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ABOUT GORDAN'S ALGORITHM FOR BINARY FORMS

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ABOUT GORDAN'S ALGORITHM FOR BINARY FORMS

MARC OLIVE

ABSTRACT. In this paper, we present a modern version of Gordan's algorithm on binary forms. Symbolic method is reinterpreted in terms of $SL(2, \mathbb{C})$ -equivariant homomorphisms defined upon Cayley operator and polarization operator. A graphical approach is thus developed to obtain Gordan's ideal, a central key to get covariant basis of binary forms. To illustrate the power of this method, we obtain for the first time a minimal covariant bases for $S_6 \oplus S_4$, $S_6 \oplus S_4 \oplus S_2$ and a minimal invariant bases of $S_8 \oplus S_4 \oplus S_4$.

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1. INTRODUCTION

Classical invariant theory was a very active research field throughout the XIXth century. As pointed out by Parshall [55], this field was initiated by Gauss' *Disquisitiones Arithmeticae* (1801) in which he studied a linear change of variables for quadratic forms with integer coefficients. About forty

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years later, Boole [11] established the main purpose of what has become today *classical invariant theory*. Cayley [23, 24] deeply investigated this field of research and developed important tools still in use nowadays, such as the *Cayley Omega operator*. During about fifteen years (until Cayley’s seventh memoir [20] in 1861) the English school of invariant theory, mainly led by Cayley and Sylvester, developed important tools to compute explicit *invariant generators* of binary forms. Thus, the role of calculation deeply influenced this first approach in invariant theory [23].

Meanwhile, a German school principally conducted by Clebsch, Aronhold and Gordan, developed their own approach, using the *symbolic method* (also used with slightly different notations by the English school). In 1868, Gordan, who was called the “King of invariant theory”, proved that the algebra of *covariants* of any binary forms is always finitely generated [34]. As a great part of the mathematical development of that time, such a result was endowed with a constructive proof: the English and the German schools were equally preoccupied by calculation and an exhibition of invariants and covariants. Despite Gordan’s constructive proof, Cayley was reluctant to make use of Gordan’s approach to obtain a new understanding of invariant theory. That’s only in 1903, with the work of Grace–Young [36], that the German approach of Gordan and al. became accessible to a wide community of mathematicians. During that time, from 1868 to 1875, Gordan’s constructive approach led to several explicit results: first, and without difficulty, Gordan [35] computed a bases for the covariants of the quintic and the sextic. Thereafter, he started the computation of a covariant bases for the septic and the octic. This work was achieved by Von Gall who exhibited a complete covariant bases for the septic [67] and for the octic [66].

In 1890, Hilbert made a critical advance in the field of *invariant theory*. Using a totally new approach [39], which is the cornerstone of today’s algebraic geometry, he proved a finiteness theorem in the very general case of a *linear reductive group* [27]. However, his first proof [39] was criticized for not being constructive [32]. Facing those critics, Hilbert produced a second proof [39], claimed to be more constructive. This effective approach is nowadays widely used to obtain a finite generating set of invariants [58, 30, 15, 16]. Let summarize here the three main steps of Hilbert’s approach [39].

The first step is to compute the *Hilbert series* of the graded algebra \mathcal{A} of invariants, which is always a rational function by the Hilbert–Serre theorem [22]. This Hilbert series¹ gives dimensions of each homogeneous space of \mathcal{A} . The second step is to exhibit a *homogeneous system of parameters* (hsop) for the algebra \mathcal{A} . Finally, the Hochster–Roberts theorem [40] ensures that the invariant algebra \mathcal{A} is *Cohen–Macaulay*². Thanks to that statement, one system of parameters (or at least the knowledge of their degree) altogether with the Hilbert series produce a bound for the degrees of generating invariants. We refer the reader to several references [64, 15, 30, 26, 27, 28] to get a general and modern approach on this subject.

¹There exists several methods to compute this Hilbert series [9, 46, 59] *a priori*.

²Meaning the algebra \mathcal{A} is a finite and free $k[\theta_1, \dots, \theta_s]$ -module, where $\{\theta_1, \dots, \theta_s\}$ is a system of parameters.

However, one major weakness of that strategy is that it depends on the knowledge of a system of parameters (or at least their degree). The Noether normalization lemma [44] ensures that such a system always exists, but as far as we know, current algorithms to obtain such a system [38] are not sufficiently effective because of the extensive use of Grobner basis. For the invariant or covariant algebra of binary forms, one has of course the concept of *nullcone* and the Mumford–Hilbert criterion [27, 13] to check that a given finite family is a system of parameters. But this criterion does not explain how to obtain a system of parameters. Furthermore, in the case of *joint invariants*, that is for the invariant algebra $\mathbf{Inv}(V)$ of $V := S_{n_1} \oplus \cdots \oplus S_{n_k}$, such a system of parameters has, in general, a complex shape. Indeed, Brion [14] showed that there exists a system of parameters which respects the multi-graduation of $\mathbf{Inv}(V)$ only in thirteen cases.

An important motivation for this article was to compare effective approaches in invariant theory since the goal was to compute invariant basis for non trivial joint invariants, such as $S_8 \oplus S_4 \oplus S_4$ or $S_6 \oplus S_4 \oplus S_2 \oplus S_2$. Those computations have important applications in *continuum mechanics* [5] in which occurs invariants of tensor spaces defined on \mathbb{R}^3 , naturally isomorphic (after complexification) to $SL(2, \mathbb{C})$ spaces of binary forms [62, 10]. For instance, to obtain invariants of the *elasticity tensor* [7], Boehler–Kirilov–Onat [10] derived from the invariant bases of S_8 (first obtained by Von Gall [66] in 1880) a generating set of invariants for the higher dimensional irreducible component of the elasticity tensor. Such an invariant bases can be used to classify the orbit space of the elasticity tensor, as pointed out by Auffray–Kolev–Petitot [6]. In a recent paper, we used a joint invariant bases of $S_6 \oplus S_2$ already obtained by von Gall [65] to obtain a new result on an invariant bases of a traceless and totally symmetric third order tensor defined on \mathbb{R}^3 [50]. Such an invariant bases is useful in piezoelectricity [69] and second-gradient of strain elasticity theory [48].

Other interests for effective computations of generating sets of invariants of binary forms arise in *geometrical arithmetic*, illustrated by the work of Lercier–Ritzenthaler [45] on hyperelliptic curves. We could also cite other areas such as *quantum informatics* with the paper of Luque [47] and *recoupling theory*, with the work of Abdesselam and Chipalkatti [2, 3, 1, 4] on $6j$ and $9j$ -symbols.

Approaches on *effective invariant theory* do not only rely on the algebraic geometry field initially developed by Hilbert. In the case of a single binary form, Olver [52] exhibits another constructive approach, which was later generalized for a single n -ary form and also specified with a “running bound” by Brini–Regonati–Creolis [12]. We could also cite Kung–Rota [43] but the combinatorial approach developed there became increasingly complex for the cases we had to deal with.

As we already noticed, a special case of Gordan’s algorithm, stated in theorem 8.1, leads to a very easy computation of one covariant bases for

$S_6 \oplus S_2$. Due to this observation, we decided to reformulate Gordan's theorem³ on binary forms in the modern language of operators and $\mathrm{SL}(2, \mathbb{C})$ equivariant homomorphisms. We also decided to represent $\mathrm{SL}(2, \mathbb{C})$ equivariant homomorphisms with *directed graphs*, in the spirit of the graphical approach developed by Olver–Shakiban [54].

The paper is organized as follows. In [section 2](#) we recall the mathematical background of classical invariant theory, and we introduce classical operators such as the Cayley operator, polarization operator and the transvectant operator. In [section 3](#), we introduce *molecule* and *molecular covariants* which are graphical representations of $\mathrm{SL}(2, \mathbb{C})$ equivariant homomorphisms constructed with the use of Cayley and polarization operators. We then give in [section 4](#) important relations between molecular covariants and transvectants. Gordan's algorithm for joint covariants, explained in [section 5](#), produces a finite generating set for $\mathbf{Cov}(S_m \oplus S_n)$, knowing a finite system of generators for the covariant algebra $\mathbf{Cov}(S_m)$ and $\mathbf{Cov}(S_n)$. A second version of Gordan's algorithm, which enables to compute a covariant bases for S_n , knowing covariant basis for S_k ($k < n$), is detailed in [section 6](#). We propose in [section 7](#) some improvement of those two algorithms and in [section 8](#) we give some illustrations of that method, by (re-)computing a minimal covariant bases for $S_6 \oplus S_2$ (already done by von Gall [65]). We also exhibit for the first time a minimal bases for the joint covariants of $S_6 \oplus S_4$ ([theorem 8.3](#)), and also a minimal bases for the joint covariants of $S_6 \oplus S_4 \oplus S_2$ (new, [theorem 8.4](#)). Then we apply the algorithm for a single binary form and give a minimal covariant bases for the binary octics. Note this was already obtained by Von Gall [67], Lercier–Ritzenthaler [45], Cröni [25] and Bedratyuk [8]. Finally we obtain for the first time a minimal invariant bases for $S_8 \oplus S_4 \oplus S_4$ ([theorem 8.11](#)). Note also that a minimal covariant bases for the binary nonics and decimics will be presented in a forthcoming paper with Lercier [51].

2. MATHEMATICAL FRAMEWORK

2.1. Covariants of binary forms.

Definition 2.1. The complex vector space of n -th degree binary forms, noted S_n , is the space of homogeneous polynomials

$$\mathbf{f}(\mathbf{x}) := a_0x^n + \binom{n}{1}a_1x^{n-1}y + \dots + \binom{n}{n-1}a_{n-1}xy^{n-1} + a_ny^n,$$

with $\mathbf{x} := (x, y) \in \mathbb{C}^2$ and $a_i \in \mathbb{C}$.

The natural $\mathrm{SL}_2(\mathbb{C})$ action on \mathbb{C}^2 induces a left action on S_n , given by

$$(g \cdot \mathbf{f})(\mathbf{x}) := \mathbf{f}(g^{-1} \cdot \mathbf{x}), \quad g \in \mathrm{SL}_2(\mathbb{C}).$$

By a space V of binary forms, we mean a direct sum

$$V := \bigoplus_{i=0}^s S_{n_i}, \quad n_i \in \mathbb{N}$$

³Note that Weyman [68] has also reformulated Gordan's method in a modern way and through algebraic geometry but unfortunately, we were unable to extract from it an effective approach. There is also a preprint of Pasechnik [56] on this method.

where the action of $\mathrm{SL}_2(\mathbb{C})$ is diagonal. One can also define an $\mathrm{SL}_2(\mathbb{C})$ action on the coordinate ring $\mathbb{C}[V \oplus \mathbb{C}^2]$ by

$$(g \cdot \mathbf{p})(\mathbf{f}, \mathbf{x}) := \mathbf{p}(g^{-1} \cdot \mathbf{f}, g^{-1} \cdot \mathbf{x}) \text{ for } g \in \mathrm{SL}_2(\mathbb{C}), \mathbf{p} \in \mathbb{C}[V \oplus \mathbb{C}^2].$$

Definition 2.2. The covariant algebra⁴ of a space V of binary forms, noted $\mathbf{Cov}(V)$, is the invariant algebra

$$\mathbf{Cov}(V) := \mathbb{C}[V \oplus \mathbb{C}^2]^{\mathrm{SL}_2(\mathbb{C})}.$$

An important result, first established by Gordan [34] and then extended by Hilbert [39] (for any linear reductive group) is the following.

Theorem 2.3. *For every space V of binary forms, the covariant algebra $\mathbf{Cov}(V)$ is finitely generated, i.e. there exists a finite set $\mathbf{h}_1, \dots, \mathbf{h}_N$ in $\mathbf{Cov}(V)$, called a bases, such that*

$$\mathbf{Cov}(V) = \mathbb{C}[\mathbf{h}_1, \dots, \mathbf{h}_N].$$

There is a natural bi-graduation on the covariant algebra $\mathbf{Cov}(V)$:

- By the **degree**, which is the polynomial degree in the coefficients of the space V ;
- By the **order** which is the polynomial degree in the variables \mathbf{x} ;

Let $\mathbf{Cov}_{d,k}(V)$ be the subspace of degree d and order k covariants, and:

$$\mathbf{C}_+ := \sum_{d+k>0} \mathbf{Cov}_{d,k}(V).$$

Then, \mathbf{C}_+ is an ideal of the graduated algebra $\mathbf{Cov}(V)$. For each $d+k > 0$, let $\delta_{d,k}$ be the codimension of $(\mathbf{C}_+^2)_{d,k} := (\mathbf{C}_+^2) \cap \mathbf{Cov}_{d,k}(V)$ in $\mathbf{Cov}_{d,k}(V)$. Since the algebra $\mathbf{Cov}(V)$ is of finite type, there exists an integer p such that $\delta_{d,k} = 0$ for $d+k \geq p$ and we can define the invariant number:

$$n(V) = \sum_{d,k} \delta_{d,k}.$$

Definition 2.4. A family $(\mathbf{p}_1, \dots, \mathbf{p}_s)$ is a *minimal* bases of $\mathbf{Cov}(V)$ if its image in the vector space $\mathbf{C}_+/\mathbf{C}_+^2$ is a bases. In that case we have $s = n(V)$.

Remark 2.5. As pointed out by Dixmier–Lazard [31], a minimal bases is obtained by taking, for each d, k , a complement bases of $(\mathbf{C}_+^2)_{d,k}$ in $\mathbf{Cov}_{d,k}(V)$. There is a long history of an explicit determination of such a minimal bases for covariant algebras. We give in Table 1 some results⁵ obtained from XIXth century to XXIth century. As we know, there is no way to get the invariant $n(V)$ but to exhibit an explicit minimal bases of $\mathbf{Cov}(V)$.

⁴For a general and modern approach on invariant and covariant algebra, we refer to the online text [42] by Kraft and Procesi.

⁵The website [http://www.win.tue.nl/~sim\\$ueb/math/invar.html](http://www.win.tue.nl/~sim$ueb/math/invar.html) gives a general overview on those results.

Algebra	$n(V)$	Explicit minimal bases
$\mathbf{Cov}(S_5)$	23	Gordan [34]
$\mathbf{Cov}(S_6)$	26	Gordan [34]
$\mathbf{Cov}(S_7)$	147	Dixmier–Lazard [31]
$\mathbf{Cov}(S_8)$	69	Cröni [25]
$\mathbf{Cov}(S_6 \oplus S_2)$	99	Von Gall [67]

TABLE 1. Minimal basis of covariant algebras.

2.2. Bidifferential operators and transvectants. Recall that S_n is an *irreducible* $\mathrm{SL}(2, \mathbb{C})$ representation [33]. The *Clebsch–Gordan decomposition* [33] of a tensor product is the $\mathrm{SL}(2, \mathbb{C})$ irreducible decomposition

$$S_n \otimes S_p \simeq \bigoplus_{r=0}^{\min(n,p)} S_{n+p-2r}.$$

We then deduce that, for each $0 \leq r \leq \min(n, p)$, there is only one (up to a scale factor) *Clebsch–Gordan projector*

$$\pi_r : S_n \otimes S_p \longrightarrow S_{n+p-2r}, \quad \mathbf{f} \otimes \mathbf{g} \mapsto (\mathbf{f}, \mathbf{g})_r := \pi_r(\mathbf{f} \otimes \mathbf{g}).$$

Such a projector is called a *transvectant*. To have an explicit formula for transvectants, we use bi-differential operators:

- the *Cayley operator* [52], which is a bi-differential operator acting on the tensor product of complex analytic functions $\mathbf{f}(\mathbf{x}_\alpha)\mathbf{g}(\mathbf{x}_\beta)$:

$$\Omega_{\alpha\beta}(\mathbf{f}(\mathbf{x}_\alpha)\mathbf{g}(\mathbf{x}_\beta)) := \frac{\partial \mathbf{f}}{\partial x_\alpha} \frac{\partial \mathbf{g}}{\partial y_\beta} - \frac{\partial \mathbf{f}}{\partial y_\alpha} \frac{\partial \mathbf{g}}{\partial x_\beta};$$

- the *polarization operator*⁶ acting on a complex analytic function $\mathbf{f}(\mathbf{x}_\alpha)$:

$$\sigma_\alpha(\mathbf{f}(\mathbf{x}_\alpha)) = x \frac{\partial \mathbf{f}}{\partial x_\alpha} + y \frac{\partial \mathbf{f}}{\partial y_\alpha}.$$

Both the Cayley and polarization operators commute with the $\mathrm{SL}_2(\mathbb{C})$ action (see [52] for instance).

Definition 2.6. Given two binary forms $\mathbf{f} \in S_n$ and $\mathbf{g} \in S_p$, their *transvectant* of index $r \geq 0$, noted $(\mathbf{f}, \mathbf{g})_r$, is defined to be

$$(\mathbf{f}, \mathbf{g})_r := \begin{cases} \Omega_{\alpha\beta}^r \sigma_\alpha^{n-r} \sigma_\beta^{p-r} (\mathbf{f}(\mathbf{x}_\alpha)\mathbf{g}(\mathbf{x}_\beta)) & \text{if } 0 \leq r \leq \min(n, p) \\ 0 & \text{else} \end{cases}.$$

Remark 2.7. Note that the definition given in [52] uses a scale factor and a trace operator:

$$(\mathbf{f}, \mathbf{g})_r = (n-r)!(p-r)! \left[\Omega_{\alpha\beta}^r \mathbf{f}(\mathbf{x}_\alpha)\mathbf{g}(\mathbf{x}_\beta) \right]_{|\mathbf{x}_\alpha=\mathbf{x}_\beta=\mathbf{x}}.$$

On the other hand, Gordan’s definition [36] corresponds to

$$\frac{1}{n!} \frac{1}{p!} (\mathbf{f}, \mathbf{g})_r.$$

⁶This operator is called *scaling process* in [52].

This last expression is very simple when applied to *powers of linear forms*. Indeed, if

$$\mathbf{a}_{\mathbf{x}_\alpha}^n := (a_0x_\alpha + a_1y_\alpha)^n, \quad \mathbf{b}_{\mathbf{x}_\beta}^p := (b_0x_\beta + b_1y_\beta)^p, \quad (ab) := a_0b_1 - a_1b_0,$$

then,

$$\frac{1}{n!} \frac{1}{p!} (\mathbf{a}_{\mathbf{x}_\alpha}^n, \mathbf{b}_{\mathbf{x}_\beta}^p)_r = (ab)^r \mathbf{a}_{\mathbf{x}}^{n-r} \mathbf{b}_{\mathbf{x}}^{p-r}.$$

Our choice of definition 2.6 has the advantage of inducing simple relations on operators and thus on transvectants (see 3.3 for instance).

Remark 2.8. Take a space of binary forms

$$V = S_{n_1} \oplus \dots \oplus S_{n_s}$$

and consider the set \mathcal{T} containing each $\mathbf{f}_i \in S_{n_i}$ and closed under transvectant operations:

$$\mathbf{f} \in \mathcal{T}, \mathbf{g} \in \mathcal{T} \Rightarrow (\mathbf{f}, \mathbf{g})_r \in \mathcal{T}, \quad \forall r \in \mathbb{N}.$$

Then as a classical result [57] the covariant algebra $\mathbf{Cov}(V)$ is generated by the (infinite) set \mathcal{T} . One important issue is then to extract a finite family from that infinite set.

3. MOLECULAR COVARIANTS

Let $\text{Sym}^d(V)$ be the space of totally symmetric tensors of order d on V . The Aronhold polarization induces an isomorphism [29] between $\mathbf{Cov}_{d,k}(V)$ and the space

$$\text{Hom}_{\text{SL}(2, \mathbb{C})}(\text{Sym}^d(V), S_k) \subset \text{Hom}_{\text{SL}(2, \mathbb{C})}(\otimes^d V, S_k).$$

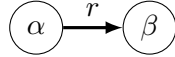
Transvectants, Cayley operator and polarization operator give natural way to obtain $\text{SL}(2, \mathbb{C})$ -equivariant homomorphisms. We already saw (definition 2.6) that the Clebsch–Gordan projector

$$\pi_r : S_n \otimes S_p \longrightarrow S_{n+p-2r},$$

can be written as

$$\Omega_{\alpha\beta}^r \sigma_\alpha^{n-r} \sigma_\beta^{p-r}.$$

Such a monomial will be represented by the colored directed graph (colored digraph)⁷:



where the *atom* α (resp. β) is colored by S_n (resp. S_p).

More generally, let $V = S_{n_1} \oplus \dots \oplus S_{n_s}$ be a space of binary forms. We are going to define equivariant multilinear maps from V to some S_k , corresponding to monomials in the symbols $\Omega_{\alpha\beta}, \sigma_\gamma, \dots$ and labelled by *molecules* (colored digraphs).

More precisely, let $\mathcal{V}(D) = \{\alpha, \beta, \dots, \varepsilon\}$ be the set of vertices of a colored digraph D and $\mathcal{E}(D)$ be its set of edges. Each vertex α of D , also called an *atom*, is colored by a factor $S(\alpha) := S_{n_i}$ of V . In that case, the *valence* of α is $\text{val}(\alpha) := n$. Define $o(e)$, $t(e)$ and $w(e)$ to be respectively the origin, the

⁷It is important to note that a digraph represents here a morphism and not a bi-differential operator as did Olver–Shakiban [53].

termination and the weight of an edge $e \in \mathcal{E}(\mathbf{D})$. Finally, we define the *free valence* $\text{val}_{\mathbf{D}}(\alpha)$ of an atom $\alpha \in \mathcal{V}(\mathbf{D})$ to be:

$$\text{val}_{\mathbf{D}}(\alpha) := \text{val}(\alpha) - \sum_{\alpha=o(e) \text{ or } \alpha=t(e)} w(e).$$

Definition 3.1. The $\text{SL}(2, \mathbb{C})$ -equivariant homomorphism $\phi_{\mathbf{D}}$ defined by the molecule \mathbf{D} is given by

$$\phi_{\mathbf{D}} := \begin{cases} \prod_{e \in \mathcal{E}(\mathbf{D})} \Omega_{o(e)t(e)}^{w(e)} \prod_{\alpha \in \mathcal{V}(\mathbf{D})} \sigma_{\alpha}^{\text{val}_{\mathbf{D}}(\alpha)} & \text{if } \text{val}_{\mathbf{D}}(\alpha) \geq 0, \quad \forall \alpha \in \mathcal{V}(\mathbf{D}) \\ 0 & \text{else} \end{cases}.$$

When $\text{val}_{\mathbf{D}}(\alpha) \geq 0$ for all $\alpha \in \mathcal{V}(\mathbf{D})$, it maps $S(\alpha) \otimes \cdots \otimes S(\varepsilon)$ to S_k , where $k = \text{val}_{\mathbf{D}}(\alpha) + \dots + \text{val}_{\mathbf{D}}(\varepsilon)$.

There exists *syzygies* on morphisms $\phi_{\mathbf{D}}$ induced by fundamental relations among operators. Let α, β, γ and δ be four atoms.

- (1) The first syzygy derives from the equality

$$\Omega_{\alpha\beta} = -\Omega_{\beta\alpha},$$

which leads to the graphical relation:

$$\begin{array}{c} \alpha \longrightarrow \beta \end{array} = - \begin{array}{c} \alpha \longleftarrow \beta \end{array} \quad (3.1)$$

- (2) The second one comes from the Plücker relation [52]:

$$\Omega_{\alpha\beta}\sigma_{\gamma} = \Omega_{\alpha\gamma}\sigma_{\beta} + \Omega_{\gamma\beta}\sigma_{\alpha}, \quad (3.2)$$

which leads to the graphical relation:

$$\begin{array}{c} \alpha \longrightarrow \beta \\ \gamma \end{array} = \begin{array}{c} \alpha \\ \searrow \\ \gamma \end{array} \begin{array}{c} \beta \\ \alpha \end{array} + \begin{array}{c} \beta \\ \nearrow \\ \gamma \end{array} \begin{array}{c} \alpha \\ \beta \end{array} \quad (3.3)$$

- (3) The third one derives also from a Plücker relation, namely

$$\Omega_{\alpha\beta}\Omega_{\gamma\delta} = \Omega_{\alpha\delta}\Omega_{\beta\gamma} + \Omega_{\alpha\gamma}\Omega_{\delta\beta},$$

which leads to the graphical relation:

$$\begin{array}{c} \alpha \longrightarrow \beta \\ \delta \longrightarrow \gamma \end{array} = \begin{array}{c} \alpha \\ \downarrow \\ \delta \end{array} \begin{array}{c} \beta \\ \alpha \end{array} + \begin{array}{c} \beta \\ \downarrow \\ \gamma \end{array} \begin{array}{c} \alpha \\ \delta \end{array} + \begin{array}{c} \alpha \longrightarrow \beta \\ \delta \longrightarrow \gamma \end{array} \quad (3.4)$$

Remark 3.2. By syzygie 3.1, we have

$$\begin{array}{c} \alpha \xrightarrow{2} \beta \end{array} = \begin{array}{c} \alpha \xleftarrow{2} \beta \end{array}$$

thus for even weighed edges, we will not specify orientation:

$$\begin{array}{c} \alpha \xrightarrow{2} \beta \end{array} := \begin{array}{c} \alpha \xrightarrow{2} \beta \end{array}$$

For each atom $\alpha \in \mathcal{V}(\mathbf{D})$, let $\mathbf{f}_\alpha \in \mathbf{S}(\alpha)$ and consider one covariant

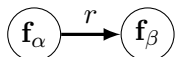
$$\phi_{\mathbf{D}} \left(\bigotimes_{\alpha \in \mathcal{V}(\mathbf{D})} \mathbf{f}_\alpha \right) \in \mathbf{Cov}(V).$$

This defines a map from the set of molecules to $\mathbf{Cov}(V)$. A *molecular covariant* \mathbf{D} is then defined to be an image of a molecule by this map, and in that case a binary form $\mathbf{f}_\alpha \in \mathbf{S}(\alpha) = \mathbf{S}_n$ is said to be an *atom* of valence n in \mathbf{D} . The following result is known as the first fundamental theorem for binary forms [43, 52].

Theorem 3.3. *Given a space $V = \mathbf{S}_{n_1} \oplus \dots \oplus \mathbf{S}_{n_s}$ of binary forms, the covariant algebra $\mathbf{Cov}(V)$ is generated by the (infinite) family of molecular covariants.*

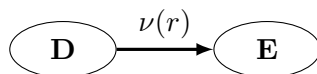
4. TRANSVECTANTS ON MOLECULAR COVARIANTS

First observe that a transvectant $(\mathbf{f}_\alpha, \mathbf{f}_\beta)_r$ is represented by a simple molecular covariant:



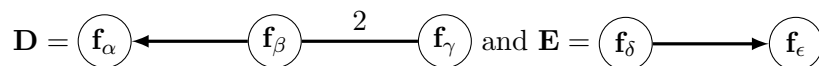
Now, to obtain general relations between iterated transvectants and molecular covariants, we need to specify some operations on molecular covariants.

Definition 4.1. Let \mathbf{D} and \mathbf{E} be two molecular covariants. Let $r \geq 0$ be an integer and $\nu(r)$ be a symbol, we define the molecular covariant $\mathbf{M}^{\nu(r)}$, graphically noted

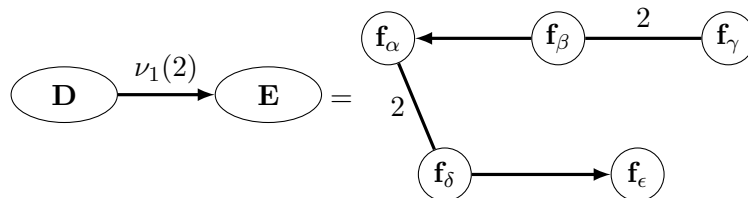


to be a new molecular covariant obtained by linking \mathbf{D} and \mathbf{E} with r edges in a given way $\nu(r)$.

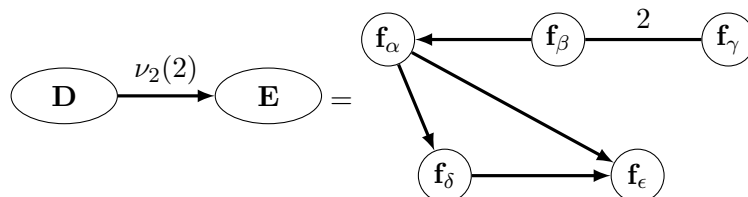
Example 4.2. Given atoms $\mathbf{f}_\alpha, \dots, \mathbf{f}_\epsilon$ of valence greater than 4, let



we can define



or



By a direct application of Leibnitz formula, we have [52]:

Proposition 4.3. Let \mathbf{D}, \mathbf{E} be two molecular covariants and $r \geq 0$ be an integer. Then the transvectant $(\mathbf{D}, \mathbf{E})_r$ is a linear combination of molecular covariants⁸ $\mathbf{M}^{\nu(r)}$ with rational positive coefficients⁹, for each possible link $\nu(r)$ between \mathbf{D} and \mathbf{E} :

$$(\mathbf{D}, \mathbf{E})_r = \sum_{\nu(r)} a_{\nu(r)} \mathbf{M}^{\nu(r)}, \quad a_{\nu(r)} \in \mathbb{Q}^+. \quad (4.1)$$

Example 4.4. Let $\mathbf{f}_\alpha, \dots, \mathbf{f}_\delta$ be atoms of valence greater than 4,

$$\mathbf{D} = \begin{array}{c} \textcircled{\mathbf{f}_\alpha} \longleftarrow \textcircled{\mathbf{f}_\beta} \xrightarrow{2} \textcircled{\mathbf{f}_\gamma} \end{array} \quad \text{and} \quad \mathbf{E} = \textcircled{\mathbf{f}_\delta}$$

We have thus:

$$\begin{aligned} (\mathbf{D}, \mathbf{E})_2 = & a_{\nu(1)} \begin{array}{c} \textcircled{\mathbf{f}_\alpha} \longleftarrow \textcircled{\mathbf{f}_\beta} \xrightarrow{2} \textcircled{\mathbf{f}_\gamma} \\ \quad \quad \quad \searrow^2 \\ \quad \quad \quad \textcircled{\mathbf{f}_\delta} \end{array} + a_{\nu(2)} \begin{array}{c} \textcircled{\mathbf{f}_\alpha} \longleftarrow \textcircled{\mathbf{f}_\beta} \xrightarrow{2} \textcircled{\mathbf{f}_\gamma} \\ \quad \quad \quad \downarrow^2 \\ \quad \quad \quad \textcircled{\mathbf{f}_\delta} \end{array} \\ & + a_{\nu(3)} \begin{array}{c} \textcircled{\mathbf{f}_\alpha} \longleftarrow \textcircled{\mathbf{f}_\beta} \xrightarrow{2} \textcircled{\mathbf{f}_\gamma} \\ \quad \quad \quad \downarrow^2 \\ \quad \quad \quad \textcircled{\mathbf{f}_\delta} \end{array} + a_{\nu(4)} \begin{array}{c} \textcircled{\mathbf{f}_\alpha} \longleftarrow \textcircled{\mathbf{f}_\beta} \xrightarrow{2} \textcircled{\mathbf{f}_\gamma} \\ \quad \quad \quad \downarrow^2 \\ \quad \quad \quad \textcircled{\mathbf{f}_\delta} \end{array} \\ & + a_{\nu(5)} \begin{array}{c} \textcircled{\mathbf{f}_\alpha} \longleftarrow \textcircled{\mathbf{f}_\beta} \xrightarrow{2} \textcircled{\mathbf{f}_\gamma} \\ \quad \quad \quad \downarrow^2 \quad \swarrow^2 \\ \quad \quad \quad \textcircled{\mathbf{f}_\delta} \end{array} + a_{\nu(6)} \begin{array}{c} \textcircled{\mathbf{f}_\alpha} \longleftarrow \textcircled{\mathbf{f}_\beta} \xrightarrow{2} \textcircled{\mathbf{f}_\gamma} \\ \quad \quad \quad \downarrow^2 \quad \swarrow^2 \\ \quad \quad \quad \textcircled{\mathbf{f}_\delta} \end{array} \end{aligned}$$

Definition 4.5. Given a molecular covariant \mathbf{D} , and an integer $k \geq 0$, we define¹⁰ $\overline{\mathbf{D}}^{\mu(k)}$ to be the molecular covariant obtained by adding k edges on \mathbf{D} in a certain way $\mu(k)$.

Example 4.6. Given atoms $\mathbf{f}_\alpha, \mathbf{f}_\beta, \mathbf{f}_\gamma$ of valence greater than 4 and the molecular covariant

$$\mathbf{D} = \begin{array}{c} \textcircled{\mathbf{f}_\alpha} \xrightarrow{2} \textcircled{\mathbf{f}_\beta} \\ \quad \quad \quad \searrow \\ \quad \quad \quad \textcircled{\mathbf{f}_\gamma} \end{array}$$

we can consider

$$\overline{\mathbf{D}}^{\mu_1(2)} = \begin{array}{c} \textcircled{\mathbf{f}_\alpha} \xrightarrow{3} \textcircled{\mathbf{f}_\beta} \\ \quad \quad \quad \searrow \quad \swarrow \\ \quad \quad \quad \textcircled{\mathbf{f}_\gamma} \end{array} \quad \text{or} \quad \overline{\mathbf{D}}^{\mu_2(2)} = \begin{array}{c} \textcircled{\mathbf{f}_\alpha} \xrightarrow{2} \textcircled{\mathbf{f}_\beta} \\ \quad \quad \quad \searrow^2 \quad \swarrow^2 \\ \quad \quad \quad \textcircled{\mathbf{f}_\gamma} \end{array}$$

⁸The covariant $\mathbf{M}^{\nu(r)}$ is called a *term* in [36].

⁹There is explicit expression of those coefficients in [49].

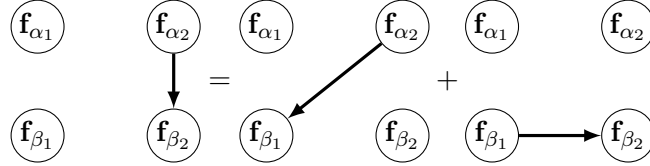
¹⁰This operation is called *convolution* in [36].

Proposition 4.7. *Let \mathbf{D}, \mathbf{E} be two molecular covariants and $r \geq 0$ be an integer. Then for every molecular covariant $\mathbf{M}^{\nu(r)}$ in the decomposition 4.1 of $(\mathbf{D}, \mathbf{E})_r$, we have*

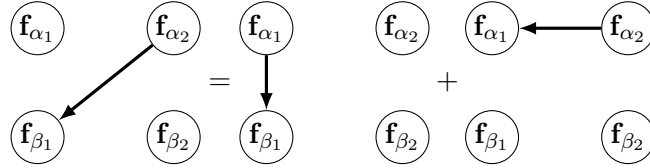
$$\mathbf{M}^{\nu(r)} = \lambda(\mathbf{D}, \mathbf{E})_r + \sum_{k_1, k_2, r'} \lambda_{k_1, k_2, r'} (\overline{\mathbf{D}}^{\mu_1(k_1)}, \overline{\mathbf{E}}^{\mu_2(k_2)})_{r'}, \quad \lambda > 0$$

with $k_1 + k_2 + r' = r$ being constant and $r' < r$.

Sketch of proof. This proof is based on induction on r . When $r = 1$ take a molecular covariant $\mathbf{M}^{\nu(1)}$ in $(\mathbf{D}, \mathbf{E})_1$. In this molecular covariant, there is a link between an atom \mathbf{f}_{α_1} in \mathbf{D} and an atom \mathbf{f}_{β_1} in \mathbf{E} . Let $\mathbf{M}^{\mu(1)}$ be another molecular covariant in $(\mathbf{D}, \mathbf{E})_1$, with a link between an atom $\mathbf{f}_{\alpha_2} \neq \mathbf{f}_{\alpha_1}$ in \mathbf{D} and an atom $\mathbf{f}_{\beta_1} \neq \mathbf{f}_{\beta_2}$ in \mathbf{E} . By relation 3.3 we have



where the last molecular covariant is a transvectant $(\mathbf{D}, \overline{\mathbf{E}}^1)_0$. By the same relation 3.3:

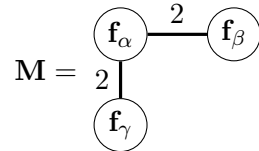


where the last molecular covariant is a transvectant $(\overline{\mathbf{D}}^1, \mathbf{E})_0$. Thus every molecular covariant of $(\mathbf{D}, \mathbf{E})_1$ is expressible in terms of $\mathbf{M}^{\nu(1)}$ and a linear combination of $(\overline{\mathbf{D}}^{a_1}, \overline{\mathbf{E}}^{a_2})_0$. All coefficients a_ν of 4.1 being positives, this concludes the case $r = 1$. \square

Example 4.8. Given $V = S_n$ ($n \geq 4$) and the molecular covariants:

$$\mathbf{D} = \left(\text{f}_\alpha \text{---}^2 \text{f}_\beta \right) \text{ and } \mathbf{E} = \left(\text{f}_\gamma \right)$$

we can consider the transvectant $(\mathbf{D}, \mathbf{E})_2$ and the molecular covariant:



By proposition 4.7:

$$\begin{aligned} \mathbf{M} &= \lambda \left(\left(\text{f}_\alpha \text{---}^2 \text{f}_\beta \right), \left(\text{f}_\gamma \right) \right)_2 + \lambda_1 \left(\left(\text{f}_\alpha \text{---}^3 \text{f}_\beta \right), \left(\text{f}_\beta \right) \right)_1 \\ &\quad + \lambda_2 \left(\left(\text{f}_\alpha \text{---}^4 \text{f}_\beta \right), \left(\text{f}_\gamma \right) \right)_0 \end{aligned}$$

Corollary 4.9. *Let V be a space of binary forms. Then every molecular covariant can be written in terms of transvectants.*

Proof. This is a simple induction on the number d of atoms in a molecular covariant \mathbf{M} . For $d = 1$, there is nothing to prove. Given $d > 1$, we can write \mathbf{M} as

$$\textcircled{\mathbf{D}} \xrightarrow{\nu(r)} \textcircled{\mathbf{f}}$$

where $\mathbf{f} \in V$, $r \geq 0$ and \mathbf{D} is a molecular covariant with $d - 1$ atoms. We conclude by induction and using proposition 4.3. \square

Remark 4.10. By corollary 4.9, we deduce theorem 3.3 from the fact that transvectants generate the covariant algebra of binary forms.

5. GORDAN'S ALGORITHM FOR JOINT COVARIANTS

Gordan's algorithm for joint covariants produces a finite generating set for $\mathbf{Cov}(V_1 \oplus V_2)$, knowing a finite family of generators for $\mathbf{Cov}(V_1)$ and $\mathbf{Cov}(V_2)$.

Let V_1 and V_2 be two spaces of binary forms and

$$\mathbf{A} := \{\mathbf{f}_1, \dots, \mathbf{f}_p\} \subset \mathbf{Cov}(V_1), \quad \mathbf{B} := \{\mathbf{g}_1, \dots, \mathbf{g}_q\} \subset \mathbf{Cov}(V_2),$$

be finite families of generators for $\mathbf{Cov}(V_1)$ and $\mathbf{Cov}(V_2)$ respectively.

Lemma 5.1. *$\mathbf{Cov}(V_1 \oplus V_2)$ is generated by transvectants*

$$(\mathbf{U}, \mathbf{V})_r,$$

with r non-negative integer and

$$\mathbf{U} := \mathbf{f}_1^{\alpha_1} \dots \mathbf{f}_p^{\alpha_p}, \quad \mathbf{V} := \mathbf{g}_1^{\beta_1} \dots \mathbf{g}_q^{\beta_q}, \quad \alpha_i, \beta_j \in \mathbb{N}$$

Proof. By theorem 3.3, $\mathbf{Cov}(V_1 \oplus V_2)$ is generated by molecular covariants \mathbf{M} with atoms in V_1 and V_2 . Just isolate in \mathbf{M} atoms $\mathbf{f}_i \in \mathbf{A}$ (resp. $\mathbf{g}_j \in \mathbf{B}$) to form a molecular covariant \mathbf{D} (resp. \mathbf{E}) such that

$$\mathbf{M} = \textcircled{\mathbf{D}} \xrightarrow{\nu(r)} \textcircled{\mathbf{E}}, \quad \mathbf{D} \in \mathbf{Cov}(V_1), \quad \mathbf{E} \in \mathbf{Cov}(V_2)$$

which is a molecular covariant $\mathbf{M}^{\nu(r)}$ in the decomposition of a transvectant $(\mathbf{D}, \mathbf{E})_r$. By proposition 4.7, $\mathbf{M}^{\nu(r)}$ is a linear combination of

$$(\mathbf{D}, \mathbf{E})_r \text{ and } (\overline{\mathbf{D}}^{\mu_1(k_1)}, \overline{\mathbf{E}}^{\mu_2(k_2)})_{r'},$$

with $\overline{\mathbf{D}}^{\mu_1(k_1)} \in \mathbf{Cov}(V_1)$ and $\overline{\mathbf{E}}^{\mu_2(k_2)} \in \mathbf{Cov}(V_2)$. By hypothesis, all covariants in $\mathbf{Cov}(V_1)$ (resp. $\mathbf{Cov}(V_2)$) can be recast using monomials in the \mathbf{f}'_i s (resp. \mathbf{g}'_j s). \square

Define a_i (resp. b_j) to be the order of the covariant \mathbf{f}_i (resp. \mathbf{g}_j). Now, to each non-vanishing transvectant

$$(\mathbf{U}, \mathbf{V})_r,$$

we can associate an integer solution $\kappa := (\boldsymbol{\alpha}, \boldsymbol{\beta}, u, v, r)$ of the linear Diophantine system

$$S(A, B) : \begin{cases} a_1\alpha_1 + \dots + a_p\alpha_p = u + r, \\ b_1\beta_1 + \dots + b_q\beta_q = v + r, \end{cases} \quad (5.1)$$

Conversely, to each integer solution κ of $S(A, B)$ we can associate a well defined transvectant $(\mathbf{U}, \mathbf{V})_r$. Recall an integer solution κ of $S(A, B)$ is *reducible* if we can decompose κ as a sum of non-trivial solutions. Recall also that a non constant covariant $\mathbf{h} \in \mathbf{Cov}_{d,k}(V)$ is said to be reducible if \mathbf{h} is in the algebra generated by covariants of degree $d' \leq d$ and order $k' \leq k$, with $d' < d$ or $k' < k$.

Lemma 5.2. *If $\kappa = (\boldsymbol{\alpha}, \boldsymbol{\beta}, u, v, r)$ is a reducible integer solution of $S(A, B)$, then there exists a reducible molecular covariant $\mathbf{M}^{\nu(r)}$ in the decomposition of $(\mathbf{U}, \mathbf{V})_r$.*

Proof. Take the integer solution $\kappa = \kappa_1 + \kappa_2$ to be reducible, with

$$\kappa_i = (\boldsymbol{\alpha}^i, \boldsymbol{\beta}^i, u^i, v^i, r^i) \text{ solution of (5.1).}$$

Thus we can write $\mathbf{U} = \mathbf{U}_1\mathbf{U}_2$ and $\mathbf{V} = \mathbf{V}_1\mathbf{V}_2$ and there exists $\nu(r), \nu_1(r^1)$ and $\nu_2(r^2)$ such that

$$\begin{array}{ccccccc} \textcircled{\mathbf{U}} & \xrightarrow{\nu(r)} & \textcircled{\mathbf{V}} & = & \textcircled{\mathbf{U}_1} & \xrightarrow{\nu_1(r^1)} & \textcircled{\mathbf{V}_1} & \textcircled{\mathbf{U}_2} & \xrightarrow{\nu_2(r^2)} & \textcircled{\mathbf{V}_2} \end{array}$$

which is a reducible molecular covariant in the decomposition of $(\mathbf{U}, \mathbf{V})_r$. \square

Remark 5.3. If an integer solution associated to a transvectant $(\mathbf{U}, \mathbf{V})_r$ is reducible, this does not imply that such a transvectant is a reducible one. Lemma 5.2 only states that such a transvectant can be decomposed in terms which *contain* a reducible transvectant. For instance, take $\mathbf{f} \in \mathbf{S}_6$, $A = B := \{\mathbf{f}\}$ and the transvectants

$$(\mathbf{f}^{\alpha_1}, \mathbf{f}^{\beta_1})_5.$$

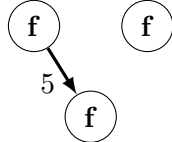
Then the solution $(\alpha_1, \beta_1, u, v, 5) = (2, 1, 7, 1, 5)$ is a reducible one:

$$(2, 1, 7, 1, 5) = (1, 1, 1, 1, 5) + (1, 0, 6, 0, 0)$$

We directly observe that the transvectant

$$(\mathbf{f}^2, \mathbf{f})_5 \quad (5.2)$$

contains the molecular covariant



which is a null covariant. Observe that property 4.7 implies that transvectant (5.2) is a linear combination of transvectants

$$((\mathbf{f}, \mathbf{f})_4, \mathbf{f})_1, \quad ((\mathbf{f}, \mathbf{f})_3, \mathbf{f})_2 = 0, \quad ((\mathbf{f}, \mathbf{f})_2, \mathbf{f})_3, \quad ((\mathbf{f}, \mathbf{f})_1, \mathbf{f})_4 = 0,$$

and one can finally show that

$$(\mathbf{f}^2, \mathbf{f})_5 = \frac{65}{66}((\mathbf{f}, \mathbf{f})_4, \mathbf{f})_1,$$

where $((\mathbf{f}, \mathbf{f})_4, \mathbf{f})_1$ is an irreducible covariant, as being in the covariant bases of S_6 (see table 8.1).

Nevertheless, we have the following result:

Lemma 5.4. *Let $\mathbf{a} := \max(a_i)$, $\mathbf{b} := \max(b_j)$ and*

$$\mathbf{U} := \mathbf{f}_1^{\alpha_1} \dots \mathbf{f}_p^{\alpha_p}, \quad \mathbf{V} := \mathbf{g}_1^{\beta_1} \dots \mathbf{g}_q^{\beta_q}.$$

Let $u = \text{Ord}(\mathbf{U}) - r$ and $v = \text{Ord}(\mathbf{V}) - r$. If

$$u + v \geq \mathbf{a} + \mathbf{b}, \tag{5.3}$$

then, the transvectant $(\mathbf{U}, \mathbf{V})_r$ is reducible.

Proof. Condition (5.3) implies that $u \geq \mathbf{a}$ or $v \geq \mathbf{b}$. Thus the transvectant $(\mathbf{U}, \mathbf{V})_r$ contains a reducible molecular covariant $\mathbf{M}^{\nu(r)}$ (the corresponding integer solution κ is thus not minimal). By virtue of proposition 4.7, this transvectant is a linear combination of the term $\mathbf{M}^{\nu(r)}$ and transvectants

$$(\overline{\mathbf{U}}^{\mu(k_1)}, \overline{\mathbf{U}}^{\mu(k_2)})_{r'},$$

where $r' < r$ and $k_1 + k_2 = r - r'$. Note that, since both families A and B are supposed to be generator sets, we have

$$\overline{\mathbf{U}}^{\mu(k_1)} = \mathbf{f}_1^{\alpha'_1} \dots \mathbf{f}_p^{\alpha'_p}, \quad \overline{\mathbf{V}}^{\mu(k_2)} = \mathbf{g}_1^{\beta'_1} \dots \mathbf{g}_q^{\beta'_q},$$

where, moreover, the order of the transvectant $(\overline{\mathbf{U}}^{\mu(k_1)}, \overline{\mathbf{V}}^{\mu(k_2)})_{r'}$ is $u' + v' = u + v$. Since we have supposed that $u + v \geq \mathbf{a} + \mathbf{b}$, we get that $u' + v' \geq \mathbf{a} + \mathbf{b}$ and the proof is achieved by a recursive argument on the index of the transvectant r . \square

Remark 5.5. The statement $u + v \geq \mathbf{a} + \mathbf{b}$ cannot be replaced by the weaker hypothesis $u \geq \mathbf{a}$ or $v \geq \mathbf{b}$. For instance in remark 5.3, for $\mathbf{f} \in S_6$ and $\mathbf{h} := (\mathbf{f}^2, \mathbf{f})_5$, we have $u = 7 \geq 6$ but \mathbf{h} is not reducible.

Lemma 5.4 is closely related to:

Corollary 5.6. *Let $\mathbf{F} \in \mathbf{Cov}(V)$ of order s and $\{\mathbf{F}_1, \dots, \mathbf{F}_k\} \subset \mathbf{Cov}(V)$ be a family of homogeneous covariants. Let t_i be the order of \mathbf{F}_i and $\mathbf{t} = \max(t_i)$. For a given integer r , if*

$$\sum_{i=1}^k t_i \geq \mathbf{a} + 2r,$$

then the transvectant $(\mathbf{F}_1 \dots \mathbf{F}_k, \mathbf{F})_r$ is reducible.

Proof. Let $\mathbf{f}_1, \dots, \mathbf{f}_p$ be a covariant bases of $\mathbf{Cov}(V)$, each \mathbf{f}_i 's being a homogeneous covariant of order a_i . Then, each covariant \mathbf{F}_j is a linear combination of monomials $\mathbf{f}_{i_1}^{\alpha_{i_1}} \dots \mathbf{f}_{i_l}^{\alpha_{i_l}}$ with $a_i \leq t_j \leq \mathbf{t}$. Thus $\mathbf{F}_1 \dots \mathbf{F}_k$ is a covariant expressible in terms of monomials \mathbf{U} in the \mathbf{f}_i 's with

$$\text{Ord}(\mathbf{U}) = \sum_{i=1}^k t_i \text{ and } \max(a_i) \leq \mathbf{t}.$$

We have also $\mathbf{F} = \mathbf{f}_{j_1}^{\beta_{j_1}} \dots \mathbf{f}_{j_m}^{\beta_{j_m}}$ with $\max(a_j) \leq s$. By lemma 5.4, each transvectant $(\mathbf{U}, \mathbf{V})_r$ is thus a reducible covariant. \square

Take back A and B to be finite generator sets of V_1 and V_2 , respectively. We know that there exists a finite family of *irreducible integer solutions* of the system $S(A, B)$ (5.1) (see [61, 60, 64] for details on linear Diophantine systems).

Theorem 5.7. *The algebra $\mathbf{Cov}(V_1 \oplus V_2)$ is generated by the finite family \mathcal{C} of transvectants*

$$(\mathbf{U}, \mathbf{V})_r$$

corresponding to irreducible solutions of the linear Diophantine system $S(A, B)$ (5.1).

Proof. Let first remark that each \mathbf{f}_i (resp. each \mathbf{g}_j) corresponds to an irreducible solution of $S(A, B)$. Thus $A \subset C$ and $B \subset C$.

From lemma 5.1, we know that $\mathbf{Cov}(V_1 \oplus V_2)$ is generated by transvectants $(\mathbf{U}, \mathbf{V})_r$ where \mathbf{U} (resp. \mathbf{V}) is a monomial in $\mathbb{C}[A]$ (resp. $\mathbb{C}[B]$) and r is a non-negative integer.

The proof is by induction on r . When $r = 0$, for $A \subset C$ and $B \subset C$, we know that the conclusion is true.

Now, let $r > 0$ and $(\mathbf{U}, \mathbf{V})_r$ be a transvectant which corresponds to a reducible integer solution¹¹

$$\kappa = \kappa_1 + \kappa_2, \quad \kappa_i \text{ irreducible.}$$

As in lemma 5.2, there exists a molecular covariant $\mathbf{M}^{\nu(r)}$ in the decomposition of $(\mathbf{U}, \mathbf{V})_r$ which can be written as

$$\mathbf{M}^{\nu(r)} = \mathbf{M}_1^{\nu_1(r_1)} \mathbf{M}_2^{\nu_2(r_2)}, \quad r_i \leq r.$$

But $\mathbf{M}_i^{\nu_i(r_i)}$ is a term in a transvectant $\tau^i = (\mathbf{U}_i, \mathbf{V}_i)_{r_i} \in C$. Then by proposition 4.7, $(\mathbf{U}, \mathbf{V})_r$ is a linear combination of a product of τ^i 's and transvectants

$$(\overline{\mathbf{U}}^\mu, \overline{\mathbf{V}}^{\mu'})_{r'}, (\overline{\mathbf{U}}_i^{\nu_i}, \overline{\mathbf{V}}_i^{\nu'_i})_{r'_i} \quad r' < r, r'_i < r \quad (5.4)$$

Now $\overline{\mathbf{U}}^\mu, \overline{\mathbf{U}}_i^{\nu_i} \in \mathbf{Cov}(V_1)$ (resp. \mathbf{V}, \mathbf{V}_i and $\mathbf{Cov}(V_2)$). Therefore the transvectants (5.4) are linear combinations of

$$(\mathbf{U}', \mathbf{V}')_{r''}, \quad r'' < r,$$

where \mathbf{U}' (resp. \mathbf{V}') is a monomial in the \mathbf{f}_i 's (resp. \mathbf{g}_j). Thus, by induction on r , the algebra $\mathbf{Cov}(V_1 \oplus V_2)$ is generated by the finite family \mathcal{C} . \square

Note that lemma 5.4 gives a bound on the order of each element of a minimal bases of joint covariants:

Corollary 5.8. *Let $V = S_{n_1} \oplus \dots \oplus S_{n_s}$. If μ_i is the maximal order of a minimal bases for S_{n_i} , then, for each element \mathbf{h} of a minimal bases for V , we get*

$$\text{ord}(\mathbf{h}) \leq \sum_{i=1}^s \mu_i.$$

Example 5.9. We can directly use theorem 5.7 to get a covariant bases of $S_3 \oplus S_4$. The same result has been obtained by Popoviciu–Brouwer [17] with more computations. Let $\mathbf{u} \in S_3$ and $\mathbf{v} \in S_4$. Recall that:

¹¹For simplicity, we can suppose that κ is the sum of two irreducible solutions.

- The algebra $\mathbf{Cov}(S_3)$ is generated by the three covariants [36]:

$$\mathbf{u} \in S_3, \quad \mathbf{h}_{2,2} := (\mathbf{u}, \mathbf{u})_2 \in S_2, \quad \mathbf{h}_{3,3} := (\mathbf{u}, \mathbf{h}_{2,2})_1 \in S_3$$

and one invariant $\Delta := (\mathbf{u}, \mathbf{h}_{3,3})_3$;

- The algebra $\mathbf{Cov}(S_4)$ is generated by the three covariants [36]:

$$\mathbf{v} \in S_4, \quad \mathbf{k}_{2,4} := (\mathbf{v}, \mathbf{v})_2 \in S_4, \quad \mathbf{k}_{3,6} := (\mathbf{v}, \mathbf{k}_{2,4})_1 \in S_6$$

and the two invariants $i := (\mathbf{v}, \mathbf{v})_4, j := (\mathbf{v}, \mathbf{k}_{2,4})_4$.

We then have to solve the linear Diophantine system

$$(S) : \begin{cases} 2\alpha_1 + 3\alpha_2 + 3\alpha_3 & = u + r \\ 4\beta_1 + 4\beta_2 + 6\beta_3 & = v + r \end{cases}. \quad (5.5)$$

Using Normaliz package in Macaulay 2 [18], this leads to 104 solutions. The associated covariants form a family of covariants of maximum total degree $d+k=18$. The Hilbert series of $\mathbf{Cov}(S_4 \oplus S_3)$, computed using Bedratyuk's Maple package [9], is given by

$$\begin{aligned} H(z) = & 1 + z^2 + 2z^3 + 5z^4 + 10z^5 + 18z^6 + 31z^7 + 55z^8 + 92z^9 \\ & + 144z^{10} + 223z^{11} + 341z^{12} + 499z^{13} + 725z^{14} + 1031z^{15} \\ & + 1436z^{16} + 1978z^{17} + 2685z^{18} + \dots \end{aligned}$$

By scripts written in Macaulay 2 [37], we reduced the family of 104 generators to a minimal set of 63 generators given in table 2, which has also been obtained by Popoviciu–Brouwer [17].

d/o	0	1	2	3	4	5	6	#	Cum
1	–	–	–	1	1	–	–	2	2
2	1	1	1	1	1	1	–	6	8
3	1	1	2	2	1	1	1	9	17
4	1	2	2	2	1	–	–	8	25
5	2	3	3	1	1	–	–	10	35
6	2	3	2	1	–	–	–	8	43
7	3	3	1	–	–	–	–	7	50
8	3	2	–	–	–	–	–	5	55
9	4	1	–	–	–	–	–	5	60
10	2	–	–	–	–	–	–	2	62
11	1	–	–	–	–	–	–	1	63
<i>Tot</i>	20	16	11	8	5	2	1		63

TABLE 2. Covariant bases of $S_3 \oplus S_4$

6. GORDAN'S ALGORITHM FOR SIMPLE COVARIANTS

There is a second version of Gordan's algorithm which enables to compute a covariant bases for S_n , knowing covariant basis for S_k , ($k < n$). The main idea is, once again, to make use of linear Diophantine system, arguing step by step modulus a chain ideal.

6.1. **Relatively complete family and Gordan's ideal.** Let

$$A = \{\mathbf{f}_1, \dots, \mathbf{f}_p\} \subset \mathbf{Cov}(S_n)$$

denote a *finite family* of covariants (not necessary a bases). We define $\mathbf{Cov}(A)$ to be the algebra generated by the set which contains the family A and closed under transvectant operations¹².

Definition 6.1. Let $I \subset \mathbf{Cov}(V)$ be an homogeneous ideal. A family $A = \{\mathbf{f}_1, \dots, \mathbf{f}_p\} \subset \mathbf{Cov}(V)$ of homogeneous covariants is *relatively complete modulo I* if every homogeneous covariant $\mathbf{h} \in \mathbf{Cov}(A)$ of degree d can be written

$$\mathbf{h} = \mathbf{p}(\mathbf{f}_1, \dots, \mathbf{f}_p) + \mathbf{h}_I \text{ with } \mathbf{h}_I \in I,$$

where $\mathbf{p}(\mathbf{f}_1, \dots, \mathbf{f}_p)$ and \mathbf{h}_I are degree d homogeneous covariants.

Remark 6.2. The notion of *relatively complete family* is weaker than the one of *generator set*. For instance, take $\mathbf{u} \in S_3$ and as in example 5.9

$$\mathbf{h}_{2,2} := (\mathbf{u}, \mathbf{u})_2 \in S_2, \quad \mathbf{h}_{3,3} := (\mathbf{u}, \mathbf{h}_{2,2})_1 \in S_3.$$

Take now the invariant $\tilde{\Delta} := (\mathbf{h}_{2,2}, \mathbf{h}_{2,2})_2$. The family $A_1 = \{\mathbf{u}, \mathbf{h}_{2,2}, \mathbf{h}_{3,3}, \tilde{\Delta}\}$ is also a covariant bases of $\mathbf{Cov}(A_1) = \mathbf{Cov}(S_3)$ and is thus a relatively complete family modulo $I = \{0\}$. Now, let

$$A_2 := \left\{ \mathbf{h}_{2,2}, \tilde{\Delta} \right\}.$$

We have $\mathbf{Cov}(A_2) \subsetneq \mathbf{Cov}(S_3)$, but A_2 is exactly a covariant bases [36] of the quadratic form $\mathbf{h}_{2,2} \in S_2$, thus A_2 is a relatively complete family modulo $I = \{0\}$ but is not a covariant bases of $\mathbf{Cov}(S_2)$.

Let now D be a molecule upon S_n (recall such a molecule represents an $SL(2, \mathbb{C})$ equivariant homomorphism), the *grade* of D , denoted $\text{gr}(D)$, is the maximum weight of the edges of D :

$$\text{gr}(D) := \max_{e \in \mathcal{E}(D)} w(e).$$

Definition 6.3. Let r be an integer; we define \mathcal{G}_r to be the set of all molecular covariants issued from a molecule D (cf. (3)) with grade at least r .

As a first observation, it is clear that $\mathcal{G}_r = \{0\}$ for $r > n$. Furthermore, we have

$$\mathcal{G}_{i+1} \subset \mathcal{G}_i \text{ for all } i. \tag{6.1}$$

Definition 6.4 (Gordan's ideals). Let r be an integer. We define the Gordan ideal I_r to be the homogeneous¹³ ideal generated by \mathcal{G}_r ; we write

$$I_r := \langle \mathcal{G}_r \rangle.$$

¹²Equivalently (by 4.3 and 4.7), we can choose the set of all molecular covariants with atoms taken in A .

¹³Such an ideal is clearly an homogeneous ideal as being generated by homogeneous elements.

Remark 6.5. Gordan's ideal I_r (for $r \leq n$) is also generated by the set of transvectants

$$(\mathbf{h}, (\mathbf{f}, \mathbf{f})_{r_1})_{r_2}, \quad r_1 \geq r, \quad r_2 \in \mathbb{N}^*,$$

where $\mathbf{h} \in \mathbf{Cov}(S_n)$ is an homogeneous covariant. This is a direct application of propositions 4.3 and 4.7. Because $(\mathbf{f}, \mathbf{f})_{r_1} = 0$ for r_1 odd, such a family can also be written as the family of transvectants

$$(\mathbf{h}, \mathbf{H}_{2k})_{r'}, \quad \mathbf{H}_{2k} := (\mathbf{f}, \mathbf{f})_{2k}, \quad 2k \geq r, \quad r' \geq 0.$$

We directly observe that:

- $I_r = \{0\}$ for all $r > n$;
- By equation 6.1, $I_{r+1} \subset I_r$ for every integer r .
- By remark 6.5:

$$I_{2k-1} = I_{2k}, \quad \forall 2k \leq n. \quad (6.2)$$

By the property 4.3, Gordan's ideals are stable by transvectant operations:

Lemma 6.6. *Let $\mathbf{h}_r \in I_r$, and $\mathbf{h} \in \mathbf{Cov}(S_n)$ be a covariant. Then for every integer $r' \geq 0$,*

$$(\mathbf{h}, \mathbf{h}_r)_{r'} \in I_r.$$

Remark 6.7. Suppose that $\Delta \in \mathbf{Cov}(S_n)$ is an invariant. Then the ideal $\langle \Delta \rangle$ is also stable by transvectant operations, since

$$(\mathbf{h}, \Delta \mathbf{k})_r = \Delta(\mathbf{h}, \mathbf{k})_r.$$

Given two finite families A and B of covariants, let $\kappa^1, \dots, \kappa^l$ be the irreducible integer solutions of the linear system $S(A, B)$ (5.1) and τ^i be the associated transvectants. Let $\mathbf{f} \in S_n$, $\Delta \in \mathbf{Cov}(S_n)$ be an invariant, $k \geq 0$ and

$$\mathbf{H}_{2k} := (\mathbf{f}, \mathbf{f})_{2k}.$$

Finally, write $J_{2k+2} = I_{2k+2}$ or $J_{2k+2} = I_{2k+2} + \langle \Delta \rangle$.

Theorem 6.8. *Suppose that A is relatively complete modulo I_{2k} and contains the binary form \mathbf{f} . Suppose also that B is relatively complete modulo J_{2k+2} and contains the covariant \mathbf{H}_{2k} . Then the family $C := \{\tau^1, \dots, \tau^l\}$ is relatively complete modulo J_{2k+2} and*

$$\mathbf{Cov}(C) = \mathbf{Cov}(A \cup B) = \mathbf{Cov}(S_n).$$

Proof. We first use the fact that $\mathbf{Cov}(A \cup B)$ is generated by the (infinite) family of transvectants

$$(\mathbf{D}, \mathbf{E})_r$$

where $\mathbf{D} \in \mathbf{Cov}(A)$ is a degree d homogeneous covariant of $\mathbf{f} \in S_n$ and $\mathbf{E} \in \mathbf{Cov}(B)$ (the proof is the same as the one for lemma 5.1). We order such a family using lexicographic order on (d, r) . By hypothesis, we can suppose that

$$\mathbf{D} = \mathbf{U} + \mathbf{h}_{2k}, \quad \mathbf{h}_{2k} \in I_{2k}$$

where \mathbf{U} is a monomial of degree d in $\mathbb{C}[A]$ and \mathbf{h}_{2k} is a homogeneous covariant of degree d . Furthermore we can write

$$\mathbf{E} = \mathbf{V} + \mathbf{h}_{2k+2}, \quad \mathbf{h}_{2k+2} \in J_{2k+2}.$$

Thus we have

$$(\mathbf{D}, \mathbf{E})_r = (\mathbf{U}, \mathbf{V})_r + (\mathbf{h}_{2k}, \mathbf{V})_r + \underbrace{(\mathbf{U}, \mathbf{h}_{2k+2})_r + (\mathbf{h}_{2k}, \mathbf{h}_{2k+2})_r}_{\in J_{2k+2}}. \quad (6.3)$$

The goal here is to prove that such a covariant can be written as

$$\mathbf{p}(C) + \mathbf{h}'_{2k+2}, \quad \mathbf{h}'_{2k+2} \in J_{2k+2}.$$

But in (6.3) we just have to focus on

$$(\mathbf{U}, \mathbf{V})_r + (\mathbf{h}_{2k}, \mathbf{V})_r.$$

We do it by induction on (d, r) . For $d = 0$, there is nothing to prove. Suppose now that our claim is true up to an integer d and take covariants

$$(\mathbf{U}, \mathbf{V})_r \text{ and } (\mathbf{h}_{2k}, \mathbf{V})_r$$

where \mathbf{U} and \mathbf{h}_{2k} are of degree $d + 1$.

- (1) If $(\mathbf{U}, \mathbf{V})_r$ corresponds to a reducible solution, using proposition 4.7 and the same argument as in the proof of theorem 5.7, this transvectant decomposes as

$$\mathbf{p}(C), \quad (\overline{\mathbf{U}}, \overline{\mathbf{V}})_{r' < r}.$$

But $\overline{\mathbf{U}}$ is of degree $d + 1$: we conclude using a direct induction on r' .

- (2) Using remark 6.5, the covariant \mathbf{h}_{2k} can be written as a linear combination of

$$(\mathbf{M}, \mathbf{H}_{2j})_{r_1}, \quad \mathbf{H}_{2j} := (\mathbf{f}, \mathbf{f})_{2j}, \quad j \geq k,$$

where the degree of \mathbf{M} in \mathbf{f} is strictly less than $d + 1$. The case $j > k$ being obvious, we only focus on the case $j = k$, and then consider transvectants

$$((\mathbf{M}, \mathbf{H}_{2k})_{r_1}, \mathbf{V})_r.$$

Using lemma B.1 on degree 3 covariant bases, such a covariant can be written as a linear combination of

$$(\mathbf{M}, (\mathbf{H}_{2k}, \mathbf{V})_{r'_1})_{r'}$$

but $(\mathbf{H}_{2k}, \mathbf{V})_{r'_1} \in \mathbf{Cov}(B)$, thus we have to consider transvectants

$$(\mathbf{M}, \mathbf{V}')_{r'}$$

where degree of \mathbf{M} in \mathbf{f} is strictly less than $d + 1$: we conclude using induction on d .

Thus, for all couple (d, r) , our claim is true. \square

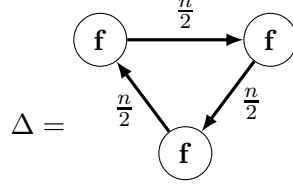
6.2. The algorithm. Take $V = S_n$ ($n > 2$) and $\mathbf{f} \in S_n$. By corollary C.1, the family $A_0 := \{\mathbf{f}\}$ is relatively complete modulo I_2 . This means that every covariant $\mathbf{h} \in \mathbf{Cov}(S_n)$ can be written as

$$\mathbf{h} = \mathbf{p}(\mathbf{f}) + \mathbf{h}_2 \text{ with } \mathbf{h}_2 \in I_2.$$

Take now the covariant $\mathbf{H}_2 = (\mathbf{f}, \mathbf{f})_2$ of order $2n - 4$:

- If $2n - 4 > n$, we take $B_0 := \{\mathbf{H}_2\}$ which is, by lemma C.3, relatively complete modulo I_4 ; applying theorem 6.8 we get a family $A_1 := C$ relatively complete modulo I_4 .

- If $2n - 4 = n$, we take $B_0 := \{\mathbf{H}_2, \Delta\}$ which is, by lemma C.4, relatively complete modulo $I_4 + \langle \Delta \rangle$; where Δ is the invariant

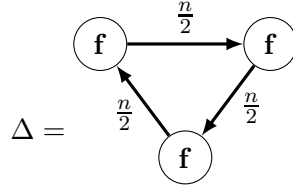


In that case, by applying theorem 6.8, we can take A_1 to be $C \cup \{\Delta\}$. A direct induction on the degree of the covariant shows that A_1 is relatively complete modulo I_4 .

- If $2n - 4 < n$, we suppose already known a covariant bases of S_{2n-4} ; we then take B_0 to be this bases, which is finite and relatively complete modulo I_4 (because relatively complete modulo $\{0\}$); we directly apply theorem 6.8 to get $A_1 := C$.

Let now be given by induction a family A_{k-1} containing \mathbf{f} , finite and relatively complete modulo I_{2k} . We consider the covariant $\mathbf{H}_{2k} = (\mathbf{f}, \mathbf{f})_{2k}$. Then:

- If \mathbf{H}_{2k} is of order $p > n$, we take $B_{k-1} := \{\mathbf{H}_{2k}\}$ which, by lemma C.3, is relatively complete modulo I_{2k+2} . By theorem 6.8 we take $A_k := C$.
- If \mathbf{H}_{2k} is of order $p = n$, we take $B_{k-1} := \{\mathbf{H}_{2k}, \Delta\}$ which, by lemma C.4, is relatively complete modulo $I_{2k+2} + \langle \Delta \rangle$; where Δ is the invariant



In that case, by applying theorem 6.8, we can take A_k to be $C \cup \{\Delta\}$. A direct induction on the degree of the covariant shows that A_k is relatively complete modulo I_{2k+2} .

- If \mathbf{H}_{2k} is of order $p < n$, we suppose already known a covariant bases of S_p ; we then take B_{k-1} to be this bases, which is relatively complete modulo I_{2k+2} (because relatively complete modulo $\{0\}$); we directly apply theorem 6.8 to get $A_k := C$.

Thus in each case, we have defined the family A_k . Now, depending on n 's parity:

- If $n = 2q$ is even, we know that the family A_{q-1} is relatively complete modulo I_{2q} ; furthermore the family B_{q-1} only contains the invariant $\Delta_q := \{\mathbf{f}, \mathbf{f}\}_{2q}$; finally we observe that A_p is given by

$$A_p := A_{p-1} \cup \{\Delta_q\}$$

and is relatively complete modulo $I_{2q+2} = \{0\}$; thus it is a covariant bases.

- If $n = 2q + 1$ is odd, the family B_{q-1} contains the quadratic form $\mathbf{H}_{2q} := \{\mathbf{f}, \mathbf{f}\}_{2q}$; we know then that the family B_{q-1} is given by the covariant \mathbf{H}_{2q} and the invariant $\delta_q := (\mathbf{H}_{2q}, \mathbf{H}_{2q})_2$. By theorem 6.8, the family $A_q := \mathbb{C}$ is relatively complete modulo $I_{2q+2} = \{0\}$ and is thus a covariant bases.

7. IMPROVMENT OF GORDAN'S ALGORITHM

Using Gordan's algorithm, one gets a finite set of generators. In general, such a family is not minimal, as shown in example 5.9 for the algebra $\mathbf{Cov}(S_3 \oplus S_4)$. A classical way to get a minimal bases once given a finite bases is to make use of *Hilbert series* [41] and then reduce the family degree per degree.

7.1. Hilbert series. Recall here that, for a \mathbb{C} graduated algebra of finite type

$$\mathcal{A} = \bigoplus_{k \geq 0} \mathcal{A}_k, \quad \mathcal{A}_0 := \mathbb{C}$$

where each homogeneous space \mathcal{A}_k is of finite dimension a_k . The Hilbert series associated to \mathcal{A} is the formal series

$$H_{\mathcal{A}}(z) := \sum_{k \geq 0} a_k z^k.$$

For covariant algebras, the Hilbert series can be computed *a priori*, using for example Bedratyuk's Maple package [9]. Suppose now that we know a finite bases \mathcal{F} (with homogeneous elements) of the algebra \mathcal{A} , which is not necessary minimal. Suppose also that we know a finite minimal family¹⁴ \mathcal{F}_k up to a degree k . To get up to degree $k + 1$:

- Compute a bases for the subspace of \mathcal{A}_{k+1} spanned by elements of \mathcal{F}_k ;
- If this dimension's subspace is strictly less than a_{k+1} , choose homogeneous elements in \mathcal{F} such that we get a subspace with dimension exactly a_{k+1} .

We obtain this way a finite minimal bases. But a major weakness of this strategy is that we work with homogeneous spaces which can be of huge dimensions. For instance, for the invariant algebra of $V = S_8 \oplus S_4 \oplus S_4$ (see subsection 8.5), Gordan's algorithm produces a degree 49 invariant, and for such a homogeneous space we have

$$\dim \mathbf{Cov}_{49,0}(V) = 103\,947\,673\,173.$$

which is far beyond our computation means.

To get one step further, we thus propose to use some algebraic tools to improve Gordan's algorithm. The main idea is to make use of *relations* on covariant algebras. Note here that this idea was suggested by R. Lercier.

¹⁴This means that for $k' < k$, we have $(\mathbb{C}[\mathcal{F}_k])_{k'} = \mathcal{A}_{k'}$ and if we take a strict subfamily $\mathcal{G} \subsetneq \mathcal{F}_k$ this property is no more true.

7.2. Relations on weighted monomials. Let

$$x_1 > x_2 > \dots > x_p$$

be indeterminates and $\mathcal{A} = \mathbb{C}[x_1, \dots, x_p]$ be a graduated algebra of finite type. Consider also the lexicographic order on monomials of \mathcal{A} . We write $\mathbf{m}_1 \mid \mathbf{m}_2$ whenever the monomial \mathbf{m}_1 divide monomial \mathbf{m}_2 .

Hypothesis 7.1. There exists a finite family $I \subset \{1, \dots, p-1\}$ of distinct integers and for each $i \in I$ a relation

$$(\mathcal{R}_i), \quad x_i^{a_i} = \sum_{k=0}^{a_i-1} x_i^k \mathbf{p}_k(x_{i+1}, \dots, x_p), \quad a_i \in \mathbb{N}^* \quad (7.1)$$

where \mathbf{p}_k is some polynomial. We write $\mathbf{m}_i := x_i^{a_i}$.

Lemma 7.2. *Under hypothesis 7.1, the algebra \mathcal{A} is generated by the family of monomials \mathbf{m} such that*

$$\mathbf{m}_i \nmid \mathbf{m}, \quad \forall i \in I$$

Proof. We first order the finite family $I = \{i_1, i_2, \dots, i_s\}$ such that

$$x_{i_1} > x_{i_2} > \dots$$

We then get a direct proof by induction on s . □

Hypothesis 7.3. There exists a finite family J and for each $j \in J$ a relation

$$(\mathcal{R}'_j), \quad x_{j_b}^{b_{j_b}} x_{j_c}^{c_{j_c}} = \mathbf{p}(x_{j_c+1}, \dots, x_p), \quad b_{j_b}, c_{j_c} \in \mathbb{N}^* \quad (7.2)$$

where $x_{j_b} > x_{j_c}$ and \mathbf{p} is some polynomial. We write $\mathbf{m}'_j := x_{j_b}^{b_{j_b}} x_{j_c}^{c_{j_c}}$.

Lemma 7.4. *Under hypotheses 7.1 and 7.3, the algebra \mathcal{A} is generated by the family of monomials \mathbf{m} such that*

$$\mathbf{m}_i \nmid \mathbf{m}, \quad \mathbf{m}'_j \nmid \mathbf{m}, \quad \forall i \in I, \quad \forall j \in J$$

Proof. We first order $J = \{j_1, \dots, j_l\}$ such that

$$\mathbf{m}'_{j_1} > \mathbf{m}'_{j_2} > \dots$$

From lemma 7.2 we can take a monomial \mathbf{m} such that $\mathbf{m}_i \nmid \mathbf{m}$ for all $i \in I$. Now Suppose that $\mathbf{m}'_j \mid \mathbf{m}$ for one given $j \in J$ which means that

$$\mathbf{m} = x_1^{r_1} \dots x_{j_b}^{r_{j_b}} \dots x_{j_c}^{r_{j_c}} \dots x_p^{r_p}, \quad r_{j_b} \geq b_{j_b}, \quad r_{j_c} \geq c_{j_c}.$$

Using relation \mathcal{R}'_j (7.2) we then have

$$\mathbf{m} = x_1^{r_1} \dots x_{j_b}^{r'_{j_b}} \dots x_{j_c}^{r'_{j_c}} \mathbf{p}(x_{j_c+1}, \dots, x_p), \quad r'_{j_b} < b_{j_b} \text{ or } r'_{j_c} < c_{j_c}.$$

We can also suppose that no monomial in $\mathbf{p}(x_{j_c+1}, \dots, x_p)$ is divided by \mathbf{m}_i , otherwise we use some relation \mathcal{R}_i (7.1).

Suppose now that the lemma is true for a given family $J = \{j_1, \dots, j_l\}$. Take $j > j_l$ and

$$(\mathcal{R}'_j), \quad x_{j_b}^{b_{j_b}} x_{j_c}^{c_{j_c}} = \mathbf{p}(x_{j_c+1}, \dots, x_p), \quad b_{j_b}, c_{j_c} \in \mathbb{N}^*.$$

Let

$$\mathbf{m} = x_1^{r_1} \dots x_{j_b}^{r_{j_b}} \dots x_{j_c}^{r_{j_c}} \dots x_p^{r_p}, \quad r_{j_b} \geq b_{j_b}, \quad r_{j_c} \geq c_{j_c}$$

such that $\mathbf{m}_i \nmid \mathbf{m}$ for all $i \in I$ and $\mathbf{m}'_{j_t} \nmid \mathbf{m}$ for all $j_t \in J$. Using relation \mathcal{R}'_j , we decompose \mathbf{m} in monomials

$$\mathbf{n} = x_1^{r_1} \dots x_{j_b}^{r'_{j_b}} \dots x_{j_c}^{r'_{j_c}} x_{j_c+1}^{r'_{j_c+1}} \dots x_p^{r'_p} \quad r'_{j_b} < b_{j_b} \text{ or } r'_{j_c} < c_{j_c}.$$

We also know that $r'_{j_b} \leq r_{j_b}$ and $r'_{j_c} \leq r_{j_c}$.

Using lemma 7.2, we can suppose that no any monomial \mathbf{m}_i divides \mathbf{n} . If now some monomial \mathbf{m}'_k divide \mathbf{m}' , we must have $k_c \geq j_c + 1$. We thus conclude using relation \mathcal{R}'_k and lemma 7.2. \square

7.2.1. *Application to joint covariant algorithm.* Let $A := \{\mathbf{f}_1, \dots, \mathbf{f}_p\}$ (resp. $B := \{\mathbf{g}_1, \dots, \mathbf{g}_q\}$) be a covariant bases of S_m (resp. S_n). We introduce an order

$$\mathbf{f}_p < \mathbf{f}_{p-1} < \dots < \mathbf{f}_1$$

on the covariants $\mathbf{f}_i \in A$ and we define a lexicographic order on $\mathbf{Cov}(S_m) = \mathbb{C}[A]$. We also suppose that there exists relations \mathcal{R}_i and \mathcal{R}'_j as in hypotheses 7.1 and 7.3.

Example 7.5. As noted in example 5.9 the algebra $\mathbf{Cov}(S_4)$ is generated by $\mathbf{v}, \mathbf{k}_{2,4}, \mathbf{k}_{3,6}$ and the two invariants i, j (where $\mathbf{v} \in S_4$). Let

$$\mathbf{k}_{3,6} > \mathbf{k}_{2,4} > \mathbf{v} > i > j$$

be an order on $\mathbf{Cov}(S_4)$. In that case we have one relation (obtained by a direct computation)

$$\mathcal{R}_1 : \quad 12\mathbf{k}_{3,6}^2 = -6\mathbf{k}_{2,4}^3 - 2j\mathbf{v}^3 + 3i\mathbf{v}^2\mathbf{k}_{2,4}. \quad (7.3)$$

Recall now that theorem 5.7 applied to $\mathbf{Cov}(S_m \oplus S_n)$ gives us a finite bases C of transvectants

$$(\mathbf{U}, \mathbf{V})_r$$

related to irreducible solutions (α, β, u, v, r) of the Diophantine system $S(A, B)$ (5.1).

Theorem 7.6. *The algebra $\mathbf{Cov}(S_m \oplus S_n)$ is generated by the finite sub-family of C*

$$(\tilde{\mathbf{U}}, \mathbf{V})_r \in C, \quad \mathbf{m}_i \nmid \tilde{\mathbf{U}}, \quad \mathbf{m}'_j \nmid \tilde{\mathbf{U}}, \quad \forall i \in I, \quad \forall j \in J.$$

Proof. Take one transvectant

$$(\mathbf{U}, \mathbf{V})_r$$

of the family C . Using lemma 7.4, we can write the monomial $\mathbf{U} \in \mathbb{C}[A]$ as a linear combination of

$$\tilde{\mathbf{U}}, \quad \mathbf{m}_i \nmid \tilde{\mathbf{U}}, \quad \mathbf{m}'_j \nmid \tilde{\mathbf{U}}, \quad \forall i \in I, \quad \forall j \in J$$

As in the proof of theorem 5.7, if the transvectant

$$(\tilde{\mathbf{U}}, \mathbf{V})_r$$

corresponds to a reducible solution $\kappa = \kappa_1 + \kappa_2$ of the Diophantine system $S(A, B)$ (5.1), it will decompose as transvectants $(\mathbf{U}', \mathbf{V})_{r_i}$ corresponding to irreducible solution κ_i and transvectants of strictly lower indexes. But we necessary have

$$\mathbf{m}_i \nmid \mathbf{U}', \quad \mathbf{m}'_j \nmid \mathbf{U}', \quad \forall i \in I, \quad \forall j \in J$$

and thus we conclude using a direct induction on r . \square

Suppose now that we also have hypotheses 7.1 and 7.3 for the algebra $\mathbf{Cov}(S_n)$, which leads to relations $\overline{\mathcal{R}}_k$ ($k \in K$) and $\overline{\mathcal{R}}'_l$ ($l \in L$), specified on monomials $\overline{\mathbf{m}}_k$ and $\overline{\mathbf{m}}'_l$. The finite family \mathbf{C} denote once again the finite family of transvectants

$$(\mathbf{U}, \mathbf{V})_r$$

related to irreducible solutions (α, β, u, v, r) of the Diophantine system $S(A, B)$ (5.1).

The same proof as in theorem 7.6 now leads to the more general result:

Theorem 7.7. *The algebra $\mathbf{Cov}(S_m \oplus S_n)$ is generated by the finite subfamily of \mathbf{C}*

$$(\tilde{\mathbf{U}}, \tilde{\mathbf{V}})_r \in \mathbf{C}, \quad \begin{cases} \mathbf{m}_i \nmid \tilde{\mathbf{U}}, & \mathbf{m}'_j \nmid \tilde{\mathbf{U}}, & \forall i \in I, & \forall j \in J \\ \overline{\mathbf{m}}_k \nmid \tilde{\mathbf{V}}, & \overline{\mathbf{m}}'_l \nmid \tilde{\mathbf{V}}, & \forall k \in K, & \forall l \in L \end{cases}.$$

7.2.2. *Application to simple covariant algorithm.* Let take the case when $V = S_n$. Recall that in that case Gordan's algorithm deals with families A_0, B_0, \dots (see subsection 6.2). Consider the case when the family B_{k-1} is one covariant bases of the binary form

$$\mathbf{H}_{2k} = (\mathbf{f}, \mathbf{f})_{2k}.$$

In that case \mathbf{H}_{2k} is of order $p < n$ and we suppose known that the related covariant basis. As for theorem 6.8, write $\Delta \in \mathbf{Cov}(S_n)$ to be an invariant and $J_{2k+2} = I_{2k+2}$ or $J_{2k+2} = I_{2k+2} + \langle \Delta \rangle$. Write $A := A_{k-1}$, $B := B_{k-1}$ and note \mathbf{C} to be the finite family of transvectants

$$(\mathbf{U}, \mathbf{V})_r$$

related to irreducible solutions (α, β, u, v, r) of the Diophantine system $S(A, B)$ (5.1).

Finally, suppose that we have hypotheses 7.1 and 7.3 on the bases B of the algebra $\mathbf{Cov}(S_p)$, with relations \mathcal{R}_i ($i \in I$) and \mathcal{R}'_j ($j \in J$) on monomials \mathbf{m}_i and \mathbf{m}'_j . In that case we have:

Theorem 7.8. *The subfamily $\tilde{\mathbf{C}}$ of \mathbf{C} given by*

$$(\mathbf{U}, \tilde{\mathbf{V}})_r \in \mathbf{C}, \quad \mathbf{m}_i \nmid \tilde{\mathbf{V}}, \quad \mathbf{m}'_j \nmid \tilde{\mathbf{V}}, \quad \forall i \in I, \quad \forall j \in J.$$

is relatively complete modulo J_{2k+2} and

$$\mathbf{Cov}(\tilde{\mathbf{C}}) = \mathbf{Cov}(A \cup B) = \mathbf{Cov}(S_n).$$

Proof. We just take back the proof of theorem 6.8 and replace every monomials \mathbf{V} with monomials $\tilde{\mathbf{V}}$; we then make use of the same ideas as in the proof of theorem 7.6. \square

7.3. Invariant's ideal and covariant relations.

Lemma 7.9. *Let $\mathbf{U} \in \mathbf{Cov}(S_m)$ be a covariant. If \mathbf{U} is in the ideal generated by invariants of S_m then every covariant*

$$(\mathbf{U}, \mathbf{V})_r, \quad r \geq 0, \quad \mathbf{V} \in S_n$$

is reducible.

Proof. We just observe that if $\mathbf{U} = \Delta \mathbf{U}'$, where $\Delta \in \mathbf{Inv}(S_m)$ and $\mathbf{U}' \in \mathbf{Cov}(S_m)$ then

$$(\Delta \mathbf{U}', \mathbf{V})_r = \Delta(\mathbf{U}', \mathbf{V})_r$$

is reducible. \square

Now, in a more general case, we take back two families A, B and the family C of transvectants

$$(\mathbf{U}, \mathbf{V})_r$$

related to irreducible solutions of the Diophantine system $S(A, B)$ (5.1).

Lemma 7.10. *Let $\hat{\mathbf{U}} = \hat{\mathbf{U}}_1 + \hat{\mathbf{U}}_2$ be monomial covariants in $\mathbf{Cov}(S_m)$, \mathbf{V} a monomial covariant in S_n and suppose that the transvectants*

$$(\hat{\mathbf{U}}_1, \mathbf{V})_r, \quad (\hat{\mathbf{U}}_2, \mathbf{V})_r$$

correspond to reducible integer solutions of the linear Diophantine system (5.1). Then the transvectant

$$(\hat{\mathbf{U}}, \mathbf{V})_r$$

is expressible in terms of transvectants of the family C and of transvectants

$$(\mathbf{U}', \mathbf{V}')_{r'}, \quad r' < r$$

where \mathbf{U}' (resp. \mathbf{V}') is a monomial in $\mathbb{C}[A]$ (resp. $\mathbb{C}[B]$).

Proof. This is a direct application of proposition 4.7 and lemma 5.2. \square

8. EFFECTIVE COMPUTATIONS

8.1. Covariant bases of $S_6 \oplus S_2$. There is a simple procedure to produce a covariant bases of $V \oplus S_2$ once we know a covariant bases of V , as detailed in the following theorem, which proof can be found in [36].

Theorem 8.1. *Let $\{\mathbf{h}_1, \dots, \mathbf{h}_s\}$ be a covariant bases of $\mathbf{Cov}(V)$, and let $\mathbf{u} \in S_2$. Then irreducible covariants of $\mathbf{Cov}(V \oplus S_2)$ are taken from the sets:*

- $\{\mathbf{h}_i, \mathbf{u}^r\}_{2r-1}$ for $i = 1 \dots s$;
- $\{\mathbf{h}_i, \mathbf{u}^r\}_{2r}$ for $i = 1 \dots s$;
- $\{\mathbf{h}_i \mathbf{h}_j, \mathbf{u}^r\}_{2r}$ where \mathbf{h}_i is of order $2p+1$ and \mathbf{h}_j is of order $2r-2p-1$.

Write now $\mathbf{h} = \mathbf{h}_{d,k}$ to be a covariant of degree d and order k , taken from the covariant bases of S_6 in table 8.1, issued from Grace–Young [36], and \mathbf{u} to be a quadratic form in S_2 . By theorem 8.1 we only have to consider covariants given by

$$\{\mathbf{h}, \mathbf{u}^r\}_{2r-1} \text{ or } \{\mathbf{h}, \mathbf{u}^r\}_{2r}.$$

Recall the covariant algebra $\mathbf{Cov}(V) := \mathbf{Cov}(S_6 \oplus S_2)$ is a multi-graded algebra:

$$\mathbf{Cov}(V) = \bigoplus_{d_1 \geq 0, d_2 \geq 0, k \geq 0} \mathbf{Cov}(V)_{d_1, d_2, k}.$$

where d_1 is the degree in the binary form $\mathbf{f} \in S_6$, d_2 is the degree in the binary form $\mathbf{u} \in S_2$ and k the degree in the variable $\mathbf{x} \in \mathbb{C}^2$. We can define the Hilbert series:

$$\mathcal{H}_{6,2}(z_1, z_2, t) := \sum_{d_1, d_2, k} \dim(\mathbf{Cov}(V)_{d_1, d_2, k}) z_1^{d_1} z_2^{d_2} t^k,$$

d/k	0	2	4	6
1				\mathbf{f}
2	$(\mathbf{f}, \mathbf{f})_6$		$\mathbf{h}_{2,4} := (\mathbf{f}, \mathbf{f})_4$	
3		$\mathbf{h}_{3,2} := (\mathbf{h}_{2,4}, \mathbf{f})_4$		$\mathbf{h}_{3,6} := (\mathbf{h}_{2,4}, \mathbf{f})_2$
4	$(\mathbf{h}_{2,4}, \mathbf{h}_{2,4})_4$		$(\mathbf{h}_{3,2}, \mathbf{f})_2$	$\mathbf{h}_{4,6} := (\mathbf{h}_{3,2}, \mathbf{f})_1$
5		$(\mathbf{h}_{2,4}, \mathbf{h}_{3,2})_2$	$(\mathbf{h}_{2,4}, \mathbf{h}_{3,2})_1$	
6	$(\mathbf{h}_{3,2}, \mathbf{h}_{3,2})_2$			$\mathbf{h}_{6,61} := (\mathbf{h}_{3,8}, \mathbf{h}_{3,2})_2$ $\mathbf{h}_{6,62} := (\mathbf{h}_{3,6}, \mathbf{h}_{3,2})_1$
7		$(\mathbf{f}, \mathbf{h}_{3,2}^2)_4$	$(\mathbf{f}, \mathbf{h}_{3,2}^2)_3$	
8		$(\mathbf{h}_{2,4}, \mathbf{h}_{3,2}^2)_3$		
9			$(\mathbf{h}_{3,8}, \mathbf{h}_{3,2}^2)_4$	
10	$(\mathbf{h}_{3,2}^3, \mathbf{f})_6$	$(\mathbf{h}_{3,2}^3, \mathbf{f})_5$		
12		$(\mathbf{h}_{3,8}, \mathbf{h}_{3,2}^3)_6$		
15	$(\mathbf{h}_{3,8}, \mathbf{h}_{3,2}^4)_8$			

d/k	8	10	12
2	$\mathbf{h}_{2,8} := (\mathbf{f}, \mathbf{f})_2$		
3	$\mathbf{h}_{3,8} := (\mathbf{h}_{2,4}, \mathbf{f})_1$		$(\mathbf{h}_{2,8}, \mathbf{f})_1$
4		$(\mathbf{h}_{2,8}, \mathbf{h}_{2,4})_1$	
5	$\mathbf{h}_{5,8} := (\mathbf{h}_{2,8}, \mathbf{h}_{3,2})_1$		

TABLE 3. Covariant bases of S_6

which has been computed using Bedratyuk's Maple package [9]. From this Hilbert series and theorem 8.1, we finally get a minimal bases of 99 covariants, already obtained by von Gall [65]. It's worth noting that, by using theorem 8.1, we only had to check invariant homogeneous space's dimensions up to degree 15. Results are summarized in table 4.

d/o	0	2	4	6	8	10	12	#	Cum
1	—	1	—	1	—	—	—	2	2
2	2	—	2	1	1	—	—	6	8
3	—	3	2	2	2	—	1	10	18
4	4	3	3	4	—	2	—	16	34
5	—	4	6	—	3	—	—	13	47
6	5	7	—	5	—	—	—	17	64
7	3	1	6	—	—	—	—	10	74
8	1	8	—	—	—	—	—	9	83
9	7	—	1	—	—	—	—	8	91
10	1	2	—	—	—	—	—	3	94
11	2	—	—	—	—	—	—	2	96
12	—	1	—	—	—	—	—	1	97
13	1	—	—	—	—	—	—	1	98
14	—	—	—	—	—	—	—	—	98
15	1	—	—	—	—	—	—	1	99
Tot	27	30	20	13	6	2	1		99

TABLE 4. Minimal covariant bases of $S_6 \oplus S_2$.

Ordre 0 : 5 invariants from S_6 , 1 invariant from S_2 and 21 joint invariants.	
Degree 2	$(\mathbf{f}, \mathbf{f})_6$ $(\mathbf{u}, \mathbf{u})_2$
Degree 4	$(\mathbf{h}_{1,6}, \mathbf{u}^3)_6$ $(\mathbf{h}_{2,4}, \mathbf{u}^2)_4$ $(\mathbf{h}_{3,2}, \mathbf{u})_2$
Degree 6	$(\mathbf{h}_{3,6}, \mathbf{u}^3)_6$ $(\mathbf{h}_{2,8}, \mathbf{u}^4)_8$ $(\mathbf{h}_{4,4}, \mathbf{u}^2)_4$ $(\mathbf{h}_{5,2}, \mathbf{u})_2$
Degree 7	$(\mathbf{h}_{5,4}, \mathbf{u}^2)_4$ $(\mathbf{h}_{3,8}, \mathbf{u}^4)_8$ $(\mathbf{h}_{4,6}, \mathbf{u}^3)_6$
Degree 8	$(\mathbf{h}_{7,2}, \mathbf{u})_2$
Degree 9	$(\mathbf{h}_{7,4}, \mathbf{u}^2)_4$ $(\mathbf{h}_{6,61}, \mathbf{u}^3)_6$ $(\mathbf{h}_{4,10}, \mathbf{u}^5)_{10}$
	$(\mathbf{h}_{5,8}, \mathbf{u}^4)_8$ $(\mathbf{h}_{8,2}, \mathbf{u})_2$ $(\mathbf{h}_{3,12}, \mathbf{u}^6)_{12}$ $(\mathbf{h}_{6,62}, \mathbf{u}^3)_6$
Degree 10	$(\mathbf{h}_{3,2}^3, \mathbf{f})_6$
Degree 11	$(\mathbf{h}_{10,2}, \mathbf{u})_2$ $(\mathbf{h}_{9,4}, \mathbf{u}^2)_4$
Degree 13	$(\mathbf{h}_{12,2}, \mathbf{u})_2$
Degree 14	$(\mathbf{h}_{3,2}^4, \mathbf{h}_{3,8})_8$
Order 2 : 1 from S_2 , 6 from S_6 and 23 joint covariants.	
Degree 1	\mathbf{u}
Degree 3	$\mathbf{h}_{3,2}$ $(\mathbf{f}, \mathbf{u}^2)_4$ $(\mathbf{h}_{2,4}, \mathbf{u})_2$
Degree 4	$(\mathbf{h}_{2,4}, \mathbf{u}^2)_3$ $(\mathbf{h}_{3,2}, \mathbf{u})_1$ $(\mathbf{f}, \mathbf{u}^3)_5$
Degree 5	$\mathbf{h}_{5,2}$ $(\mathbf{h}_{2,8}, \mathbf{u}^3)_6$ $(\mathbf{h}_{4,4}, \mathbf{u})_2$ $(\mathbf{h}_{3,6}, \mathbf{u}^2)_4$
Degree 6	$(\mathbf{h}_{2,8}, \mathbf{u}^4)_7$ $(\mathbf{h}_{4,4}, \mathbf{u}^2)_3$ $(\mathbf{h}_{5,2}, \mathbf{u})_1$ $(\mathbf{h}_{3,6}, \mathbf{u}^3)_5$
	$(\mathbf{h}_{5,4}, \mathbf{u})_2$ $(\mathbf{h}_{3,8}, \mathbf{u}^3)_6$ $(\mathbf{h}_{4,6}, \mathbf{u}^2)_4$
Degree 7	$\mathbf{h}_{7,2}$
Degree 8	$\mathbf{h}_{8,2}$ $(\mathbf{h}_{7,2}, \mathbf{u})_1$ $(\mathbf{h}_{7,4}, \mathbf{u})_2$ $(\mathbf{h}_{6,6b}, \mathbf{u}^2)_4$
	$(\mathbf{h}_{5,8}, \mathbf{u}^3)_6$ $(\mathbf{h}_{3,12}, \mathbf{u}^5)_{10}$ $(\mathbf{h}_{4,10}, \mathbf{u}^4)_8$ $(\mathbf{h}_{6,6a}, \mathbf{u}^2)_4$
Degree 10	$\mathbf{h}_{10,2}$ $(\mathbf{h}_{9,4}, \mathbf{u})_2$
Degree 12	$\mathbf{h}_{12,2}$
Order 4 : 5 covariants from S_6 and 15 joint covariants.	
Degree 2	$\mathbf{h}_{2,4}$ $(\mathbf{f}, \mathbf{u})_2$
Degree 3	$(\mathbf{h}_{2,4}, \mathbf{u})_1$ $(\mathbf{f}, \mathbf{u}^2)_3$
Degree 4	$\mathbf{h}_{4,4}$ $(\mathbf{h}_{3,6}, \mathbf{u})_2$ $(\mathbf{h}_{2,8}, \mathbf{u}^2)_4$
Degree 5	$\mathbf{h}_{5,4}$ $(\mathbf{h}_{3,8}, \mathbf{u}^2)_4$ $(\mathbf{h}_{3,6}, \mathbf{u}^2)_3$ $(\mathbf{h}_{4,4}, \mathbf{u})_1$ $(\mathbf{h}_{4,6}, \mathbf{u})_2$ $(\mathbf{h}_{2,8}, \mathbf{u}^3)_5$
Degree 7	$\mathbf{h}_{7,4}$ $(\mathbf{h}_{6,61}, \mathbf{u})_2$ $(\mathbf{h}_{3,12}, \mathbf{u}^4)_8$ $(\mathbf{h}_{4,10}, \mathbf{u}^3)_6$ $(\mathbf{h}_{6,62}, \mathbf{u})_2$ $(\mathbf{h}_{5,8}, \mathbf{u}^2)_4$
Degree 9	$\mathbf{h}_{9,4}$
Order 6 : 5 covariants from S_6 and 8 joint covariants.	
Degree 1	\mathbf{f}
Degree 2	$(\mathbf{f}, \mathbf{u})_1$
Degree 3	$\mathbf{h}_{3,6}$ $(\mathbf{h}_{2,8}, \mathbf{u})_2$

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Degree 4	$\mathbf{h}_{4,6}$	$(\mathbf{h}_{2,8}, \mathbf{u}^2)_3$	$(\mathbf{h}_{3,6}, \mathbf{u})_1$	$(\mathbf{h}_{3,8}, \mathbf{u})_2$	
Degree 7	$\mathbf{h}_{6,61}$	$\mathbf{h}_{6,62}$	$(\mathbf{h}_{5,8}, \mathbf{u})_2$	$(\mathbf{h}_{4,10}, \mathbf{u}^2)_4$	$(\mathbf{h}_{3,12}, \mathbf{u}^3)_6$
Order 8 : 3 covariants from S_6 and 3 joint covariants.					
Degree 2	$\mathbf{h}_{2,8}$				
Degree 3	$\mathbf{h}_{3,8}$	$(\mathbf{h}_{2,8}, \mathbf{u})_1$			
Degree 5	$\mathbf{h}_{5,8}$	$(\mathbf{h}_{4,10}, \mathbf{u})_2$	$(\mathbf{h}_{3,12}, \mathbf{u}^2)_4$		
Order 10 : 1 covariant from S_6 and 1 joint covariant.					
Degree 3	$\mathbf{h}_{4,10}$	$(\mathbf{h}_{3,12}, \mathbf{u})_2$			
Order 12 : 1 covariant from S_6 .					
Degree 3	$\mathbf{h}_{3,12}$				

8.2. Covariant bases of $S_6 \oplus S_4$. Taking $\mathbf{f} \in S_6$ and $\mathbf{v} \in S_4$ we take generators $\mathbf{h}_{d,k}$ of $\mathbf{Cov}(S_6)$ given by 8.1 and we write

$$\begin{aligned} \mathbf{v} \in S_4, \quad \mathbf{k}_{2,4} &:= (\mathbf{v}, \mathbf{v})_2, \quad \mathbf{k}_{3,6} := (\mathbf{v}, \mathbf{k}_{2,4})_1, \\ i &:= (\mathbf{v}, \mathbf{v})_4, \quad j := (\mathbf{v}, \mathbf{k}_{2,4})_4. \end{aligned}$$

We already know one relation on $\mathbf{Cov}(S_4)$ (see example 7.5):

$$\mathcal{R}_1 : \quad 12\mathbf{k}_{3,6}^2 = -6\mathbf{k}_{2,4}^3 - 2j\mathbf{v}^3 + 3iv^2\mathbf{k}_{2,4}. \quad (8.1)$$

and thus we have hypothesis 7.1 on that algebra.

Let put on $\mathbf{Cov}(S_6)$ the order

$$\begin{aligned} \mathbf{h}_{2,0} &> \mathbf{h}_{4,0} > \mathbf{h}_{6,0} > \mathbf{h}_{10,0} > \mathbf{h}_{15,0} > \mathbf{h}_{3,2} > \mathbf{h}_{2,4} > \mathbf{h}_{4,4} > \mathbf{f} > \mathbf{h}_{3,6} > \\ \mathbf{h}_{4,6} &> \mathbf{h}_{5,4} > \mathbf{h}_{2,8} > \mathbf{h}_{6,61} > \mathbf{h}_{6,62} > \mathbf{h}_{3,8} > \mathbf{h}_{7,4} > \mathbf{h}_{5,2} > \mathbf{h}_{7,2} > \\ \mathbf{h}_{9,4} &> \mathbf{h}_{12,2} > \mathbf{h}_{10,2} > \mathbf{h}_{8,2} > \mathbf{h}_{5,8} > \mathbf{h}_{4,10} > \mathbf{h}_{3,12}. \end{aligned}$$

Lemma 8.2. *There exists, in the algebra $\mathbf{Cov}(S_6)$, 12 relations as in hypothesis 7.1 and one relation as in hypothesis 7.3. The monomials \mathbf{m}_i and \mathbf{m}_j^l occurring in those relations are*

$$\begin{aligned} \mathbf{h}_{12,2}^2, \quad \mathbf{h}_{10,2}^2, \quad \mathbf{h}_{8,2}^2, \quad \mathbf{h}_{7,2}^2, \quad \mathbf{h}_{9,4}^2, \quad \mathbf{h}_{7,4}^2, \quad \mathbf{h}_{5,4}^2 \\ \mathbf{h}_{6,61}^2, \quad \mathbf{h}_{6,62}^2, \quad \mathbf{h}_{3,12}^2, \quad \mathbf{h}_{5,8}^2, \quad \mathbf{h}_{4,10}^2, \quad \mathbf{h}_{12,2}\mathbf{h}_{10,2}. \end{aligned}$$

Proof. To get one relation \mathcal{R} on a monomial \mathbf{m} , we consider the homogeneous space $\mathbf{Cov}_{d,k}$ associated to the monomial \mathbf{m} . Take for instance the monomial $\mathbf{h}_{3,12}^2 \in \mathbf{Cov}_{6,24}$. By a direct computation in Macaulay2 [37], we get that the space $\mathbf{Cov}_{6,24}$ is spanned by the family

$$\{\mathbf{h}_{2,0}\mathbf{f}^4; \mathbf{h}_{2,4}\mathbf{h}_{1,6}^2\mathbf{h}_{2,8}; \mathbf{f}^3\mathbf{h}_{3,6}; \mathbf{h}_{2,8}^3; \mathbf{h}_{3,12}^2\}.$$

Now, writing $\mathbf{f} = \mathbf{f}(a_0, \dots, a_6, x, y)$ we compute the exact expression of those monomials in (a_i, x, y) and, by computing a kernel, we directly obtain the relation

$$36\mathbf{h}_{3,12}^2 + \mathbf{h}_{2,0}\mathbf{f}^4 - 6\mathbf{f}^3\mathbf{h}_{3,6} - 9\mathbf{h}_{2,4}\mathbf{f}^2\mathbf{h}_{2,8} + 18\mathbf{h}_{2,8}^3 = 0.$$

We do in the same way for all other relations. □

We are now able to compute a minimal covariant bases of $\mathbf{Cov}(S_6 \oplus S_4)$:

- As a first step, we use Normaliz package [18] developed in Macaulay2 [37] to compute irreducible solutions of the linear Diophantine system associated to the covariant basis A and B of $\mathbf{Cov}(S_6)$ and $\mathbf{Cov}(S_4)$. To get such a system, we only have to deal with non-invariant covariant of each bases, which then leads to a system of $21 + 3 + 3 = 27$ unknowns

$$\begin{cases} 2(\alpha_1 + \dots + \alpha_6) + 4(\alpha_7 + \dots + \alpha_{11}) + 6(\alpha_{12} + \dots + \alpha_{16}) + \\ \quad 8(\alpha_{17} + \alpha_{18} + \alpha_{19}) + 10\alpha_{20} + 12\alpha_{21} & = u + r . \\ 4(\beta_1 + \beta_2) + 6\beta_3 & = v + r \end{cases}$$

We thus obtain on this first step a finite family of 1732 transvectants.

- By the known relations (8.1) and by lemma 8.2, we can use theorem 7.7 and thus we get a first reduction process that leads to a finite family of 1134 transvectants.
- We compute now degree per degree, making use of the *multigraduated Hilbert series* of the algebra

$$\mathbf{Cov}(S_6 \oplus S_4) = \bigoplus_{d_1, d_2, k} \mathbf{Cov}_{d_1, d_2, k}(S_6 \oplus S_4)$$

where d_1 is on degree on $\mathbf{f} \in S_6$, d_2 is the degree on $\mathbf{v} \in S_4$ and k is the order of the covariant. Such a multigraduated series can be directly computed using Bedratyuk's Maple package [9].

We then organize all the 1134 transvectants using orders, then using degrees, which is summarized in table 5.

Order	0	2	4	6	8	10	12	14	16
# of trans.	365	462	144	78	46	24	10	4	1
Degrees	3-30	2-30	1-23	1-15	2-12	3-9	3-8	4-7	6

TABLE 5. Finite generating family after reduction process.

Theorem 8.3. *The covariant algebra $\mathbf{Cov}(S_6 \oplus S_4)$ is generated by a minimal bases of 194 elements, summarized in table 7.*

Proof. We have to get a minimal family using the 1134 transvectants summarized in table 5. We do this order per order, then degree per degree, using multigraduated Hilbert series of the algebra $\mathbf{Cov}(S_6 \oplus S_4)$. For instance, we have 365 order 0 covariants (which are invariants) from degree 3 up to degree 30. Furthermore, for high level degree, we know exactly which homogeneous spaces $\mathbf{Cov}_{d_1, d_2, 0}(S_6 \oplus S_4)$ occur (table 6) in the given family.

Degree	30	28	27	26
d_1	27	25	24	23
d_2	3	3	3	3
Dimension	639	518	534	413

TABLE 6. Dimension of homogenous spaces for high degree invariants.

Then, using scripts written in Macaulay2 [37] and algorithm explained in subsection 7.1, we get a finite minimal family of 60 invariants. We then do in the same way for each order. \square

d/o	0	2	4	6	8	10	12	#	Cum
1	—	—	1	1	—	—	—	2	2
2	2	1	3	1	2	—	—	9	11
3	2	4	4	5	3	1	1	20	31
4	4	6	9	5	2	1	—	27	58
5	4	12	11	3	1	—	—	31	89
6	9	14	6	2	—	—	—	31	120
7	9	17	2	—	—	—	—	28	148
8	9	7	1	—	—	—	—	17	165
9	8	3	1	—	—	—	—	12	177
10	5	2	—	—	—	—	—	7	184
11	3	1	—	—	—	—	—	4	188
12	2	1	—	—	—	—	—	3	191
13	1	—	—	—	—	—	—	1	192
14	1	—	—	—	—	—	—	1	193
15	1	—	—	—	—	—	—	1	194
<i>Tot</i>	60	68	38	17	8	2	1		194

TABLE 7. Minimal covariant bases of $S_6 \oplus S_4$

We give now transvectants expression of this minimal covariant bases.

Order 0 : 5 invariants from S_6 , 2 invariants from S_4 and 53 joint invariants	
Degree 3	$(\mathbf{h}_{2,4}, \mathbf{v})_4$
Degree 4	$(\mathbf{h}_{2,8}, \mathbf{v}^2)_8$ $(\mathbf{h}_{2,4}, \mathbf{k}_{2,4})_4$ $(\mathbf{f}, \mathbf{k}_{3,6})_6$
Degree 5	$(\mathbf{h}_{3,8}, \mathbf{v}^2)_8$ $(\mathbf{h}_{4,4}, \mathbf{v})_4$ $(\mathbf{h}_{2,8}, \mathbf{v} \cdot \mathbf{k}_{2,4})_8$ $(\mathbf{f}^2, \mathbf{v}^3)_{12}$
Degree 6	$(\mathbf{h}_{3,8}, \mathbf{v} \cdot \mathbf{k}_{2,4})_8$ $(\mathbf{f}^2, \mathbf{v}^2 \cdot \mathbf{k}_{2,4})_{12}$ $(\mathbf{h}_{2,8}, \mathbf{k}_{2,4}^2)_8$ $(\mathbf{h}_{3,6}, \mathbf{k}_{3,6})_6$
	$(\mathbf{h}_{3,12}, \mathbf{v}^3)_{12}$ $(\mathbf{h}_{5,4}, \mathbf{v})_4$
	$(\mathbf{h}_{4,4}, \mathbf{k}_{2,4})_4$ $(\mathbf{h}_{3,2} \cdot \mathbf{f}, \mathbf{v}^2)_8$
Degree 7	$(\mathbf{h}_{3,2}^2, \mathbf{v})_4$ $(\mathbf{h}_{5,4}, \mathbf{k}_{2,4})_4$ $(\mathbf{h}_{5,8}, \mathbf{v}^2)_8$ $(\mathbf{f} \cdot \mathbf{h}_{3,6}, \mathbf{v}^3)_{12}$
	$(\mathbf{f}^2, \mathbf{v} \cdot \mathbf{k}_{2,4}^2)_{12}$ $(\mathbf{h}_{3,2} \cdot \mathbf{f}, \mathbf{v} \cdot \mathbf{k}_{2,4})_8$
	$(\mathbf{h}_{4,6}, \mathbf{k}_{3,6})_6$ $(\mathbf{h}_{3,12}, \mathbf{v}^2 \cdot \mathbf{k}_{2,4})_{12}$ $(\mathbf{h}_{3,8}, \mathbf{k}_{2,4}^2)_8$

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Degree 8	$(\mathbf{h}_{3,2}\mathbf{h}_{2,4}, \mathbf{k}_{3,6})_6 \quad (\mathbf{h}_{3,12}, \mathbf{v} \cdot \mathbf{k}_{2,4}^2)_{12} \quad (\mathbf{h}_{3,2}\mathbf{h}_{3,6}, \mathbf{v}^2)_8 \quad (\mathbf{h}_{3,2}^2, \mathbf{k}_{2,4})_4$
	$(\mathbf{h}_{7,4}, \mathbf{v})_4 \quad (\mathbf{f} \cdot \mathbf{h}_{4,6}, \mathbf{v}^3)_{12}$
	$(\mathbf{f} \cdot \mathbf{h}_{3,6}, \mathbf{v}^2 \cdot \mathbf{k}_{2,4})_{12} \quad (\mathbf{h}_{3,2} \cdot \mathbf{f}, \mathbf{k}_{2,4}^2)_8 \quad (\mathbf{h}_{5,8}, \mathbf{v} \cdot \mathbf{k}_{2,4})_8$
Degree 9	$(\mathbf{h}_{7,4}, \mathbf{k}_{2,4})_4 \quad (\mathbf{h}_{3,2} \cdot \mathbf{h}_{5,2}, \mathbf{v})_4 \quad (\mathbf{h}_{5,2} \cdot \mathbf{f}, \mathbf{v} \cdot \mathbf{k}_{2,4})_8$
	$(\mathbf{h}_{3,12}, \mathbf{k}_{2,4}^3)_{12} \quad (\mathbf{h}_{3,2} \cdot \mathbf{h}_{2,8}, \mathbf{v} \cdot \mathbf{k}_{3,6})_{10}$
	$(\mathbf{f}\mathbf{h}_{4,6}, \mathbf{v}^2 \cdot \mathbf{k}_{2,4})_{12} \quad (\mathbf{h}_{3,6}^2, \mathbf{v}^3)_{12} \quad (\mathbf{h}_{3,2} \cdot \mathbf{h}_{4,6}, \mathbf{v}^2)_8$
Degree 10	$(\mathbf{h}_{9,4}, \mathbf{v})_4 \quad (\mathbf{h}_{3,2} \cdot \mathbf{h}_{2,8}, \mathbf{k}_{2,4}\mathbf{k}_{3,6})_{10} \quad (\mathbf{h}_{5,2} \cdot \mathbf{h}_{3,6}, \mathbf{v}^2)_8 \quad (\mathbf{f} \cdot \mathbf{h}_{6,61}, \mathbf{v}^3)_{12}$
Degree 11	$(\mathbf{h}_{5,2}^2, \mathbf{v})_4 \quad (\mathbf{f} \cdot \mathbf{h}_{6,62}, \mathbf{v}^2 \cdot \mathbf{k}_{2,4})_{12} \quad (\mathbf{h}_{3,2} \cdot \mathbf{h}_{6,61}, \mathbf{v}^2)_8$
Degree 12	$(\mathbf{h}_{3,2}\mathbf{h}_{8,2}, \mathbf{v})_4 \quad (\mathbf{h}_{3,2}\mathbf{h}_{6,62}, \mathbf{v}\mathbf{k}_{2,4})_8$
Degree 13	$(\mathbf{h}_{8,2}\mathbf{h}_{3,6}, \mathbf{v}^2)_8$
Degree 14	$(\mathbf{h}_{3,2}\mathbf{h}_{10,2}, \mathbf{v})_4$
Order 2 : 6 covariants from S_6 and 62 joint covariants.	
Degree 2	$(\mathbf{f}, \mathbf{v})_4$
Degree 3	$\mathbf{h}_{3,2} \quad (\mathbf{h}_{2,4}, \mathbf{v})_3 \quad (\mathbf{f}, \mathbf{k}_{2,4})_4 \quad (\mathbf{f}, \mathbf{v}^2)_6$
Degree 4	$(\mathbf{h}_{2,4}, \mathbf{k}_{2,4})_3 \quad (\mathbf{f}, \mathbf{v} \cdot \mathbf{k}_{2,4})_6 \quad (\mathbf{f}, \mathbf{k}_{3,6})_5 \quad (\mathbf{h}_{2,8}, \mathbf{v}^2)_7 \quad (\mathbf{h}_{3,2}, \mathbf{v})_2 \quad (\mathbf{h}_{3,6}, \mathbf{v})_4$
Degree 5	$\mathbf{h}_{5,2} \quad (\mathbf{h}_{3,6}, \mathbf{k}_{2,4})_4 \quad (\mathbf{h}_{4,4}, \mathbf{v})_3 \quad (\mathbf{h}_{3,6}, \mathbf{v}^2)_6$
	$(\mathbf{h}_{2,8}, \mathbf{v} \cdot \mathbf{k}_{2,4})_7 \quad (\mathbf{f}, \mathbf{k}_{2,4}^2)_6 \quad (\mathbf{h}_{3,2}, \mathbf{k}_{2,4})_2$
	$(\mathbf{f}^2, \mathbf{v}^3)_{11} \quad (\mathbf{h}_{4,6}, \mathbf{v})_4 \quad (\mathbf{h}_{2,8}, \mathbf{k}_{3,6})_6 \quad (\mathbf{h}_{2,4}, \mathbf{k}_{3,6})_4 \quad (\mathbf{h}_{3,8}, \mathbf{v}^2)_7$
Degree 6	$(\mathbf{f}^2, \mathbf{v}^2 \cdot \mathbf{k}_{2,4})_{11} \quad (\mathbf{h}_{2,8}, \mathbf{v} \cdot \mathbf{k}_{3,6})_8 \quad (\mathbf{h}_{3,2} \cdot \mathbf{f}, \mathbf{v}^2)_7 \quad (\mathbf{h}_{2,8}, \mathbf{k}_{2,4}^2)_7$
	$(\mathbf{h}_{4,4}, \mathbf{k}_{2,4})_3 \quad (\mathbf{h}_{4,10}, \mathbf{v}^2)_8 \quad (\mathbf{h}_{3,12}, \mathbf{v}^3)_{11} \quad (\mathbf{h}_{5,2}, \mathbf{v})_2$
	$(\mathbf{h}_{4,6}, \mathbf{v}^2)_6 \quad (\mathbf{h}_{3,6}, \mathbf{v} \cdot \mathbf{k}_{2,4})_6 \quad (\mathbf{h}_{4,6}, \mathbf{k}_{2,4})_4 \quad (\mathbf{h}_{3,8}, \mathbf{v} \cdot \mathbf{k}_{2,4})_7$
	$(\mathbf{h}_{3,8}, \mathbf{k}_{3,6})_6 \quad (\mathbf{h}_{5,4}, \mathbf{v})_3$
Degree 7	$\mathbf{h}_{7,2} \quad (\mathbf{h}_{2,8}, \mathbf{k}_{2,4} \cdot \mathbf{k}_{3,6})_8 \quad (\mathbf{h}_{6,62}, \mathbf{v})_4 \quad (\mathbf{h}_{3,12}, \mathbf{v}^2 \cdot \mathbf{k}_{2,4})_{11} \quad (\mathbf{h}_{4,10}, \mathbf{v}^3)_{10} \quad (\mathbf{h}_{6,61}, \mathbf{v})_4$
	$(\mathbf{f} \cdot \mathbf{h}_{3,6}, \mathbf{v}^3)_{11} \quad (\mathbf{h}_{3,2}^2, \mathbf{v})_3 \quad (\mathbf{h}_{5,2}, \mathbf{k}_{2,4})_2 \quad (\mathbf{h}_{3,8}, \mathbf{v} \cdot \mathbf{k}_{3,6})_8 \quad (\mathbf{h}_{2,4}^2, \mathbf{k}_{3,6})_6 \quad (\mathbf{h}_{5,8}, \mathbf{v}^2)_7$
	$(\mathbf{h}_{4,6}, \mathbf{v} \cdot \mathbf{k}_{2,4})_6 \quad (\mathbf{f}^2, \mathbf{v} \cdot \mathbf{k}_{2,4}^2)_{11} \quad (\mathbf{h}_{5,4}, \mathbf{k}_{2,4})_3 \quad (\mathbf{h}_{4,10}, \mathbf{v} \cdot \mathbf{k}_{2,4})_8 \quad (\mathbf{h}_{4,6}, \mathbf{k}_{3,6})_5$
Degree 8	$\mathbf{h}_{8,2} \quad (\mathbf{h}_{3,2} \cdot \mathbf{h}_{3,6}, \mathbf{v}^2)_7 \quad (\mathbf{h}_{7,2}, \mathbf{v})_2 \quad (\mathbf{h}_{3,2}^2, \mathbf{k}_{2,4})_3$
	$(\mathbf{h}_{6,61}, \mathbf{k}_{2,4})_4 \quad (\mathbf{h}_{6,62}, \mathbf{v}^2)_6 \quad (\mathbf{h}_{4,10}, \mathbf{k}_{2,4}^2)_8$
Degree 9	$(\mathbf{h}_{8,2}, \mathbf{v})_2 \quad (\mathbf{h}_{3,2}^2, \mathbf{k}_{3,6})_4 \quad (\mathbf{h}_{3,2} \cdot \mathbf{h}_{5,2}, \mathbf{v})_3$
Degree 10	$\mathbf{h}_{10,2} \quad (\mathbf{h}_{5,2} \cdot \mathbf{h}_{3,6}, \mathbf{v}^2)_7$
Degree 11	$(\mathbf{h}_{5,2}^2, \mathbf{v})_3$
Degree 12	$\mathbf{h}_{12,2}$
Order 4 : 2 covariants from S_4 , 5 covariants from S_6 and 31 joint covariants.	
Degree 1	\mathbf{v}
Degree 2	$\mathbf{k}_{2,4} \quad \mathbf{h}_{2,4} \quad (\mathbf{f}, \mathbf{v})_3$
Degree 3	$(\mathbf{h}_{2,4}, \mathbf{v})_2 \quad (\mathbf{f}, \mathbf{v}^2)_5 \quad (\mathbf{h}_{2,8}, \mathbf{v})_4 \quad (\mathbf{f}, \mathbf{k}_{2,4})_3$
Degree 4	$\mathbf{h}_{4,4} \quad (\mathbf{h}_{3,8}, \mathbf{v})_4 \quad (\mathbf{h}_{3,2}, \mathbf{v})_1 \quad (\mathbf{h}_{2,8}, \mathbf{k}_{2,4})_4$

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	$(\mathbf{h}_{2,4}, \mathbf{k}_{2,4})_2$ $(\mathbf{h}_{2,8}, \mathbf{v}^2)_6$
	$(\mathbf{f}, \mathbf{k}_{3,6})_4$ $(\mathbf{f}, \mathbf{v} \cdot \mathbf{k}_{2,4})_5$ $(\mathbf{h}_{3,6}, \mathbf{v})_3$
Degree 5	$\mathbf{h}_{5,4}$ $(\mathbf{h}_{2,8}, \mathbf{v} \cdot \mathbf{k}_{2,4})_6$ $(\mathbf{h}_{3,12}, \mathbf{v}^2)_8$ $(\mathbf{h}_{4,6}, \mathbf{v})_3$ $(\mathbf{h}_{3,2}, \mathbf{k}_{2,4})_1$
	$(\mathbf{h}_{3,6}, \mathbf{k}_{2,4})_3$ $(\mathbf{h}_{4,4}, \mathbf{v})_2$ $(\mathbf{h}_{3,8}, \mathbf{k}_{2,4})_4$ $(\mathbf{h}_{2,8}, \mathbf{k}_{3,6})_5$ $(\mathbf{f}, \mathbf{k}_{2,4}^2)_5$ $(\mathbf{h}_{3,6}, \mathbf{v}^2)_5$
Degree 6	$(\mathbf{h}_{5,2}, \mathbf{v})_1$ $(\mathbf{h}_{3,12}, \mathbf{v} \cdot \mathbf{k}_{2,4})_8$ $(\mathbf{h}_{5,8}, \mathbf{v})_4$ $(\mathbf{h}_{5,4}, \mathbf{v})_2$ $(\mathbf{h}_{4,6}, \mathbf{k}_{2,4})_3$ $(\mathbf{h}_{3,6}, \mathbf{k}_{3,6})_4$
Degree 7	$\mathbf{h}_{7,4}$ $(\mathbf{h}_{5,8}, \mathbf{k}_{2,4})_4$
Degree 8	$(\mathbf{h}_{7,4}, \mathbf{v})_2$
Degree 9	$\mathbf{h}_{9,4}$
Order 6 : 1 covariant form S_4 , 5 covariants from S_6 and 11 joint covariants.	
Degree 1	\mathbf{f}
Degree 2	$(\mathbf{f}, \mathbf{v})_2$
Degree 3	$\mathbf{h}_{3,6}$ $\mathbf{k}_{3,6}$ $(\mathbf{f}, \mathbf{k}_{2,4})_2$ $(\mathbf{h}_{2,4}, \mathbf{v})_1$ $(\mathbf{h}_{2,8}, \mathbf{v})_3$
Degree 4	$\mathbf{h}_{4,6}$ $(\mathbf{h}_{2,8}, \mathbf{k}_{2,4})_3$ $(\mathbf{h}_{3,8}, \mathbf{v})_3$ $(\mathbf{h}_{2,8}, \mathbf{v}^2)_5$ $(\mathbf{h}_{2,4}, \mathbf{k}_{2,4})_1$
Degree 5	$(\mathbf{h}_{4,10}, \mathbf{v})_4$ $(\mathbf{h}_{4,4}, \mathbf{v})_1$ $(\mathbf{h}_{4,6}, \mathbf{v})_2$
Degree 6	$\mathbf{h}_{6,61}$ $\mathbf{h}_{6,62}$
Order 8 : 3 covariants form S_6 , 5 joint covariants.	
Degree 2	$\mathbf{h}_{2,8}$ $(\mathbf{f}, \mathbf{v})_1$
Degree 3	$\mathbf{h}_{3,8}$ $(\mathbf{f}, \mathbf{k}_{2,4})_1$ $(\mathbf{h}_{2,8}, \mathbf{v})_2$
Degree 4	$(\mathbf{h}_{3,8}, \mathbf{v})_2$ $(\mathbf{h}_{3,12}, \mathbf{v})_4$
Degree 5	$\mathbf{h}_{5,8}$
Order 10 : 1 covariant from S_6 and 1 joint covariant.	
Degree 3	$(\mathbf{h}_{2,8}, \mathbf{k}_{1,4})_1$
Degree 4	$\mathbf{h}_{4,10}$
Order 12 : 1 covariant from S_6 .	
Degree 3	$\mathbf{h}_{3,12}$

8.3. Covariant bases of $S_6 \oplus S_4 \oplus S_2$. Now we have a covariant bases of $\mathbf{Cov}(S_6 \oplus S_4)$ from theorem 8.3, we can use theorem 5.7 with $V = S_6 \oplus S_4$. We thus have:

Theorem 8.4. *The covariant algebra $\mathbf{Cov}(S_6 \oplus S_4 \oplus S_2)$ is generated by a minimal bases of 494 elements, summarized in table 8.*

d/o	0	2	4	6	8	10	12	#	Cum
1	—	1	1	1	—	—	—	3	3
2	3	2	5	2	2	—	—	14	17
3	4	10	9	8	4	1	1	37	54
4	12	19	20	10	3	2	—	66	120
5	15	38	24	6	3	—	—	86	206
6	37	46	12	5	—	—	—	100	306
7	42	31	7	—	—	—	—	80	386
8	38	15	1	—	—	—	—	54	440
9	22	4	1	—	—	—	—	27	467
10	9	3	—	—	—	—	—	12	479
11	6	1	—	—	—	—	—	7	486
12	3	1	—	—	—	—	—	4	490
13	2	—	—	—	—	—	—	2	492
14	1	—	—	—	—	—	—	1	493
15	1	—	—	—	—	—	—	1	494
Tot	195	171	80	32	12	3	1		494

TABLE 8. Covariant bases of $S_6 \oplus S_4 \oplus S_2$

We give now transvectants expression of this minimal covariant bases.

195 invariants: 5 from S_6 , 2 from S_4 , 1 from S_2 , 21 joint invariants of $S_6 \oplus S_2$ given in 8.1, 53 joint invariants of $S_6 \oplus S_4$ given in 8.2. There is left 113 invariants.	
Degree 3	$(\mathbf{v}, \mathbf{u}^2)_4$ $((\mathbf{f}, \mathbf{v})_4, \mathbf{u})_2$
Degree 4	$(\mathbf{k}_{2,4}, \mathbf{u}^2)_4$ $((\mathbf{f}, \mathbf{v})_3, \mathbf{u}^2)_4$ $((\mathbf{f}, \mathbf{v}^2)_6, \mathbf{u})_2$ $((\mathbf{f}, \mathbf{k}_{2,4})_4, \mathbf{u})_2$ $((\mathbf{h}_{2,4}, \mathbf{v})_3, \mathbf{u})_2$
Degree 5	$((\mathbf{f}, \mathbf{v})_2, \mathbf{u}^3)_6$ $((\mathbf{f}, \mathbf{k}_{2,4})_3, \mathbf{u}^2)_4$ $((\mathbf{f}, \mathbf{v}^2)_5, \mathbf{u}^2)_4$ $((\mathbf{f}, \mathbf{k}_{3,6})_5, \mathbf{u})_2$
	$((\mathbf{f}, \mathbf{v} \cdot \mathbf{k}_{2,4})_6, \mathbf{u})_2$ $((\mathbf{h}_{2,8}, \mathbf{v})_4, \mathbf{u}^2)_4$ $((\mathbf{h}_{2,4}, \mathbf{v})_2, \mathbf{u}^2)_4$ $((\mathbf{h}_{2,8}, \mathbf{v}^2)_7, \mathbf{u})_2$
	$((\mathbf{h}_{2,4}, \mathbf{k}_{2,4})_3, \mathbf{u})_2$ $((\mathbf{h}_{3,6}, \mathbf{v})_4, \mathbf{u})_2$
	$((\mathbf{h}_{3,2}, \mathbf{v})_2, \mathbf{u})_2$
Degree 6	$(\mathbf{k}_{3,6}, \mathbf{u}^3)_6$ $((\mathbf{f}, \mathbf{v})_1, \mathbf{u}^4)_8$ $((\mathbf{f}, \mathbf{k}_{2,4})_2, \mathbf{u}^3)_6$ $((\mathbf{f}, \mathbf{k}_{3,6})_4, \mathbf{u}^2)_4$
	$((\mathbf{f}, \mathbf{v} \cdot \mathbf{k}_{2,4})_5, \mathbf{u}^2)_4$ $((\mathbf{f}, \mathbf{k}_{2,4}^2)_6, \mathbf{u})_2$ $((\mathbf{h}_{2,4}, \mathbf{v})_1, \mathbf{u}^3)_6$ $((\mathbf{h}_{2,8}, \mathbf{v})_3, \mathbf{u}^3)_6$
	$((\mathbf{h}_{2,8}, \mathbf{v}^2)_6, \mathbf{u}^2)_4$ $((\mathbf{h}_{2,8}, \mathbf{k}_{2,4})_4, \mathbf{u}^2)_4$ $((\mathbf{h}_{2,4}, \mathbf{k}_{2,4})_2, \mathbf{u}^2)_4$ $((\mathbf{h}_{2,8}, \mathbf{v} \cdot \mathbf{k}_{2,4})_7, \mathbf{u})_2$
	$((\mathbf{h}_{2,8}, \mathbf{k}_{3,6})_6, \mathbf{u})_2$ $((\mathbf{f}^2, \mathbf{v}^3)_{11}, \mathbf{u})_2$ $((\mathbf{h}_{2,4}, \mathbf{k}_{3,6})_4, \mathbf{u})_2$ $((\mathbf{h}_{3,6}, \mathbf{v})_3, \mathbf{u}^2)_4$
	$((\mathbf{h}_{3,8}, \mathbf{v})_4, \mathbf{u}^2)_4$ $((\mathbf{h}_{3,2}, \mathbf{v})_1, \mathbf{u}^2)_4$ $((\mathbf{h}_{3,6}, \mathbf{k}_{2,4})_4, \mathbf{u})_2$ $((\mathbf{h}_{3,6}, \mathbf{v}^2)_6, \mathbf{u})_2$
	$((\mathbf{h}_{3,2}, \mathbf{k}_{2,4})_2, \mathbf{u})_2$ $((\mathbf{h}_{3,8}, \mathbf{v}^2)_7, \mathbf{u})_2$ $((\mathbf{h}_{4,4}, \mathbf{v})_3, \mathbf{u})_2$ $((\mathbf{h}_{4,6}, \mathbf{v})_4, \mathbf{u})_2$
Degree 7	$((\mathbf{f}, \mathbf{k}_{2,4})_1, \mathbf{u}^4)_8$ $((\mathbf{f}, \mathbf{k}_{2,4}^2)_5, \mathbf{u}^2)_4$ $((\mathbf{h}_{2,8}, \mathbf{v})_2, \mathbf{u}^4)_8$ $((\mathbf{h}_{2,8}, \mathbf{v}^2)_5, \mathbf{u}^3)_6$

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	$((\mathbf{h}_{2,8}, \mathbf{k}_{2,4})_3, \mathbf{u}^3)_6$	$((\mathbf{h}_{2,4}, \mathbf{k}_{2,4})_1, \mathbf{u}^3)_6$	$((\mathbf{h}_{2,8}, \mathbf{v} \cdot \mathbf{k}_{2,4})_6, \mathbf{u}^2)_4$	$((\mathbf{h}_{2,8}, \mathbf{k}_{3,6})_5, \mathbf{u}^2)_4$
	$((\mathbf{h}_{2,8}, \mathbf{k}_{2,4}^2)_7, \mathbf{u})_2$	$((\mathbf{f}^2, \mathbf{v}^2 \cdot \mathbf{k}_{2,4})_{11}, \mathbf{u})_2$	$((\mathbf{h}_{2,8}, \mathbf{v} \cdot \mathbf{k}_{3,6})_8, \mathbf{u})_2$	$((\mathbf{h}_{3,8}, \mathbf{v})_3, \mathbf{u}^3)_6$
	$((\mathbf{h}_{3,2}, \mathbf{k}_{2,4})_1, \mathbf{u}^2)_4$	$((\mathbf{h}_{3,8}, \mathbf{k}_{2,4})_4, \mathbf{u}^2)_4$	$((\mathbf{h}_{3,6}, \mathbf{v}^2)_5, \mathbf{u}^2)_4$	$((\mathbf{h}_{3,12}, \mathbf{v}^2)_8, \mathbf{u}^2)_4$
	$((\mathbf{h}_{3,6}, \mathbf{k}_{2,4})_3, \mathbf{u}^2)_4$	$((\mathbf{h}_{3,6}, \mathbf{v} \cdot \mathbf{k}_{2,4})_6, \mathbf{u})_2$	$((\mathbf{h}_{3,12}, \mathbf{v}^3)_{11}, \mathbf{u})_2$	$((\mathbf{h}_{3,8}, \mathbf{v} \cdot \mathbf{k}_{2,4})_7, \mathbf{u})_2$
	$((\mathbf{h}_{3,8}, \mathbf{k}_{3,6})_6, \mathbf{u})_2$	$((\mathbf{h}_{4,6}, \mathbf{v})_3, \mathbf{u}^2)_4$	$((\mathbf{h}_{4,4}, \mathbf{v})_2, \mathbf{u}^2)_4$	$((\mathbf{h}_{4,6}, \mathbf{k}_{2,4})_4, \mathbf{u})_2$
	$((\mathbf{h}_{4,6}, \mathbf{v}^2)_6, \mathbf{u})_2$	$((\mathbf{h}_{4,4}, \mathbf{k}_{2,4})_3, \mathbf{u})_2$	$((\mathbf{h}_{4,10}, \mathbf{v}^2)_8, \mathbf{u})_2$	$((\mathbf{h}_{3,2} \cdot \mathbf{f}, \mathbf{v}^2)_7, \mathbf{u})_2$
	$((\mathbf{h}_{5,4}, \mathbf{v})_3, \mathbf{u})_2$	$((\mathbf{h}_{5,2}, \mathbf{v})_2, \mathbf{u})_2$		
Degree 8	$((\mathbf{h}_{2,8}, \mathbf{v})_1, \mathbf{u}^5)_{10}$	$((\mathbf{h}_{2,8}, \mathbf{k}_{2,4} \cdot \mathbf{k}_{3,6})_8, \mathbf{u})_2$	$((\mathbf{f}^2, \mathbf{v} \cdot \mathbf{k}_{2,4}^2)_{11}, \mathbf{u})_2$	$((\mathbf{h}_{3,12}, \mathbf{v})_4, \mathbf{u}^4)_8$
	$((\mathbf{h}_{3,8}, \mathbf{v})_2, \mathbf{u}^4)_8$	$((\mathbf{h}_{3,12}, \mathbf{v} \cdot \mathbf{k}_{2,4})_8, \mathbf{u}^2)_4$	$((\mathbf{h}_{3,6}, \mathbf{k}_{3,6})_4, \mathbf{u}^2)_4$	$((\mathbf{h}_{3,8}, \mathbf{v} \cdot \mathbf{k}_{3,6})_8, \mathbf{u})_2$
	$((\mathbf{h}_{3,12}, \mathbf{v}^2 \cdot \mathbf{k}_{2,4})_{11}, \mathbf{u})_2$	$((\mathbf{h}_{4,10}, \mathbf{v})_4, \mathbf{u}^3)_6$	$((\mathbf{h}_{4,6}, \mathbf{v})_2, \mathbf{u}^3)_6$	$((\mathbf{h}_{4,4}, \mathbf{v})_1, \mathbf{u}^3)_6$
	$((\mathbf{h}_{4,6}, \mathbf{k}_{2,4})_3, \mathbf{u}^2)_4$	$((\mathbf{h}_{4,6}, \mathbf{v} \cdot \mathbf{k}_{2,4})_6, \mathbf{u})_2$	$((\mathbf{h}_{4,10}, \mathbf{v}^3)_{10}, \mathbf{u})_2$	$((\mathbf{f} \cdot \mathbf{h}_{3,6}, \mathbf{v}^3)_{11}, \mathbf{u})_2$
	$((\mathbf{h}_{4,10}, \mathbf{v} \cdot \mathbf{k}_{2,4})_8, \mathbf{u})_2$	$((\mathbf{h}_{4,6}, \mathbf{k}_{3,6})_5, \mathbf{u})_2$	$((\mathbf{h}_{2,4}^2, \mathbf{k}_{3,6})_6, \mathbf{u})_2$	$((\mathbf{h}_{5,8}, \mathbf{v})_4, \mathbf{u}^2)_4$
	$((\mathbf{h}_{5,2}, \mathbf{v})_1, \mathbf{u}^2)_4$	$((\mathbf{h}_{5,4}, \mathbf{v})_2, \mathbf{u}^2)_4$	$((\mathbf{h}_{5,2}, \mathbf{k}_{2,4})_2, \mathbf{u})_2$	$((\mathbf{h}_{5,8}, \mathbf{v}^2)_7, \mathbf{u})_2$
	$((\mathbf{h}_{5,4}, \mathbf{k}_{2,4})_3, \mathbf{u})_2$	$((\mathbf{h}_{6,62}, \mathbf{v})_4, \mathbf{u})_2$	$((\mathbf{h}_{3,2}^2, \mathbf{v})_3, \mathbf{u})_2$	$((\mathbf{h}_{6,61}, \mathbf{v})_4, \mathbf{u})_2$
Degree 9	$((\mathbf{h}_{4,10}, \mathbf{k}_{2,4}^2)_8, \mathbf{u})_2$	$((\mathbf{h}_{5,8}, \mathbf{k}_{2,4})_4, \mathbf{u}^2)_4$	$((\mathbf{h}_{3,2} \cdot \mathbf{h}_{3,6}, \mathbf{v}^2)_7, \mathbf{u})_2$	$((\mathbf{h}_{6,61}, \mathbf{k}_{2,4})_4, \mathbf{u})_2$
	$((\mathbf{h}_{3,2}^2, \mathbf{k}_{2,4})_3, \mathbf{u})_2$	$((\mathbf{h}_{6,62}, \mathbf{v}^2)_6, \mathbf{u})_2$	$((\mathbf{h}_{7,2}, \mathbf{v})_2, \mathbf{u})_2$	
Degree 10	$((\mathbf{h}_{3,2}^2, \mathbf{k}_{3,6})_4, \mathbf{u})_2$	$((\mathbf{h}_{7,4}, \mathbf{v})_2, \mathbf{u}^2)_4$	$((\mathbf{h}_{8,2}, \mathbf{v})_2, \mathbf{u})_2$	$((\mathbf{h}_{3,2} \cdot \mathbf{h}_{5,2}, \mathbf{v})_3, \mathbf{u})_2$
Degree 11	$((\mathbf{h}_{5,2} \cdot \mathbf{h}_{3,6}, \mathbf{v}^2)_7, \mathbf{u})_2$			
Degree 12	$((\mathbf{h}_{5,2}^2, \mathbf{v})_3, \mathbf{u})_2$			
171 covariants of order 2:6 from S_6 , 1 from S_2 , 23 joint covariants of $S_6 \oplus S_2$ given in 8.1, 62 joint covariants of $S_6 \oplus S_4$ given in 8.2. There is left 79 covariants given below:				
Degree 2	$(\mathbf{v}, \mathbf{u})_2$			
Degree 3	$(\mathbf{v}, \mathbf{u}^2)_3$	$(\mathbf{k}_{2,4}, \mathbf{u})_2$	$((\mathbf{f}, \mathbf{v})_4, \mathbf{u})_1$	$((\mathbf{f}, \mathbf{v})_3, \mathbf{u})_2$
Degree 4	$((1, \mathbf{k}_{2,4})_0, \mathbf{u}^2)_3$	$((\mathbf{f}, \mathbf{v})_2, \mathbf{u}^2)_4$	$((\mathbf{f}, \mathbf{v})_3, \mathbf{u}^2)_3$	$((\mathbf{f}, \mathbf{k}_{2,4})_3, \mathbf{u})_2$
	$((\mathbf{f}, \mathbf{v}^2)_5, \mathbf{u})_2$	$((\mathbf{f}, \mathbf{k}_{2,4})_4, \mathbf{u})_1$	$((\mathbf{f}, \mathbf{v}^2)_6, \mathbf{u})_1$	$((\mathbf{h}_{2,4}, \mathbf{v})_2, \mathbf{u})_2$
	$((\mathbf{h}_{2,8}, \mathbf{v})_4, \mathbf{u})_2$	$((\mathbf{h}_{2,4}, \mathbf{v})_3, \mathbf{u})_1$		
Degree 5	$(\mathbf{k}_{3,6}, \mathbf{u}^2)_4$	$((\mathbf{f}, \mathbf{v})_2, \mathbf{u}^3)_5$	$((\mathbf{f}, \mathbf{v})_1, \mathbf{u}^3)_6$	$((\mathbf{f}, \mathbf{k}_{2,4})_2, \mathbf{u}^2)_4$
	$((\mathbf{f}, \mathbf{k}_{2,4})_3, \mathbf{u}^2)_3$	$((\mathbf{f}, \mathbf{k}_{3,6})_5, \mathbf{u})_1$	$((\mathbf{f}, \mathbf{v} \cdot \mathbf{k}_{2,4})_5, \mathbf{u})_2$	$((\mathbf{f}, \mathbf{v} \cdot \mathbf{k}_{2,4})_6, \mathbf{u})_1$
	$((\mathbf{f}, \mathbf{k}_{3,6})_4, \mathbf{u})_2$	$((\mathbf{h}_{2,8}, \mathbf{v})_4, \mathbf{u}^2)_3$	$((\mathbf{h}_{2,4}, \mathbf{v})_1, \mathbf{u}^2)_4$	$((\mathbf{h}_{2,8}, \mathbf{v})_3, \mathbf{u}^2)_4$
	$((\mathbf{h}_{2,4}, \mathbf{v})_2, \mathbf{u}^2)_3$	$((\mathbf{h}_{2,8}, \mathbf{v}^2)_7, \mathbf{u})_1$	$((\mathbf{h}_{2,4}, \mathbf{k}_{2,4})_3, \mathbf{u})_1$	$((\mathbf{h}_{2,8}, \mathbf{k}_{2,4})_4, \mathbf{u})_2$
	$((\mathbf{h}_{2,8}, \mathbf{v}^2)_6, \mathbf{u})_2$	$((\mathbf{h}_{2,4}, \mathbf{k}_{2,4})_2, \mathbf{u})_2$	$((\mathbf{h}_{3,2}, \mathbf{v})_1, \mathbf{u})_2$	$((\mathbf{h}_{3,6}, \mathbf{v})_4, \mathbf{u})_1$
	$((\mathbf{h}_{3,2}, \mathbf{v})_2, \mathbf{u})_1$	$((\mathbf{h}_{3,6}, \mathbf{v})_3, \mathbf{u})_2$	$((\mathbf{h}_{3,8}, \mathbf{v})_4, \mathbf{u})_2$	
Degree 6	$((\mathbf{f}, \mathbf{k}_{2,4})_1, \mathbf{u}^3)_6$	$((\mathbf{f}, \mathbf{k}_{2,4}^2)_5, \mathbf{u})_2$	$((\mathbf{f}, \mathbf{k}_{2,4}^2)_6, \mathbf{u})_1$	$((\mathbf{h}_{2,8}, \mathbf{v})_2, \mathbf{u}^3)_6$
	$((\mathbf{h}_{2,8}, \mathbf{v}^2)_5, \mathbf{u}^2)_4$	$((\mathbf{h}_{2,8}, \mathbf{k}_{2,4})_4, \mathbf{u}^2)_3$	$((\mathbf{h}_{2,8}, \mathbf{k}_{2,4})_3, \mathbf{u}^2)_4$	$((\mathbf{h}_{2,4}, \mathbf{k}_{2,4})_1, \mathbf{u}^2)_4$
	$((\mathbf{h}_{2,8}, \mathbf{v} \cdot \mathbf{k}_{2,4})_7, \mathbf{u})_1$	$((\mathbf{h}_{2,8}, \mathbf{k}_{3,6})_5, \mathbf{u})_2$	$((\mathbf{h}_{2,8}, \mathbf{k}_{3,6})_6, \mathbf{u})_1$	$((\mathbf{h}_{2,8}, \mathbf{v} \cdot \mathbf{k}_{2,4})_6, \mathbf{u})_2$
	$((\mathbf{h}_{3,8}, \mathbf{v})_3, \mathbf{u}^2)_4$	$((\mathbf{h}_{3,6}, \mathbf{v})_3, \mathbf{u}^2)_3$	$((\mathbf{h}_{3,2}, \mathbf{k}_{2,4})_1, \mathbf{u})_2$	$((\mathbf{h}_{3,8}, \mathbf{k}_{2,4})_4, \mathbf{u})_2$

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	$((\mathbf{h}_{3,6}, \mathbf{v}^2)_5, \mathbf{u})_2$ $((\mathbf{h}_{3,6}, \mathbf{v}^2)_6, \mathbf{u})_1$ $((\mathbf{h}_{3,6}, \mathbf{k}_{2,4})_3, \mathbf{u})_2$ $((\mathbf{h}_{3,2}, \mathbf{k}_{2,4})_2, \mathbf{u})_1$
	$((\mathbf{h}_{3,12}, \mathbf{v}^2)_8, \mathbf{u})_2$ $((\mathbf{h}_{3,8}, \mathbf{v}^2)_7, \mathbf{u})_1$ $((\mathbf{h}_{4,6}, \mathbf{v})_3, \mathbf{u})_2$ $((\mathbf{h}_{4,6}, \mathbf{v})_4, \mathbf{u})_1$
	$((\mathbf{h}_{4,4}, \mathbf{v})_2, \mathbf{u})_2$
Degree 7	$((\mathbf{h}_{2,8}, \mathbf{v})_1, \mathbf{u}^4)_8$ $((\mathbf{h}_{3,8}, \mathbf{v})_2, \mathbf{u}^3)_6$ $((\mathbf{h}_{3,12}, \mathbf{v})_4, \mathbf{u}^3)_6$ $((\mathbf{h}_{3,12}, \mathbf{v} \cdot \mathbf{k}_{2,4})_8, \mathbf{u})_2$
	$((\mathbf{h}_{3,6}, \mathbf{k}_{3,6})_4, \mathbf{u})_2$ $((\mathbf{h}_{4,6}, \mathbf{v})_2, \mathbf{u}^2)_4$ $((\mathbf{h}_{4,4}, \mathbf{v})_1, \mathbf{u}^2)_4$ $((\mathbf{h}_{4,10}, \mathbf{v})_4, \mathbf{u}^2)_4$
	$((\mathbf{h}_{4,6}, \mathbf{v}^2)_6, \mathbf{u})_1$ $((\mathbf{h}_{4,6}, \mathbf{k}_{2,4})_3, \mathbf{u})_2$ $((\mathbf{h}_{5,4}, \mathbf{v})_2, \mathbf{u})_2$ $((\mathbf{h}_{5,8}, \mathbf{v})_4, \mathbf{u})_2$
	$((\mathbf{h}_{5,4}, \mathbf{v})_3, \mathbf{u})_1$ $((\mathbf{h}_{5,2}, \mathbf{v})_1, \mathbf{u})_2$
Degree 8	$((\mathbf{h}_{5,8}, \mathbf{k}_{2,4})_4, \mathbf{u})_2$
Degree 9	$((\mathbf{h}_{7,4}, \mathbf{v})_2, \mathbf{u})_2$
80 covariants of order 4 : 5 from S_6 , 2 from S_4 , 15 joint covariants of $S_6 \oplus S_2$ given in 8.1, 31 joint covariants of $S_6 \oplus S_4$ given in 8.2. There is left 27 covariants given below:	
Degree 2	$(\mathbf{v}, \mathbf{u})_1$
Degree 3	$(\mathbf{k}_{2,4}, \mathbf{u})_1$ $((\mathbf{f}, \mathbf{v})_2, \mathbf{u})_2$ $((\mathbf{f}, \mathbf{v})_3, \mathbf{u})_1$
Degree 4	$(\mathbf{k}_{3,6}, \mathbf{u})_2$ $((\mathbf{f}, \mathbf{v})_2, \mathbf{u}^2)_3$ $((\mathbf{f}, \mathbf{v})_1, \mathbf{u}^2)_4$ $((\mathbf{f}, \mathbf{k}_{2,4})_3, \mathbf{u})_1$
	$((\mathbf{f}, \mathbf{k}_{2,4})_2, \mathbf{u})_2$ $((\mathbf{h}_{2,8}, \mathbf{v})_4, \mathbf{u})_1$ $((\mathbf{h}_{2,4}, \mathbf{v})_2, \mathbf{u})_1$ $((\mathbf{h}_{2,8}, \mathbf{v})_3, \mathbf{u})_2$
	$((\mathbf{h}_{2,4}, \mathbf{v})_1, \mathbf{u})_2$
Degree 5	$((\mathbf{f}, \mathbf{k}_{2,4})_1, \mathbf{u}^2)_4$ $((\mathbf{h}_{2,8}, \mathbf{v})_2, \mathbf{u}^2)_4$ $((\mathbf{h}_{2,8}, \mathbf{k}_{2,4})_4, \mathbf{u})_1$ $((\mathbf{h}_{2,8}, \mathbf{v}^2)_5, \mathbf{u})_2$
	$((\mathbf{h}_{2,8}, \mathbf{k}_{2,4})_3, \mathbf{u})_2$ $((\mathbf{h}_{2,4}, \mathbf{k}_{2,4})_1, \mathbf{u})_2$ $((\mathbf{h}_{3,6}, \mathbf{v})_3, \mathbf{u})_1$ $((\mathbf{h}_{3,8}, \mathbf{v})_3, \mathbf{u})_2$
Degree 6	$((\mathbf{h}_{2,8}, \mathbf{v})_1, \mathbf{u}^3)_6$ $((\mathbf{h}_{3,8}, \mathbf{v})_2, \mathbf{u}^2)_4$ $((\mathbf{h}_{3,12}, \mathbf{v})_4, \mathbf{u}^2)_4$ $((\mathbf{h}_{4,10}, \mathbf{v})_4, \mathbf{u})_2$
	$((\mathbf{h}_{4,4}, \mathbf{v})_1, \mathbf{u})_2$ $((\mathbf{h}_{4,6}, \mathbf{v})_2, \mathbf{u})_2$
32 covariants of order 6 : 5 from S_6 , 1 from S_4 , 8 joint covariants of $S_6 \oplus S_2$ given in 8.1, 11 joint covariants of $S_6 \oplus S_4$ given in 8.2. There is left 7 covariants given below:	
Degree 3	$((\mathbf{f}, \mathbf{v})_2, \mathbf{u})_1$ $((\mathbf{f}, \mathbf{v})_1, \mathbf{u})_2$
Degree 4	$((\mathbf{f}, \mathbf{k}_{2,4})_1, \mathbf{u})_2$ $((\mathbf{h}_{2,8}, \mathbf{v})_2, \mathbf{u})_2$
Degree 5	$((\mathbf{h}_{2,8}, \mathbf{v})_1, \mathbf{u}^2)_4$ $((\mathbf{h}_{3,12}, \mathbf{v})_4, \mathbf{u})_2$ $((\mathbf{h}_{3,8}, \mathbf{v})_2, \mathbf{u})_2$
12 covariants of order 8 : 3 from S_6 , 3 joint covariants of $S_6 \oplus S_2$ given in 8.1, 5 joint covariants of $S_6 \oplus S_4$ given in 8.2. There is left 1 covariant given below:	
degree 4	$((\mathbf{h}_{2,8}, \mathbf{v})_1, \mathbf{u})_2$

There is left 3 covariants of order 10 : 1 from S_6 , 1 joint covariant of $S_6 \oplus S_2$ given in 8.1 and 1 joint covariant of $S_6 \oplus S_4$ given in 8.2. Finally there is 1 covariant of order 12 taken from S_6 .

8.4. Covariant bases of S_8 . We apply here Gordan's algorithm for a simple binary form.

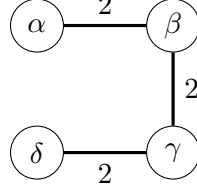
- (1) As a first step $A_0 = \{\mathbf{f}\}$ for $\mathbf{f} \in S_8$. The family B_0 only contains the covariant

$$\mathbf{h}_{2,12} := \{\mathbf{f}, \mathbf{f}\}_2 \in S_{12}.$$

- (2) To obtain A_1 we have to consider transvectants

$$(\mathbf{f}^a, \mathbf{h}_{2,12}^b)_r,$$

which contain no reducible molecular covariants modulo I_4 . From (6.2) we deduce that necessarily $r \leq 2$. Take now a molecule



Using lemma A.4 with $e_0 = 2$ and $e_1 = 2$, this molecule is of grade 3 and thus by (6.2) of grade 4.

We can deduce from all this that the family A_1 is

$$\mathbf{f}, \mathbf{h}_{2,12}, \mathbf{h}_{3,18} := \{\mathbf{f}, \mathbf{h}_{2,12}\}_1$$

and family B_1 only contains the covariant

$$\mathbf{h}_{2,8} := \{\mathbf{f}, \mathbf{f}\}_4 \in S_8.$$

(3) To get the system A_2 we have to consider transvectants

$$(\mathbf{f}^{a_1} \mathbf{h}_{2,12}^{a_2} \mathbf{h}_{3,18}^{a_3}, \mathbf{h}_{2,8}^b)_r.$$

The same kind of argument as above, using lemma such as lemma A.4 leads to [36, 35]:

Lemma 8.5. *The family A_2 is given by the seven covariants*

$$\mathbf{f}, \mathbf{h}_{2,8} = (\mathbf{f}, \mathbf{f})_4, \quad \mathbf{h}_{2,12} = (\mathbf{f}, \mathbf{f})_2, \quad \mathbf{h}_{3,12} := (\mathbf{f}, \mathbf{h}_{2,8})_2, \quad \mathbf{h}_{3,14} := (\mathbf{f}, \mathbf{h}_{2,8})_1$$

$$\mathbf{h}_{3,18} := (\mathbf{f}, \mathbf{h}_{2,12})_1, \quad \mathbf{h}_{4,18} := (\mathbf{h}_{2,12}, \mathbf{h}_{2,8})_1$$

Recall also that we have to consider the invariant

$$(\mathbf{f}, \mathbf{h}_{2,8})_8.$$

Now, the family B_2 is given by one covariant bases of

$$\mathbf{h}_{2,4} := (\mathbf{f}, \mathbf{f})_6 \in S_4.$$

As seen above in subsection 8.2, such a covariant bases is given by:

$$\mathbf{h}_{2,4}, \quad \mathbf{h}_{4,4} := (\mathbf{h}_{2,4}, \mathbf{h}_{2,4})_2, \quad \mathbf{h}_{6,6} := (\mathbf{h}_{2,4}, (\mathbf{h}_{2,4}, \mathbf{h}_{2,4})_2)_1$$

and two invariants

$$\mathbf{h}_{4,0} := (\mathbf{h}_{2,4}, \mathbf{h}_{2,4})_4, \quad \mathbf{h}_{6,0} := (\mathbf{h}_{2,4}, (\mathbf{h}_{2,4}, \mathbf{h}_{2,4})_2)_4.$$

(4) To get the family B_3 , we have to consider transvectants

$$(\mathbf{f}^{a_1} \mathbf{h}_{2,8}^{a_2} \mathbf{h}_{2,12}^{a_3} \mathbf{h}_{3,12}^{a_4} \mathbf{h}_{3,14}^{a_5} \mathbf{h}_{3,18}^{a_6} \mathbf{h}_{4,18}^{a_7}, \mathbf{h}_{2,4}^{b_1} \mathbf{h}_{4,4}^{b_2} \mathbf{h}_{6,6}^{b_3})_r$$

which is associated to the integer system

$$\begin{cases} 8a_1 + 8a_2 + 12a_3 + 12a_4 + 14a_5 + 18a_6 + 18a_7 & = u + r \\ 4b_1 + 4b_2 + 6b_3 & = v + r \end{cases} \quad (8.2)$$

We also make use of the relation (8.1) in $\mathbf{Cov}(S_4)$, thus we can apply theorem 7.8. With computations made in Macaulay2 [37], we finally get a covariant bases of S_8 given bellow.

8 invariants	
Degree 2	$\mathbf{h}_{2,0} := (\mathbf{f}, \mathbf{f})_8$
Degree 3	$(\mathbf{f}, \mathbf{h}_{2,8})_8$
Degree 4	$(\mathbf{h}_{2,4}, \mathbf{h}_{2,4})_4$
Degree 5	$(\mathbf{f}, \mathbf{h}_{2,4}^2)_8$
Degree 6	$(\mathbf{h}_{4,4}, \mathbf{h}_{2,4})_4$
Degree 7	$(\mathbf{f}, \mathbf{h}_{2,4}\mathbf{h}_{4,4})_8$
Degree 8	$(\mathbf{h}_{2,12}, \mathbf{h}_{2,4}^3)_{12}$
Degree 9	$(\mathbf{h}_{3,12}, \mathbf{h}_{2,4}^3)_{12}$
Degree 10	$(\mathbf{h}_{2,12}, \mathbf{h}_{2,4}^2\mathbf{h}_{4,4})_{12}$
14 covariants of order 2.	
Degree 5	$(\mathbf{f}, \mathbf{h}_{2,4}^2)_7$
Degree 6	$(\mathbf{h}_{2,8}, \mathbf{h}_{2,4}^2)_7$
Degree 7	$(\mathbf{f}, \mathbf{h}_{6,6})_6 \quad (\mathbf{f}, \mathbf{h}_{2,4}\mathbf{h}_{4,4})_7$
Degree 8	$(\mathbf{h}_{2,12}, \mathbf{h}_{2,4}^3)_{11} \quad (\mathbf{h}_{2,8}, \mathbf{h}_{6,6})_6$
Degree 9	$(\mathbf{h}_{3,14}, \mathbf{h}_{2,4}^3)_{12} \quad (\mathbf{h}_{3,12}, \mathbf{h}_{2,4}^3)_{11} \quad (\mathbf{f}, \mathbf{h}_{4,4}^2)_7$
Degree 10	$(\mathbf{h}_{2,12}, \mathbf{h}_{2,4}\mathbf{h}_{6,6})_{10} \quad (\mathbf{h}_{2,12}, \mathbf{h}_{2,4}^2\mathbf{h}_{4,4})_{11}$
Degree 11	$(\mathbf{h}_{3,18}, \mathbf{h}_{2,4}^4)_{16} \quad (\mathbf{h}_{3,14}, \mathbf{h}_{2,4}^2\mathbf{h}_{4,4})_{12}$
Degree 12	$(\mathbf{h}_{4,18}, \mathbf{h}_{2,4}^4)_{16}$
13 covariants of order 4.	
Degree 2	$\mathbf{h}_{2,4} := (\mathbf{f}, \mathbf{f})_6$
Degree 3	$(\mathbf{f}, \mathbf{h}_{2,4})_4$
Degree 4	$\mathbf{h}_{4,4} := (\mathbf{h}_{2,4}, \mathbf{h}_{2,4})_2$ $(\mathbf{h}_{2,8}, \mathbf{h}_{2,4})_4$
Degree 5	$(\mathbf{f}, \mathbf{h}_{4,4})_4 \quad (\mathbf{f}, \mathbf{h}_{2,4}^2)_6$
Degree 6	$(\mathbf{h}_{2,12}, \mathbf{h}_{2,4}^2)_8 \quad (\mathbf{h}_{2,8}, \mathbf{h}_{4,4})_4$
Degree 7	$(\mathbf{h}_{3,12}, \mathbf{h}_{2,4}^2)_8 \quad (\mathbf{f}, \mathbf{h}_{6,6})_5$
Degree 8	$(\mathbf{h}_{2,12}, \mathbf{h}_{2,4}\mathbf{h}_{4,4})_8 \quad (\mathbf{h}_{2,12}, \mathbf{h}_{2,4}^3)_{10}$
Degree 9	$(\mathbf{h}_{3,14}, \mathbf{h}_{2,4}^3)_{11}$
12 covariants of order 6.	
Degree 3	$(\mathbf{f}, \mathbf{h}_{2,4})_3$
Degree 4	$(\mathbf{h}_{2,8}, \mathbf{h}_{2,4})_3$
Degree 5	$(\mathbf{f}, \mathbf{h}_{4,4})_3 \quad (\mathbf{f}, \mathbf{h}_{2,4}^2)_5$
Degree 6	$\mathbf{h}_{6,6} := (\mathbf{h}_{4,4}, \mathbf{h}_{2,4})_1 \quad (\mathbf{h}_{2,12}, \mathbf{h}_{2,4}^2)_7 \quad (\mathbf{h}_{2,8}, \mathbf{h}_{4,4})_3$
Degree 7	$(\mathbf{h}_{3,14}, \mathbf{h}_{2,4}^2)_8 \quad (\mathbf{h}_{3,12}, \mathbf{h}_{2,4}^2)_7 \quad (\mathbf{f}, \mathbf{h}_{6,6})_4$

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Degree 8	$(\mathbf{h}_{2,12}, \mathbf{h}_{6,6})_6$ $(\mathbf{h}_{2,12}, \mathbf{h}_{2,4}\mathbf{h}_{4,4})_7$
6 covariants of order 8.	
Degree 1	\mathbf{f}
Degree 2	$\mathbf{h}_{2,8}$
Degree 3	$(\mathbf{f}, \mathbf{h}_{2,4})_2$
Degree 4	$(\mathbf{h}_{2,12}, \mathbf{h}_{2,4})_4$
Degree 5	$(\mathbf{f}, \mathbf{h}_{4,4})_2$
Degree 6	$(\mathbf{h}_{2,12}, \mathbf{h}_{4,4})_4$
7 covariants of order 10.	
Degree 3	$(\mathbf{f}, \mathbf{h}_{2,4})_1$
Degree 4	$(\mathbf{h}_{2,12}, \mathbf{h}_{2,4})_3$ $(\mathbf{h}_{2,8}, \mathbf{h}_{2,4})_1$
Degree 5	$(\mathbf{h}_{3,14}, \mathbf{h}_{2,4})_4$ $(\mathbf{h}_{3,12}, \mathbf{h}_{2,4})_3$
	$(\mathbf{f}, \mathbf{h}_{4,4})_1$
Degree 6	$(\mathbf{h}_{2,12}, \mathbf{h}_{4,4})_3$
3 covariants of order 12.	
Degree 2	$\mathbf{h}_{2,12}$
Degree 3	$\mathbf{h}_{3,12} := (\mathbf{f}, \mathbf{h}_{2,8})_2$
Degree 4	$(\mathbf{h}_{2,12}, \mathbf{h}_{2,4})_2$
3 covariants of order 14.	
Degree 3	$\mathbf{h}_{3,14} := (\mathbf{f}, \mathbf{h}_{2,8})_1$
Degree 4	$(\mathbf{h}_{2,12}, \mathbf{h}_{2,4})_1$
Degree 5	$(\mathbf{h}_{3,12}, \mathbf{h}_{2,4})_1$
2 covariants of order 18.	
Degree 3	$\mathbf{h}_{3,18} := (\mathbf{f}, \mathbf{h}_{2,12})_1$
Degree 4	$(\mathbf{h}_{2,12}, \mathbf{h}_{2,8})_1$

8.5. Invariant bases of $S_8 \oplus S_4 \oplus S_4$. For such an invariant bases, we apply theorem 7.6 with $V_1 = S_8$ and $V_2 = S_4 \oplus S_4$. Note here that we make use of a different covariant bases¹⁵ of $\mathbf{Cov}(S_8)$ than the one given in subsection 8.4.

All the covariant basis of $\mathbf{Cov}(S_8)$ and $\mathbf{Cov}(S_4 \oplus S_4)$ are given in tables 9 and 10. In those tables, \mathbf{h}_n (resp. \mathbf{f}_n) denotes the covariant of $S_4 \oplus S_4$ (resp. S_8) defined by line numbered n .

¹⁵Such a covariant bases had been used to obtain a covariant bases of $\mathbf{Cov}(S_{10})$ [51].

d/o	0	2	4	6	#	Cum
1	-	-	2	-	2	2
2	3	1	3	1	8	10
3	4	2	2	4	12	22
4	1	3	-	-	4	26
5	-	2	-	-	2	28
Tot	8	8	7	5		28

Number	Covariant	(d_1, d_2, o)	Number	Covariant	(d_1, d_2, o)
1	\mathbf{v}_1	(1, 0, 4)	15	$(\mathbf{v}_1, \mathbf{h}_8)_3$	(1, 2, 2)
2	\mathbf{v}_2	(0, 1, 4)	16	$(\mathbf{v}_2, \mathbf{h}_7)_3$	(2, 1, 2)
3	$(\mathbf{v}_1, \mathbf{v}_1)_4$	(2, 0, 0)	17	$(\mathbf{v}_1, \mathbf{h}_8)_2$	(1, 2, 4)
4	$(\mathbf{v}_2, \mathbf{v}_2)_4$	(0, 2, 0)	18	$(\mathbf{v}_2, \mathbf{h}_7)_2$	(2, 1, 4)
5	$(\mathbf{v}_1, \mathbf{v}_2)_4$	(1, 1, 0)	19	$(\mathbf{v}_1, \mathbf{h}_7)_1$	(3, 0, 6)
6	$(\mathbf{v}_1, \mathbf{v}_2)_3$	(1, 1, 2)	20	$(\mathbf{v}_2, \mathbf{h}_8)_1$	(0, 3, 6)
7	$(\mathbf{v}_1, \mathbf{v}_1)_2$	(2, 0, 4)	21	$(\mathbf{v}_1, \mathbf{h}_8)_1$	(1, 2, 6)
8	$(\mathbf{v}_2, \mathbf{v}_2)_2$	(0, 2, 4)	22	$(\mathbf{v}_2, \mathbf{h}_7)_1$	(2, 1, 6)
9	$(\mathbf{v}_1, \mathbf{v}_2)_2$	(1, 1, 4)	23	$(\mathbf{h}_7, \mathbf{h}_8)_4$	(2, 2, 0)
10	$(\mathbf{v}_1, \mathbf{v}_2)_1$	(1, 1, 6)	24	$(\mathbf{h}_7, \mathbf{h}_8)_3$	(2, 2, 2)
11	$(\mathbf{v}_1, \mathbf{h}_7)_4$	(3, 0, 0)	25	$(\mathbf{h}_{19}, \mathbf{v}_2)_4$	(3, 1, 2)
12	$(\mathbf{v}_2, \mathbf{h}_8)_4$	(0, 3, 0)	26	$(\mathbf{v}_1, \mathbf{h}_{20})_4$	(1, 3, 2)
13	$(\mathbf{v}_1, \mathbf{h}_8)_4$	(1, 2, 0)	27	$(\mathbf{v}_1^2, \mathbf{h}_{20})_6$	(2, 3, 2)
14	$(\mathbf{v}_2, \mathbf{h}_7)_4$	(2, 1, 0)	28	$(\mathbf{h}_{19}, \mathbf{v}_2^2)_6$	(3, 2, 2)

TABLE 9. Covariant bases of $\mathbf{Cov}(S_4 \oplus S_4)$.

Number	Covariant	(d,o)	Number	Covariant	(d,o)	Number	Covariant	(d,o)
1	\mathbf{f}	(1, 8)	26	$(\mathbf{f}_{21}, \mathbf{f})_8$	(5, 4)	51	$(\mathbf{f}_{41}, \mathbf{f})_4$	(7, 6)
2	$(\mathbf{f}, \mathbf{f})_8$	(2, 0)	27	$(\mathbf{f}_{20}, \mathbf{f})_7$	(5, 4)	52	$(\mathbf{f}_7 \mathbf{f}_{16}, \mathbf{f})_8$	(8, 0)
3	$(\mathbf{f}, \mathbf{f})_6$	(2, 4)	28	$(\mathbf{f}_{22}, \mathbf{f})_8$	(5, 6)	53	$(\mathbf{f}_{51}, \mathbf{f})_6$	(8, 2)
4	$(\mathbf{f}, \mathbf{f})_4$	(2, 8)	29	$(\mathbf{f}_{21}, \mathbf{f})_7$	(5, 6)	54	$(\mathbf{f}_{50}, \mathbf{f})_6$	(8, 2)
5	$(\mathbf{f}, \mathbf{f})_2$	(2, 12)	30	$(\mathbf{f}_{22}, \mathbf{f})_7$	(5, 8)	55	$(\mathbf{f}_{51}, \mathbf{f})_5$	(8, 4)
6	$(\mathbf{f}_4, \mathbf{f})_8$	(3, 0)	31	$(\mathbf{f}_{23}, \mathbf{f})_8$	(5, 10)	56	$(\mathbf{f}_{50}, \mathbf{f})_5$	(8, 4)
7	$(\mathbf{f}_5, \mathbf{f})_8$	(3, 4)	32	$(\mathbf{f}_{22}, \mathbf{f})_6$	(5, 10)	57	$(\mathbf{f}_{51}, \mathbf{f})_4$	(8, 6)
8	$(\mathbf{f}_5, \mathbf{f})_7$	(3, 6)	33	$(\mathbf{f}_{21}, \mathbf{f})_5$	(5, 10)	58	$(\mathbf{f}_{50}, \mathbf{f})_4$	(8, 6)
9	$(\mathbf{f}_5, \mathbf{f})_6$	(3, 8)	34	$(\mathbf{f}_{23}, \mathbf{f})_6$	(5, 14)	59	$(\mathbf{f}_{15} \mathbf{f}_{16}, \mathbf{f})_8$	(9, 0)
10	$(\mathbf{f}_5, \mathbf{f})_5$	(3, 10)	35	$(\mathbf{f}_3 \mathbf{f}_7, \mathbf{f})_8$	(6, 0)	60	$(\mathbf{f}_{58}, \mathbf{f})_6$	(9, 2)
11	$(\mathbf{f}_5, \mathbf{f})_4$	(3, 12)	36	$(\mathbf{f}_{33}, \mathbf{f})_8$	(6, 2)	61	$(\mathbf{f}_{57}, \mathbf{f})_6$	(9, 2)
12	$(\mathbf{f}_5, \mathbf{f})_3$	(3, 14)	37	$(\mathbf{f}_{33}, \mathbf{f})_7$	(6, 4)	62	$(\mathbf{f}_{16} \mathbf{f}_{17}, \mathbf{f})_8$	(9, 2)
13	$(\mathbf{f}_5, \mathbf{f})_1$	(3, 18)	38	$(\mathbf{f}_{32}, \mathbf{f})_7$	(6, 4)	63	$(\mathbf{f}_{58}, \mathbf{f})_5$	(9, 4)
14	$(\mathbf{f}_9, \mathbf{f})_8$	(4, 0)	39	$(\mathbf{f}_{34}, \mathbf{f})_8$	(6, 6)	64	$(\mathbf{f}_{17} \mathbf{f}_{25}, \mathbf{f})_8$	(10, 0)
15	$(\mathbf{f}_{11}, \mathbf{f})_8$	(4, 4)	40	$(\mathbf{f}_{33}, \mathbf{f})_6$	(6, 6)	65	$(\mathbf{f}_{17} \mathbf{f}_{27}, \mathbf{f})_8$	(10, 2)
16	$(\mathbf{f}_{10}, \mathbf{f})_7$	(4, 4)	41	$(\mathbf{f}_{32}, \mathbf{f})_6$	(6, 6)	66	$(\mathbf{f}_{17} \mathbf{f}_{26}, \mathbf{f})_8$	(10, 2)
17	$(\mathbf{f}_{12}, \mathbf{f})_8$	(4, 6)	42	$(\mathbf{f}_{34}, \mathbf{f})_7$	(6, 8)	67	$(\mathbf{f}_{27} \mathbf{f}_{29}, \mathbf{f})_8$	(11, 2)
18	$(\mathbf{f}_{12}, \mathbf{f})_7$	(4, 8)	43	$(\mathbf{f}_{34}, \mathbf{f})_6$	(6, 10)	68	$(\mathbf{f}_{27} \mathbf{f}_{28}, \mathbf{f})_8$	(11, 2)
19	$(\mathbf{f}_{13}, \mathbf{f})_8$	(4, 10)	44	$(\mathbf{f}_7^2, \mathbf{f})_8$	(7, 0)	69	$(\mathbf{f}_{29} \mathbf{f}_{38}, \mathbf{f})_8$	(12, 2)
20	$(\mathbf{f}_{12}, \mathbf{f})_6$	(4, 10)	45	$(\mathbf{f}_{43}, \mathbf{f})_8$	(7, 2)			
21	$(\mathbf{f}_{13}, \mathbf{f})_7$	(4, 12)	46	$(\mathbf{f}_{42}, \mathbf{f})_7$	(7, 2)			
22	$(\mathbf{f}_{13}, \mathbf{f})_6$	(4, 14)	47	$(\mathbf{f}_{43}, \mathbf{f})_7$	(7, 4)			
23	$(\mathbf{f}_{13}, \mathbf{f})_4$	(4, 18)	48	$(\mathbf{f}_{42}, \mathbf{f})_6$	(7, 4)			
24	$(\mathbf{f}_3^2, \mathbf{f})_8$	(5, 0)	49	$(\mathbf{f}_{43}, \mathbf{f})_6$	(7, 6)			
25	$(\mathbf{f}_{20}, \mathbf{f})_8$	(5, 2)	50	$(\mathbf{f}_{42}, \mathbf{f})_5$	(7, 6)			

TABLE 10. Covariant bases of $\mathbf{Cov}(S_8)$.

To get an invariant bases of the invariant algebra $\mathbf{Inv}(S_8 \oplus S_4 \oplus S_4)$, we use the same strategy as the one used for the computation of a covariant bases of $\mathbf{Cov}(S_6 \oplus S_4)$ (see [subsection 8.2](#)). We give now details of the three steps of this strategy.

Resolution of the associated linear Diophantine system. To apply Gordan's algorithm given in theorem [7.6](#), we have to solve a linear Diophantine system associated to covariant *orders* (and thus excluding invariants) given by tables [9](#) and [10](#).

We thus have a linear Diophantine system with 81 unknowns:

$$(S_e) : \begin{cases} 2x_{1,1} + \dots + 2x_{14,1} + 4x_{1,2} + \dots + 4x_{13,2} + 6x_{1,3} + \dots + 6x_{12,3} + \\ 8x_{1,4} + \dots + 8x_{6,4} + 10x_{1,5} + \dots + 10x_{7,5} + 12x_{1,6} + \dots + 12x_{3,6} + \\ 14x_{1,7} + \dots + 14x_{3,7} + 18x_{1,8} + 18x_{2,8} = r \\ 2y_{1,1} + \dots + 2y_{8,1} + 4y_{1,2} + \dots + 4y_{7,2} + 6y_{1,3} + \dots + 6y_{5,3} = r \end{cases}$$

To solve such a system, we use Clausen–Fortenbacher's result [21] on reduced systems. First note $(a_1, a_2, \dots, a_8, b_1, b_2, b_3)$ the irreducible solutions of the reduced system:

$$(S') : \begin{cases} 2X_1 + 4X_2 + 6X_3 + 8X_4 + 10X_5 + 12X_6 + 14X_7 + 18X_8 = r \\ 2Y_1 + 4Y_2 + 6Y_3 = r \end{cases}$$

We then get irreducible solutions of the initial system (S_e) by solving all the systems

$$\begin{cases} x_{1,1} + \dots + x_{14,1} = a_1 \\ \dots \\ x_{1,8} + x_{2,8} = a_8 \\ y_{1,1} + \dots + y_{8,1} = b_1 \\ y_{1,2} + \dots + y_{7,2} = b_2 \\ y_{1,3} + \dots + y_{5,3} = b_3 \end{cases}$$

Note also that to get irreducible solutions of the reduced system (S') , we used Normaliz package [18] in Macaulay2 [37]. Finally we have:

Lemma 8.6. *The system (S_e) has 695 754 irreducible integer solutions, corresponding to invariants from degree 3 to degree 49.*

Relations on $\mathbf{Cov}(S_8)$ and $\mathbf{Cov}(S_4 \oplus S_4)$. Take here the minimal bases of $\mathbf{Cov}(S_8)$ given in table 10 and take the lexicographic order:

$$\begin{aligned} & \mathbf{h}_{69} > \mathbf{h}_{67} > \mathbf{h}_{68} > \mathbf{h}_{66} > \mathbf{h}_{65} > \mathbf{h}_{64} > \mathbf{h}_{63} > \mathbf{h}_{62} > \mathbf{h}_{61} > \mathbf{h}_{60} > \mathbf{h}_{59} > \mathbf{h}_{58} > \\ & \mathbf{h}_{57} > \mathbf{h}_{55} > \mathbf{h}_{56} > \mathbf{h}_{53} > \mathbf{h}_{54} > \mathbf{h}_{52} > \mathbf{h}_{49} > \mathbf{h}_{50} > \mathbf{h}_{51} > \mathbf{h}_{48} > \mathbf{h}_{47} > \mathbf{h}_{45} > \\ & \mathbf{h}_{46} > \mathbf{h}_{43} > \mathbf{h}_{42} > \mathbf{h}_{40} > \mathbf{h}_{41} > \mathbf{h}_{39} > \mathbf{h}_{37} > \mathbf{h}_{38} > \mathbf{h}_{36} > \mathbf{h}_{34} > \mathbf{h}_{32} > \mathbf{h}_{33} > \\ & \mathbf{h}_{31} > \mathbf{h}_{30} > \mathbf{h}_{28} > \mathbf{h}_{29} > \mathbf{h}_{27} > \mathbf{h}_{26} > \mathbf{h}_{25} > \mathbf{h}_{23} > \mathbf{h}_{22} > \mathbf{h}_{21} > \mathbf{h}_{19} > \mathbf{h}_{20} > \\ & \mathbf{h}_{18} > \mathbf{h}_{17} > \mathbf{h}_{15} > \mathbf{h}_{16} > \mathbf{h}_{13} > \mathbf{h}_{12} > \mathbf{h}_{11} > \mathbf{h}_{10} > \mathbf{h}_9 > \mathbf{h}_8 > \mathbf{h}_7 > \mathbf{h}_5 > \\ & \mathbf{h}_4 > \mathbf{h}_3 > \mathbf{h}_1 \end{aligned}$$

An algorithm developed by Lercier [51] leads to:

Lemma 8.7. *There exists 1723 relations \mathcal{R}_i and \mathcal{R}'_j which verify hypotheses 7.1 and 7.3.*

By theorem 7.6, we thus reduce the family of 695 754 invariants to a family of 508 021 invariants.

Using scripts written in Macaulay 2 and direct computations, we found:

Lemma 8.8. *There exists 179 monomial covariants $\mathbf{m} \in \mathbf{Cov}(S_4 \oplus S_4)$ contained in the invariant ideal of $\mathbf{Cov}(S_4 \oplus S_4)$. Furthermore, there exists 98 relations*

$$\mathbf{V} = \sum \mathbf{V}_i, \quad \mathbf{V}, \mathbf{V}_i \text{ monomials in } \mathbf{Cov}(S_4 \oplus S_4)$$

From this lemma 8.8 and from lemmas 7.9 and 7.10, we get:

Lemma 8.9. *All invariants from degree 22 to degree 49 are reducibles.*

Proof. We have to consider invariants given as transvectants

$$(\mathbf{U}, \mathbf{V})_r, \quad r \leq 0$$

where \mathbf{U} (resp. \mathbf{V}) is a monomial in $\mathbf{Cov}(S_8)$ (resp. $\mathbf{Cov}(S_4 \oplus S_4)$). From lemma 7.9, we know that every time one monomial \mathbf{m} (given by one of the 179 first relations of lemma 8.8) divide \mathbf{V} , then the invariant is a reducible one. We get here a first reduction process. For instance, for degree 26 invariants, we initially have 20 392 invariants, and this first reduction leads to 1822 invariants. We now use lemma 7.10 for a second reduction process. For degree 26 invariant, we have for example to consider the invariant

$$(\mathbf{f}_{18}\mathbf{f}_{12}\mathbf{f}_{22}, \mathbf{h}_{10}^3\mathbf{h}_{19}^3)_{36}. \quad (8.3)$$

In that case, we have the relation

$$12\mathbf{h}_{19}^2 + 6\mathbf{h}_7^3 + 2\mathbf{h}_{11}\mathbf{h}_1^3 - 3\mathbf{h}_3\mathbf{h}_1^2\mathbf{h}_7 = 0$$

which leads to consider invariants

$$(\mathbf{f}_{18}\mathbf{f}_{12}\mathbf{f}_{22}, \mathbf{h}_{10}^3\mathbf{h}_{19}\mathbf{h}_7^3)_{36}, \quad (\mathbf{f}_{18}\mathbf{f}_{12}\mathbf{f}_{22}, \mathbf{h}_{10}^3\mathbf{h}_{19}\mathbf{h}_{11}\mathbf{h}_1^3)_{36}, \quad (\mathbf{f}_{18}\mathbf{f}_{12}\mathbf{f}_{22}, \mathbf{h}_{10}^3\mathbf{h}_{19}\mathbf{h}_3\mathbf{h}_1^2\mathbf{h}_7)_{36}$$

where \mathbf{h}_{11} and \mathbf{h}_3 are invariants (thus the two last transvectants are reducible). By a direct computation, we can check that the transvectant

$$(\mathbf{f}_{18}\mathbf{f}_{12}\mathbf{f}_{22}, \mathbf{h}_{10}^3\mathbf{h}_{19}\mathbf{h}_7^3)_{36}$$

correspond to a reducible integer solution. Using lemma 7.10 we thus deduce that transvectant (8.3) is expressible in terms of reduced invariant and lower index transvectants. We then use the same arguments for lower index transvectants, which are all reducible. \square

Now there still remain 257 770 invariants, from degree 3 to degree 21. Direct computation in the algebra $\mathbf{Cov}(S_8)$ leads to:

Lemma 8.10. *There exists 4085 monomial covariants $\overline{\mathbf{m}} \in \mathbf{Cov}(S_8)$ contained in the invariant ideal of $\mathbf{Cov}(S_8)$. Furthermore, there exists 964 relations*

$$\mathbf{U} = \sum \mathbf{U}_i, \quad \mathbf{U}, \mathbf{U}_i \text{ monomials in } \mathbf{Cov}(S_8)$$

Using those relations and lemmas 7.9 and 7.10 thus lead to a third reduction process. We can now make use of the multigraduate Hilbert series of $\mathbf{Inv}(S_8 \oplus S_4 \oplus S_4)$ to get our final result.

Finite minimal bases of $\mathbf{Inv}(S_8 \oplus S_4 \oplus S_4)$.

Theorem 8.11. *The invariant algebra $\mathbf{Inv}(S_8 \oplus S_4 \oplus S_4)$ is generated by a minimal bases of 297 invariants, resumed¹⁶ in table 11.*

¹⁶We note here $\mathbf{Inv}_j(V_1 \oplus V_2)$ a set of joint invariants of degree $d_1 > 0$ and $d_2 > 0$ in V_1 and V_2 .

Degree	$\mathbf{Inv}(S_8)$	$\mathbf{Inv}(S_4)$	$\mathbf{Inv}_j(S_4 \oplus S_4)$	$\mathbf{Inv}_j(S_8 \oplus S_4)$	$\mathbf{Inv}_j(S_8 \oplus S_4 \oplus S_4)$
1	—	1	—	—	—
2	1	—	1	—	—
3	1	—	2	2	1
4	1	—	1	4	6
5	1	—	—	7	18
6	1	—	—	10	36
7	1	—	—	11	53
8	1	—	—	10	45
9	1	—	—	5	10
10	1	—	—	2	2
11	—	—	—	2	3
<i>Tot</i>	9	1	4	53	174

TABLE 11. Minimal bases of $\mathbf{Inv}(S_8 \oplus S_4 \oplus S_4)$

Proof. Let d_1 be the invariant degree in $\mathbf{f} \in S_8$, d_2 the degree in $\mathbf{v}_1 \in S_4$ and d_3 the degree in $\mathbf{v}_2 \in S_4$. Thus we have

$$\mathbf{Inv}(S_8 \oplus S_4 \oplus S_4) = \bigoplus_{d_1, d_2, d_3 \geq 0} \mathbf{Inv}_{d_1, d_2, d_3}(S_8 \oplus S_4 \oplus S_4)$$

Using multigraduated Hilbert series computed by Bedratyuk's Maple package [9], we can compute the minimal bases degree per degree, as explained in subsection 7.1. For instance, we have left 740 degree 12 invariants. Note that for each of those invariants, we know the associated homogeneous spaces, given in table 12.

d_1, d_2, d_3	Dimension	d_1, d_2, d_3	Dimension	d_1, d_2, d_3	Dimension
4, 4, 4	1004	3, 9, 0	44	8, 4, 0	176
6, 3, 3	1003	5, 7, 0	126	6, 5, 1	494
8, 2, 2	544	5, 6, 1	414	6, 4, 2	871
10, 1, 1	135	4, 6, 2	611	7, 4, 1	488
4, 8, 0	91	4, 5, 3	872	9, 3, 0	131
3, 4, 5	695	4, 7, 1	290	10, 2, 0	95
3, 8, 1	157	5, 5, 2	788	7, 2, 3	747
3, 7, 2	350	5, 3, 4	1046	8, 3, 1	404
3, 6, 3	558	7, 5, 0	176	9, 1, 2	271

TABLE 12. Homogeneous spaces in degree 12

Using scripts written in Macaulay2 [37], we thus checked all homogeneous spaces for the finite family already obtained. This leads to no irreducible

invariants for this degree. Such computations had thus been done to homogeneous spaces up to degree 21. \square

We now give joint invariants of $S_8 \oplus S_4$. For that purpose, we write $\mathbf{v} \in S_4$ and

$$\mathbf{k}_{2,4} := (\mathbf{v}, \mathbf{v})_2, \quad \mathbf{k}_{3,6} := (\mathbf{v}, \mathbf{k}_{2,4})_1.$$

53 joint invariants of $\mathbf{Inv}_j(S_8 \oplus S_4)$.	
Degree 3	$(\mathbf{f}_3, \mathbf{v})_4 \quad (\mathbf{f}_1, \mathbf{v}^2)_8$
Degree 4	$(\mathbf{f}_1, \mathbf{v} \cdot \mathbf{k}_{2,4})_8 \quad (\mathbf{f}_4, \mathbf{v}^2)_8 \quad (\mathbf{f}_3, \mathbf{k}_{2,4})_4 \quad (\mathbf{f}_7, \mathbf{v})_4$
Degree 5	$(\mathbf{f}_1, \mathbf{k}_{2,4}^2)_8 \quad (\mathbf{f}_4, \mathbf{v} \cdot \mathbf{k}_{2,4})_8 \quad (\mathbf{f}_5, \mathbf{v}^3)_{12} \quad (\mathbf{f}_7, \mathbf{k}_{2,4})_4$ $(\mathbf{f}_9, \mathbf{v}^2)_8 \quad (\mathbf{f}_{15}, \mathbf{v})_4 \quad (\mathbf{f}_{16}, \mathbf{v})_4$
Degree 6	$(\mathbf{f}_4, \mathbf{k}_{2,4}^2)_8 \quad (\mathbf{f}_5, \mathbf{v}^2 \cdot \mathbf{k}_{2,4})_{12} \quad (\mathbf{f}_{11}, \mathbf{v}^3)_{12} \quad (\mathbf{f}_9, \mathbf{v} \cdot \mathbf{k}_{2,4})_8$ $(\mathbf{f}_8, \mathbf{k}_{3,6})_6 \quad (\mathbf{f}_{15}, \mathbf{k}_{2,4})_4 \quad (\mathbf{f}_{18}, \mathbf{v}^2)_8 \quad (\mathbf{f}_{16}, \mathbf{k}_{2,4})_4$ $(\mathbf{f}_{26}, \mathbf{v})_4 \quad (\mathbf{f}_{27}, \mathbf{v})_4$
Degree 7	$(\mathbf{f}_5, \mathbf{v} \cdot \mathbf{k}_{2,4}^2)_{12} \quad (\mathbf{f}_{10}, \mathbf{v} \cdot \mathbf{k}_{3,6})_{10} \quad (\mathbf{f}_{11}, \mathbf{v}^2 \cdot \mathbf{k}_{2,4})_{12} \quad (\mathbf{f}_{18}, \mathbf{v} \cdot \mathbf{k}_{2,4})_8$ $(\mathbf{f}_{17}, \mathbf{k}_{3,6})_6 \quad (\mathbf{f}_{21}, \mathbf{v}^3)_{12} \quad (\mathbf{f}_{30}, \mathbf{v}^2)_8 \quad (\mathbf{f}_{27}, \mathbf{k}_{2,4})_4$ $(\mathbf{f}_{26}, \mathbf{k}_{2,4})_4 \quad (\mathbf{f}_{37}, \mathbf{v})_4 \quad (\mathbf{f}_{38}, \mathbf{v})_4$
Degree 8	$(\mathbf{f}_{47}, \mathbf{v})_4 \quad (\mathbf{f}_{48}, \mathbf{v})_4 \quad (\mathbf{f}_{37}, \mathbf{k}_{2,4})_4 \quad (\mathbf{f}_{38}, \mathbf{k}_{2,4})_4$ $(\mathbf{f}_{42}, \mathbf{v}^2)_8 \quad (\mathbf{f}_{29}, \mathbf{k}_{3,6})_6 \quad (\mathbf{f}_{30}, \mathbf{v} \cdot \mathbf{k}_{2,4})_8 \quad (\mathbf{f}_{20}, \mathbf{v} \cdot \mathbf{k}_{3,6})_{10}$ $(\mathbf{f}_{21}, \mathbf{v}^2 \cdot \mathbf{k}_{2,4})_{12} \quad (\mathbf{f}_{11}, \mathbf{v} \cdot \mathbf{k}_{2,4}^2)_{12}$
Degree 9	$(\mathbf{f}_8^2, \mathbf{v}^3)_{12} \quad (\mathbf{f}_{48}, \mathbf{k}_{2,4})_4 \quad (\mathbf{f}_{47}, \mathbf{k}_{2,4})_4 \quad (\mathbf{f}_{55}, \mathbf{v})_4 \quad (\mathbf{f}_{56}, \mathbf{v})_4$
Degree 10	$(\mathbf{f}_{56}, \mathbf{k}_{2,4})_4 \quad (\mathbf{f}_{63}, \mathbf{v})_4$
Degree 11	$(\mathbf{f}_{63}, \mathbf{k}_{2,4})_4 \quad (\mathbf{f}_{25}^2, \mathbf{v})_4$

Finally, we give joint invariants of $S_8 \oplus S_4 \oplus S_4$. Recall here that \mathbf{h}_n is defined to be the number n covariant in the covariant bases of $\mathbf{Cov}(S_4 \oplus S_4)$, given in table 9.

174 joints invariant of $\mathbf{Inv}_j(S_8 \oplus S_4 \oplus S_4)$.	
Degree 3	$(\mathbf{f}_1, \mathbf{h}_1 \cdot \mathbf{h}_2)_8$
Degree 4	$(\mathbf{f}_1, \mathbf{h}_1 \cdot \mathbf{h}_8)_8 \quad (\mathbf{f}_1, \mathbf{h}_2 \cdot \mathbf{h}_9)_8 \quad (\mathbf{f}_1, \mathbf{h}_2 \cdot \mathbf{h}_7)_8 \quad (\mathbf{f}_1, \mathbf{h}_1 \cdot \mathbf{h}_9)_8$ $(\mathbf{f}_3, \mathbf{h}_9)_4 \quad (\mathbf{f}_4, \mathbf{h}_1 \cdot \mathbf{h}_2)_8$
Degree 5	$(\mathbf{f}_1, \mathbf{h}_8 \cdot \mathbf{h}_9)_8 \quad (\mathbf{f}_1, \mathbf{h}_2 \cdot \mathbf{h}_{17})_8 \quad (\mathbf{f}_1, \mathbf{h}_7 \cdot \mathbf{h}_8)_8 \quad (\mathbf{f}_1, \mathbf{h}_2 \cdot \mathbf{h}_{18})_8$ $(\mathbf{f}_1, \mathbf{h}_9^2)_8 \quad (\mathbf{f}_1, \mathbf{h}_7 \cdot \mathbf{h}_9)_8 \quad (\mathbf{f}_1, \mathbf{h}_1 \cdot \mathbf{h}_{18})_8 \quad (\mathbf{f}_4, \mathbf{h}_1 \cdot \mathbf{h}_8)_8$ $(\mathbf{f}_4, \mathbf{h}_2 \cdot \mathbf{h}_9)_8 \quad (\mathbf{f}_5, \mathbf{h}_1 \cdot \mathbf{h}_2^2)_{12} \quad (\mathbf{f}_3, \mathbf{h}_{17})_4 \quad (\mathbf{f}_4, \mathbf{h}_2 \cdot \mathbf{h}_7)_8$ $(\mathbf{f}_3, \mathbf{h}_{18})_4 \quad (\mathbf{f}_4, \mathbf{h}_1 \cdot \mathbf{h}_9)_8 \quad (\mathbf{f}_5, \mathbf{h}_1^2 \cdot \mathbf{h}_2)_{12} \quad (\mathbf{f}_9, \mathbf{h}_1 \cdot \mathbf{h}_2)_8$ $(\mathbf{f}_7, \mathbf{h}_9)_4 \quad (\mathbf{f}_8, \mathbf{h}_{10})_6$
Degree 6	$(\mathbf{f}_1, \mathbf{h}_8 \cdot \mathbf{h}_{17})_8 \quad (\mathbf{f}_1, \mathbf{h}_2 \cdot \mathbf{h}_6^2)_8 \quad (\mathbf{f}_1, \mathbf{h}_9 \cdot \mathbf{h}_{17})_8 \quad (\mathbf{f}_1, \mathbf{h}_9 \cdot \mathbf{h}_{18})_8$

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	$(\mathbf{f}_1, \mathbf{h}_1 \cdot \mathbf{h}_6^2)_8$ $(\mathbf{f}_1, \mathbf{h}_7 \cdot \mathbf{h}_{18})_8$ $(\mathbf{f}_4, \mathbf{h}_2 \cdot \mathbf{h}_{17})_8$ $(\mathbf{f}_5, \mathbf{h}_1 \cdot \mathbf{h}_2 \cdot \mathbf{h}_8)_{12}$ $(\mathbf{f}_4, \mathbf{h}_8 \cdot \mathbf{h}_9)_8$ $(\mathbf{f}_5, \mathbf{h}_2^2 \cdot \mathbf{h}_9)_{12}$ $(\mathbf{f}_4, \mathbf{h}_2 \cdot \mathbf{h}_{18})_8$ $(\mathbf{f}_5, \mathbf{h}_1 \cdot \mathbf{h}_2 \cdot \mathbf{h}_9)_{12}$ $(\mathbf{f}_5, \mathbf{h}_1^2 \cdot \mathbf{h}_8)_{12}$ $(\mathbf{f}_4, \mathbf{h}_9^2)_8$ $(\mathbf{f}_4, \mathbf{h}_7 \cdot \mathbf{h}_8)_8$ $(\mathbf{f}_5, \mathbf{h}_2^2 \cdot \mathbf{h}_7)_{12}$ $(\mathbf{f}_5, \mathbf{h}_1^2 \cdot \mathbf{h}_9)_{12}$ $(\mathbf{f}_4, \mathbf{h}_7 \cdot \mathbf{h}_9)_8$ $(\mathbf{f}_4, \mathbf{h}_1 \cdot \mathbf{h}_{18})_8$ $(\mathbf{f}_5, \mathbf{h}_1 \cdot \mathbf{h}_2 \cdot \mathbf{h}_7)_{12}$ $(\mathbf{f}_9, \mathbf{h}_1 \cdot \mathbf{h}_8)_8$ $(\mathbf{f}_8, \mathbf{h}_{21})_6$ $(\mathbf{f}_{10}, \mathbf{h}_2 \cdot \mathbf{h}_{10})_{10}$ $(\mathbf{f}_8, \mathbf{h}_2 \cdot \mathbf{h}_6)_6$ $(\mathbf{f}_9, \mathbf{h}_2 \cdot \mathbf{h}_9)_8$ $(\mathbf{f}_{11}, \mathbf{h}_1 \cdot \mathbf{h}_2^2)_{12}$ $(\mathbf{f}_{11}, \mathbf{h}_1^2 \cdot \mathbf{h}_2)_{12}$ $(\mathbf{f}_{10}, \mathbf{h}_1 \cdot \mathbf{h}_{10})_{10}$ $(\mathbf{f}_9, \mathbf{h}_2 \cdot \mathbf{h}_7)_8$ $(\mathbf{f}_9, \mathbf{h}_1 \cdot \mathbf{h}_9)_8$ $(\mathbf{f}_8, \mathbf{h}_1 \cdot \mathbf{h}_6)_6$ $(\mathbf{f}_8, \mathbf{h}_{22})_6$ $(\mathbf{f}_{16}, \mathbf{h}_9)_4$ $(\mathbf{f}_{17}, \mathbf{h}_{10})_6$ $(\mathbf{f}_{18}, \mathbf{h}_1 \cdot \mathbf{h}_2)_8$ $(\mathbf{f}_{15}, \mathbf{h}_9)_4$
Degree 7	$(\mathbf{f}_5, \mathbf{h}_2^2 \cdot \mathbf{h}_{17})_{12}$ $(\mathbf{f}_5, \mathbf{h}_1 \cdot \mathbf{h}_8^2)_{12}$ $(\mathbf{f}_5, \mathbf{h}_2 \cdot \mathbf{h}_8 \cdot \mathbf{h}_9)_{12}$ $(\mathbf{f}_5, \mathbf{h}_2^2 \cdot \mathbf{h}_{18})_{12}$ $(\mathbf{f}_5, \mathbf{h}_1 \cdot \mathbf{h}_8 \cdot \mathbf{h}_9)_{12}$ $(\mathbf{f}_5, \mathbf{h}_2 \cdot \mathbf{h}_7 \cdot \mathbf{h}_8)_{12}$ $(\mathbf{f}_5, \mathbf{h}_2 \cdot \mathbf{h}_9^2)_{12}$ $(\mathbf{f}_5, \mathbf{h}_1 \cdot \mathbf{h}_9^2)_{12}$ $(\mathbf{f}_5, \mathbf{h}_2 \cdot \mathbf{h}_7 \cdot \mathbf{h}_9)_{12}$ $(\mathbf{f}_5, \mathbf{h}_1 \cdot \mathbf{h}_2 \cdot \mathbf{h}_{18})_{12}$ $(\mathbf{f}_5, \mathbf{h}_1 \cdot \mathbf{h}_7 \cdot \mathbf{h}_8)_{12}$ $(\mathbf{f}_5, \mathbf{h}_1 \cdot \mathbf{h}_7 \cdot \mathbf{h}_9)_{12}$ $(\mathbf{f}_5, \mathbf{h}_1^2 \cdot \mathbf{h}_{18})_{12}$ $(\mathbf{f}_5, \mathbf{h}_2 \cdot \mathbf{h}_7^2)_{12}$ $(\mathbf{f}_{10}, \mathbf{h}_2 \cdot \mathbf{h}_{21})_{10}$ $(\mathbf{f}_{10}, \mathbf{h}_1 \cdot \mathbf{h}_{20})_{10}$ $(\mathbf{f}_{11}, \mathbf{h}_2^2 \cdot \mathbf{h}_9)_{12}$ $(\mathbf{f}_{11}, \mathbf{h}_1 \cdot \mathbf{h}_2 \cdot \mathbf{h}_8)_{12}$ $(\mathbf{f}_{10}, \mathbf{h}_2^2 \cdot \mathbf{h}_6)_{10}$ $(\mathbf{f}_{12}, \mathbf{h}_2^2 \cdot \mathbf{h}_{10})_{14}$ $(\mathbf{f}_{10}, \mathbf{h}_1 \cdot \mathbf{h}_2 \cdot \mathbf{h}_6)_{10}$ $(\mathbf{f}_{11}, \mathbf{h}_1^2 \cdot \mathbf{h}_8)_{12}$ $(\mathbf{f}_{10}, \mathbf{h}_1 \cdot \mathbf{h}_{21})_{10}$ $(\mathbf{f}_{10}, \mathbf{h}_2 \cdot \mathbf{h}_{22})_{10}$ $(\mathbf{f}_{12}, \mathbf{h}_1 \cdot \mathbf{h}_2 \cdot \mathbf{h}_{10})_{14}$ $(\mathbf{f}_{11}, \mathbf{h}_1 \cdot \mathbf{h}_2 \cdot \mathbf{h}_9)_{12}$ $(\mathbf{f}_{11}, \mathbf{h}_2^2 \cdot \mathbf{h}_7)_{12}$ $(\mathbf{f}_9, \mathbf{h}_9^2)_8$ $(\mathbf{f}_{10}, \mathbf{h}_1 \cdot \mathbf{h}_{22})_{10}$ $(\mathbf{f}_{11}, \mathbf{h}_1 \cdot \mathbf{h}_2 \cdot \mathbf{h}_7)_{12}$ $(\mathbf{f}_{11}, \mathbf{h}_1^2 \cdot \mathbf{h}_9)_{12}$ $(\mathbf{f}_{10}, \mathbf{h}_1^2 \cdot \mathbf{h}_6)_{10}$ $(\mathbf{f}_{10}, \mathbf{h}_2 \cdot \mathbf{h}_{19})_{10}$ $(\mathbf{f}_{12}, \mathbf{h}_1^2 \cdot \mathbf{h}_{10})_{14}$ $(\mathbf{f}_{21}, \mathbf{h}_1 \cdot \mathbf{h}_2^2)_{12}$ $(\mathbf{f}_{18}, \mathbf{h}_2 \cdot \mathbf{h}_9)_8$ $(\mathbf{f}_{17}, \mathbf{h}_{21})_6$ $(\mathbf{f}_{17}, \mathbf{h}_2 \cdot \mathbf{h}_6)_6$ $(\mathbf{f}_{20}, \mathbf{h}_2 \cdot \mathbf{h}_{10})_{10}$ $(\mathbf{f}_{19}, \mathbf{h}_2 \cdot \mathbf{h}_{10})_{10}$ $(\mathbf{f}_{18}, \mathbf{h}_1 \cdot \mathbf{h}_8)_8$ $(\mathbf{f}_{17}, \mathbf{h}_1 \cdot \mathbf{h}_6)_6$ $(\mathbf{f}_{18}, \mathbf{h}_1 \cdot \mathbf{h}_9)_8$ $(\mathbf{f}_{17}, \mathbf{h}_{22})_6$ $(\mathbf{f}_{20}, \mathbf{h}_1 \cdot \mathbf{h}_{10})_{10}$ $(\mathbf{f}_{21}, \mathbf{h}_1^2 \cdot \mathbf{h}_2)_{12}$ $(\mathbf{f}_{19}, \mathbf{h}_1 \cdot \mathbf{h}_{10})_{10}$ $(\mathbf{f}_{18}, \mathbf{h}_2 \cdot \mathbf{h}_7)_8$ $(\mathbf{f}_{29}, \mathbf{h}_{10})_6$ $(\mathbf{f}_{30}, \mathbf{h}_1 \cdot \mathbf{h}_2)_8$ $(\mathbf{f}_{26}, \mathbf{h}_9)_4$ $(\mathbf{f}_{27}, \mathbf{h}_9)_4$ $(\mathbf{f}_{28}, \mathbf{h}_{10})_6$
Degree 8	$(\mathbf{f}_{37}, \mathbf{h}_9)_4$ $(\mathbf{f}_{38}, \mathbf{h}_9)_4$ $(\mathbf{f}_{40}, \mathbf{h}_{10})_6$ $(\mathbf{f}_{41}, \mathbf{h}_{10})_6$ $(\mathbf{f}_{42}, \mathbf{h}_1 \cdot \mathbf{h}_2)_8$ $(\mathbf{f}_{29}, \mathbf{h}_{21})_6$ $(\mathbf{f}_{30}, \mathbf{h}_1 \cdot \mathbf{h}_8)_8$ $(\mathbf{f}_{30}, \mathbf{h}_2 \cdot \mathbf{h}_9)_8$ $(\mathbf{f}_{31}, \mathbf{h}_2 \cdot \mathbf{h}_{10})_{10}$ $(\mathbf{f}_{32}, \mathbf{h}_2 \cdot \mathbf{h}_{10})_{10}$ $(\mathbf{f}_{33}, \mathbf{h}_2 \cdot \mathbf{h}_{10})_{10}$ $(\mathbf{f}_{29}, \mathbf{h}_{22})_6$ $(\mathbf{f}_{30}, \mathbf{h}_1 \cdot \mathbf{h}_9)_8$ $(\mathbf{f}_{30}, \mathbf{h}_2 \cdot \mathbf{h}_7)_8$ $(\mathbf{f}_{31}, \mathbf{h}_1 \cdot \mathbf{h}_{10})_{10}$ $(\mathbf{f}_{32}, \mathbf{h}_1 \cdot \mathbf{h}_{10})_{10}$ $(\mathbf{f}_{33}, \mathbf{h}_1 \cdot \mathbf{h}_{10})_{10}$ $(\mathbf{f}_{20}, \mathbf{h}_2 \cdot \mathbf{h}_{22})_{10}$ $(\mathbf{f}_{20}, \mathbf{h}_1 \cdot \mathbf{h}_2 \cdot \mathbf{h}_6)_{10}$ $(\mathbf{f}_{21}, \mathbf{h}_1^2 \cdot \mathbf{h}_8)_{12}$ $(\mathbf{f}_{21}, \mathbf{h}_1 \cdot \mathbf{h}_2 \cdot \mathbf{h}_9)_{12}$ $(\mathbf{f}_{21}, \mathbf{h}_2^2 \cdot \mathbf{h}_7)_{12}$ $(\mathbf{f}_{22}, \mathbf{h}_1 \cdot \mathbf{h}_2 \cdot \mathbf{h}_{10})_{14}$ $(\mathbf{f}_{20}, \mathbf{h}_1 \cdot \mathbf{h}_{22})_{10}$ $(\mathbf{f}_{20}, \mathbf{h}_1^2 \cdot \mathbf{h}_6)_{10}$ $(\mathbf{f}_{21}, \mathbf{h}_1^2 \cdot \mathbf{h}_9)_{12}$ $(\mathbf{f}_{21}, \mathbf{h}_1 \cdot \mathbf{h}_2 \cdot \mathbf{h}_7)_{12}$ $(\mathbf{f}_{22}, \mathbf{h}_1^2 \cdot \mathbf{h}_{10})_{14}$ $(\mathbf{f}_{11}, \mathbf{h}_2 \cdot \mathbf{h}_7 \cdot \mathbf{h}_9)_{12}$ $(\mathbf{f}_{12}, \mathbf{h}_1^2 \cdot \mathbf{h}_2 \cdot \mathbf{h}_6)_{14}$ $(\mathbf{f}_{13}, \mathbf{h}_1^2 \cdot \mathbf{h}_2 \cdot \mathbf{h}_{10})_{18}$ $(\mathbf{f}_{11}, \mathbf{h}_2 \cdot \mathbf{h}_7^2)_{12}$ $(\mathbf{f}_{12}, \mathbf{h}_{13} \cdot \mathbf{h}_6)_{14}$ $(\mathbf{f}_{13}, \mathbf{h}_{13} \cdot \mathbf{h}_{10})_{18}$ $(\mathbf{f}_{11}, \mathbf{h}_2 \cdot \mathbf{h}_9^2)_{12}$ $(\mathbf{f}_{12}, \mathbf{h}_1 \cdot \mathbf{h}_2^2 \cdot \mathbf{h}_6)_{14}$ $(\mathbf{f}_{13}, \mathbf{h}_1 \cdot \mathbf{h}_2^2 \cdot \mathbf{h}_{10})_{18}$ $(\mathbf{f}_{20}, \mathbf{h}_2 \cdot \mathbf{h}_{21})_{10}$ $(\mathbf{f}_{20}, \mathbf{h}_2^2 \cdot \mathbf{h}_6)_{10}$ $(\mathbf{f}_{21}, \mathbf{h}_1 \cdot \mathbf{h}_2 \cdot \mathbf{h}_8)_{12}$ $(\mathbf{f}_{21}, \mathbf{h}_2^2 \cdot \mathbf{h}_9)_{12}$

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	$(\mathbf{f}_{22}, \mathbf{h}_2^2 \cdot \mathbf{h}_{10})_{14}$	$(\mathbf{f}_{11}, \mathbf{h}_2 \cdot \mathbf{h}_8 \cdot \mathbf{h}_9)_{12}$	$(\mathbf{f}_{12}, \mathbf{h}_{2^3} \cdot \mathbf{h}_6)_{14}$	
	$(\mathbf{f}_{13}, \mathbf{h}_{2^3} \cdot \mathbf{h}_{10})_{18}$			
Degree 9	$(\mathbf{f}_1 \cdot \mathbf{f}_{25}, \mathbf{h}_2 \cdot \mathbf{h}_{10})_{10}$	$(\mathbf{f}_{43}, \mathbf{h}_2 \cdot \mathbf{h}_{10})_{10}$	$(\mathbf{f}_8^2, \mathbf{h}_1 \cdot \mathbf{h}_2^2)_{12}$	$(\mathbf{f}_1 \cdot \mathbf{f}_{25}, \mathbf{h}_1 \cdot \mathbf{h}_{10})_{10}$
	$(\mathbf{f}_8^2, \mathbf{h}_1^2 \cdot \mathbf{h}_2)_{12}$	$(\mathbf{f}_{43}, \mathbf{h}_1 \cdot \mathbf{h}_{10})_{10}$	$(\mathbf{f}_3 \cdot \mathbf{f}_{25}, \mathbf{h}_{10})_6$	$(\mathbf{f}_{51}, \mathbf{h}_{10})_6$
	$(\mathbf{f}_{48}, \mathbf{h}_9)_4$	$(\mathbf{f}_{47}, \mathbf{h}_9)_4$		
Degree 10	$(\mathbf{f}_{54}, \mathbf{h}_6)_2$	$(\mathbf{f}_{56}, \mathbf{h}_9)_4$		
Degree 11	$(\mathbf{f}_{61}, \mathbf{h}_6)_2$	$(\mathbf{f}_{62}, \mathbf{h}_6)_2$	$(\mathbf{f}_{63}, \mathbf{h}_9)_4$	

APPENDIX A. THE STROH FORMULA AND SOME COROLLARIES

The following general algebraic relation was obtained by Stroh [63] (see also [36]).

Lemma A.1. *Let u_1, u_2 and u_3 be three commutative variables such that*

$$u_1 + u_2 + u_3 = 0.$$

Then we have

$$\begin{aligned} (-1)^{k_2} \sum_{i=0}^{k_1} \binom{g}{i} \binom{k_1 + k_3 - i}{k_3} u_3^{g-i} u_1^i + (-1)^{k_3} \sum_{i=0}^{k_2} \binom{g}{i} \binom{k_2 + k_1 - i}{k_1} u_1^{g-i} u_2^i + \\ (-1)^{k_1} \sum_{i=0}^{k_3} \binom{g}{i} \binom{k_3 + k_2 - i}{k_2} u_2^{g-i} u_3^i = 0, \quad (\text{A.1}) \end{aligned}$$

with $k_1 + k_2 + k_3 = g - 1$.

This formula leads to new degree three relations on molecules. Let $V = \mathbb{S}_n$ and (e_0, e_1, e_2) be three integers such that $e_i + e_j \leq n$ ($i \neq j$). Define:

$$D(e_0, e_1, e_2) := \begin{array}{ccc} \alpha & \xrightarrow{e_0} & \beta \\ & \swarrow e_2 & \searrow e_1 \\ & \gamma & \end{array} \quad \text{with weight } w = e_0 + e_1 + e_2, \quad (\text{A.2})$$

Note that $D(e_0, e_1, e_2) \in \text{Hom}_{\text{SL}(2, \mathbb{C})}(\mathbb{S}_n \otimes \mathbb{S}_n \otimes \mathbb{S}_n, \mathbb{S}_{3n-2w})$.

Lemma A.2. *Let $w \leq n$ and $m_1, m_2, m_3 \geq 1$ be integers such that $m_1 + m_2 + m_3 = w + 1$. Then the molecule $D(e_0, e_1, e_2)$ is a linear combination of*

$$D(w - i_1, i_1, 0), \quad D(0, w - i_2, i_2), \quad D(i_3, 0, w - i_3),$$

with $i_s = 0 \dots m_s - 1$,

Sketch of proof. Using Clebsch–Gordan decomposition, first observe that

$$\dim \text{Hom}_{\text{SL}(2, \mathbb{C})}(\mathbb{S}_n \otimes \mathbb{S}_n \otimes \mathbb{S}_n, \mathbb{S}_{3n-2w}) = w + 1$$

Suppose that we have a linear relation

$$\sum_{i=0}^w \lambda_i D(w - i, i, 0) = 0.$$

Taking $\mathbf{f}_\alpha = x_\alpha^n$, $\mathbf{f}_\beta = y_\beta^n$ and $\mathbf{f}_\gamma = y_\gamma^n$ leads to $\lambda_0 = 0$; and by induction we get $\lambda_i = 0$ for all i . Thus $\mathcal{F}_1 := \{\mathbf{D}(w-i, i, 0), i = 0 \dots w\}$ is a bases of $\text{Hom}_{\text{SL}(2, \mathbb{C})}(\mathbb{S}_n \otimes \mathbb{S}_n \otimes \mathbb{S}_n, \mathbb{S}_{3n-2w})$. There is the same statement for $\mathcal{F}_2 := \{\mathbf{D}(0, w-i, i), i = 0 \dots w\}$ and $\mathcal{F}_3 := \{\mathbf{D}(i, 0, w-i), i = 0 \dots w\}$.

Let

$$u_1 = \Omega_{\alpha\beta}\sigma_\gamma, \quad u_2 = \Omega_{\beta\gamma}\sigma_\alpha, \quad u_3 = \Omega_{\gamma\alpha}\sigma_\beta$$

Those are commutative variables verifying $u_1 + u_2 + u_3 = 0$. Now, taking the family

$$\mathcal{F} := \{\mathbf{D}(w-i_1, i_1, 0), \mathbf{D}(0, w-i_2, i_2), \mathbf{D}(i_3, 0, w-i_3), \quad i_s = 0 \dots m_s - 1\}$$

lemma A.1 with $k_1 = m_1, k_2 = m_2, k_3 = m_3 + 1$ (for $m_3 < w$) and $g = w + 3$ induces that $\mathbf{D}(m_3 + 1, 0, w - m_3 - 1) \in \mathcal{F}_3$ is generated by the family \mathcal{F} . By induction, \mathcal{F}_3 and thus all molecules are generated by \mathcal{F} . \square

Lemma A.3. *Let $\mathbf{D}(e_0, e_1, e_2)$ be given by A.2.*

(1) *If $w \leq n$ then*

$$\mathbf{D}(e_0, e_1, e_2) \text{ is of grade } r \geq \frac{2}{3}w.$$

(2) *If $w > n$ then*

$$\mathbf{D}(e_0, e_1, e_2) \text{ is of grade } r \geq n - \frac{w}{3}.$$

Sketch of proof. The detailed proof is in [36]. Just consider here the case when $w \leq n$ with $w = 3k - 1$. Taking $m_1 = m_2 = m_3 = m$ in lemma A.2 leads to a family \mathcal{F} whose molecules are of grade at least $2k$. We use the same kind of arguments for $w = 3k + 2$ and $w = 3k$. \square

A special case of A.3 is:

Lemma A.4. *Let $\mathbf{D}(e_0, e_1, e_2)$ be given by A.2 with $e_i + e_j \leq n$ ($i \neq j$). Suppose that*

$$e_0 \leq \frac{n}{2} \text{ and } e_1 + e_2 > \frac{e_0}{2},$$

then

$$\mathbf{D}(e_0, e_1, e_2) \text{ is of grade } e_0 + 1,$$

unless $e_0 = e_1 = e_2 = \frac{n}{2}$.

APPENDIX B. DEGREE THREE COVARIANT BASIS

Take n, p and q be three non negative integers. By Clebsch–Gordan decomposition, we first know that we have an $\text{SL}(2, \mathbb{C})$ decomposition

$$\mathbb{S}_n \otimes \mathbb{S}_p \simeq \bigoplus_{i=0}^{\min(n,p)} \mathbb{S}_{n+p-2i}$$

thus

$$\mathbb{S}_n \otimes \mathbb{S}_p \otimes \mathbb{S}_q \simeq \bigoplus_{i=0}^{\min(n,p)} \mathbb{S}_{n+p-2i} \otimes \mathbb{S}_q$$

The same argument leads to

$$S_n \otimes S_p \otimes S_q \simeq \bigoplus_{j=0}^{\min(p,q)} S_n \otimes S_{p+q-2j}$$

We define a triplet (n, p, i) to be *admissible* if $0 \leq i \leq \min(n, p)$: this means that the irreducible component S_{n+p-2i} appears in the $\mathrm{SL}(2, \mathbb{C})$ decomposition of $S_n \otimes S_p$.

Lemma B.1. *Let r be an integer, i_1, i_2, j_1, j_2 be integers such that*

$$\begin{cases} (n, p, i_1), (n + p - 2i_1, q, i_2), (p, q, j_1), (n, p + q - 2j_1, j_2) & \text{are admissible} \\ n + p + q - 2(i_1 + i_2) = r \\ n + p + q - 2(j_1 + j_2) = r \end{cases} .$$

Then both sets

$$\phi_{i_1, i_2} : \mathbf{f} \otimes \mathbf{g} \otimes \mathbf{h} \mapsto (\mathbf{f}, (\mathbf{g}, \mathbf{h})_{i_1})_{i_2}, \quad S_n \otimes S_p \otimes S_q \longrightarrow S_{n+p+q-2(i_1+i_2)}$$

and

$$\psi_{j_1, j_2} : \mathbf{f} \otimes \mathbf{g} \otimes \mathbf{h} \mapsto ((\mathbf{f}, \mathbf{g})_{j_1}, \mathbf{h})_{j_2}, \quad S_n \otimes S_p \otimes S_q \longrightarrow S_{n+p+q-2(j_1+j_2)}$$

are vector basis of

$$\mathrm{Hom}_{\mathrm{SL}(2, \mathbb{C})}(S_n \otimes S_p \otimes S_q, S_r).$$

Proof. This lemma is the dual version of [19, lemma 2.6.1], the only idea being that for finite dimensional linear $\mathrm{SL}(2, \mathbb{C})$ representations we have

$$\mathrm{Hom}_{\mathrm{SL}(2, \mathbb{C})}(E_1 \oplus E_2, F) \simeq \mathrm{Hom}_{\mathrm{SL}(2, \mathbb{C})}(E_1, F) \oplus \mathrm{Hom}_{\mathrm{SL}(2, \mathbb{C})}(E_2, F)$$

Now, given integers i_1, i_2, r as in the hypothesis, we have

$$S_n \otimes S_p \otimes S_q \simeq \bigoplus_{i_1=0}^{\min(n,p)} S_{n+p-2i_1} \otimes S_q$$

and ϕ_{i_1, i_2} is a non nul vector of the one dimensional space

$$\mathrm{Hom}_{\mathrm{SL}(2, \mathbb{C})}(S_{n+p-2i_1} \otimes S_q, S_r).$$

We do similarly for integers j_1, j_2, r . □

APPENDIX C. RELATIVELY COMPLETE FAMILIES OF A SINGLE BINARY FORM

We give here results about reduction of some families modulo an ideal. We take a space S_n of binary forms and Gordan's ideal I_r (see definition 6.4). By (6.2), every molecular covariant of grade 1 is thus in I_2 , and then:

Corollary C.1. *The family $A_0 := \{\mathbf{f}\}$ is relatively complete modulo I_2*

The following lemma is about degree three molecular covariants, and is used in the following:

Lemma C.2. *Let V be a space of binary forms, α , β and γ be three atoms of respective valence n , p , q . Let r be an integer such that $r \leq \min(n, p, q)$; then*

$$\begin{array}{c} \alpha \end{array} \xrightarrow{r} \begin{array}{c} \beta \\ \gamma \end{array} = \sum_{i=0}^r \binom{r}{i} \begin{array}{c} \alpha \\ \gamma \end{array} \xrightarrow{i} \begin{array}{c} \beta \\ \gamma \end{array} \xrightarrow{r-i} \begin{array}{c} \beta \\ \gamma \end{array} \quad (\text{C.1})$$

Proof. Starting with relation (3.2):

$$\Omega_{\alpha\beta}\sigma_{\gamma} = \Omega_{\alpha\gamma}\sigma_{\beta} + \Omega_{\gamma\beta}\sigma_{\alpha},$$

we get

$$\Omega_{\alpha\beta}\sigma_{\gamma}^r = \sum_{i=0}^r \binom{r}{i} \Omega_{\alpha\gamma}^i \Omega_{\gamma\beta}^{r-i} \sigma_{\beta}^i \sigma_{\gamma}^{r-i},$$

and we just have to multiply each side of the equation by $\sigma_{\alpha}^{n-r} \sigma_{\beta}^{p-r} \sigma_{\gamma}^{q-r}$. \square

Recall here that, for $\mathbf{f} \in \mathbf{S}_n$ and for a given integer $k \geq 0$, we have

$$\mathbf{H}_{2k} := (\mathbf{f}, \mathbf{f})_{2k}.$$

Lemma C.3. *If $2n - 4k > n$, where $2n - 4k$ is the order of \mathbf{H}_{2k} , then the family $\mathbf{B} = \{\mathbf{H}_{2k}\}$ is relatively complete modulo I_{2k+2} .*

Proof. We have to consider molecular covariants containing

$$\mathbf{D} := \begin{array}{ccc} \begin{array}{c} \mathbf{f}_{\alpha} \end{array} & \xrightarrow{2k} & \begin{array}{c} \mathbf{f}_{\beta} \end{array} \\ & & \downarrow r \\ \begin{array}{c} \mathbf{f}_{\delta} \end{array} & \xrightarrow{2k} & \begin{array}{c} \mathbf{f}_{\gamma} \end{array} \end{array} \quad \text{with } 1 \leq r \leq 2k$$

all symbol being equivalent. When $r > k$, the molecular covariant

$$\begin{array}{ccc} \begin{array}{c} \mathbf{f}_{\alpha} \end{array} & \xrightarrow{e_0 = 2k} & \begin{array}{c} \mathbf{f}_{\beta} \end{array} \\ & & \searrow e_1 = r \\ & & \begin{array}{c} \mathbf{f}_{\gamma} \end{array} \end{array}$$

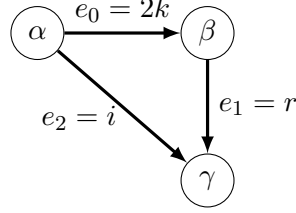
is of grade $2k + 1$ by lemma A.4. Thus the molecular covariant associated to \mathbf{D} is in $I_{2k+1} = I_{2k+2}$ by (6.2).

When $r < k$, by relation (C.1), \mathbf{D} decomposes as a linear combination of

$$\begin{array}{ccc} \begin{array}{c} \mathbf{f}_{\alpha} \end{array} & \xrightarrow{2k} & \begin{array}{c} \mathbf{f}_{\beta} \end{array} \\ & \searrow i & \downarrow r \\ \begin{array}{c} \mathbf{f}_{\delta} \end{array} & \xrightarrow{2k-i} & \begin{array}{c} \mathbf{f}_{\gamma} \end{array} \end{array} \quad \text{with } 0 \leq i \leq 2k$$

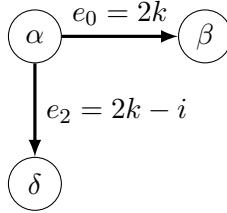
Now:

- If $i \geq k$, we consider the molecule



of weight $w = 2k + r + i \geq 3k + r > 3k$. Since $2k + r + i \leq n$, by lemma A.3 this molecule is of grade $r \geq \frac{2}{3}w > 2k$;

- If $i < k$, we consider the molecule

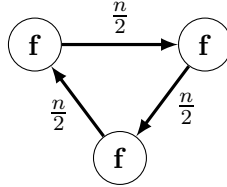


and we conclude using lemma A.4.

□

In the same way:

Lemma C.4. *If $n = 4k$, then \mathbf{H}_{2k} is of order n and the family $\mathbf{B} = \{\mathbf{H}_{2k}\}$ is relatively complete modulo $I_{2k+2} + \langle \Delta \rangle$ where Δ is the invariant given by:*



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