin{pmatrix} -y_A^2 - y_B^2 \\ x_A y_A + x_B y_B \\ 0 \end{pmatrix}$$

which is non null since $(y_A, y_B) \neq (0, 0)$ and since \mathbf{A} and \mathbf{B} are linearly independent. So $h_P^{(j)} = 0$.

- Assume now that $P \in \mathcal{C} \cap \ell_\infty \setminus \{I, J\}$ and $\mathcal{T}_P^{(j)} \neq \ell_\infty$. Then $y_A + iy_B \neq 0$ and $y_A - iy_B \neq 0$ (since $I, J \notin \mathcal{T}_P^{(j)}$) and so $y_A^2 + y_B^2 \neq 0$ which together with (17) implies that $h_P^{(j)} = 0$ as in the previous case.
- Assume that $P \in \mathcal{C} \cap \ell_\infty \setminus \{I, J\}$ and $\mathcal{T}_P^{(i)} = \ell_\infty$. Assume that $M(1, 0, 0) = (1, i, 0)$. Hence $\mathbf{A} + i\mathbf{B} = (1, 0, 0)$. Then $y_A = y_B = 0$, $x_A + ix_B = 1$, $z_A + iz_B = 0$. So $z_A^2 + z_B^2 = 0$ and $z_A x_A + z_B x_B = z_A \neq 0$ (since $z_B = iz_A$ and $x_B = i(x_A - 1)$). Observe that $P \neq J$ implies also that $x_A - ix_B \neq 0$. So that $x_A^2 + x_B^2 \neq 0$. Hence, due to (17), $G_j(x) \wedge (W_{\mathbf{A},j}(x) \cdot \mathbf{A} + W_{\mathbf{B},j}(x) \cdot \mathbf{B})$ is equal to

$$\begin{pmatrix} x \\ g_j(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} (x_A^2 + x_B^2)g'_j(x) + z_A(xg'_j(x) - g_j(x)) \\ 0 \\ z_A g'_j(x) \end{pmatrix}.$$

Therefore we have $h_P^{(j)} = \text{val}_x((x_A^2 + x_B^2)g'_j(x)) = i_P^{(j)} - 1$.

- Assume that $P = I$ and that $\mathcal{T}_P^{(j)} = \ell_\infty$. Take M such that $M(\mathbf{O}) = (1, i, 0)$, $\mathbf{B} = (1, 0, 0)$ and so $\mathbf{A} = (-i, 0, 1)$. Due to (17), $G_j(x) \wedge (W_{\mathbf{A},j}(x) \cdot \mathbf{A} + W_{\mathbf{B},j}(x) \cdot \mathbf{B})$ is equal to

$$\begin{pmatrix} x \\ g_j(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} -i(xg'_j(x) - g_j(x)) \\ 0 \\ ig'_j(x) + (xg'_j(x) - g_j(x)) \end{pmatrix}. \quad (18)$$

Now observe that each coordinate has valuation at least equal to $i_P^{(j)} = \text{val } g_j$ and that the term of degree $i_P^{(j)}$ of the second coordinate is the term of degree $i_P^{(j)}$ of

$$-i(xg'_j(x) - g_j(x)) + xig'_j(x) = ig_j(x) \neq 0$$

which is non null. Therefore $h_P^{(j)} = i_P^{(j)}$.

- Assume finally that $P = I$ and that $\mathcal{T}_P^{(j)} \neq \ell_\infty$. Take M such that $M(\mathbf{O}) = (1, i, 0)$, $\mathbf{B} = (0, 1, 0)$ and so $\mathbf{A} = (0, -i, 1)$. Due to (17), $G_j(x) \wedge (W_{\mathbf{A},j}(x) \cdot \mathbf{A} + W_{\mathbf{B},j}(x) \cdot \mathbf{B})$ is equal to

$$\begin{pmatrix} x \\ g_j(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ -i(xg_j'(x) - g_j(x)) \\ -i + (xg_j'(x) - g_j(x)) \end{pmatrix}. \quad (19)$$

Now observe that each coordinate has valuation at least equal to 1 and that the term of degree 1 of the second coordinate is $ix \neq 0$. Hence $i_P^{(j)} = 1$.

Observe that the case $P = J$ can be treated in the same way than the case $P = I$.

Theorem 5 follows from (16) and from the previous computation of h_P .

APPENDIX A. NORMAL CLASS OF SURFACES LINKED TO NORMAL CLASS OF PLANAR CURVES

We consider here two particular cases of surfaces the normal class of which is naturally related to the normal class of a curve.

We start with the case of cylinders.

We call **cylinder of base** $\mathcal{C} = V(G) \subset \mathbb{P}^2$ **and of axis** $V(x, y) \subset \mathbb{P}^3$ the surface $V(F) \subset \mathbb{P}^3$, with $F(x, y, z, t) := G(x, y, t)$.

Proposition 25. *Let $\mathcal{S} = V(F) \subset \frac{1}{2}\mathbb{P}^3$ be a cylinder of axis $V(x, y) \subset \mathbb{P}^3$ and of base $\mathcal{C} = V(G) \subset \mathbb{P}^2$. Then*

$$c_\nu(\mathcal{S}) = c_\nu(\mathcal{C}).$$

Proof. The normal line of \mathcal{S} at $m[x_1 : y_1 : z_1 : t_1] \in \mathcal{S}$ is contained in the plane $V(t_1z - z_1t)$. Hence, for any $P[x_0 : y_0 : z_0 : 1] \in \mathbb{P}^3 \setminus \mathcal{H}_\infty$, we have the following equivalences:

$$\begin{aligned} P \in \mathcal{N}_m \mathcal{S} &\Leftrightarrow P \in (m, n_{\mathcal{S}}(m)) \\ &\Leftrightarrow P \in (m, n_{\mathcal{S}}(m)), \quad m \notin \mathcal{H}_\infty \\ &\Leftrightarrow P \in (m, n_{\mathcal{S}}(m)), \quad t_1 \neq 0, \quad z_1 = z_0 t_1 \\ &\Leftrightarrow t_1 \neq 0, \quad z_1 = z_0 t_1, \quad P_1 \in (m_1, n_{\mathcal{C}}(m_1)) \\ &\Leftrightarrow t_1 \neq 0, \quad z_1 = z_0 t_1, \quad P_1 \in \mathcal{N}_{m_1} \mathcal{C}, \end{aligned}$$

with $m_1[x_1 : y_1 : t_1]$ and $P_1[x_0 : y_0 : 1]$ in \mathbb{P}^2 . Hence $\#\{m \in \mathcal{S} : P \in \mathcal{N}_m \mathcal{S}\} = \#\{m_1 \in \mathcal{C} : P_1 \in \mathcal{N}_{m_1} \mathcal{C}\}$ and the result follows. \square

Another case when the normal class of a surface corresponds to the normal class of a planar surface is the case of revolution surfaces.

We call **algebraic surface of revolution of axis** $V(x, y)$ a surface $V(F) \subset \mathbb{P}^3$, where $F(x, y, z, t) := G(x^2 + y^2, z, t)$, where $G(u, v, w) \in \mathbb{C}[u, v, w]$ is such that $G(u^2, v, w)$ is homogeneous.

Proposition 26. *Let $\mathcal{S} = V(F) \subset \mathbb{P}^3$ be an irreducible surface of degree $d_{\mathcal{S}} \geq 2$. We suppose that \mathcal{S} is an algebraic surface of revolution of axis $V(x, y)$, then $c_\nu(\mathcal{S}) = c_\nu(\mathcal{C})$ where $\mathcal{C} = V(G(x^2, y, z)) \subset \mathbb{P}^2$ where $G \in \mathbb{C}[u, v, w]$ is such that $F(x, y, z, t) = G(x^2 + y^2, z, t)$.*

Proof. We observe that, for every $m[x_1 : y_1 : z_1 : t_1] \in \mathcal{S}$, we have

$$n_{\mathcal{S}}(m) = [F_x(\mathbf{m}) : F_y(\mathbf{m}) : F_z(\mathbf{m}) : 0] = [2x_1 G_u(\mathbf{m}_3) : 2y_1 G_u(\mathbf{m}_3) : G_v(\mathbf{m}_3) : 0],$$

with $\mathbf{m}_3 = (x_1^2 + y_1^2, z_1, t_1)$. The normal line of \mathcal{S} at $m[x_1 : y_1 : z_1 : t_1] \in \mathcal{S}$ is contained in the plane $V(x_1y - y_1x)$ if $(x_1, y_1) \neq 0$. Hence, for any $P[x_0 : y_0 : z_0 : 1] \in \mathbb{P}^3 \setminus \mathcal{H}_\infty$ with $x_0^2 + y_0^2 \neq 0$ and $x_0 \neq 0$, we have the following equivalences:

$$\begin{aligned} P \in \mathcal{N}_m \mathcal{S} &\Leftrightarrow P \in (m, n_{\mathcal{S}}(m)), \quad m \notin \mathcal{H}_\infty \\ &\Leftrightarrow P \in (m, n_{\mathcal{S}}(m)), \quad t_1 \neq 0, \quad x_1y_0 = x_0y_1. \end{aligned}$$

We observe that $x_1y_0 = x_0y_1$ implies that $x_1^2 + y_1^2 \neq 0$ or $(x_1, y_1) = 0$. Hence we have

$$P \in \mathcal{N}_{\mathcal{S}}(m) \Leftrightarrow t_1 \neq 0, \quad y_1 = \frac{x_1y_0}{x_0}, \quad P_1 \in (m_1, N_1),$$

with $P_1[x_0 : z_0 : 1]$, $m_1[x_1 : z_1 : t_1]$ and $N_1 = [2x_1G_u(\mathbf{m}_3) : G_z(\mathbf{m}_3) : 0]$ in \mathbb{P}^2 . Now let δ be a complex number such that $\delta^2x_0^2 = x_0^2 + y_0^2$. We then have

$$\begin{aligned} N_1 \left[\frac{2}{\delta}(x_1\delta)G_u((x_1\delta)^2, z_1, t_1) : G_v((x_1\delta)^2, z_1, t_1) : 0 \right], \\ m_1 \left[\frac{1}{\delta}x_1\delta : z_1 : t_1 \right] \quad \text{and} \quad P_1 \left[\frac{1}{\delta}x_0\delta : z_0 : 1 \right]. \end{aligned}$$

Now we set $m_2[x_1\delta : z_1 : t_1]$ and $P_2[x_0\delta : y_0 : 1]$. We observe that

$$n_{\mathcal{C}}(m_2) [2(x_1\delta)G_u((x_1\delta)^2, z_1, t_1) : G_v((x_1\delta)^2, z_1, t_1) : 0],$$

so that

$$\begin{aligned} P \in \mathcal{N}_m \mathcal{S} &\Leftrightarrow t_1 \neq 0, \quad y_1 = \frac{x_1y_0}{x_0}, \quad P_2 \in (m_2, n_{\mathcal{C}}(m_2)) \\ &\Leftrightarrow t_1 \neq 0, \quad z_1 = z_0t_1, \quad P_2 \in \mathcal{N}_{m_2} \mathcal{C}. \end{aligned}$$

Hence $\#\{m \in \mathcal{S} : P \in \mathcal{N}_{\mathcal{S}}(m)\} = \#\{m_2 \in \mathcal{C} : P_2 \in \mathcal{N}_{\mathcal{C}}(m_2)\}$ and this number does not depend on the choice of δ such that $\delta^2x_0^2 = x_0^2 + y_0^2$. The result follows. \square

REFERENCES

- [1] F. Catanese and C. Trifogli. *Focal loci of algebraic varieties. I. Special issue in honor of Robin Hartshorne*. Comm. Algebra 28 (2000), no. 12, pp. 6017–6057.
- [2] I. V. Dolgachev. *Classical algebraic geometry. A modern view*. Cambridge University Press, Cambridge (2012) xii+639 pp.
- [3] D. Eisenbud and J. Harris. *Intersection Theory in Algebraic Geometry* (2011).
- [4] B. Fantechi. *The Evolute of a Plane Algebraic Curve*, (1992) UTM 408, University of Trento.
- [5] G. H. Halphen. *Mémoire sur les points singuliers des courbes algébriques planes*. Académie des Sciences t. XXVI (1889) No 2.
- [6] A. Josse, F. Pène. *On the degree of caustics by reflection*. Commun. Algebra (2014) vol 42, p. 2442–2475.
- [7] A. Josse, F. Pène. *On the class of caustics by reflection*. accepted for publication in Ann. Sc. Norm. Sup. Pisa. DOI Number: 10.2422/2036-2145.201304_003.
- [8] A. Josse, F. Pène. *On caustics by reflection of algebraic surfaces*. arXiv:1304.3883
- [9] G. Salmon G. *A treatise on higher plane curves: Intended as a sequel to a treatise on conic sections*. Elibron classics (1934).
- [10] C. Trifogli. *Focal Loci of Algebraic Hypersurfaces: a General Theory*, Geom. Dedicata 70 (1998), pp. 1–26.
- [11] C. T. C. Wall. *Singular Points of Plane Curves*. Cambridge University Press. 2004.
- [12] C. T. C. Wall. *Plücker formulae for singular space curves*. Math. Proc. Cambridge Philos. Soc. 151 (2011), no. 1, 129–143.
- [13] F. L. Zak, *Tangents and secants of algebraic varieties*, Translations of Mathematical Monographs, 127 (1993), American Mathematical Society, Providence, RI.

UNIVERSITÉ EUROPÉENNE DE BRETAGNE, UNIVERSITÉ DE BREST, LABORATOIRE DE MATHÉMATIQUES DE BRETAGNE ATLANTIQUE, UMR CNRS 6205, 29238 BREST CEDEX, FRANCE

E-mail address: `alfrederic.josse@univ-brest.fr`

UNIVERSITÉ EUROPÉENNE DE BRETAGNE, UNIVERSITÉ DE BREST, LABORATOIRE DE MATHÉMATIQUES DE BRETAGNE ATLANTIQUE, UMR CNRS 6205, 29238 BREST CEDEX, FRANCE

E-mail address: `francoise.pene@univ-brest.fr`