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Marc Olive

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# ABOUT GORDAN'S ALGORITHM FOR BINARY FORMS

MARC OLIVE

ABSTRACT. In this paper, we present a modern version of Gordan's algorithm on binary forms. Symbolic method is reinterpreted in terms of  $\mathrm{SL}_2(\mathbb{C})$ -equivariant homomorphisms defined upon Cayley operator and polarization operator. A graphical approach is thus developed to obtain Gordan's ideal, a central key to get covariant bases of binary forms. To illustrate the power of this method, we obtain for the first time a minimal covariant basis for  $S_6 \oplus S_4$  and  $S_6 \oplus S_4 \oplus S_2$ .

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## 1. INTRODUCTION

*Classical invariant theory* was a very active research field throughout the XIX<sup>th</sup> century. As pointed out by Parshall [49], the birth of this field can be found in Gauss' *Disquisitiones Arithmeticae* (1801). In this book, he studied a linear change of variables in a quadratic form with integer coefficients. About forty years later, Boole [12] established the main purpose of what has become today *classical invariant theory*. Cayley [22, 23] deeply investigated this field of research and developed important tools still in use nowadays, such as the *Cayley Omega operator*. During about fifteen years (until 1861 and Cayley's seventh memoir [20]) the English school of invariant theory, mainly led by Cayley and Sylvester, developed important tools to compute explicit *invariant generators* of binary forms. Thus, the role of calculation deeply influenced this first approach in invariant theory [22].

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At that time, a German school mainly conducted by Clebsch, Aronhold and Gordan, developed their own approach, using the *symbolic method*, also used with slightly different notations by the English school. In 1868, Gordan, who was called the “King of invariant theory”, proved that the algebra of *covariants* of any binary forms is always finitely generated [32]. As a great part of the mathematical development of that time, such a result was endowed with a constructive proof: the English and the German schools were equally preoccupied by calculation and an exhibition of invariants and covariants. Despite Gordan’s constructive proof, Cayley was reluctant to make use of Gordan’s approach to obtain a new understanding of invariant theory. That’s only in 1903, with the work of Grace–Young [34], that the German approach of Gordan and al. became accessible to a wide community of mathematicians. During that time, from 1868 to 1875, Gordan’s constructive approach led to several explicit results: first, and without difficulty, Gordan [33] computed a basis for the covariants of the quintic and the sextic. Thereafter, he started the computation of a covariant basis for the septic and the octic. This work was achieved by Von Gall who exhibited a complete covariant basis for the septic [60] and for the octic [59].

In 1890, Hilbert made a critical advance in the field of *invariant theory*. Using a totally new approach [37], which is the cornerstone of today’s algebraic geometry, he proved the finiteness theorem in the very general case of a *reductive group*. However, his first proof [37] was criticized for not being constructive. Facing these critics, Hilbert produced a second proof [37], claimed to be more constructive. This effective approach is nowadays widely used to obtain a finite generating set of invariants [51, 29, 16, 17]. As pointed out by Hilbert himself in [37], the main scope of this approach can be summarized in three steps.

The first step is to compute the *Hilbert series* of the graded algebra  $\mathcal{A}$  of invariants, which is always a rational function by the Hilbert–Serre theorem [21]. This Hilbert series gives the dimensions of each homogeneous space  $\mathcal{A}_i$  of  $\mathcal{A}$ . There exists several methods to compute this Hilbert series [10, 43, 52] *a priori*. The second step is to exhibit a *homogeneous system of parameters* (hsop) for the algebra  $\mathcal{A}$ . Finally, the Hochster–Roberts theorem [38] ensures that the algebra  $\mathcal{A}$  is *Cohen–Macaulay*<sup>1</sup>. Thanks to that statement, the system of parameters (or at least the knowledge of their degree) altogether with the Hilbert series produce a bound for the degree of generating invariants. We refer the reader to several references [57, 16, 29, 25, 26, 27] to get a general and modern approach on this subject.

One major weakness of this strategy is however that we need to compute a system of parameters (or at least their degree). The Noether normalization lemma [41] ensures that such a system always exists, but as far as we know, current algorithms to get such a system [36] are not sufficiently effective because of the extensive use of Grobner bases. For the invariant or covariant algebra of binary forms, one has of course the concept of *nullcone* and the Mumford–Hilbert criterion [26, 14] to check that a given finite family is a

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<sup>1</sup>Meaning the algebra  $\mathcal{A}$  is a finite and free  $k[\theta_1, \dots, \theta_s]$ -module, where  $\{\theta_1, \dots, \theta_s\}$  is a system of parameters

system of parameters. But this criterion does not explain how to obtain a system of parameters. Furthermore, in the case of *joint invariants*, that is for the invariant algebra of  $V := S_{n_1} \oplus \cdots \oplus S_{n_k}$ , such a system of parameters has, in general, a complex shape. Indeed, Brion [15] showed there exists a system of parameters that respects the multi-graduation of  $\mathbf{Inv}(V)$  for only thirteen cases.

Let's point out that an important motivation for this work was to compare effective approaches in invariant theory since we had to compute invariant bases for non trivial joint invariants, such as  $S_8 \oplus S_4 \oplus S_4$  or  $S_6 \oplus S_4 \oplus S_2 \oplus S_2$ . These computations are issued from *continuum mechanics* [5], where *constitutive laws* (such as the generalized Hooke's law) involve moduli spaces of tensors by the three-dimensional orthogonal group. For instance, to obtain invariants of the *elasticity tensor* [8], Boehler–Kirilov–Onat used a classical isomorphism between complex  $SO(3, \mathbb{R})$  linear representations and  $SL(2, \mathbb{C})$  linear representations on binary forms [55, 11]. Doing so, they derived from the invariant basis of  $S_8$  (first obtained by Von Gall [59] in 1880), a generating set of invariants for the higher dimensional irreducible component of the elasticity tensor. Such an invariant basis can be used to classify the orbit space of the elasticity tensor, as pointed out by Auffray–Kolev–Petitot [6]. In a forthcoming paper, we will present a new result, useful in continuum mechanics [7]. This result is derived from the knowledge of a basis for joint invariants of  $S_6 \oplus S_2$ , already obtained by von Gall [58] and checked again in this paper.

Other interests for effective computations of generating sets for invariants of binary forms arise in *geometrical arithmetic*, illustrated by the work of Lercier–Ritzenthaler [42] on hyperelliptic curves. But we could also cite other areas such as *quantum informatics* with the paper of Luque [44] and *recoupling theory*, with the work of Abdesselam and Chipalkatti [2, 3, 1, 4] on  $6j$  and  $9j$ -symbols.

The algebraic geometry approach first developed by Hilbert is not the only constructive one. In the case of a single binary form, Olver [46] exhibits another constructive approach, which was generalized for a single  $n$ -ary form and also specified with a “running bound” by Brini–Regonati–Creolis [13]. We could also cite Kung–Rota [40] but the combinatorial approach developed there became increasingly complex for the cases we had to deal with.

A special case of Gordan's algorithm, stated in theorem 7.1 of our present paper, leads to a very simple computation of the covariant basis for  $S_6 \oplus S_2$ . Due to this observation, we decided to reformulate Gordan's theorem<sup>2</sup> on binary forms in the modern language of operators and  $SL(2, \mathbb{C})$  equivariant homomorphism. We also decided to represent  $SL(2, \mathbb{C})$  equivariant homomorphisms with *directed graphs*, in the spirit of the graphical approach developed by Olver–Shakiban [48]. Understanding Gordan's algorithm allowed us to obtain for the first time a covariant basis of  $S_6 \oplus S_4$  (in subsection 7.2) and of  $S_6 \oplus S_4 \oplus S_2$  (in subsection 7.3). A minimal covariant basis for the binary nonics will be presented in a forthcoming paper with Lercier [45].

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<sup>2</sup>Note that Weyman [61] has also reformulated Gordan's method in a modern way and through algebraic geometry but unfortunately, we were unable to extract from it an effective approach. There is also a preprint of Pasechnik [50] on this method.

The paper is organized as follows. In [section 2](#) we recall the mathematical background from classical invariant theory, and we introduce classical operators such as the Cayley operator, polarization operators and the transvectant operator. In [section 4](#), we introduce *molecule* and *molecular covariants* which are graphical representations of  $\mathrm{SL}(2, \mathbb{C})$  equivariant morphisms constructed with the use of Cayley and polarization operators. Gordan's algorithm for joint covariants produces a finite generating set for  $\mathbf{Cov}(S_m \oplus S_n)$ , knowing a finite system of generators for  $\mathbf{Cov}(S_m)$  and  $\mathbf{Cov}(S_n)$ . It is explained in [section 5](#). There is a second version of Gordan's algorithm which enables to compute a covariant basis for  $S_n$ , knowing covariant bases for  $S_k$ , ( $k < n$ ). This method is detailed in [section 6](#). In [subsection 7.1](#), we illustrate Gordan's algorithm for joint covariants by (re-)computing a minimal covariant basis for  $S_6 \oplus S_2$  (already done by von Gall [[58](#)]). In [subsection 7.2](#), we exhibit for the first time a minimal basis for the joint covariants of  $S_6 \oplus S_4$ , and in [subsection 7.3](#) a minimal basis for the joint covariants of  $S_6 \oplus S_4 \oplus S_2$  (new). Finally, in [subsection 7.4](#) we apply the algorithm for a single binary form and give a minimal covariant basis for the binary octics. This was already obtained by Von Gall [[60](#)], Lercier–Ritzenthaler [[42](#)], Cröni [[24](#)] and Bedratyuk [[9](#)].

## 2. COVARIANTS OF BINARY FORMS

**Definition 2.1.** The complex vector space of  $n$ th degree binary forms, noted  $S_n$ , is the space of homogeneous polynomials

$$\mathbf{f}(\mathbf{x}) = a_0x^n + \binom{n}{1}a_1x^{n-1}y + \dots + \binom{n}{n-1}a_{n-1}xy^{n-1} + a_ny^n,$$

with  $a_i \in \mathbb{C}$ .

The natural  $\mathrm{SL}_2(\mathbb{C})$  action on  $\mathbb{C}^2$  induces a left action on  $S_n$ , given by

$$(g \cdot \mathbf{f})(\mathbf{x}) := \mathbf{f}(g^{-1} \cdot \mathbf{x}) \text{ for } g \in \mathrm{SL}_2(\mathbb{C}), \mathbf{x} = (x, y) \in \mathbb{C}^2.$$

By a space  $V$  of binary forms, we mean a direct sum

$$V := \bigoplus_{i=0}^s S_{n_i},$$

where the action of  $\mathrm{SL}_2(\mathbb{C})$  is diagonal. The action of  $\mathrm{SL}_2(\mathbb{C})$  on the coordinate ring  $\mathbb{C}[V \oplus \mathbb{C}^2]$  is defined by

$$(g \cdot p)(\mathbf{f}, \mathbf{x}) := p(g^{-1} \cdot \mathbf{f}, g^{-1} \cdot \mathbf{x}) \text{ for } g \in \mathrm{SL}_2(\mathbb{C}), p \in \mathbb{C}[V \oplus \mathbb{C}^2].$$

**Definition 2.2.** The covariant algebra<sup>3</sup> of a space  $V$  of binary forms, noted  $\mathbf{Cov}(V)$ , is the invariant algebra

$$\mathbf{Cov}(V) := \mathbb{C}[V \oplus \mathbb{C}^2]^{\mathrm{SL}_2(\mathbb{C})}.$$

An important result, first established by Gordan [[32](#)] and then extended by Hilbert [[37](#)] (for any reductive group) is the following.

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<sup>3</sup>For a general and modern approach on invariant and covariant algebra, we refer to the online text [[39](#)] by Kraft and Procesi.

**Theorem 2.3.** *For every space  $V$  of binary forms, the covariant algebra  $\mathbf{Cov}(V)$  is finitely generated, i.e. there exists a finite set  $\mathbf{h}_1, \dots, \mathbf{h}_N$  in  $\mathbf{Cov}(V)$ , called a basis, such that*

$$\mathbf{Cov}(V) = \mathbb{C}[\mathbf{h}_1, \dots, \mathbf{h}_N].$$

There is a natural bi-graduation on the covariant algebra  $\mathbf{Cov}(V)$ :

- By the **degree**, which is the polynomial degree in the coefficients of the space  $V$  ;
- By the **order** which is the polynomial degree in the variables  $\mathbf{x}$  ;

Let  $\mathbf{Cov}_{d,k}(V)$  be the subspace of degree  $d$  and order  $k$  covariants, and:

$$C_+ := \sum_{d+k>0} \mathbf{Cov}_{d,k}(V).$$

Then,  $C_+$  is an ideal of the graduated algebra  $\mathbf{Cov}(V)$ . For each  $d+k > 0$ , let  $\delta_{d,k}$  be the codimension of  $(C_+^2)_i$  in  $\mathbf{Cov}_{d,k}$ . Since the algebra  $\mathbf{Cov}(V)$  is of finite type, there exists an integer  $p$  such that  $\delta_{d,k} = 0$  for  $d+k \geq p$  and we can define the invariant number:

$$n(V) = \sum_{d,k} \delta_{d,k}.$$

**Definition 2.4.** A family  $(\mathbf{p}_1, \dots, \mathbf{p}_s)$  is a *minimal* basis of  $\mathbf{Cov}(V)$  if its image in the vector space  $C_+/C_+^2$  is a basis. In that case we have  $s = n(V)$

*Remark 2.5.* As pointed out by Dixmier–Lazard [30], a minimal basis is obtained by taking, for each  $d, k$ , a complement basis of  $(C_+^2)_i$  in  $\mathbf{Cov}_{d,k}$ .

A first way to generate a covariant is by mean of the *Cayley operator* [46], which is a bi-differential operator acting on the tensor product of complex analytic functions  $\mathbf{f}(\mathbf{x}_\alpha)\mathbf{g}(\mathbf{x}_\beta)$

$$\Omega_{\alpha\beta}(\mathbf{f}(\mathbf{x}_\alpha)\mathbf{g}(\mathbf{x}_\beta)) := \frac{\partial \mathbf{f}}{\partial x_\alpha} \frac{\partial \mathbf{g}}{\partial y_\beta} - \frac{\partial \mathbf{f}}{\partial y_\alpha} \frac{\partial \mathbf{g}}{\partial x_\beta}.$$

We also introduce the *polarization operator*<sup>4</sup>:

$$\sigma_\alpha := x \frac{\partial}{\partial x_\alpha} + y \frac{\partial}{\partial y_\alpha}.$$

The Cayley and polarization operators commute with the  $\mathrm{SL}_2(\mathbb{C})$  action (see [46] for instance).

**Definition 2.6.** Given two binary forms  $\mathbf{f} \in S_n$  and  $\mathbf{g} \in S_p$ , their *transvectant* of index  $r$  is defined by

$$(\mathbf{f}, \mathbf{g})_r := \Omega_{\alpha\beta}^r \sigma_\alpha^{n-r} \sigma_\beta^{p-r} (\mathbf{f}(\mathbf{x}_\alpha)\mathbf{g}(\mathbf{x}_\beta)).$$

Recall that  $S_n$  is an *irreducible*  $\mathrm{SL}(2, \mathbb{C})$  representation [31]. The *Clebsch–Gordan decomposition* [31] is the decomposition of a tensor product

$$S_n \otimes S_p \simeq \bigoplus_{r=0}^{\min(n,p)} S_{n+p-2r}.$$

---

<sup>4</sup>This operator is called *scaling process* in [46]

The transvectant of index  $r$  corresponds to the *Clebsch-Gordan projector*

$$\pi_r : \mathbb{S}_n \otimes \mathbb{S}_p \longrightarrow \mathbb{S}_{n+p-2r}, \quad \mathbf{f} \otimes \mathbf{g} \mapsto (\mathbf{f}, \mathbf{g})_r. \quad (2.1)$$

*Remark 2.7.* There are different definitions of transvectants in the literature, each one differs from another by a scaling factor. The definition given in [46] uses a trace operator:

$$[\Omega_{\alpha\beta}^r \mathbf{f}(\mathbf{x}_\alpha) \mathbf{g}(\mathbf{x}_\beta)]_{|\mathbf{x}_\alpha=\mathbf{x}_\beta=\mathbf{x}} = \frac{1}{(n-r)!} \frac{1}{(p-r)!} (\mathbf{f}, \mathbf{g})_r.$$

Gordan's definition [34] corresponds to

$$\frac{1}{n!} \frac{1}{p!} (\mathbf{f}, \mathbf{g})_r.$$

This last expression is very simple when applied to *powers of linear forms*. Indeed, if

$$\mathbf{a}_{\mathbf{x}_\alpha}^n := (a_0 x_\alpha + a_1 y_\alpha)^n, \quad \mathbf{b}_{\mathbf{x}_\beta}^p := (b_0 x_\beta + b_1 y_\beta)^p,$$

then,

$$\frac{1}{n!} \frac{1}{p!} (\mathbf{a}_{\mathbf{x}_\alpha}^n, \mathbf{b}_{\mathbf{x}_\beta}^p)_r = (ab)^r \mathbf{a}_{\mathbf{x}}^{n-r} \mathbf{b}_{\mathbf{x}}^{p-r}, \quad (ab) := a_0 b_1 - a_1 b_0.$$

Our choice of definition 2.6 has the advantage of inducing simple relations (see 3.3 for instance) on operators and thus on transvectants.

Transvectants give also a canonical way to obtain covariants of binary forms:

**Theorem 2.8.** *Given a space  $V$  of binary forms, the covariant algebra  $\mathbf{Cov}(V)$  is generated by the (infinite) family of iterated transvectants:*

$$(\mathbf{f}_1, \mathbf{f}_2)_{r_1}, \quad ((\mathbf{f}_1, \mathbf{f}_2)_{r_1}, \mathbf{f}_2)_{r_2}, \dots \quad \mathbf{f}_i \in V, \quad r_i \in \mathbb{N}.$$

### 3. MOLECULAR COVARIANTS

Let  $\text{Sym}^d(V)$  be the space of totally symmetric tensors of order  $d$  on  $V$ . The *Aronhold polarization* induces an isomorphism [28] between  $\mathbf{Cov}_{d,k}(V)$  and the space

$$\text{Hom}_{\text{SL}(2, \mathbb{C})}(\text{Sym}^d(V), \mathbb{S}_k) \subset \text{Hom}_{\text{SL}(2, \mathbb{C})}(\otimes^d V, \mathbb{S}_k).$$

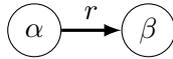
Transvectants, Cayley operators and polarization operators give natural way to obtain  $\text{SL}(2, \mathbb{C})$ -equivariant homomorphism. For instance, the Clebsch-Gordan projector

$$\pi_r : \mathbb{S}_n \otimes \mathbb{S}_p \longrightarrow \mathbb{S}_{n+p-2r},$$

can be written as

$$\Omega_{\alpha\beta}^r \sigma_\alpha^{n-r} \sigma_\beta^{p-r}.$$

Such a monomial will be represented by the colored directed graph (colored digraph)<sup>5</sup>:



<sup>5</sup>It is important to note that a digraph  $D$  represents here a morphism and not a bi-differential operator as did Olver-Shakiban [47].

where the *atom*  $\alpha$  (resp.  $\beta$ ) is colored by  $S_n$  (resp.  $S_p$ ).

More generally, let  $V = S_{n_1} \oplus \cdots \oplus S_{n_s}$  be a space of binary forms. We are going to define equivariant multilinear maps from  $V$  to some  $S_k$ , corresponding to monomials in the symbols  $\Omega_{\alpha\beta}, \sigma_\gamma, \dots$  and labelled by *molecules* (colored digraphs).

More precisely, let  $\mathcal{V}(\mathbf{D}) = \{\alpha, \beta, \dots, \varepsilon\}$  be the set of vertices of a colored digraph  $\mathbf{D}$  and  $\mathcal{E}(\mathbf{D})$  be its set of edges. Each vertex  $\alpha$  of  $\mathbf{D}$ , also called an *atom*, is colored by a factor  $S(\alpha) := S_n$  of  $V$ . In that case, the *valence* of  $\alpha$  is  $\text{val}(\alpha) := n$ . Define  $o(e)$ ,  $t(e)$  and  $w(e)$  to be respectively the origin, the termination and the weight of an edge  $e \in \mathcal{E}(\mathbf{D})$ . Finally, we define the valence of  $\alpha$  in the digraph  $\mathbf{D}$  to be the *free valence* of the atom  $\alpha \in \mathcal{V}(\mathbf{D})$ :

$$\text{val}_{\mathbf{D}}(\alpha) := \text{val}(\alpha) - \sum_{\alpha=o(e) \text{ or } \alpha=t(e)} w(e).$$

**Definition 3.1.** The  $\text{SL}(2, \mathbb{C})$ -equivariant homomorphism  $\phi_{\mathbf{D}}$  defined by the molecule  $\mathbf{D}$  is given by

$$\phi_{\mathbf{D}} := \prod_{e \in \mathcal{E}(\mathbf{D})} \Omega_{o(e)t(e)}^{w(e)} \prod_{\alpha \in \mathcal{V}(\mathbf{D})} \sigma_{\alpha}^{\text{val}_{\mathbf{D}}(\alpha)}.$$

It maps  $S(\alpha) \otimes \cdots \otimes S(\varepsilon)$  to  $S_k$ , where  $k = \text{val}_{\mathbf{D}}(\alpha) + \dots + \text{val}_{\mathbf{D}}(\varepsilon)$ .

There exists *syzygies* on morphisms  $\phi_{\mathbf{D}}$  induced by fundamental relations among operators. Let  $\alpha, \beta, \gamma$  and  $\delta$  be four symbols associated to valence  $n_1, n_2, n_3$  and  $n_4$ .

- (1) The first syzygy derives from the equality

$$\Omega_{\alpha\beta} = -\Omega_{\beta\alpha}$$

which leads to the graphical relation:

$$\begin{array}{c} \alpha \longrightarrow \beta \end{array} = - \begin{array}{c} \alpha \longleftarrow \beta \end{array} \quad (3.1)$$

- (2) The second one comes from the Plücker relation [46]:

$$\Omega_{\alpha\beta}\sigma_{\gamma} = \Omega_{\alpha\gamma}\sigma_{\beta} + \Omega_{\gamma\beta}\sigma_{\alpha}, \quad (3.2)$$

which leads to the graphical relation:

$$\begin{array}{c} \alpha \longrightarrow \beta \\ \gamma \end{array} = \begin{array}{c} \alpha \\ \searrow \\ \gamma \end{array} \begin{array}{c} \beta \\ \circlearrowleft \end{array} + \begin{array}{c} \alpha \\ \circlearrowleft \end{array} \begin{array}{c} \beta \\ \nearrow \\ \gamma \end{array} \quad (3.3)$$

- (3) The third one derives also from a Plücker relation, namely

$$\Omega_{\alpha\beta}\Omega_{\gamma\delta} = \Omega_{\alpha\delta}\Omega_{\beta\gamma} + \Omega_{\alpha\gamma}\Omega_{\delta\beta},$$

which leads to the graphical relation:

$$\begin{array}{c} \alpha \longrightarrow \beta \\ \delta \longrightarrow \gamma \end{array} = \begin{array}{c} \alpha \\ \downarrow \\ \delta \end{array} \begin{array}{c} \beta \\ \circlearrowleft \end{array} + \begin{array}{c} \beta \\ \downarrow \\ \gamma \end{array} \begin{array}{c} \alpha \\ \circlearrowleft \end{array} + \begin{array}{c} \alpha \longrightarrow \beta \\ \delta \longrightarrow \gamma \end{array} \quad (3.4)$$

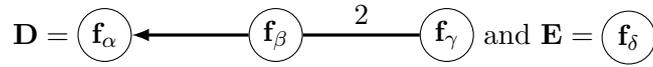


**Proposition 4.3.** *Let  $\mathbf{D}$  and  $\mathbf{E}$  be two molecular covariants,  $r \geq 0$  be an integer, the transvectant  $(\mathbf{D}, \mathbf{E})_r$  can be obtained as a linear combination of molecular covariants<sup>6</sup>  $\mathbf{M}^{\nu(r)}$  with rational positive coefficients, for each possible link  $\nu(r)$  between  $\mathbf{D}$  and  $\mathbf{E}$ :*

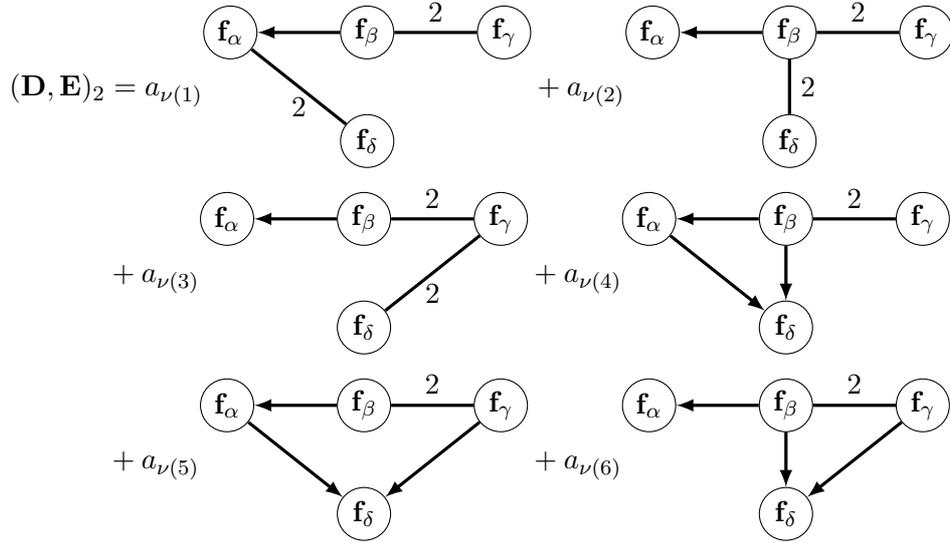
$$(\mathbf{D}, \mathbf{E})_r = \sum_{\nu(r)} a_{\nu(r)} \mathbf{M}^{\nu(r)} \tag{4.1}$$

where  $a_{\nu(r)} \in \mathbb{Q}^+$ .

*Exemple 4.4.* Let  $\mathbf{f}_\alpha, \dots$  be atoms of valence greater than 4,

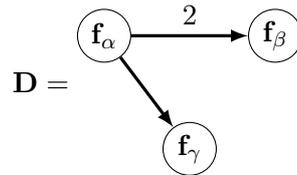


We thus have:



**Definition 4.5.** Given a molecular covariant  $\mathbf{D}$ , and an integer  $k \geq 0$ , we define<sup>7</sup>  $\overline{\mathbf{D}}^{\mu(k)}$  as the molecular covariant obtained by adding  $k$  edges on  $\mathbf{D}$  in a certain way  $\mu(k)$ .

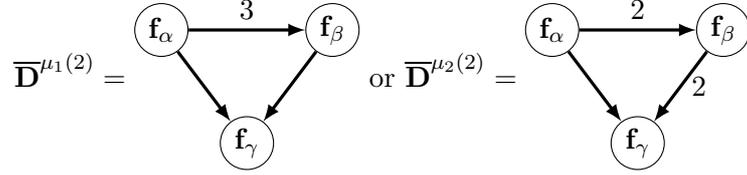
*Exemple 4.6.* Given atoms  $\mathbf{f}_\alpha, \mathbf{f}_\beta, \mathbf{f}_\gamma$  of valence greater than 4 and the molecular covariant



<sup>6</sup>The covariant  $\mathbf{M}^{\nu(r)}$  is called a *term* in [34].

<sup>7</sup>This operation is called *convolution* in [34].

We can consider

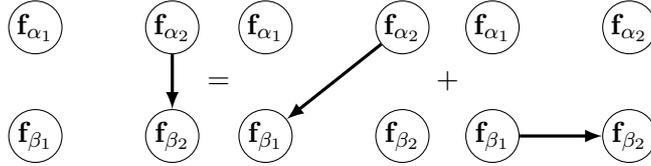


**Proposition 4.7.** *Given two molecular covariants  $\mathbf{D}$  and  $\mathbf{E}$ , an integer  $r$  and a molecular covariant  $\mathbf{M}^{\nu(r)}$  in the decomposition 4.1 of  $(\mathbf{D}, \mathbf{E})_r$ , then  $\mathbf{M}^{\nu(r)}$  is a linear combination of*

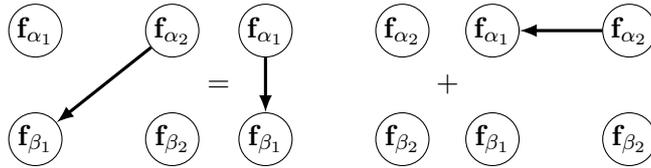
$$(\mathbf{D}, \mathbf{E})_r \text{ and } (\bar{\mathbf{D}}^{\mu_1(k_1)}, \bar{\mathbf{E}}^{\mu_2(k_2)})_{r'},$$

with  $k_1 + k_2 + r' = r$  being constant and  $r' < r$ .

*Sketch of proof.* We do by induction on  $r$ . As an illustration, take the case when  $r = 1$  and a molecular covariant  $\mathbf{M}^{\nu(1)}$  in  $(\mathbf{D}, \mathbf{E})_1$ . In this molecular covariant, there is a link between an atom  $\mathbf{f}_{\alpha_1}$  in  $\mathbf{D}$  and an atom  $\mathbf{f}_{\beta_1}$  in  $\mathbf{E}$ . Let  $\mathbf{M}^{\mu(1)}$  be another molecular covariant in  $(\mathbf{D}, \mathbf{E})_1$ , with a link between an atom  $\mathbf{f}_{\alpha_2} \neq \mathbf{f}_{\alpha_1}$  in  $\mathbf{D}$  and an atom  $\mathbf{f}_{\beta_1} \neq \mathbf{f}_{\beta_2}$  in  $\mathbf{E}$ . By relation 3.3 we have



where the last molecular covariant is a transvectant  $(\mathbf{D}, \bar{\mathbf{E}}^1)_0$ . By the same relation 3.3:



where the last molecular covariant is a transvectant  $(\bar{\mathbf{D}}^1, \mathbf{E})_0$ . Thus every molecular covariant of  $(\mathbf{D}, \mathbf{E})_1$  is expressible in terms of  $\mathbf{M}^{\nu(1)}$  and a linear combination of  $(\bar{\mathbf{D}}^{a_1}, \bar{\mathbf{E}}^{a_2})_0$ . All coefficients  $a_\nu$  of 4.1 being positives, this conclude the case  $r = 1$ .  $\square$

*Exemple 4.8.* Given  $V = S_n$  ( $n \geq 4$ ) and the molecular covariants:

$$\mathbf{D} = \begin{array}{c} \textcircled{\mathbf{f}_\alpha} \text{---} 2 \text{---} \textcircled{\mathbf{f}_\beta} \\ \text{---} \end{array} \text{ and } \mathbf{E} = \textcircled{\mathbf{f}_\gamma}$$

we can consider the transvectant  $(\mathbf{D}, \mathbf{E})_2$  and the molecular covariant:

$$\mathbf{M} = \begin{array}{c} \textcircled{\mathbf{f}_\alpha} \text{---} 2 \text{---} \textcircled{\mathbf{f}_\beta} \\ | \\ \textcircled{\mathbf{f}_\gamma} \end{array}$$

By proposition 4.7:

$$\begin{aligned} \mathbf{M} = & \mu_1 \left( \begin{array}{c} \textcircled{\mathbf{f}_\alpha} \text{---}^2 \textcircled{\mathbf{f}_\beta}, \textcircled{\mathbf{f}_\gamma} \end{array} \right)_2 + \mu_2 \left( \begin{array}{c} \textcircled{\mathbf{f}_\alpha} \text{---}^3 \textcircled{\mathbf{f}_\beta}, \textcircled{\mathbf{f}_\beta} \end{array} \right)_1 \\ & + \mu_3 \left( \begin{array}{c} \textcircled{\mathbf{f}_\alpha} \text{---}^4 \textcircled{\mathbf{f}_\beta}, \textcircled{\mathbf{f}_\gamma} \end{array} \right)_0 \end{aligned}$$

*Remark 4.9.* A molecular covariant  $\mathbf{M}$  with  $d$  atoms can be obtained as a molecular covariant  $\mathbf{M}^{\nu(r)}$  in the decomposition of the transvectant  $(\mathbf{D}, \mathbf{E})_r$ , where  $\mathbf{D}$  (resp.  $\mathbf{E}$ ) is constructed on  $d_1 < d$  (resp.  $d_2 < d$ ) atoms. By proposition 4.7 and by induction on  $d$ , we deduce that all molecular covariants can be expressible in terms of transvectants. Thus theorem 3.2 implies theorem 2.8.

## 5. GORDAN'S ALGORITHM FOR JOINT COVARIANTS

Gordan's algorithm for joint covariants produces a finite generating set for  $\mathbf{Cov}(V_1 \oplus V_2)$ , knowing a finite system of generators for  $\mathbf{Cov}(V_1)$  and  $\mathbf{Cov}(V_2)$ .

**Definition 5.1.** Let  $A = \{\mathbf{f}_1, \mathbf{f}_2, \dots\} \subset \mathbf{Cov}(V)$  be a **covariant** family taken from a space  $V$  of binary forms. Then  $\mathbf{Cov}(A)$  is defined to be the algebra generated by the iterated transvectants<sup>8</sup>

$$(\mathbf{f}_1, \mathbf{f}_2)_{r_1}, \quad ((\mathbf{f}_1, \mathbf{f}_2)_{r_1}, \mathbf{f}_3)_{r_2}, \dots, \quad \mathbf{f}_i \in A, \quad r_i \in \mathbb{N}.$$

Note that for every family  $A$  and  $B$ :

$$A \subset B \Rightarrow \mathbf{Cov}(A) \subset \mathbf{Cov}(B). \quad (5.1)$$

By theorem 3.2:

**Lemma 5.2.** *Let  $V = S_n$ ,  $\mathbf{f} \in V$  and  $A$  a family containing  $\mathbf{f}$ , then  $\mathbf{Cov}(A) = \mathbf{Cov}(V)$ .*

And by (5.1):

**Lemma 5.3.** *Let  $A_1$  and  $A_2$  be two families of  $\mathbf{Cov}(V)$ . If*

$$A_1 \subset A_2 \subset \mathbf{Cov}(A_1)$$

*then  $\mathbf{Cov}(A_1) = \mathbf{Cov}(A_2)$*

**Definition 5.4.** A covariant family  $A$  of  $V$  is said to be *complete* if  $\mathbf{Cov}(A) = \mathbb{C}[A]$ .

*Remark 5.5.* The notion of complete family is weaker than the one of a covariant basis. For instance, let  $\mathbf{f} \in S_3$ ,

$$\mathbf{H} := (\mathbf{f}, \mathbf{f})_2, \quad \mathbf{T} := (\mathbf{f}, \mathbf{H})_1 \text{ and } \Delta := (\mathbf{H}, \mathbf{H})_2.$$

As a classical result, the family  $A_1 = \{\mathbf{f}, \mathbf{H}, \mathbf{T}, \Delta\}$  is a covariant basis of  $\mathbf{Cov}(A_1) = \mathbf{Cov}(S_3)$  and is thus a complete family. Now, let

$$A_2 := \{\mathbf{H}, \Delta\}.$$

We have  $\mathbf{Cov}(A_2) \subsetneq \mathbf{Cov}(V)$ , but  $A_2$  is exactly the covariant basis [34] of the quadratic form  $\mathbf{H} \in S_2$ , thus  $A_2$  is a complete family but is not a covariant basis of  $\mathbf{Cov}(V)$ .

<sup>8</sup>Or equivalently, by all molecular covariants which atoms are in the family  $A$ .

From now on, let  $V_1$  and  $V_2$  be two spaces of binary forms,

$$A := \{\mathbf{f}_1, \dots, \mathbf{f}_p\} \subset \mathbf{Cov}(V_1), \quad B := \{\mathbf{g}_1, \dots, \mathbf{g}_q\} \subset \mathbf{Cov}(V_2),$$

be two finite and complete families of covariants of binary forms. Write  $a_i = \text{Ord}(\mathbf{f}_i)$  (resp.  $b_j = \text{Ord}(\mathbf{g}_j)$ ) to be the order of  $\mathbf{f}_i$  (resp.  $\mathbf{g}_j$ ). Write  $\mathbf{U}$  (resp.  $\mathbf{V}$ ) to be a monomial in  $\mathbb{C}[A]$  (resp.  $\mathbb{C}[B]$ ):

$$\mathbf{U} := \mathbf{f}_1^{\alpha_1} \dots \mathbf{f}_p^{\alpha_p}, \quad \mathbf{V} := \mathbf{g}_1^{\beta_1} \dots \mathbf{g}_q^{\beta_q}.$$

To each non-vanishing transvectant

$$(\mathbf{U}, \mathbf{V})_r,$$

we can associate an integer solution  $\kappa := (\boldsymbol{\alpha}, \boldsymbol{\beta}, u, v, r)$  of the linear system

$$S(A, B) : \begin{cases} a_1\alpha_1 + \dots + a_p\alpha_p = u + r, \\ b_1\beta_1 + \dots + b_q\beta_q = v + r, \end{cases} \quad (5.2)$$

Reciprocally, to each integer solution  $\kappa$  of  $S(A, B)$  we can associate a well defined transvectant  $(\mathbf{U}, \mathbf{V})_r$ .

**Lemma 5.6.** *If  $\kappa$  is a reducible integer solution of  $(S)$ , then  $\mathcal{F}(\kappa)$  contains a non connected molecular covariant.*

*Proof.* Take the integer solution  $\kappa = \kappa_1 + \kappa_2$  to be reducible, with

$$\kappa_i = (\boldsymbol{\alpha}^i, \boldsymbol{\beta}^i, u^i, v^i, r^i) \text{ solution of (5.2).}$$

Thus  $\mathbf{U} = \mathbf{U}_1\mathbf{U}_2$  and  $\mathbf{V} = \mathbf{V}_1\mathbf{V}_2$  and there exists  $\nu(r), \nu_1(r^1)$  and  $\nu_2(r^2)$  such that

$$\begin{array}{ccccccc} \textcircled{\mathbf{U}} & \xrightarrow{\nu(r)} & \textcircled{\mathbf{V}} & = & \textcircled{\mathbf{U}_1} & \xrightarrow{\nu_1(r^1)} & \textcircled{\mathbf{V}_1} & \textcircled{\mathbf{U}_2} & \xrightarrow{\nu_2(r^2)} & \textcircled{\mathbf{V}_2} \end{array}$$

which is a non connected covariant molecular occurring in  $\mathcal{F}(\kappa)$ .  $\square$

*Remark 5.7.* If an integer solution associated to a transvectant  $(\mathbf{U}, \mathbf{V})_r$  is reducible, this does not imply that such a transvectant is a reducible one. For instance, take  $\mathbf{f} \in S_6$ ,  $A = B := \{\mathbf{f}\}$  and the transvectants

$$(\mathbf{f}^{\alpha_1}\mathbf{f}^{\alpha_2}, \mathbf{f}^{\beta_1})_5.$$

Then the solution  $(\alpha_1, \alpha_2, \beta_1, u, v, 5) = (1, 1, 1, 7, 1, 5)$  is a reducible one, and the transvectant

$$(\mathbf{f}^2, \mathbf{f})_5 \quad (5.3)$$

contains the molecular covariant

$$\begin{array}{c} \textcircled{\mathbf{f}} \xrightarrow{5} \textcircled{\mathbf{f}} \\ \\ \textcircled{\mathbf{f}} \end{array}$$

which is a null covariant. Thus property 4.7 implies that transvectant (5.3) is a linear combination of transvectants

$$((\mathbf{f}, \mathbf{f})_4, \mathbf{f})_1, \quad ((\mathbf{f}, \mathbf{f})_3, \mathbf{f})_2 = 0, \quad ((\mathbf{f}, \mathbf{f})_2, \mathbf{f})_3, \quad ((\mathbf{f}, \mathbf{f})_1, \mathbf{f})_4 = 0,$$

and one can finally show that

$$(\mathbf{f}^2, \mathbf{f})_5 = \frac{65}{66}((\mathbf{f}, \mathbf{f})_4, \mathbf{f})_1,$$

where  $((\mathbf{f}, \mathbf{f})_4, \mathbf{f})_1$  is a non reducible covariant, as being in the covariant basis of  $S_6$  (see table 7.1).

Nevertheless, we have the result:

**Lemma 5.8.** *Let  $\mathbf{a} := \max(a_i)$ ,  $\mathbf{b} := \max(b_j)$  and*

$$\mathbf{U} := \mathbf{f}_1^{\alpha_1} \dots \mathbf{f}_p^{\alpha_p}, \quad \mathbf{V} := \mathbf{g}_1^{\beta_1} \dots \mathbf{g}_q^{\beta_q}.$$

*Let  $u = \text{Ord}(\mathbf{U}) - r$  and  $v = \text{Ord}(\mathbf{V}) - r$ . If*

$$u + v \geq \mathbf{a} + \mathbf{b}, \tag{5.4}$$

*then, the transvectant  $(\mathbf{U}, \mathbf{V})_r$  is reducible.*

*Proof.* Condition (5.4) implies that  $u \geq \mathbf{a}$  or  $v \geq \mathbf{b}$  and thus that the transvectant  $(\mathbf{U}, \mathbf{V})_r$  contains a reducible molecular covariant  $\mathbf{T}$  (the corresponding integer solution  $\kappa$  is thus not minimal). By virtue of proposition 4.7, the transvectant is a linear combination the term  $\mathbf{T}$  and transvectants

$$(\overline{\mathbf{U}}^{\mu(k_1)}, \overline{\mathbf{U}}^{\mu(k_2)})_{r'},$$

where  $r' < r$  and  $k_1 + k_2 = r - r'$ . Note that, since both families  $\mathbf{A}$  and  $\mathbf{B}$  are supposed to be complete, we have

$$\overline{\mathbf{U}}^{\mu(k_1)} = \mathbf{f}_1^{\alpha'_1} \dots \mathbf{f}_p^{\alpha'_p}, \quad \overline{\mathbf{V}}^{\mu(k_2)} = \mathbf{g}_1^{\beta'_1} \dots \mathbf{g}_q^{\beta'_q},$$

where, moreover, the order of the transvectant  $(\overline{\mathbf{U}}^{\mu(k_1)}, \overline{\mathbf{V}}^{\mu(k_2)})_{r'}$  is of order  $u' + v' = u + v$ . Since we have supposed that  $u + v \geq \mathbf{a} + \mathbf{b}$ , we get that  $u' + v' \geq \mathbf{a} + \mathbf{b}$  and the proof is achieved by a recursive argument on the index of the transvectant  $r$ .  $\square$

*Remark 5.9.* The statement  $u + v \geq \mathbf{a} + \mathbf{b}$  can't be replaced by the hypothesis  $u \geq \mathbf{a}$  or  $v \geq \mathbf{b}$ . Taking back the example given in remark 5.7, for  $\mathbf{f} \in S_6$  and  $\mathbf{h} := (\mathbf{f}^2, \mathbf{f})_5$ , we have  $u = 7 \geq 6$  but  $\mathbf{h}$  is not reducible.

Lemma 5.8 is closely related to:

**Corollary 5.10.** *Let  $\mathbf{F} \in \mathbf{Cov}(V)$  of order  $s$  and  $\{\mathbf{F}_1, \dots, \mathbf{F}_k\} \subset \mathbf{Cov}(V)$  be a family of homogeneous covariants. Let  $t_i$  be the order of  $\mathbf{F}_i$  and  $\mathbf{t} = \max(t_i)$ . For a given integer  $r$ , if*

$$\sum_{i=1}^k t_i \geq \mathbf{a} + 2r,$$

*then the transvectant  $(\mathbf{F}_1 \dots \mathbf{F}_k, \mathbf{F})_r$  is reducible.*

*Proof.* Let  $\mathbf{f}_1, \dots, \mathbf{f}_p$  be a covariant basis of  $\mathbf{Cov}(V)$ , each  $\mathbf{f}_i$ 's being a homogeneous covariant of order  $a_i$ . Then, each covariant  $\mathbf{F}_j$  is a linear combination of monomials  $\mathbf{f}_{i_1}^{\alpha_{i_1}} \dots \mathbf{f}_{i_l}^{\alpha_{i_l}}$  with  $a_i \leq t_j \leq \mathbf{t}$ . Thus  $\mathbf{F}_1 \dots \mathbf{F}_k$  is a covariant expressible in terms of monomials  $\mathbf{U}$  in the  $\mathbf{f}_i$ 's with

$$\text{Ord}(\mathbf{U}) = \sum_{i=1}^k t_i \text{ and } \max(a_i) \leq \mathbf{t}.$$

We have also  $\mathbf{F} = \mathbf{f}_{j_1}^{\beta_{j_1}} \dots \mathbf{f}_{j_m}^{\beta_{j_m}}$  with  $\max(a_j) \leq s$ . By lemma 5.8, each transvectant  $(\mathbf{U}, \mathbf{V})_r$  is thus a reducible covariant.  $\square$

We know there exists a finite family of irreducible integer solutions of the system  $S(A, B)$ (5.2) (see [54, 53, 57] for details). Let  $C := \{\tau^1, \dots, \tau^l\}$  be the set of transvectants associated to these irreducible solutions.

**Theorem 5.11.** *The algebra  $\mathbf{Cov}(A \cup B)$  is generated by the finite and complete family  $C = \{\tau^1, \dots, \tau^l\}$ .*

*Proof.* Let first remark that each  $\mathbf{f}_i$  (resp. each  $\mathbf{g}_j$ ) correspond to an irreducible solution of  $S(A, B)$ , then  $A \subset C$  and  $B \subset C$ .

From theorem 3.2 we first have to prove that each molecular covariant  $\mathbf{M} \in \mathbf{Cov}(A \cup B)$  is in a finite algebra. But, using definition 4.1 we can write the molecular covariant  $\mathbf{M}$  as

$$\mathbf{M} = \text{Oval}(\mathbf{D}) \xrightarrow{\nu(r)} \text{Oval}(\mathbf{E})$$

with molecular covariants  $\mathbf{D} \in \mathbf{Cov}(A)$  and  $\mathbf{E} \in \mathbf{Cov}(B)$ ;  $r$  being some integer. Thus, by proposition 4.7, all covariants of  $\mathbf{Cov}(A \cup B)$  is a linear combination of transvectants

$$(\mathbf{D}, \mathbf{E})_r, \quad (\overline{\mathbf{D}}^{\mu_1}, \overline{\mathbf{E}}^{\mu_2})_r$$

Since  $A$  is complete, we can suppose all molecular covariants  $\mathbf{D}, \overline{\mathbf{D}}^{\mu_1}$  to be a monomial expression on the  $\mathbf{f}_i$ 's. In the same way we can suppose  $\mathbf{E}, \overline{\mathbf{E}}^{\mu_2}$  to be monomial expression  $\mathbf{V}$  on the  $\mathbf{g}_j$ 's. We then have to consider covariants taken from the family  $(\mathbf{U}, \mathbf{V})_r$ . We do now by induction on  $r$ . In the case  $r = 0$ , we have that  $A \subset C$  and  $B \subset C$ , thus the assumption is true.

Let  $(\mathbf{U}, \mathbf{V})_r$  be a transvectant which corresponds to a reducible integer solution  $\kappa$ . By proposition 4.7,  $(\mathbf{U}, \mathbf{V})_r$  is a linear combination of a product of  $\tau^i$ 's and transvectants

$$(\overline{\mathbf{U}}^{\mu_1(k_1)}, \overline{\mathbf{V}}^{\mu_2(k_2)})_{r'}, \quad r' < r \quad (5.5)$$

But  $\overline{\mathbf{U}}^{\mu_1(k_1)} \in \mathbf{Cov}(A)$  (resp.  $\mathbf{V}$  and  $B$ ) and since  $A$  (resp.  $B$ ) is complete we know that the transvectants (5.5) are a linear combination of

$$(\mathbf{U}', \mathbf{V}')_{r'}, \quad r' < r,$$

where  $\mathbf{U}'$  (resp.  $\mathbf{V}'$ ) is a monomial in the  $\mathbf{f}_i$ 's (resp.  $\mathbf{g}_j$ ). Thus, by induction on  $r$ , the algebra  $\mathbf{Cov}(A \cup B)$  is generated by the finite family  $C$ .

Now, to prove  $C$  is a complete family, just remark that

$$A \cup B \subset C \subset \mathbf{Cov}(A \cup B),$$

and then

$$\mathbf{Cov}(C) = \mathbf{Cov}(A \cup B) = \mathbb{C}[C],$$

$\square$

One direct application of theorem 5.11 is about joint covariants. Indeed, this theorem gives a constructive approach to get a covariant basis of  $V_1 \oplus V_2$ , once we know a covariant basis of each space  $V_1$  and  $V_2$ .

Note that lemma 5.8 gives a bound for the order of each element of a minimal basis of joint covariants:

**Corollary 5.12.** *Let  $V = S_{n_1} \oplus \cdots \oplus S_{n_s}$ . If  $\mu_i$  is the maximal order of a minimal basis for  $S_{n_i}$ , then, for each element  $\mathbf{h}$  of a minimal basis for  $V$ , we get*

$$\text{ord}(\mathbf{h}) \leq \sum_{i=1}^s \mu_i.$$

*Example 5.13.* We can directly use theorem 5.11 to get a covariant basis of  $S_3 \oplus S_4$ . The same result has been obtained by Popoviciu–Brouwer [18] with more computations. Let  $\mathbf{u} \in S_3$  and  $\mathbf{v} \in S_4$ . Recall that:

- The algebra  $\mathbf{Cov}(S_3)$  is generated by the three covariants  $\mathbf{u} \in S_3$ ,  $\mathbf{h}_{2,2} := (\mathbf{u}, \mathbf{u})_2 \in S_2$ ,  $\mathbf{h}_{3,3} := (\mathbf{u}, \mathbf{h}_{2,2})_1 \in S_3$  and one invariant  $\Delta := (\mathbf{u}, \mathbf{t})_3$  ;
- The algebra  $\mathbf{Cov}(S_4)$  is generated by the three covariants  $\mathbf{v} \in S_4$ ,  $\mathbf{k}_{2,4} := (\mathbf{v}, \mathbf{v})_2 \in S_4$ ,  $\mathbf{k}_{3,6} := (\mathbf{v}, \mathbf{H}_{2,4})_1 \in S_6$  and the two invariants  $i := (\mathbf{v}, \mathbf{v})_4$ ,  $j := (\mathbf{v}, \mathbf{H})_4$  ;

We then have to solve the linear diophantine system

$$(S) : \begin{cases} 2\alpha_1 + 3\alpha_2 + 3\alpha_3 &= u + r \\ 4\beta_1 + 4\beta_2 + 6\beta_3 &= v + r \end{cases} \quad (5.6)$$

Using Normaliz package in Macaulay 2 [19], this leads to 104 solutions. The associated covariants form a family of covariants of maximum total degree 18 (the total degree of a covariant being the sum of its degree and its order). The Hilbert series of  $\mathbf{Cov}(S_4 \oplus S_3)$  is given by

$$\begin{aligned} H(z) &= 1 + z^2 + 2z^3 + 5z^4 + 10z^5 + 18z^6 + 31z^7 + 55z^8 + 92z^9 \\ &\quad + 144z^{10} + 223z^{11} + 341z^{12} + 499z^{13} + 725z^{14} + 1031z^{15} \\ &\quad + 1436z^{16} + 1978z^{17} + 2685z^{18} + \dots \end{aligned}$$

Using scripts written in Macaulay 2 [35], we reduce the family of 104 generators to a minimal set of 63 generators given in table 1, which has also been obtained by Popoviciu–Brouwer [18].

$d/o$	0	1	2	3	4	5	6	#	Cum
1	—	—	—	1	1	—	—	2	2
2	1	1	1	1	1	1	—	6	8
3	1	1	2	2	1	1	1	9	17
4	1	2	2	2	1	—	—	8	25
5	2	3	3	1	1	—	—	10	35
6	2	3	2	1	—	—	—	8	43
7	3	3	1	—	—	—	—	7	50
8	3	2	—	—	—	—	—	5	55
9	4	1	—	—	—	—	—	5	60
10	2	—	—	—	—	—	—	2	62
11	1	—	—	—	—	—	—	1	63
Tot	20	16	11	8	5	2	1		63

TABLE 1. Covariant basis of  $S_3 \oplus S_4$

*Remark 5.14.* An important reduction of the integer system (5.6) can occur. As noted in example 5.13 the algebra  $\mathbf{Cov}(S_4)$  is generated by  $\mathbf{v}, \mathbf{k}_{2,4}, \mathbf{k}_{3,6}$  and the two invariants  $i, j$ . But from the relation

$$12\mathbf{k}_{3,6}^2 = -6\mathbf{k}_{2,4}^3 - 2j\mathbf{v}^3 + 3i\mathbf{v}^2\mathbf{k}_{2,4}, \quad (5.7)$$

we deduce that  $\mathbf{Cov}(S_4)$  is a finite  $\mathbb{C}[i, j, \mathbf{v}, \mathbf{k}_{2,4}]$ -module:

$$\mathbf{Cov}(S_4) = \mathbb{C}[i, j, \mathbf{v}, \mathbf{k}_{2,4}] + \mathbf{TC}[i, j, \mathbf{v}, \mathbf{k}_{2,4}].$$

From now on, suppose there exists a subfamily

$$A_0 := \{\mathbf{f}_{l+1}, \dots, \mathbf{f}_p\} \subset A$$

such that  $\mathbb{C}[A]$  is a  $\mathbb{C}[A_0]$ -module of finite type generated by monomials  $\eta_1, \dots, \eta_s$  taken from the family  $\mathbf{f}_1, \dots, \mathbf{f}_l$ . To the monomials

$$\eta_1 = \mathbf{f}_1^{u_1^1} \dots \mathbf{f}_l^{u_l^1}, \eta_2 = \mathbf{f}_1^{u_1^2} \dots \mathbf{f}_l^{u_l^2}, \dots (u_i^j \neq 0)$$

we associate the set

$$\mathcal{I}(A) := \{\boldsymbol{\alpha}, \quad \alpha_1 \leq u_1^1, \dots, \alpha_l \leq u_l^1, \alpha_1 \leq u_1^2, \dots\}$$

Suppose also there exists a subfamily  $B_0 := \{\mathbf{g}_{k+1}, \dots, \mathbf{g}_q\}$  such that  $\mathbb{C}[B]$  is a  $\mathbb{C}[B_0]$ -module of finite type generated by monomials  $\xi_1, \dots, \xi_m$  taken from the family  $\mathbf{g}_1, \dots, \mathbf{g}_k$ . We thus have another set  $\mathcal{I}(B)$  associated to the monomials  $\xi_j$ . Consider the reduced system

$$S^*(A, B) : \begin{cases} a_1\alpha_1 + \dots + a_p\alpha_p = u + r, \\ b_1\beta_1 + \dots + b_q\beta_q = v + r, \end{cases} \quad \boldsymbol{\alpha} \in \mathcal{I}(A), \quad \boldsymbol{\beta} \in \mathcal{I}(B) \quad (5.8)$$

and  $\kappa^1, \dots, \kappa^n$  be its irreducible solutions. Let  $\boldsymbol{\tau}^1, \dots, \boldsymbol{\tau}^n$  be their associated transvectant.

**Theorem 5.15.** *The algebra  $\mathbf{Cov}(A \cup B)$  is generated by the finite and complete family  $C = \{\boldsymbol{\tau}^1, \dots, \boldsymbol{\tau}^n\}$ .*

*Proof.* We have to take back the proof of theorem 5.11. First observe then that each  $\mathbf{f}_i$  (resp. each  $\mathbf{g}_j$ ) correspond to an irreducible solution of  $S^*(A, B)$  (5.8). Thus we know that  $A \subset C$  and  $B \subset C$ . Now, we can write a covariant  $\mathbf{M} \in \mathbf{Cov}(A \cup B)$  as

$$\mathbf{M} = \text{Oval}(\mathbf{D}) \xrightarrow{\nu(r)} \text{Oval}(\mathbf{E})$$

with a molecular covariant  $\mathbf{D} \in \mathbf{Cov}(A)$  and  $\mathbf{E} \in \mathbf{Cov}(B)$ ;  $r$  being some integer. From the hypothesis, we can write covariants  $\mathbf{D}$  and  $\mathbf{E}$  as

$$\mathbf{D} = \sum_i \eta_i \mathbf{F}_i, \quad \mathbf{E} = \sum_j \xi_j \mathbf{G}_j, \quad (5.9)$$

where  $\mathbf{F}_i$ 's (resp.  $\mathbf{G}_j$ 's) are monomials in  $\mathbf{f}_{p+i}, i \geq 1$  (resp.  $\mathbf{g}_{k+j}, j \geq 1$ ). Thus we have to consider transvectants

$$(\eta_i \mathbf{F}_i, \xi_j \mathbf{G}_j)_r,$$

which corresponds to a solution  $\kappa$  of the system  $S^*(A, B)$ . Now, if  $\kappa$  is a reducible solution of  $S^*(A, B)$ , we have  $\kappa = \kappa_1 + \kappa_2$  where each  $\kappa_i$  is a solution of  $S^*(A, B)$  (a direct consequence of  $\kappa_i$  being lower than  $\kappa$ ). As

in the proof of 5.11, by proposition 4.7, we can write transvectant (5) as a linear combination of products of  $\tau^{i'}$ 's and transvectants

$$(\overline{\eta_i \mathbf{F}_i^{\mu_1(k_1)}}, \overline{\xi_j \mathbf{G}_j^{\mu_2(k_2)}})_{r'}, \quad r' < r. \quad (5.10)$$

But  $\overline{\eta_i \mathbf{F}_i^{\mu_1(k_1)}} \in \mathbf{Cov}(A)$  (resp.  $\xi_j \mathbf{G}_j$  and B) and by hypothesis on A and B the transvectant (5.10) is a linear combination of

$$(\eta_i \mathbf{F}'_i, \xi_j \mathbf{G}'_j)_{r'}, \quad r' < r + 1,$$

and we thus conclude the proof by induction.  $\square$

## 6. GORDAN'S ALGORITHM FOR SIMPLE COVARIANTS

There is a second version of Gordan's algorithm which enables to compute a covariant basis for  $S_n$ , knowing covariant bases for  $S_k$ , ( $k < n$ ).

**Definition 6.1.** Let  $I \subset \mathbf{Cov}(V)$  be an homogeneous ideal. A family  $A = \{\mathbf{f}_1, \dots, \mathbf{f}_p\} \subset \mathbf{Cov}(V)$  of homogeneous covariants is *relatively complete modulo I* if every homogeneous covariant  $\mathbf{h} \in \mathbf{Cov}(A)$  of degree  $d$  can be written

$$\mathbf{h} = p(\mathbf{f}_1, \dots, \mathbf{f}_p) + \mathbf{h}_I \text{ with } \mathbf{h}_I \in I,$$

where  $p(\mathbf{f}_1, \dots, \mathbf{f}_p)$  and  $\mathbf{h}_I$  are homogeneous covariants of degree  $d$ .

Given molecule D upon a space  $V$  of binary forms, the *grade* of D, denoted  $\text{gr}(D)$  is the maximum weight of the edges of D :

$$\text{gr}(D) := \max_{e \in \mathcal{E}(D)} w(e).$$

**Definition 6.2.** Let  $r$  be an integer ; we define  $\mathcal{G}_r(V)$  to be the set of all molecular covariants with grade at least  $r$ .

As a first observation, it is clear that for  $V = S_n$ , we have  $\mathcal{G}_r(S_n) = \emptyset$  for  $r > n$ . Furthermore, we have

$$\mathcal{G}_{i+1}(V) \subset \mathcal{G}_i(V) \text{ for all } i. \quad (6.1)$$

**Definition 6.3** (Gordan's ideals). Let  $r$  be an integer. We define the Gordan ideal  $I_r(V)$  to be the homogeneous<sup>9</sup> ideal generated by  $\mathcal{G}_r(V)$  ; we write

$$I_r(V) := \langle \mathcal{G}_r(V) \rangle.$$

We observe directly that:

- $I_r(S_n) = \{0\}$  for all  $r > n$  ;
- By equation 6.1, we have  $I_{r+1}(V) \subset I_r(V)$  for every integer  $r$ .

By the property 4.3:

**Lemma 6.4.** If  $\mathbf{h}_r \in I_r(V)$ , for every covariant  $\mathbf{h} \in \mathbf{Cov}(V)$  and for every integer  $j \geq 0$ ,

$$\{\mathbf{h}_r, \mathbf{h}\}_j \in I_r(V).$$

*Remark 6.5.* For every invariant  $\Delta \in \mathbf{Cov}(S_n)$ , the ideal  $\langle \Delta \rangle$  is stable by transvection, since

$$(\mathbf{h}, \Delta \mathbf{k})_r = \Delta(\mathbf{h}, \mathbf{k})_r.$$

<sup>9</sup>Such an ideal is an homogeneous ideal as being generated by homogeneous elements

Given two finite families  $A$  and  $B$  of covariants, let  $\kappa^1, \dots, \kappa^l$  be the irreducible integer solutions of the linear system  $S(A, B)$  (5.2) and  $\tau^i$  be the associated transvectants. Let  $\mathbf{f} \in S_n$ ,  $\Delta \in \mathbf{Cov}(V)$  be an invariant,  $k \geq 0$  be a given integer and

$$\mathbf{H}_{2k} := \{\mathbf{f}, \mathbf{f}\}_{2k}.$$

Finally, write  $J = I_{2k+2}$  or  $J = I_{2k+2} + \langle \Delta \rangle$ .

**Theorem 6.6.** *Suppose  $A$  is relatively complete modulo  $I_{2k}$  and contains the binary form  $\mathbf{f}$ . Suppose also that  $B$  is relatively complete modulo  $J$  and contains the covariants  $\mathbf{H}_{2k}$ . Then the family  $C := \{\tau^1, \dots, \tau^l\}$  is relatively complete modulo  $J$  and*

$$\mathbf{Cov}(C) = \mathbf{Cov}(A \cup B) = \mathbf{Cov}(S_n).$$

Before getting to the proof of this theorem, we have to consider two previous lemmas.

**Lemma 6.7.** *Taking the same hypothesis as in theorem 6.6. Suppose  $\mathbf{h}_{2k} \in I_{2k}$  is a covariant of degree  $d$  in  $\mathbf{f}$ , and let  $\mathbf{V}$  be a monomial in the  $\mathbf{g}_j$ 's in  $B$ . For a given integer  $r \geq 0$ , the transvectant*

$$(\mathbf{h}_{2k}, \mathbf{V})_r, \tag{6.2}$$

is a linear combination of

$$(\mathbf{U}, \mathbf{V}')_{r'} \text{ and } \mathbf{h}_J \in J,$$

where  $\mathbf{U}$  is a monomial in  $A$  with degree strictly less than  $d$ .

*Proof.* We do it by induction on  $d$ , starting with  $d = 2$ . In this case observe  $\mathbf{h}_{2k} = \mathbf{H}_{2k}$ , thus

$$(\mathbf{H}_{2k}, \mathbf{V})_r \in \mathbf{Cov}(B),$$

and  $B$  being relatively complete modulo  $J$ , this covariant can be written as a linear combination of monomials  $\mathbf{V}' = (1, \mathbf{V}')_0$  and  $\mathbf{h} \in J$ . Now, for a given degree  $d$ , we may suppose  $\mathbf{h}_{2k}$  is a molecular covariant

$$\begin{array}{ccc} \textcircled{\mathbf{M}} & \xrightarrow{\nu(r_1)} & \textcircled{\mathbf{H}_{2k}} \end{array}$$

where degree of  $\mathbf{M}$  in  $\mathbf{f}$  is  $d' < d$ . Thus by proposition 4.3, the transvectant (6.2) is a linear combination of molecular covariants

$$\begin{array}{ccc} \textcircled{\mathbf{M}} & \xrightarrow{\nu(r_1)} & \textcircled{\mathbf{H}_{2k}} \\ \textcircled{\hspace{1.5cm}} & & \xrightarrow{\nu(r)} & \textcircled{\mathbf{V}} \end{array}$$

But such a molecular covariant is a term in the transvectant

$$(\mathbf{M}, \mathbf{N})_{r_1'} \text{ where } \mathbf{N} := \begin{array}{ccc} \textcircled{\mathbf{H}_{2k}} & \xrightarrow{\nu(r_2)} & \textcircled{\mathbf{V}} \end{array}$$

Then, by proposition 4.7, transvectant (6.2) is a linear combination of

$$(\overline{\mathbf{M}}^{\mu_1}, \overline{\mathbf{N}}^{\mu_2})_{r'},$$

where  $\overline{\mathbf{M}}^{\mu_1}$  is a covariant in  $\mathbf{Cov}(A)$  of degree  $d' < d$  and  $\overline{\mathbf{N}}^{\mu_2}$  is a covariant of  $\mathbf{Cov}(B)$ . Since  $A$  and  $B$  are relatively complete, we thus can write

$$\overline{\mathbf{M}}^{\mu_1} = \mathbf{U}_1 + \mathbf{h}_{2k},$$

where  $\mathbf{U}_1$  and  $\mathbf{h}_{2k}$  are of degree  $d' < d$  and

$$\overline{\mathbf{N}}^{\mu_2} = \mathbf{V}_1 + \mathbf{h}_J, \quad \mathbf{h}_J \in J.$$

Transvectant (6.2) is then a linear combination of transvectants

$$\begin{aligned} & (\mathbf{U}_1, \mathbf{V}_1)_{r'} \\ & (\mathbf{h}_{2k}, \mathbf{h}_J)_{r'} \in J \text{ by lemma 6.4 and remark 6.5} \\ & (\mathbf{U}_1, \mathbf{h}_J)_{r'} \in J \\ & (\mathbf{h}_{2k}, \mathbf{V}_1)_{r'} \end{aligned}$$

and by induction we can decompose the last one into a linear combination of

$$(\mathbf{U}_2, \mathbf{V}^l)_{r''}, \quad \mathbf{h} \in J,$$

with  $\mathbf{U}_2$  being a monomial in  $A$  with degree strictly less than  $d$ .  $\square$

**Lemma 6.8.** *Taking the same hypothesis as in theorem 6.6. Let  $\mathbf{U}$  (resp.  $\mathbf{V}$ ) be a monomial in the  $\mathbf{f}_i$ 's in  $A$  (resp. in the  $\mathbf{g}_j$ 's in  $B$ ), then for every integer  $r \geq 0$*

$$(\mathbf{U}, \mathbf{V})_r = \mathbf{p}(\tau^1, \dots, \tau^l) + \mathbf{h}_{2k+2}, \quad \mathbf{h}_{2k+2} \in J,$$

*Proof.* We first observe that  $A \subset C$  and  $B \subset C$  since each  $\mathbf{f}_i$  and  $\mathbf{g}_j$  corresponds to a minimal solution of the linear system  $S(A, B)$ .

Then, we argue by induction on the degree  $d$  of  $\mathbf{U}$  in  $\mathbf{f} \in \mathbf{Cov}(V)$  and by induction on  $r$ .

- In the case  $d = 1$  we can only have  $\mathbf{U} = \mathbf{f}$ 
  - For  $r = 0$  we have to consider a product  $\mathbf{fV}$  which is a product of  $\tau^i$ 's;
  - For a given  $r$ , if  $(\mathbf{f}, \mathbf{V})_r$  corresponds to a reducible solution of  $S(A, B)$ , by proposition 4.7 we have

$$(\mathbf{f}, \mathbf{V})_r = \mathbf{T}_1 \mathbf{T}_2 + \sum (\overline{\mathbf{f}}^{\mu_1}, \overline{\mathbf{V}}^{\mu_2})_{r' < r}$$

But we can only have  $\overline{\mathbf{f}}^{\mu_1} = \mathbf{f}$  and thus by induction on  $r$  the lemma is true for  $\mathbf{U} = \mathbf{f}$ .

- In the case  $d = 2$ ;
  - For  $r = 0$  we have to consider a product  $\mathbf{UV}$  which is a product of  $\tau^i$ 's
  - For a given  $r$ , if  $(\mathbf{U}, \mathbf{V})_r$  correspond to a reducible solution of  $S(A, B)$ , by proposition 4.7 we have

$$(\mathbf{U}, \mathbf{V})_r = \mathbf{T}_1 \mathbf{T}_2 + \sum (\overline{\mathbf{U}}^{\mu_1}, \overline{\mathbf{V}}^{\mu_2})_{r' < r}. \quad (6.3)$$

But each  $\overline{\mathbf{U}}^{\mu_1}$  is of degree 2. Since  $A$  is relatively complete modulo  $I_{2k}$  we thus have

$$\overline{\mathbf{U}}^{\mu_1} = \mathbf{U}_1 + \mathbf{h}_{2k},$$

each of the covariants being of degree 2 in  $\mathbf{f}$ . From this we can suppose that  $\mathbf{h}_2 = \mathbf{H}_{2k} \in \mathbf{B}$ . We can also write

$$\overline{\mathbf{V}}^{\mu_2} = \mathbf{V}_1 + \mathbf{h}_{2k+2} + \Delta\mathbf{h}.$$

In (6.3) we then have to consider transvectants

$$\begin{aligned} & (\mathbf{U}_1, \mathbf{V}_1)_{r' < r}, \\ & (\mathbf{U}_1, \mathbf{h}_{2k+2})_{r' < r} \in J, \\ & (\mathbf{U}_1, \Delta\mathbf{h})_{r' < r} = \Delta(\mathbf{U}_1, \mathbf{h})_{r' < r} \in J, \\ & (\mathbf{H}_{2k}, \mathbf{V}_1)_{r' < r} \in \mathbf{Cov}(\mathbf{B}), \\ & (\mathbf{H}_{2k}, \mathbf{h}_{2k+2} + \Delta\mathbf{h})_{r' < r} \in J, \end{aligned}$$

where  $\mathbf{U}_1$  is a degree 2 transvectant of lower indexes. By induction, this prove the case when  $d = 2$ .

- For a given  $d$  the same ideas as in the case  $d = 2$  will occur;
  - For  $r = 0$  we have to consider a product  $\mathbf{UV}$  which is a product of  $\tau^i$ ;
  - For a given  $r$ , if  $(\mathbf{U}, \mathbf{V})_r$  correspond to a reducible solution of  $S(\mathbf{A}, \mathbf{B})$ , by proposition 4.7 we have

$$(\mathbf{U}, \mathbf{V})_r = \mathbf{T}_1\mathbf{T}_2 + \sum (\overline{\mathbf{U}}^{\mu_1}, \overline{\mathbf{V}}^{\mu_2})_{r' < r} \quad (6.4)$$

But each  $\overline{\mathbf{U}}^{\mu_1}$  is of degree  $d$ . Since  $\mathbf{A}$  is relatively complete modulo  $I_{2k}$  we thus have

$$\overline{\mathbf{U}}^{\mu_1} = \mathbf{U}_1 + \mathbf{h}_{2k},$$

each of the covariants being of degree  $d$  in  $\mathbf{f}$ . We also write

$$\overline{\mathbf{V}}^{\mu_2} = \mathbf{V}_1 + \mathbf{h}_{2k+2} + \Delta\mathbf{h}.$$

Thus we have to consider transvectants

$$\begin{aligned} & (\mathbf{U}_1, \mathbf{V}_1)_{r' < r}, \\ & (\mathbf{U}_1, \mathbf{h}_{2k+2} + \Delta\mathbf{h})_{r' < r} \in J, \\ & (\mathbf{h}_{2k}, \mathbf{V}_1)_{r' < r}, \\ & (\mathbf{h}_{2k}, \mathbf{h}_{2k+2} + \Delta\mathbf{h})_{r' < r} \in J. \end{aligned} \quad (6.5)$$

By lemma 6.7, we can write transvectant (6.5) as a linear combination of

$$(\mathbf{U}', \mathbf{V}')_{r'}, \quad \mathbf{h} \in J$$

with  $\mathbf{U}'$  being a monomial in  $\mathbf{A}$  with degree strictly less than  $d$ . We thus can conclude by induction on  $d$ . □

*Proof of theorem 6.6.* Since  $\mathbf{A} \subset \mathbf{C}$  and  $\mathbf{f} \in \mathbf{A}$ , we know that

$$\mathbf{Cov}(\mathbf{S}_n) = \mathbf{Cov}(\mathbf{A}) = \mathbf{Cov}(\mathbf{A} \cup \mathbf{B}) = \mathbf{Cov}(\mathbf{C}).$$

As already stated in the proof of theorem 5.11,  $\mathbf{Cov}(\mathbf{C})$  is generated by transvectants

$$(\mathbf{D}, \mathbf{E})_r.$$

where  $\mathbf{D} \in \mathbf{Cov}(A)$  and  $\mathbf{E} \in \mathbf{Cov}(B)$ . By hypothesis, we can suppose that

$$\begin{aligned} \mathbf{D} &= \mathbf{U} + \mathbf{h}_{2k}, & \mathbf{E} &= \mathbf{U} + \mathbf{h}_J, \\ \mathbf{h}_{2k} &\in I_{2k}, & \mathbf{h}_J &\in J. \end{aligned}$$

Thus we have to consider transvectants

$$(\mathbf{U}, \mathbf{V})_r, \tag{6.6}$$

$$(\mathbf{U}, \mathbf{h}_J)_r \in J \text{ by lemma 6.4 and remark 6.5,}$$

$$(\mathbf{h}_{2k}, \mathbf{V})_r, \tag{6.7}$$

$$(\mathbf{h}_{2k}, \mathbf{h}_J)_r \in J. \tag{6.8}$$

We conclude with lemmas 6.7 and 6.8.  $\square$

In the case family A or B can be associated to a reduced system  $S^*(A, B)$  (5.8), we define  $\kappa^1, \dots, \kappa^n$  to be the irreducible solutions of  $S^*(A, B)$  and  $\tau^i$  to be their associated transvectants. In all the proofs to get theorem 6.6, we can write monomials  $\mathbf{U}$  or  $\mathbf{V}$  to be

$$\mathbf{U} = \eta \mathbf{U}', \quad \mathbf{V} = \xi \mathbf{V}'.$$

And we thus get:

**Theorem 6.9.** *Given the same hypothesis as in theorem 6.6, the family  $C := \{\tau^1, \dots, \tau^n\}$  is relatively complete modulo  $J$  and*

$$\mathbf{Cov}(C) = \mathbf{Cov}(A \cup B) = \mathbf{Cov}(S_n).$$

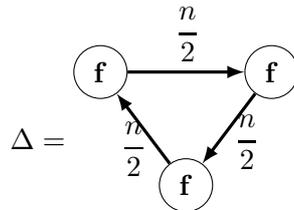
### The algorithm

Take  $V = S_n$  ( $n > 2$ ) be a space of a single binary form and  $\mathbf{f} \in S_n$ . By corollary B.2, the family  $A_0 := \{\mathbf{f}\}$  is relatively complete family modulo  $I_2$ . This means that every covariant  $\mathbf{h} \in \mathbf{Cov}(S_n)$  can be written as

$$\mathbf{h} = p(\mathbf{f}) + \mathbf{h}_2 \text{ with } \mathbf{h}_2 \in I_2$$

Take the covariant  $\mathbf{H}_2 = (\mathbf{f}, \mathbf{f})_2$  of order  $2n - 4$ . Then

- If  $2n - 4 > n$ , we take  $B_1 := \{\mathbf{H}_2\}$  which, by lemma B.4, is relatively complete modulo  $I_4$ ; applying theorem 6.6 leads us to a family  $A_1 := C$  relatively complete modulo  $I_4$ .
- If  $2n - 4 = n$ , we take  $B_1 := \{\mathbf{H}_2, \Delta\}$  which, by lemma B.5, is relatively complete modulo  $I_4 + \langle \Delta \rangle$ ; where  $\Delta$  is the invariant



In that case, by applying theorem 6.6, we can take  $A_1$  to be  $C \cup \{\Delta\}$ . A direct induction on the degree of the covariant shows that  $A_1$  is relatively complete modulo  $I_4$ .

- If  $2n - 4 < n$ , we suppose already known a covariant basis of  $S_{2n-4}$ ; we then take  $B_1$  to be this basis, which is finite and complete, thus finite and relatively complete modulo  $I_4$ ; we directly apply theorem 6.6 to get  $A_1 := C$ .

Let now be given by induction a family  $A_k$  containing  $\mathbf{f}$  to be finite and relatively complete modulo  $I_{2k}$ , we consider the covariant  $\mathbf{H}_{2k} = (\mathbf{f}, \mathbf{f})_{2k}$ . Then

- If  $\mathbf{H}_{2k}$  is of order  $p > n$ , we take  $B_k := \{\mathbf{H}_{2k}\}$  which, by lemma B.4, is relatively complete modulo  $I_{2k+2}$ . By theorem 6.6 we take  $A_{k+1} := C$ .
- If  $\mathbf{H}_{2k}$  is of order  $p = n$ , we take  $B_k := \{\mathbf{H}_{2k}, \Delta\}$  which, by lemma B.5, is relatively complete modulo  $I_{2k+2} + \langle \Delta \rangle$ ; where  $\Delta$  is the invariant

$$\Delta = \begin{array}{ccc} & \frac{n}{2} & \\ & \longrightarrow & \\ \textcircled{\mathbf{f}} & & \textcircled{\mathbf{f}} \\ & \nwarrow \quad \nearrow & \\ & \frac{n}{2} & \\ & \textcircled{\mathbf{f}} & \end{array}$$

In that case, by applying theorem 6.6, we can take  $A_{k+1}$  to be  $C \cup \{\Delta\}$ . A direct induction on the degree of the covariant shows that  $A_{k+1}$  is relatively complete modulo  $I_{2k+2}$ .

- If  $\mathbf{H}_{2k}$  is of order  $p < n$ , we suppose already known a covariant basis of  $S_p$ ; we then take  $B_k$  to be this basis, which is finite and complete, thus finite and relatively complete modulo  $I_{2k+2}$ ; we directly apply theorem 6.6 to get  $A_{k+1} := C$ .

Thus in each case we get the construction of the family  $A_{k+1}$ .

Now, depending on  $n$ 's parity:

- If  $n = 2q$  is even, we know that the family  $A_{q-1}$  is relatively complete modulo  $I_{2q}$ ; furthermore the family  $B_{q-1}$  only contains the invariant  $\Delta_q := \{\mathbf{f}, \mathbf{f}\}_{2q}$ ; finally we observe that  $A_p$  is given by

$$A_p := A_{p-1} \cup \{\Delta_q\}$$

and it is relatively complete modulo  $I_{2q+2} = \{0\}$  and is thus a covariant basis.

- If  $n = 2q + 1$  is odd, the family  $B_{q-1}$  contains the quadratic form  $\mathbf{H}_{2q} := \{\mathbf{f}, \mathbf{f}\}_{2q}$ ; we then know that the family  $B_{q-1}$  is given by the covariant  $\mathbf{H}_{2q}$  and the invariant  $\delta_q := \{\mathbf{H}_{2q}, \mathbf{H}_{2q}\}_2$ . By theorem 6.6, the family  $A_q := C$  is relatively complete modulo  $I_{2q+2} = \{0\}$  and is thus a covariant basis.

## 7. EFFECTIVE COMPUTATIONS

**7.1. Covariant basis of  $S_6 \oplus S_2$ .** There is a simple procedure to get a basis covariant of  $V \oplus S_2$  once we know a covariant basis of  $V$ , as detailed in theorem 7.1, which proof is given in [34].

**Theorem 7.1.** *Let  $\{\mathbf{h}_1, \dots, \mathbf{h}_s\}$  be a covariant basis of  $\mathbf{Cov}(V)$ , and let  $\mathbf{u} \in \mathbf{S}_2$ . Then irreducible covariants of  $\mathbf{Cov}(V \oplus \mathbf{S}_2)$  are taken from one of this set:*

- $\{\mathbf{h}_i, \mathbf{u}^r\}_{2r-1}$  for  $i = 1 \dots s$  ;
- $\{\mathbf{h}_i, \mathbf{u}^r\}_{2r}$  for  $i = 1 \dots s$  ;
- $\{\mathbf{h}_i \mathbf{h}_j, \mathbf{u}^r\}_{2r}$  where  $\mathbf{h}_i$  is of order  $2p+1$  and  $\mathbf{h}_j$  is of order  $2r-2p-1$ .

We write  $\mathbf{h}_{d,o}$  to be a covariant of degree  $d$  and order  $o$ , taken from the covariant basis of  $\mathbf{S}_6$  in table 7.1, issue from Grace–Young [34], and  $\mathbf{u}$  to be a quadratic form in  $\mathbf{S}_2$ . By theorem 7.1 we only have to consider covariants given by

$$\{\mathbf{h}, \mathbf{u}^r\}_{2r-1} \text{ or } \{\mathbf{h}, \mathbf{u}^r\}_{2r}.$$

D/O	0	2	4	6
1				$\mathbf{f}$
2	$(\mathbf{f}, \mathbf{f})_6$		$\mathbf{h}_{2,4} := (\mathbf{f}, \mathbf{f})_4$	
3		$\mathbf{h}_{3,2} := (\mathbf{h}_{2,4}, \mathbf{f})_4$		$\mathbf{h}_{3,6} := (\mathbf{h}_{2,4}, \mathbf{f})_2$
4	$(\mathbf{h}_{2,4}, \mathbf{h}_{2,4})_4$		$(\mathbf{h}_{3,2}, \mathbf{f})_2$	$\mathbf{h}_{4,6} := (\mathbf{h}_{3,2}, \mathbf{f})_1$
5		$(\mathbf{h}_{2,4}, \mathbf{h}_{3,2})_2$	$(\mathbf{h}_{2,4}, \mathbf{h}_{3,2})_1$	
6	$(\mathbf{h}_{3,2}, \mathbf{h}_{3,2})_2$			$\mathbf{h}_{6,61} := (\mathbf{h}_{3,8}, \mathbf{h}_{3,2})_2$ $\mathbf{h}_{6,62} := (\mathbf{h}_{3,6}, \mathbf{h}_{3,2})_1$
7		$(\mathbf{f}, \mathbf{h}_{3,2}^2)_4$	$(\mathbf{f}, \mathbf{h}_{3,2}^2)_3$	
8		$(\mathbf{h}_{2,4}, \mathbf{h}_{3,2}^2)_3$		
9			$(\mathbf{h}_{3,8}, \mathbf{h}_{3,2}^2)_4$	
10	$(\mathbf{h}_{3,2}^3, \mathbf{f})_6$	$(\mathbf{h}_{3,2}^3, \mathbf{f})_5$		
12		$(\mathbf{h}_{3,8}, \mathbf{h}_{3,2}^3)_6$		
15	$(\mathbf{h}_{3,8}, \mathbf{h}_{3,2}^4)_8$			

D/O	8	10	12
2	$\mathbf{h}_{2,8} := (\mathbf{f}, \mathbf{f})_2$		
3	$\mathbf{h}_{3,8} := (\mathbf{h}_{2,4}, \mathbf{f})_1$		$(\mathbf{h}_{2,8}, \mathbf{f})_1$
4		$(\mathbf{h}_{2,8}, \mathbf{h}_{2,4})_1$	
5	$\mathbf{h}_{5,8} := (\mathbf{h}_{2,8}, \mathbf{h}_{3,2})_1$		

TABLE 2. Covariant basis of  $\mathbf{S}_6$

Recall the covariant algebra  $\mathbf{Cov}(V) := \mathbf{Cov}(\mathbf{S}_6 \oplus \mathbf{S}_2)$  is a multi-graded algebra:

$$\mathbf{Cov}(V) = \bigoplus_{d_1 \geq 0, d_2 \geq 0, o \geq 0} \mathbf{Cov}(V)_{d_1, d_2, o}.$$

where  $d_1$  is the degree in the binary form  $\mathbf{f} \in \mathbf{S}_6$ ,  $d_2$  is the degree in the binary form  $\mathbf{u} \in \mathbf{S}_2$  and  $o$  the degree in the variable  $\mathbf{x} \in \mathbb{C}^2$ . We can define the Hilbert series:

$$\mathcal{H}_{6,2}(z_1, z_2, t) := \sum_{d_1, d_2, o} \dim(\mathbf{Cov}(V)_{d_1, d_2, o}) z_1^{d_1} z_2^{d_2} t^o,$$

which has been computed using maple package of Bedratyuk [10]. From this Hilbert series and theorem 7.1, we finally get a minimal basis of 99 covariants: it's worth noting that, by using this algorithm, we had to check

invariant homogeneous space's dimensions up to degree 15. We summarize the results in table 3.

$d/o$	0	2	4	6	8	10	12	#	Cum
1	—	1	—	1	—	—	—	2	2
2	2	—	2	1	1	—	—	6	8
3	—	3	2	2	2	—	1	10	18
4	4	3	3	4	—	2	—	16	34
5	—	4	6	—	3	—	—	13	47
6	5	7	—	5	—	—	—	17	64
7	3	1	6	—	—	—	—	10	74
8	1	8	—	—	—	—	—	9	83
9	7	—	1	—	—	—	—	8	91
10	1	2	—	—	—	—	—	3	94
11	2	—	—	—	—	—	—	2	96
12	—	1	—	—	—	—	—	1	97
13	1	—	—	—	—	—	—	1	98
14	—	—	—	—	—	—	—	—	98
15	1	—	—	—	—	—	—	1	99
<i>Tot</i>	27	30	20	13	6	2	1		99

TABLE 3. Minimal covariant basis of  $S_6 \oplus S_2$

27 invariants	
5 invariants from $S_6$ , 1 invariant from $S_2$ and 21 joint invariants.	
Degree 2	$(\mathbf{f}, \mathbf{f})_6$ $(\mathbf{u}, \mathbf{u})_2$
Degree 4	$(\mathbf{h}_{1,6}, \mathbf{u}^3)_6$ $(\mathbf{h}_{2,4}, \mathbf{u}^2)_4$ $(\mathbf{h}_{3,2}, \mathbf{u})_2$
Degree 6	$(\mathbf{h}_{3,6}, \mathbf{u}^3)_6$ $(\mathbf{h}_{2,8}, \mathbf{u}^4)_8$ $(\mathbf{h}_{4,4}, \mathbf{u}^2)_4$ $(\mathbf{h}_{5,2}, \mathbf{u})_2$
Degree 7	$(\mathbf{h}_{5,4}, \mathbf{u}^2)_4$ $(\mathbf{h}_{3,8}, \mathbf{u}^4)_8$ $(\mathbf{h}_{4,6}, \mathbf{u}^3)_6$
Degree 8	$(\mathbf{h}_{7,2}, \mathbf{u})_2$
Degree 9	$(\mathbf{h}_{7,4}, \mathbf{u}^2)_4$ $(\mathbf{h}_{6,61}, \mathbf{u}^3)_6$ $(\mathbf{h}_{4,10}, \mathbf{u}^5)_{10}$
	$(\mathbf{h}_{5,8}, \mathbf{u}^4)_8$ $(\mathbf{h}_{8,2}, \mathbf{u})_2$ $(\mathbf{h}_{3,12}, \mathbf{u}^6)_{12}$ $(\mathbf{h}_{6,62}, \mathbf{u}^3)_6$
Degree 10	$(\mathbf{h}_{3,2}^3, \mathbf{f})_6$
Degree 11	$(\mathbf{h}_{10,2}, \mathbf{u})_2$ $(\mathbf{h}_{9,4}, \mathbf{u}^2)_4$
Degree 13	$(\mathbf{h}_{12,2}, \mathbf{u})_2$
Degree 14	$(\mathbf{h}_{3,2}^4, \mathbf{h}_{3,8})_8$
30 covariants of order 2	
1 from $S_2$ , 6 from $S_6$ and 23 joint covariants.	
Degree 1	$\mathbf{u}$

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Degree 3	$\mathbf{h}_{3,2} \quad (\mathbf{f}, \mathbf{u}^2)_4 \quad (\mathbf{h}_{2,4}, \mathbf{u})_2$
Degree 4	$(\mathbf{h}_{2,4}, \mathbf{u}^2)_3 \quad (\mathbf{h}_{3,2}, \mathbf{u})_1 \quad (\mathbf{f}, \mathbf{u}^3)_5$
Degree 5	$\mathbf{h}_{5,2} \quad (\mathbf{h}_{2,8}, \mathbf{u}^3)_6 \quad (\mathbf{h}_{4,4}, \mathbf{u})_2 \quad (\mathbf{h}_{3,6}, \mathbf{u}^2)_4$
Degree 6	$(\mathbf{h}_{2,8}, \mathbf{u}^4)_7 \quad (\mathbf{h}_{4,4}, \mathbf{u}^2)_3 \quad (\mathbf{h}_{5,2}, \mathbf{u})_1 \quad (\mathbf{h}_{3,6}, \mathbf{u}^3)_5$
	$(\mathbf{h}_{5,4}, \mathbf{u})_2 \quad (\mathbf{h}_{3,8}, \mathbf{u}^3)_6 \quad (\mathbf{h}_{4,6}, \mathbf{u}^2)_4$
Degree 7	$\mathbf{h}_{7,2}$
Degree 8	$\mathbf{h}_{8,2} \quad (\mathbf{h}_{7,2}, \mathbf{u})_1 \quad (\mathbf{h}_{7,4}, \mathbf{u})_2 \quad (\mathbf{h}_{6,6b}, \mathbf{u}^2)_4$
	$(\mathbf{h}_{5,8}, \mathbf{u}^3)_6 \quad (\mathbf{h}_{3,12}, \mathbf{u}^5)_{10} \quad (\mathbf{h}_{4,10}, \mathbf{u}^4)_8 \quad (\mathbf{h}_{6,6a}, \mathbf{u}^2)_4$
Degree 10	$\mathbf{h}_{10,2} \quad (\mathbf{h}_{9,4}, \mathbf{u})_2$
Degree 12	$\mathbf{h}_{12,2}$
20 covariants of order 4 5 covariants from $S_6$ and 15 joint covariants.	
Degree 2	$\mathbf{h}_{2,4} \quad (\mathbf{f}, \mathbf{u})_2$
Degree 3	$(\mathbf{h}_{2,4}, \mathbf{u})_1 \quad (\mathbf{f}, \mathbf{u}^2)_3$
Degree 4	$\mathbf{h}_{4,4} \quad (\mathbf{h}_{3,6}, \mathbf{u})_2 \quad (\mathbf{h}_{2,8}, \mathbf{u}^2)_4$
Degree 5	$\mathbf{h}_{5,4} \quad (\mathbf{h}_{3,8}, \mathbf{u}^2)_4 \quad (\mathbf{h}_{3,6}, \mathbf{u}^2)_3 \quad (\mathbf{h}_{4,4}, \mathbf{u})_1 \quad (\mathbf{h}_{4,6}, \mathbf{u})_2 \quad (\mathbf{h}_{2,8}, \mathbf{u}^3)_5$
Degree 7	$\mathbf{h}_{7,4} \quad (\mathbf{h}_{6,61}, \mathbf{u})_2 \quad (\mathbf{h}_{3,12}, \mathbf{u}^4)_8 \quad (\mathbf{h}_{4,10}, \mathbf{u}^3)_6 \quad (\mathbf{h}_{6,62}, \mathbf{u})_2 \quad (\mathbf{h}_{5,8}, \mathbf{u}^2)_4$
Degree 9	$\mathbf{h}_{9,4}$
13 covariants of order 6: 5 covariants from $S_6$ and 8 joint covariants.	
Degree 1	$\mathbf{f}$
Degree 2	$(\mathbf{f}, \mathbf{u})_1$
Degree 3	$\mathbf{h}_{3,6} \quad (\mathbf{h}_{2,8}, \mathbf{u})_2$
Degree 4	$\mathbf{h}_{4,6} \quad (\mathbf{h}_{2,8}, \mathbf{u}^2)_3 \quad (\mathbf{h}_{3,6}, \mathbf{u})_1 \quad (\mathbf{h}_{3,8}, \mathbf{u})_2$
Degree 7	$\mathbf{h}_{6,61} \quad \mathbf{h}_{6,62} \quad (\mathbf{h}_{5,8}, \mathbf{u})_2 \quad (\mathbf{h}_{4,10}, \mathbf{u}^2)_4 \quad (\mathbf{h}_{3,12}, \mathbf{u}^3)_6$
6 covariants of order 8: 3 covariants from $S_6$ and 3 joint covariants.	
Degree 2	$\mathbf{h}_{2,8}$
Degree 3	$\mathbf{h}_{3,8} \quad (\mathbf{h}_{2,8}, \mathbf{u})_1$
Degree 5	$\mathbf{h}_{5,8} \quad (\mathbf{h}_{4,10}, \mathbf{u})_2 \quad (\mathbf{h}_{3,12}, \mathbf{u}^2)_4$
2 covariants of order 10: 1 covariant from $S_6$ and 1 joint covariant.	
Degree 3	$\mathbf{h}_{4,10} \quad (\mathbf{h}_{3,12}, \mathbf{u})_2$
1 covariant of order 12 from $S_6$ .	
Degree 3	$\mathbf{h}_{3,12}$

**7.2. Covariant basis of  $S_6 \oplus S_4$ .** Taking  $\mathbf{f} \in S_6$  and  $\mathbf{v} \in S_4$  we take generators of  $\mathbf{Cov}(S_6)$  given by 7.1 and we write

$$\begin{aligned} \mathbf{v} \in S_4 \quad \mathbf{k}_{2,4} &:= (\mathbf{v}, \mathbf{v})_2 & \mathbf{k}_{3,6} &:= (\mathbf{v}, \mathbf{k}_{2,4})_1 \\ i &:= (\mathbf{v}, \mathbf{v})_4 \quad j &:= (\mathbf{v}, \mathbf{k}_{2,4})_4 \end{aligned}$$

From relation (5.7) we have:

**Lemma 7.2.**  $\mathbf{Cov}(S_4)$  is a finite  $\mathbb{C}[i, j, \mathbf{v}, \mathbf{k}_{2,4}]$ -module:

$$\mathbf{Cov}(S_4) = \mathbb{C}[i, j, \mathbf{v}, \mathbf{k}_{2,4}] + \mathbf{TC}[i, j, \mathbf{v}, \mathbf{k}_{2,4}].$$

We can also find an interesting result about  $\mathbf{Cov}(S_6)$  by getting relations as:

$$\begin{aligned} 36\mathbf{h}_{4,10}^2 &= -\mathbf{h}_{2,0}\mathbf{h}_{2,4}^2\mathbf{f}^2 + 6\mathbf{h}_{2,4}^2\mathbf{f}\mathbf{h}_{3,6} - 3\mathbf{h}_{2,4}^3\mathbf{h}_{2,8} - 6\mathbf{h}_{3,2}\mathbf{h}_{2,4}\mathbf{f}\mathbf{h}_{2,8} \\ &+ 3\mathbf{h}_{2,0}\mathbf{h}_{2,4}\mathbf{h}_{2,8}^2 - 9\mathbf{h}_{4,4}\mathbf{h}_{2,8}^2. \end{aligned}$$

Now:

**Lemma 7.3.** Take  $\mathbf{h}_{d,k}$  to be the covariant of degree  $d$  and order  $k$  in  $\mathbf{Cov}(S_6)$  given by table 7.1. Let

$$\begin{aligned} A_0 &:= \{\mathbf{h}_{2,0}, \mathbf{h}_{4,0}, \mathbf{h}_{6,0}, \mathbf{h}_{10,0}, \mathbf{h}_{15,0}, \mathbf{h}_{3,2}, \mathbf{h}_{5,2}, \mathbf{h}_{2,4}, \mathbf{h}_{4,4}, \mathbf{f}, \\ &\mathbf{h}_{3,6}, \mathbf{h}_{4,6}, \mathbf{h}_{2,8}, \mathbf{h}_{3,8}\} \end{aligned}$$

Then  $\mathbf{Cov}(S_6)$  is a finite  $\mathbb{C}[A_0]$ -module generated by the monomials

$$\mathbf{h}_{7,2}^{u_1}\mathbf{h}_{8,2}^{u_2}\mathbf{h}_{10,2}^{u_3}\mathbf{h}_{12,2}^{u_4}\mathbf{h}_{5,4}^{u_5}\mathbf{h}_{7,4}^{u_6}\mathbf{h}_{9,4}^{u_7}\mathbf{h}_{6,61}^{u_8}\mathbf{h}_{6,62}^{u_9}\mathbf{h}_{5,8}^{u_{10}}\mathbf{h}_{4,10}^{u_{11}}\mathbf{h}_{3,12}^{u_{12}}$$

with

$$u_i \leq 1, \quad \forall i, \quad \text{and } u_3 + u_4 \leq 1$$

From lemmas 7.2 and 7.3, the reduced integer system  $S^*(A, B)$  and theorem 6.9 leads to 1072 generators. Observe also that we know in which space  $\mathbf{Cov}(S_6 \oplus S_4)_{d_1, d_2, o}$  each covariant of these solutions belong, where  $d_1$  is the degree in  $S_6$ ,  $d_2$  is the degree in  $S_4$  and  $o$  is the order of the covariant. Hilbert series of  $\mathbf{Cov}(S_6 \oplus S_4)$  have been computed using Maple package by Bedratyuk [10].

From the 1072 original generators, we had to check 339 invariants. The maximum total degree  $d_1 + d_2$  is 24 and, for that degree, only one invariant occurs, of degrees  $d_1 = 21$  and  $d_2 = 3$ . From Hilbert series, we only had to check a space of dimension

$$\dim(\mathbf{Cov}(S_6 \oplus S_4))_{21,3,0} = 324.$$

As other examples, we also had to check invariant spaces with degrees and dimension given in table 5.

After reduction, this leads to the 53 joint invariants. In order 2, we find 68 covariants : 6 covariants from  $S_6$  and 62 joint covariants. From the 1072 original generators, we had 433 order 2 covariants, the maximum total degree  $d_1 + d_2$  being 24, and for that degree, only one covariant occurs, of degrees  $d_1 = 21$  and  $d_2 = 3$ . From Hilbert series, we only had to check a space of dimension 1063. Finally, results are summarized in table 4.

$d/o$	0	2	4	6	8	10	12	#	$Cum$
1	—	—	1	1	—	—	—	2	2
2	2	1	3	1	2	—	—	9	11
3	2	4	4	5	3	1	1	20	31
4	4	6	9	5	2	1	—	27	58
5	4	12	11	3	1	—	—	31	89
6	9	14	6	2	—	—	—	31	120
7	9	17	2	—	—	—	—	28	148
8	9	7	1	—	—	—	—	17	165
9	8	3	1	—	—	—	—	12	177
10	5	2	—	—	—	—	—	7	184
11	3	1	—	—	—	—	—	4	188
12	2	1	—	—	—	—	—	3	191
13	1	—	—	—	—	—	—	1	192
14	1	—	—	—	—	—	—	1	193
15	1	—	—	—	—	—	—	1	194
<i>Tot</i>	60	68	38	17	8	2	1		194

TABLE 4. Minimal covariant basis of  $S_6 \oplus S_4$

Degree	$d_1$	$d_2$	dim	Degree	$d_1$	$d_2$	dim
23	20	3	335	21	18	3	258
22	19	3	248		17	4	354
	18	4	498		16	5	621
	17	5	650		15	6	747
	16	6	1005				
	15	7	1142				

TABLE 5. Dimensions of homogeneous space from  $\mathbf{Cov}(S_6 \oplus S_4)$

53 joint invariants	
Degree 3	$(\mathbf{h}_{2,4}, \mathbf{v})_4$
Degree 4	$(\mathbf{h}_{2,8}, \mathbf{v}^2)_8$ $(\mathbf{h}_{2,4}, \mathbf{k}_{2,4})_4$ $(\mathbf{f}, \mathbf{k}_{3,6})_6$
Degree 5	$(\mathbf{h}_{3,8}, \mathbf{v}^2)_8$ $(\mathbf{h}_{4,4}, \mathbf{v})_4$ $(\mathbf{h}_{2,8}, \mathbf{v} \cdot \mathbf{k}_{2,4})_8$ $(\mathbf{f}^2, \mathbf{v}^3)_{12}$
Degree 6	$(\mathbf{h}_{3,8}, \mathbf{v} \cdot \mathbf{k}_{2,4})_8$ $(\mathbf{f}^2, \mathbf{v}^2 \cdot \mathbf{k}_{2,4})_{12}$ $(\mathbf{h}_{2,8}, \mathbf{k}_{2,4}^2)_8$ $(\mathbf{h}_{3,6}, \mathbf{k}_{3,6})_6$
	$(\mathbf{h}_{3,12}, \mathbf{v}^3)_{12}$ $(\mathbf{h}_{5,4}, \mathbf{v})_4$

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	$(\mathbf{h}_{4,4}, \mathbf{k}_{2,4})_4$ $(\mathbf{h}_{3,2} \cdot \mathbf{f}, \mathbf{v}^2)_8$
Degree 7	$(\mathbf{h}_{3,2}^2, \mathbf{v})_4$ $(\mathbf{h}_{5,4}, \mathbf{k}_{2,4})_4$ $(\mathbf{h}_{5,8}, \mathbf{v}^2)_8$ $(\mathbf{f} \cdot \mathbf{h}_{3,6}, \mathbf{v}^3)_{12}$
	$(\mathbf{f}^2, \mathbf{v} \cdot \mathbf{k}_{2,4}^2)_{12}$ $(\mathbf{h}_{3,2} \cdot \mathbf{f}, \mathbf{v} \cdot \mathbf{k}_{2,4})_8$
	$(\mathbf{h}_{4,6}, \mathbf{k}_{3,6})_6$ $(\mathbf{h}_{3,12}, \mathbf{v}^2 \cdot \mathbf{k}_{2,4})_{12}$ $(\mathbf{h}_{3,8}, \mathbf{k}_{2,4}^2)_8$
Degree 8	$(\mathbf{h}_{3,2} \mathbf{h}_{2,4}, \mathbf{k}_{3,6})_6$ $(\mathbf{h}_{3,12}, \mathbf{v} \cdot \mathbf{k}_{2,4}^2)_{12}$ $(\mathbf{h}_{3,2} \mathbf{h}_{3,6}, \mathbf{v}^2)_8$ $(\mathbf{h}_{3,2}^2, \mathbf{k}_{2,4})_4$
	$(\mathbf{h}_{7,4}, \mathbf{v})_4$ $(\mathbf{f} \cdot \mathbf{h}_{4,6}, \mathbf{v}^3)_{12}$
	$(\mathbf{f} \cdot \mathbf{h}_{3,6}, \mathbf{v}^2 \cdot \mathbf{k}_{2,4})_{12}$ $(\mathbf{h}_{3,2} \cdot \mathbf{f}, \mathbf{k}_{2,4}^2)_8$ $(\mathbf{h}_{5,8}, \mathbf{v} \cdot \mathbf{k}_{2,4})_8$
Degree 9	$(\mathbf{h}_{7,4}, \mathbf{k}_{2,4})_4$ $(\mathbf{h}_{3,2} \cdot \mathbf{h}_{5,2}, \mathbf{v})_4$ $(\mathbf{h}_{5,2} \cdot \mathbf{f}, \mathbf{v} \cdot \mathbf{k}_{2,4})_8$
	$(\mathbf{h}_{3,12}, \mathbf{k}_{2,4}^3)_{12}$ $(\mathbf{h}_{3,2} \cdot \mathbf{h}_{2,8}, \mathbf{v} \cdot \mathbf{k}_{3,6})_{10}$
	$(\mathbf{f} \mathbf{h}_{4,6}, \mathbf{v}^2 \cdot \mathbf{k}_{2,4})_{12}$ $(\mathbf{h}_{3,6}^2, \mathbf{v}^3)_{12}$ $(\mathbf{h}_{3,2} \cdot \mathbf{h}_{4,6}, \mathbf{v}^2)_8$
Degree 10	$(\mathbf{h}_{9,4}, \mathbf{v})_4$ $(\mathbf{h}_{3,2} \cdot \mathbf{h}_{2,8}, \mathbf{k}_{2,4} \mathbf{k}_{3,6})_{10}$ $(\mathbf{h}_{5,2} \cdot \mathbf{h}_{3,6}, \mathbf{v}^2)_8$ $(\mathbf{f} \cdot \mathbf{h}_{6,61}, \mathbf{v}^3)_{12}$
Degree 11	$(\mathbf{h}_{5,2}^2, \mathbf{v})_4$ $(\mathbf{f} \cdot \mathbf{h}_{6,62}, \mathbf{v}^2 \cdot \mathbf{k}_{2,4})_{12}$ $(\mathbf{h}_{3,2} \cdot \mathbf{h}_{6,61}, \mathbf{v}^2)_8$
Degree 12	$(\mathbf{h}_{3,2} \mathbf{h}_{8,2}, \mathbf{v})_4$ $(\mathbf{h}_{3,2} \mathbf{h}_{6,62}, \mathbf{v} \mathbf{k}_{2,4})_8$
Degree 13	$(\mathbf{h}_{8,2} \mathbf{h}_{3,6}, \mathbf{v}^2)_8$
Degree 14	$(\mathbf{h}_{3,2} \mathbf{h}_{10,2}, \mathbf{v})_4$
Order 2 : 62 joint covariants.	
Degree 2	$(\mathbf{f}, \mathbf{v})_4$
Degree 3	$\mathbf{h}_{3,2}$ $(\mathbf{h}_{2,4}, \mathbf{v})_3$ $(\mathbf{f}, \mathbf{k}_{2,4})_4$ $(\mathbf{f}, \mathbf{v}^2)_6$
Degree 4	$(\mathbf{h}_{2,4}, \mathbf{k}_{2,4})_3$ $(\mathbf{f}, \mathbf{v} \cdot \mathbf{k}_{2,4})_6$ $(\mathbf{f}, \mathbf{k}_{3,6})_5$ $(\mathbf{h}_{2,8}, \mathbf{v}^2)_7$ $(\mathbf{h}_{3,2}, \mathbf{v})_2$ $(\mathbf{h}_{3,6}, \mathbf{v})_4$
Degree 5	$\mathbf{h}_{5,2}$ $(\mathbf{h}_{3,6}, \mathbf{k}_{2,4})_4$ $(\mathbf{h}_{4,4}, \mathbf{v})_3$ $(\mathbf{h}_{3,6}, \mathbf{v}^2)_6$
	$(\mathbf{h}_{2,8}, \mathbf{v} \cdot \mathbf{k}_{2,4})_7$ $(\mathbf{f}, \mathbf{k}_{2,4}^2)_6$ $(\mathbf{h}_{3,2}, \mathbf{k}_{2,4})_2$
	$(\mathbf{f}^2, \mathbf{v}^3)_{11}$ $(\mathbf{h}_{4,6}, \mathbf{v})_4$ $(\mathbf{h}_{2,8}, \mathbf{k}_{3,6})_6$ $(\mathbf{h}_{2,4}, \mathbf{k}_{3,6})_4$ $(\mathbf{h}_{3,8}, \mathbf{v}^2)_7$
Degree 6	$(\mathbf{f}^2, \mathbf{v}^2 \cdot \mathbf{k}_{2,4})_{11}$ $(\mathbf{h}_{2,8}, \mathbf{v} \cdot \mathbf{k}_{3,6})_8$ $(\mathbf{h}_{3,2} \cdot \mathbf{f}, \mathbf{v}^2)_7$ $(\mathbf{h}_{2,8}, \mathbf{k}_{2,4}^2)_7$
	$(\mathbf{h}_{4,4}, \mathbf{k}_{2,4})_3$ $(\mathbf{h}_{4,10}, \mathbf{v}^2)_8$ $(\mathbf{h}_{3,12}, \mathbf{v}^3)_{11}$ $(\mathbf{h}_{5,2}, \mathbf{v})_2$
	$(\mathbf{h}_{4,6}, \mathbf{v}^2)_6$ $(\mathbf{h}_{3,6}, \mathbf{v} \cdot \mathbf{k}_{2,4})_6$ $(\mathbf{h}_{4,6}, \mathbf{k}_{2,4})_4$ $(\mathbf{h}_{3,8}, \mathbf{v} \cdot \mathbf{k}_{2,4})_7$
	$(\mathbf{h}_{3,8}, \mathbf{k}_{3,6})_6$ $(\mathbf{h}_{5,4}, \mathbf{v})_3$
Degree 7	$\mathbf{h}_{7,2}$ $(\mathbf{h}_{2,8}, \mathbf{k}_{2,4} \cdot \mathbf{k}_{3,6})_8$ $(\mathbf{h}_{6,62}, \mathbf{v})_4$ $(\mathbf{h}_{3,12}, \mathbf{v}^2 \cdot \mathbf{k}_{2,4})_{11}$ $(\mathbf{h}_{4,10}, \mathbf{v}^3)_{10}$ $(\mathbf{h}_{6,61}, \mathbf{v})_4$
	$(\mathbf{f} \cdot \mathbf{h}_{3,6}, \mathbf{v}^3)_{11}$ $(\mathbf{h}_{3,2}^2, \mathbf{v})_3$ $(\mathbf{h}_{5,2}, \mathbf{k}_{2,4})_2$ $(\mathbf{h}_{3,8}, \mathbf{v} \cdot \mathbf{k}_{3,6})_8$ $(\mathbf{h}_{2,4}^2, \mathbf{k}_{3,6})_6$ $(\mathbf{h}_{5,8}, \mathbf{v}^2)_7$
	$(\mathbf{h}_{4,6}, \mathbf{v} \cdot \mathbf{k}_{2,4})_6$ $(\mathbf{f}^2, \mathbf{v} \cdot \mathbf{k}_{2,4}^2)_{11}$ $(\mathbf{h}_{5,4}, \mathbf{k}_{2,4})_3$ $(\mathbf{h}_{4,10}, \mathbf{v} \cdot \mathbf{k}_{2,4})_8$ $(\mathbf{h}_{4,6}, \mathbf{k}_{3,6})_5$
Degree 8	$\mathbf{h}_{8,2}$ $(\mathbf{h}_{3,2} \cdot \mathbf{h}_{3,6}, \mathbf{v}^2)_7$ $(\mathbf{h}_{7,2}, \mathbf{v})_2$ $(\mathbf{h}_{3,2}^2, \mathbf{k}_{2,4})_3$
	$(\mathbf{h}_{6,61}, \mathbf{k}_{2,4})_4$ $(\mathbf{h}_{6,62}, \mathbf{v}^2)_6$ $(\mathbf{h}_{4,10}, \mathbf{k}_{2,4}^2)_8$
Degree 9	$(\mathbf{h}_{8,2}, \mathbf{v})_2$ $(\mathbf{h}_{3,2}^2, \mathbf{k}_{3,6})_4$ $(\mathbf{h}_{3,2} \cdot \mathbf{h}_{5,2}, \mathbf{v})_3$
Degree 10	$\mathbf{h}_{10,2}$ $(\mathbf{h}_{5,2} \cdot \mathbf{h}_{3,6}, \mathbf{v}^2)_7$
Degree 11	$(\mathbf{h}_{5,2}^2, \mathbf{v})_3$
Degree 12	$\mathbf{h}_{12,2}$
38 covariants of order 4:	

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2 covariants from $S_4$ , 5 covariants from $S_6$ and 31 joint covariants.	
Degree 1	$\mathbf{v}$
Degree 2	$\mathbf{k}_{2,4} \quad \mathbf{h}_{2,4} \quad (\mathbf{f}, \mathbf{v})_3$
Degree 3	$(\mathbf{h}_{2,4}, \mathbf{v})_2 \quad (\mathbf{f}, \mathbf{v}^2)_5 \quad (\mathbf{h}_{2,8}, \mathbf{v})_4 \quad (\mathbf{f}, \mathbf{k}_{2,4})_3$
Degree 4	$\mathbf{h}_{4,4} \quad (\mathbf{h}_{3,8}, \mathbf{v})_4 \quad (\mathbf{h}_{3,2}, \mathbf{v})_1 \quad (\mathbf{h}_{2,8}, \mathbf{k}_{2,4})_4$
	$(\mathbf{h}_{2,4}, \mathbf{k}_{2,4})_2 \quad (\mathbf{h}_{2,8}, \mathbf{v}^2)_6$
	$(\mathbf{f}, \mathbf{k}_{3,6})_4 \quad (\mathbf{f}, \mathbf{v} \cdot \mathbf{k}_{2,4})_5 \quad (\mathbf{h}_{3,6}, \mathbf{v})_3$
Degree 5	$\mathbf{h}_{5,4} \quad (\mathbf{h}_{2,8}, \mathbf{v} \cdot \mathbf{k}_{2,4})_6 \quad (\mathbf{h}_{3,12}, \mathbf{v}^2)_8 \quad (\mathbf{h}_{4,6}, \mathbf{v})_3 \quad (\mathbf{h}_{3,2}, \mathbf{k}_{2,4})_1$
	$(\mathbf{h}_{3,6}, \mathbf{k}_{2,4})_3 \quad (\mathbf{h}_{4,4}, \mathbf{v})_2 \quad (\mathbf{h}_{3,8}, \mathbf{k}_{2,4})_4 \quad (\mathbf{h}_{2,8}, \mathbf{k}_{3,6})_5 \quad (\mathbf{f}, \mathbf{k}_{2,4}^2)_5 \quad (\mathbf{h}_{3,6}, \mathbf{v}^2)_5$
Degree 6	$(\mathbf{h}_{5,2}, \mathbf{v})_1 \quad (\mathbf{h}_{3,12}, \mathbf{v} \cdot \mathbf{k}_{2,4})_8 \quad (\mathbf{h}_{5,8}, \mathbf{v})_4 \quad (\mathbf{h}_{5,4}, \mathbf{v})_2 \quad (\mathbf{h}_{4,6}, \mathbf{k}_{2,4})_3 \quad (\mathbf{h}_{3,6}, \mathbf{k}_{3,6})_4$
Degree 7	$\mathbf{h}_{7,4} \quad (\mathbf{h}_{5,8}, \mathbf{k}_{2,4})_4$
Degree 8	$(\mathbf{h}_{7,4}, \mathbf{v})_2$
Degree 9	$\mathbf{h}_{9,4}$
17 covariants of order 6: 1 covariant form $S_4$ , 5 covariants from $S_6$ and 11 joint covariants.	
Degree 1	$\mathbf{f}$
Degree 2	$(\mathbf{f}, \mathbf{v})_2$
Degree 3	$\mathbf{h}_{3,6} \quad \mathbf{k}_{3,6} \quad (\mathbf{f}, \mathbf{k}_{2,4})_2 \quad (\mathbf{h}_{2,4}, \mathbf{v})_1 \quad (\mathbf{h}_{2,8}, \mathbf{v})_3$
Degree 4	$\mathbf{h}_{4,6} \quad (\mathbf{h}_{2,8}, \mathbf{k}_{2,4})_3 \quad (\mathbf{h}_{3,8}, \mathbf{v})_3 \quad (\mathbf{h}_{2,8}, \mathbf{v}^2)_5 \quad (\mathbf{h}_{2,4}, \mathbf{k}_{2,4})_1$
Degree 5	$(\mathbf{h}_{4,10}, \mathbf{v})_4 \quad (\mathbf{h}_{4,4}, \mathbf{v})_1 \quad (\mathbf{h}_{4,6}, \mathbf{v})_2$
Degree 6	$\mathbf{h}_{6,61} \quad \mathbf{h}_{6,62}$
8 covariants of order 8: 3 covariant form $S_6$ , 5 joint covariants.	
Degree 2	$\mathbf{h}_{2,8} \quad (\mathbf{f}, \mathbf{v})_1$
Degree 3	$\mathbf{h}_{3,8} \quad (\mathbf{f}, \mathbf{k}_{2,4})_1 \quad (\mathbf{h}_{2,8}, \mathbf{v})_2$
Degree 4	$(\mathbf{h}_{3,8}, \mathbf{v})_2 \quad (\mathbf{h}_{3,12}, \mathbf{v})_4$
Degree 5	$\mathbf{h}_{5,8}$
2 covariants of order 10: 1 covariant from $S_6$ and 1 joint covariant.	
Degree 3	$(\mathbf{h}_{2,8}, \mathbf{k}_{1,4})_1$
Degree 4	$\mathbf{h}_{4,10}$
1 covariants of order 12 taken from $S_6$ .	
Degree 3	$\mathbf{h}_{3,12}$

7.3. **Covariant bases of  $S_6 \oplus S_4 \oplus S_2$ .** We directly use theorem 5.11 with  $V = S_6 \oplus S_4$ . We summarize the result in the table 6

$d/o$	0	2	4	6	8	10	12	#	Cum
1	—	1	1	1	—	—	—	3	3
2	3	2	5	2	2	—	—	14	17
3	4	10	9	8	4	1	1	37	54
4	12	19	20	10	3	2	—	66	120
5	15	38	24	6	3	—	—	86	206
6	37	46	12	5	—	—	—	100	306
7	42	31	7	—	—	—	—	80	386
8	38	15	1	—	—	—	—	54	440
9	22	4	1	—	—	—	—	27	467
10	9	3	—	—	—	—	—	12	479
11	6	1	—	—	—	—	—	7	486
12	3	1	—	—	—	—	—	4	490
13	2	—	—	—	—	—	—	2	492
14	1	—	—	—	—	—	—	1	493
15	1	—	—	—	—	—	—	1	494
<i>Tot</i>	195	171	80	32	12	3	1		494

TABLE 6. Covariant basis of  $S_6 \oplus S_4 \oplus S_2$ 

195 invariants: 5 from $S_6$ , 2 from $S_4$ , 1 from $S_2$ , 21 joint invariants of $S_6 \oplus S_2$ given in 7.1, 53 joint invariants of $S_6 \oplus S_4$ given in 7.2. There is left 113 invariants.	
Degree 3	$(\mathbf{v}, \mathbf{u}^2)_4$ $((\mathbf{f}, \mathbf{v})_4, \mathbf{u})_2$
Degree 4	$(\mathbf{k}_{2,4}, \mathbf{u}^2)_4$ $((\mathbf{f}, \mathbf{v})_3, \mathbf{u}^2)_4$ $((\mathbf{f}, \mathbf{v}^2)_6, \mathbf{u})_2$ $((\mathbf{f}, \mathbf{k}_{2,4})_4, \mathbf{u})_2$ $((\mathbf{h}_{2,4}, \mathbf{v})_3, \mathbf{u})_2$
Degree 5	$((\mathbf{f}, \mathbf{v})_2, \mathbf{u}^3)_6$ $((\mathbf{f}, \mathbf{k}_{2,4})_3, \mathbf{u}^2)_4$ $((\mathbf{f}, \mathbf{v}^2)_5, \mathbf{u}^2)_4$ $((\mathbf{f}, \mathbf{k}_{3,6})_5, \mathbf{u})_2$
	$((\mathbf{f}, \mathbf{v} \cdot \mathbf{k}_{2,4})_6, \mathbf{u})_2$ $((\mathbf{h}_{2,8}, \mathbf{v})_4, \mathbf{u}^2)_4$ $((\mathbf{h}_{2,4}, \mathbf{v})_2, \mathbf{u}^2)_4$ $((\mathbf{h}_{2,8}, \mathbf{v}^2)_7, \mathbf{u})_2$
	$((\mathbf{h}_{2,4}, \mathbf{k}_{2,4})_3, \mathbf{u})_2$ $((\mathbf{h}_{3,6}, \mathbf{v})_4, \mathbf{u})_2$
	$((\mathbf{h}_{3,2}, \mathbf{v})_2, \mathbf{u})_2$
Degree 6	$(\mathbf{k}_{3,6}, \mathbf{u}^3)_6$ $((\mathbf{f}, \mathbf{v})_1, \mathbf{u}^4)_8$ $((\mathbf{f}, \mathbf{k}_{2,4})_2, \mathbf{u}^3)_6$ $((\mathbf{f}, \mathbf{k}_{3,6})_4, \mathbf{u}^2)_4$
	$((\mathbf{f}, \mathbf{v} \cdot \mathbf{k}_{2,4})_5, \mathbf{u}^2)_4$ $((\mathbf{f}, \mathbf{k}_{2,4}^2)_6, \mathbf{u})_2$ $((\mathbf{h}_{2,4}, \mathbf{v})_1, \mathbf{u}^3)_6$ $((\mathbf{h}_{2,8}, \mathbf{v})_3, \mathbf{u}^3)_6$
	$((\mathbf{h}_{2,8}, \mathbf{v}^2)_6, \mathbf{u}^2)_4$ $((\mathbf{h}_{2,8}, \mathbf{k}_{2,4})_4, \mathbf{u}^2)_4$ $((\mathbf{h}_{2,4}, \mathbf{k}_{2,4})_2, \mathbf{u}^2)_4$ $((\mathbf{h}_{2,8}, \mathbf{v} \cdot \mathbf{k}_{2,4})_7, \mathbf{u})_2$
	$((\mathbf{h}_{2,8}, \mathbf{k}_{3,6})_6, \mathbf{u})_2$ $((\mathbf{f}^2, \mathbf{v}^3)_{11}, \mathbf{u})_2$ $((\mathbf{h}_{2,4}, \mathbf{k}_{3,6})_4, \mathbf{u})_2$ $((\mathbf{h}_{3,6}, \mathbf{v})_3, \mathbf{u}^2)_4$
	$((\mathbf{h}_{3,8}, \mathbf{v})_4, \mathbf{u}^2)_4$ $((\mathbf{h}_{3,2}, \mathbf{v})_1, \mathbf{u}^2)_4$ $((\mathbf{h}_{3,6}, \mathbf{k}_{2,4})_4, \mathbf{u})_2$ $((\mathbf{h}_{3,6}, \mathbf{v}^2)_6, \mathbf{u})_2$
	$((\mathbf{h}_{3,2}, \mathbf{k}_{2,4})_2, \mathbf{u})_2$ $((\mathbf{h}_{3,8}, \mathbf{v}^2)_7, \mathbf{u})_2$ $((\mathbf{h}_{4,4}, \mathbf{v})_3, \mathbf{u})_2$ $((\mathbf{h}_{4,6}, \mathbf{v})_4, \mathbf{u})_2$
Degree 7	$((\mathbf{f}, \mathbf{k}_{2,4})_1, \mathbf{u}^4)_8$ $((\mathbf{f}, \mathbf{k}_{2,4}^2)_5, \mathbf{u}^2)_4$ $((\mathbf{h}_{2,8}, \mathbf{v})_2, \mathbf{u}^4)_8$ $((\mathbf{h}_{2,8}, \mathbf{v}^2)_5, \mathbf{u}^3)_6$
	$((\mathbf{h}_{2,8}, \mathbf{k}_{2,4})_3, \mathbf{u}^3)_6$ $((\mathbf{h}_{2,4}, \mathbf{k}_{2,4})_1, \mathbf{u}^3)_6$ $((\mathbf{h}_{2,8}, \mathbf{v} \cdot \mathbf{k}_{2,4})_6, \mathbf{u}^2)_4$ $((\mathbf{h}_{2,8}, \mathbf{k}_{3,6})_5, \mathbf{u}^2)_4$

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	$((\mathbf{h}_{2,8}, \mathbf{k}_{2,4}^2)_7, \mathbf{u})_2$	$((\mathbf{f}^2, \mathbf{v}^2 \cdot \mathbf{k}_{2,4})_{11}, \mathbf{u})_2$	$((\mathbf{h}_{2,8}, \mathbf{v} \cdot \mathbf{k}_{3,6})_8, \mathbf{u})_2$	$((\mathbf{h}_{3,8}, \mathbf{v})_3, \mathbf{u}^3)_6$
	$((\mathbf{h}_{3,2}, \mathbf{k}_{2,4})_1, \mathbf{u}^2)_4$	$((\mathbf{h}_{3,8}, \mathbf{k}_{2,4})_4, \mathbf{u}^2)_4$	$((\mathbf{h}_{3,6}, \mathbf{v}^2)_5, \mathbf{u}^2)_4$	$((\mathbf{h}_{3,12}, \mathbf{v}^2)_8, \mathbf{u}^2)_4$
	$((\mathbf{h}_{3,6}, \mathbf{k}_{2,4})_3, \mathbf{u}^2)_4$	$((\mathbf{h}_{3,6}, \mathbf{v} \cdot \mathbf{k}_{2,4})_6, \mathbf{u})_2$	$((\mathbf{h}_{3,12}, \mathbf{v}^3)_{11}, \mathbf{u})_2$	$((\mathbf{h}_{3,8}, \mathbf{v} \cdot \mathbf{k}_{2,4})_7, \mathbf{u})_2$
	$((\mathbf{h}_{3,8}, \mathbf{k}_{3,6})_6, \mathbf{u})_2$	$((\mathbf{h}_{4,6}, \mathbf{v})_3, \mathbf{u}^2)_4$	$((\mathbf{h}_{4,4}, \mathbf{v})_2, \mathbf{u}^2)_4$	$((\mathbf{h}_{4,6}, \mathbf{k}_{2,4})_4, \mathbf{u})_2$
	$((\mathbf{h}_{4,6}, \mathbf{v}^2)_6, \mathbf{u})_2$	$((\mathbf{h}_{4,4}, \mathbf{k}_{2,4})_3, \mathbf{u})_2$	$((\mathbf{h}_{4,10}, \mathbf{v}^2)_8, \mathbf{u})_2$	$((\mathbf{h}_{3,2} \cdot \mathbf{f}, \mathbf{v}^2)_7, \mathbf{u})_2$
	$((\mathbf{h}_{5,4}, \mathbf{v})_3, \mathbf{u})_2$	$((\mathbf{h}_{5,2}, \mathbf{v})_2, \mathbf{u})_2$		
Degree 8	$((\mathbf{h}_{2,8}, \mathbf{v})_1, \mathbf{u}^5)_{10}$	$((\mathbf{h}_{2,8}, \mathbf{k}_{2,4} \cdot \mathbf{k}_{3,6})_8, \mathbf{u})_2$	$((\mathbf{f}^2, \mathbf{v} \cdot \mathbf{k}_{2,4}^2)_{11}, \mathbf{u})_2$	$((\mathbf{h}_{3,12}, \mathbf{v})_4, \mathbf{u}^4)_8$
	$((\mathbf{h}_{3,8}, \mathbf{v})_2, \mathbf{u}^4)_8$	$((\mathbf{h}_{3,12}, \mathbf{v} \cdot \mathbf{k}_{2,4})_8, \mathbf{u}^2)_4$	$((\mathbf{h}_{3,6}, \mathbf{k}_{3,6})_4, \mathbf{u}^2)_4$	$((\mathbf{h}_{3,8}, \mathbf{v} \cdot \mathbf{k}_{3,6})_8, \mathbf{u})_2$
	$((\mathbf{h}_{3,12}, \mathbf{v}^2 \cdot \mathbf{k}_{2,4})_{11}, \mathbf{u})_2$	$((\mathbf{h}_{4,10}, \mathbf{v})_4, \mathbf{u}^3)_6$	$((\mathbf{h}_{4,6}, \mathbf{v})_2, \mathbf{u}^3)_6$	$((\mathbf{h}_{4,4}, \mathbf{v})_1, \mathbf{u}^3)_6$
	$((\mathbf{h}_{4,6}, \mathbf{k}_{2,4})_3, \mathbf{u}^2)_4$	$((\mathbf{h}_{4,6}, \mathbf{v} \cdot \mathbf{k}_{2,4})_6, \mathbf{u})_2$	$((\mathbf{h}_{4,10}, \mathbf{v}^3)_{10}, \mathbf{u})_2$	$((\mathbf{f} \cdot \mathbf{h}_{3,6}, \mathbf{v}^3)_{11}, \mathbf{u})_2$
	$((\mathbf{h}_{4,10}, \mathbf{v} \cdot \mathbf{k}_{2,4})_8, \mathbf{u})_2$	$((\mathbf{h}_{4,6}, \mathbf{k}_{3,6})_5, \mathbf{u})_2$	$((\mathbf{h}_{2,4}^2, \mathbf{k}_{3,6})_6, \mathbf{u})_2$	$((\mathbf{h}_{5,8}, \mathbf{v})_4, \mathbf{u}^2)_4$
	$((\mathbf{h}_{5,2}, \mathbf{v})_1, \mathbf{u}^2)_4$	$((\mathbf{h}_{5,4}, \mathbf{v})_2, \mathbf{u}^2)_4$	$((\mathbf{h}_{5,2}, \mathbf{k}_{2,4})_2, \mathbf{u})_2$	$((\mathbf{h}_{5,8}, \mathbf{v}^2)_7, \mathbf{u})_2$
	$((\mathbf{h}_{5,4}, \mathbf{k}_{2,4})_3, \mathbf{u})_2$	$((\mathbf{h}_{6,62}, \mathbf{v})_4, \mathbf{u})_2$	$((\mathbf{h}_{3,2}^2, \mathbf{v})_3, \mathbf{u})_2$	$((\mathbf{h}_{6,61}, \mathbf{v})_4, \mathbf{u})_2$
Degree 9	$((\mathbf{h}_{4,10}, \mathbf{k}_{2,4}^2)_8, \mathbf{u})_2$	$((\mathbf{h}_{5,8}, \mathbf{k}_{2,4})_4, \mathbf{u}^2)_4$	$((\mathbf{h}_{3,2} \cdot \mathbf{h}_{3,6}, \mathbf{v}^2)_7, \mathbf{u})_2$	$((\mathbf{h}_{6,61}, \mathbf{k}_{2,4})_4, \mathbf{u})_2$
	$((\mathbf{h}_{3,2}^2, \mathbf{k}_{2,4})_3, \mathbf{u})_2$	$((\mathbf{h}_{6,62}, \mathbf{v}^2)_6, \mathbf{u})_2$	$((\mathbf{h}_{7,2}, \mathbf{v})_2, \mathbf{u})_2$	
Degree 10	$((\mathbf{h}_{3,2}^2, \mathbf{k}_{3,6})_4, \mathbf{u})_2$	$((\mathbf{h}_{7,4}, \mathbf{v})_2, \mathbf{u}^2)_4$	$((\mathbf{h}_{8,2}, \mathbf{v})_2, \mathbf{u})_2$	$((\mathbf{h}_{3,2} \cdot \mathbf{h}_{5,2}, \mathbf{v})_3, \mathbf{u})_2$
Degree 11	$((\mathbf{h}_{5,2} \cdot \mathbf{h}_{3,6}, \mathbf{v}^2)_7, \mathbf{u})_2$			
Degree 12	$((\mathbf{h}_{5,2}^2, \mathbf{v})_3, \mathbf{u})_2$			
171 covariants of order 2:6 from $S_6$ , 1 from $S_2$ , 23 joint covariants of $S_6 \oplus S_2$ given in 7.1, 62 joint covariants of $S_6 \oplus S_4$ given in 7.2. There is left 79 covariants given below:				
Degree 2	$(\mathbf{v}, \mathbf{u})_2$			
Degree 3	$(\mathbf{v}, \mathbf{u}^2)_3$	$(\mathbf{k}_{2,4}, \mathbf{u})_2$	$((\mathbf{f}, \mathbf{v})_4, \mathbf{u})_1$	$((\mathbf{f}, \mathbf{v})_3, \mathbf{u})_2$
Degree 4	$((1, \mathbf{k}_{2,4})_0, \mathbf{u}^2)_3$	$((\mathbf{f}, \mathbf{v})_2, \mathbf{u}^2)_4$	$((\mathbf{f}, \mathbf{v})_3, \mathbf{u}^2)_3$	$((\mathbf{f}, \mathbf{k}_{2,4})_3, \mathbf{u})_2$
	$((\mathbf{f}, \mathbf{v}^2)_5, \mathbf{u})_2$	$((\mathbf{f}, \mathbf{k}_{2,4})_4, \mathbf{u})_1$	$((\mathbf{f}, \mathbf{v}^2)_6, \mathbf{u})_1$	$((\mathbf{h}_{2,4}, \mathbf{v})_2, \mathbf{u})_2$
	$((\mathbf{h}_{2,8}, \mathbf{v})_4, \mathbf{u})_2$	$((\mathbf{h}_{2,4}, \mathbf{v})_3, \mathbf{u})_1$		
Degree 5	$(\mathbf{k}_{3,6}, \mathbf{u}^2)_4$	$((\mathbf{f}, \mathbf{v})_2, \mathbf{u}^3)_5$	$((\mathbf{f}, \mathbf{v})_1, \mathbf{u}^3)_6$	$((\mathbf{f}, \mathbf{k}_{2,4})_2, \mathbf{u}^2)_4$
	$((\mathbf{f}, \mathbf{k}_{2,4})_3, \mathbf{u}^2)_3$	$((\mathbf{f}, \mathbf{k}_{3,6})_5, \mathbf{u})_1$	$((\mathbf{f}, \mathbf{v} \cdot \mathbf{k}_{2,4})_5, \mathbf{u})_2$	$((\mathbf{f}, \mathbf{v} \cdot \mathbf{k}_{2,4})_6, \mathbf{u})_1$
	$((\mathbf{f}, \mathbf{k}_{3,6})_4, \mathbf{u})_2$	$((\mathbf{h}_{2,8}, \mathbf{v})_4, \mathbf{u}^2)_3$	$((\mathbf{h}_{2,4}, \mathbf{v})_1, \mathbf{u}^2)_4$	$((\mathbf{h}_{2,8}, \mathbf{v})_3, \mathbf{u}^2)_4$
	$((\mathbf{h}_{2,4}, \mathbf{v})_2, \mathbf{u}^2)_3$	$((\mathbf{h}_{2,8}, \mathbf{v}^2)_7, \mathbf{u})_1$	$((\mathbf{h}_{2,4}, \mathbf{k}_{2,4})_3, \mathbf{u})_1$	$((\mathbf{h}_{2,8}, \mathbf{k}_{2,4})_4, \mathbf{u})_2$
	$((\mathbf{h}_{2,8}, \mathbf{v}^2)_6, \mathbf{u})_2$	$((\mathbf{h}_{2,4}, \mathbf{k}_{2,4})_2, \mathbf{u})_2$	$((\mathbf{h}_{3,2}, \mathbf{v})_1, \mathbf{u})_2$	$((\mathbf{h}_{3,6}, \mathbf{v})_4, \mathbf{u})_1$
	$((\mathbf{h}_{3,2}, \mathbf{v})_2, \mathbf{u})_1$	$((\mathbf{h}_{3,6}, \mathbf{v})_3, \mathbf{u})_2$	$((\mathbf{h}_{3,8}, \mathbf{v})_4, \mathbf{u})_2$	
Degree 6	$((\mathbf{f}, \mathbf{k}_{2,4})_1, \mathbf{u}^3)_6$	$((\mathbf{f}, \mathbf{k}_{2,4}^2)_5, \mathbf{u})_2$	$((\mathbf{f}, \mathbf{k}_{2,4}^2)_6, \mathbf{u})_1$	$((\mathbf{h}_{2,8}, \mathbf{v})_2, \mathbf{u}^3)_6$
	$((\mathbf{h}_{2,8}, \mathbf{v}^2)_5, \mathbf{u}^2)_4$	$((\mathbf{h}_{2,8}, \mathbf{k}_{2,4})_4, \mathbf{u}^2)_3$	$((\mathbf{h}_{2,8}, \mathbf{k}_{2,4})_3, \mathbf{u}^2)_4$	$((\mathbf{h}_{2,4}, \mathbf{k}_{2,4})_1, \mathbf{u}^2)_4$
	$((\mathbf{h}_{2,8}, \mathbf{v} \cdot \mathbf{k}_{2,4})_7, \mathbf{u})_1$	$((\mathbf{h}_{2,8}, \mathbf{k}_{3,6})_5, \mathbf{u})_2$	$((\mathbf{h}_{2,8}, \mathbf{k}_{3,6})_6, \mathbf{u})_1$	$((\mathbf{h}_{2,8}, \mathbf{v} \cdot \mathbf{k}_{2,4})_6, \mathbf{u})_2$
	$((\mathbf{h}_{3,8}, \mathbf{v})_3, \mathbf{u}^2)_4$	$((\mathbf{h}_{3,6}, \mathbf{v})_3, \mathbf{u}^2)_3$	$((\mathbf{h}_{3,2}, \mathbf{k}_{2,4})_1, \mathbf{u})_2$	$((\mathbf{h}_{3,8}, \mathbf{k}_{2,4})_4, \mathbf{u})_2$
	$((\mathbf{h}_{3,6}, \mathbf{v}^2)_5, \mathbf{u})_2$	$((\mathbf{h}_{3,6}, \mathbf{v}^2)_6, \mathbf{u})_1$	$((\mathbf{h}_{3,6}, \mathbf{k}_{2,4})_3, \mathbf{u})_2$	$((\mathbf{h}_{3,2}, \mathbf{k}_{2,4})_2, \mathbf{u})_1$

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	$((\mathbf{h}_{3,12}, \mathbf{v}^2)_8, \mathbf{u})_2$ $((\mathbf{h}_{3,8}, \mathbf{v}^2)_7, \mathbf{u})_1$ $((\mathbf{h}_{4,6}, \mathbf{v})_3, \mathbf{u})_2$ $((\mathbf{h}_{4,6}, \mathbf{v})_4, \mathbf{u})_1$
	$((\mathbf{h}_{4,4}, \mathbf{v})_2, \mathbf{u})_2$
Degree 7	$((\mathbf{h}_{2,8}, \mathbf{v})_1, \mathbf{u}^4)_8$ $((\mathbf{h}_{3,8}, \mathbf{v})_2, \mathbf{u}^3)_6$ $((\mathbf{h}_{3,12}, \mathbf{v})_4, \mathbf{u}^3)_6$ $((\mathbf{h}_{3,12}, \mathbf{v} \cdot \mathbf{k}_{2,4})_8, \mathbf{u})_2$
	$((\mathbf{h}_{3,6}, \mathbf{k}_{3,6})_4, \mathbf{u})_2$ $((\mathbf{h}_{4,6}, \mathbf{v})_2, \mathbf{u}^2)_4$ $((\mathbf{h}_{4,4}, \mathbf{v})_1, \mathbf{u}^2)_4$ $((\mathbf{h}_{4,10}, \mathbf{v})_4, \mathbf{u}^2)_4$
	$((\mathbf{h}_{4,6}, \mathbf{v}^2)_6, \mathbf{u})_1$ $((\mathbf{h}_{4,6}, \mathbf{k}_{2,4})_3, \mathbf{u})_2$ $((\mathbf{h}_{5,4}, \mathbf{v})_2, \mathbf{u})_2$ $((\mathbf{h}_{5,8}, \mathbf{v})_4, \mathbf{u})_2$
	$((\mathbf{h}_{5,4}, \mathbf{v})_3, \mathbf{u})_1$ $((\mathbf{h}_{5,2}, \mathbf{v})_1, \mathbf{u})_2$
Degree 8	$((\mathbf{h}_{5,8}, \mathbf{k}_{2,4})_4, \mathbf{u})_2$
Degree 9	$((\mathbf{h}_{7,4}, \mathbf{v})_2, \mathbf{u})_2$
80 covariants of order 4 : 5 from $S_6$ , 2 from $S_4$ , 15 joint covariants of $S_6 \oplus S_2$ given in 7.1, 31 joint covariants of $S_6 \oplus S_4$ given in 7.2. There is left 27 covariants given below:	
Degree 2	$(\mathbf{v}, \mathbf{u})_1$
Degree 3	$(\mathbf{k}_{2,4}, \mathbf{u})_1$ $((\mathbf{f}, \mathbf{v})_2, \mathbf{u})_2$ $((\mathbf{f}, \mathbf{v})_3, \mathbf{u})_1$
Degree 4	$(\mathbf{k}_{3,6}, \mathbf{u})_2$ $((\mathbf{f}, \mathbf{v})_2, \mathbf{u}^2)_3$ $((\mathbf{f}, \mathbf{v})_1, \mathbf{u}^2)_4$ $((\mathbf{f}, \mathbf{k}_{2,4})_3, \mathbf{u})_1$
	$((\mathbf{f}, \mathbf{k}_{2,4})_2, \mathbf{u})_2$ $((\mathbf{h}_{2,8}, \mathbf{v})_4, \mathbf{u})_1$ $((\mathbf{h}_{2,4}, \mathbf{v})_2, \mathbf{u})_1$ $((\mathbf{h}_{2,8}, \mathbf{v})_3, \mathbf{u})_2$
	$((\mathbf{h}_{2,4}, \mathbf{v})_1, \mathbf{u})_2$
Degree 5	$((\mathbf{f}, \mathbf{k}_{2,4})_1, \mathbf{u}^2)_4$ $((\mathbf{h}_{2,8}, \mathbf{v})_2, \mathbf{u}^2)_4$ $((\mathbf{h}_{2,8}, \mathbf{k}_{2,4})_4, \mathbf{u})_1$ $((\mathbf{h}_{2,8}, \mathbf{v}^2)_5, \mathbf{u})_2$
	$((\mathbf{h}_{2,8}, \mathbf{k}_{2,4})_3, \mathbf{u})_2$ $((\mathbf{h}_{2,4}, \mathbf{k}_{2,4})_1, \mathbf{u})_2$ $((\mathbf{h}_{3,6}, \mathbf{v})_3, \mathbf{u})_1$ $((\mathbf{h}_{3,8}, \mathbf{v})_3, \mathbf{u})_2$
Degree 6	$((\mathbf{h}_{2,8}, \mathbf{v})_1, \mathbf{u}^3)_6$ $((\mathbf{h}_{3,8}, \mathbf{v})_2, \mathbf{u}^2)_4$ $((\mathbf{h}_{3,12}, \mathbf{v})_4, \mathbf{u}^2)_4$ $((\mathbf{h}_{4,10}, \mathbf{v})_4, \mathbf{u})_2$
	$((\mathbf{h}_{4,4}, \mathbf{v})_1, \mathbf{u})_2$ $((\mathbf{h}_{4,6}, \mathbf{v})_2, \mathbf{u})_2$
32 covariants of order 6 : 5 from $S_6$ , 1 from $S_4$ , 8 joint covariants of $S_6 \oplus S_2$ given in 7.1, 11 joint covariants of $S_6 \oplus S_4$ given in 7.2. There is left 7 covariants given below:	
Degree 3	$((\mathbf{f}, \mathbf{v})_2, \mathbf{u})_1$ $((\mathbf{f}, \mathbf{v})_1, \mathbf{u})_2$
Degree 4	$((\mathbf{f}, \mathbf{k}_{2,4})_1, \mathbf{u})_2$ $((\mathbf{h}_{2,8}, \mathbf{v})_2, \mathbf{u})_2$
Degree 5	$((\mathbf{h}_{2,8}, \mathbf{v})_1, \mathbf{u}^2)_4$ $((\mathbf{h}_{3,12}, \mathbf{v})_4, \mathbf{u})_2$ $((\mathbf{h}_{3,8}, \mathbf{v})_2, \mathbf{u})_2$
12 covariants of order 8 : 3 from $S_6$ , 3 joint covariants of $S_6 \oplus S_2$ given in 7.1, 5 joint covariants of $S_6 \oplus S_4$ given in 7.2. There is left 1 covariant given below:	
degree 4	$((\mathbf{h}_{2,8}, \mathbf{v})_1, \mathbf{u})_2$

There is left 3 covariants of order 10 : 1 from  $S_6$ , 1 joint covariant of  $S_6 \oplus S_2$  given in 7.1 and 1 joint covariant of  $S_6 \oplus S_4$  given in 7.2. Finally there is 1 covariant of order 12 taken from  $S_6$ .

**7.4. Covariant bases of  $S_8$ .** We apply here Gordan's algorithm for a simple binary form.

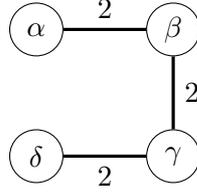
- (1) As a first step  $A_0 = \{\mathbf{f}\}$  for  $\mathbf{f} \in S_8$ . The family  $B_0$  contains only

$$\mathbf{h}_{2,12} := \{\mathbf{f}, \mathbf{f}\}_2 \in S_{12}.$$

- (2) To obtain  $A_1$  we have to consider transvectants

$$(\mathbf{f}^a, \mathbf{h}_{2,12}^b)_r,$$

with no reducible molecular covariants modulo  $I_4$ . From lemma B.1 we deduce that necessarily  $r \leq 2$ . Take now a molecule



By lemma A.4 with  $e_0 = 2$  and  $e_1 = 2$ , this molecule is of grade 3 and thus by lemma B.1 of grade 4.

We can deduce from all this that  $A_1$  is the family

$$\mathbf{f}, \mathbf{h}_{2,12}, \mathbf{h}_{3,18} := \{\mathbf{f}, \mathbf{h}_{2,12}\}_1.$$

The family  $B_1$  is sthe form

$$\mathbf{h}_{2,8} := \{\mathbf{f}, \mathbf{f}\}_4 \in S_8.$$

(3) To get  $A_2$  we have to consider transvectants

$$(\mathbf{f}^{a_1} \mathbf{h}_{2,12}^{a_2} \mathbf{h}_{3,18}^{a_3}, \mathbf{h}_{2,8}^b)_r.$$

The same kind of argument as above, using lemma such as lemma A.4 leads to [34, 33]:

**Lemma 7.4.** *The family  $A_2$  is given by the seven covariants*

$$\mathbf{f}, \mathbf{h}_{2,8} = (\mathbf{f}, \mathbf{f})_4, \quad \mathbf{h}_{2,12} = (\mathbf{f}, \mathbf{f})_2, \quad \mathbf{h}_{3,12} := (\mathbf{f}, \mathbf{h}_{2,8})_2, \quad \mathbf{h}_{3,14} := (\mathbf{f}, \mathbf{h}_{2,8})_1$$

$$\mathbf{h}_{3,18} := (\mathbf{f}, \mathbf{h}_{2,12})_1, \quad \mathbf{h}_{4,18} := (\mathbf{h}_{2,12}, \mathbf{h}_{2,8})_1$$

Recall we have to take the invariant

$$(\mathbf{f}, \mathbf{h}_{2,8})_8.$$

The family  $B_2$  is given by the covariant basis of

$$\mathbf{h}_{2,4} := (\mathbf{f}, \mathbf{f})_6 \in S_4.$$

As seen above in 7.2, a covariant basis is given by:

$$\mathbf{h}_{2,4}, \quad \mathbf{h}_{4,4} := (\mathbf{h}_{2,4}, \mathbf{h}_{2,4})_2, \quad \mathbf{h}_{6,6} := (\mathbf{h}_{2,4}, (\mathbf{h}_{2,4}, \mathbf{h}_{2,4})_2)_1$$

and two invariants

$$\mathbf{h}_{4,0} := (\mathbf{h}_{2,4}, \mathbf{h}_{2,4})_4, \quad \mathbf{h}_{6,0} := (\mathbf{h}_{2,4}, (\mathbf{h}_{2,4}, \mathbf{h}_{2,4})_2)_4$$

(4) To get family  $B_3$ , we have to consider transvectants

$$(\mathbf{f}^{a_1} \mathbf{h}_{2,8}^{a_2} \mathbf{h}_{2,12}^{a_3} \mathbf{h}_{3,12}^{a_4} \mathbf{h}_{3,14}^{a_5} \mathbf{h}_{3,18}^{a_6} \mathbf{h}_{4,18}^{a_7}, \mathbf{h}_{2,4}^{b_1} \mathbf{h}_{4,4}^{b_2} \mathbf{h}_{6,6}^{b_3})_r$$

which is associated to the integer system

$$\begin{cases} 8a_1 + 8a_2 + 12a_3 + 12a_4 + 14a_5 + 18a_6 + 18a_7 & = u + r \\ 4b_1 + 4b_2 + 6b_3 & = v + r \end{cases}. \quad (7.1)$$

We also make use of the relation taken from  $\mathbf{Cov}(S_4)$ :

$$12\mathbf{h}_{6,6}^2 + 6\mathbf{h}_{4,4}^3 + 2\mathbf{h}_{6,0}\mathbf{h}_{2,4}^3 - 3\mathbf{h}_{2,4}^2\mathbf{h}_{4,4}\mathbf{h}_{4,0} = 0.$$

With computations in Macaulay2 [35], we finally get a covariant

basis of  $S_8$  given bellow.

8 invariants	
Degree 2	$\mathbf{h}_{2,0} := (\mathbf{f}, \mathbf{f})_8$
Degree 3	$(\mathbf{f}, \mathbf{h}_{2,8})_8$
Degree 4	$(\mathbf{h}_{2,4}, \mathbf{h}_{2,4})_4$
Degree 5	$(\mathbf{f}, \mathbf{h}_{2,4}^2)_8$
Degree 6	$(\mathbf{h}_{4,4}, \mathbf{h}_{2,4})_4$
Degree 7	$(\mathbf{f}, \mathbf{h}_{2,4}\mathbf{h}_{4,4})_8$
Degree 8	$(\mathbf{h}_{2,12}, \mathbf{h}_{2,4}^3)_{12}$
Degree 9	$(\mathbf{h}_{3,12}, \mathbf{h}_{2,4}^3)_{12}$
Degree 10	$(\mathbf{h}_{2,12}, \mathbf{h}_{2,4}^2\mathbf{h}_{4,4})_{12}$
14 covariants of order 2.	
Degree 5	$(\mathbf{f}, \mathbf{h}_{2,4}^2)_7$
Degree 6	$(\mathbf{h}_{2,8}, \mathbf{h}_{2,4}^2)_7$
Degree 7	$(\mathbf{f}, \mathbf{h}_{6,6})_6 \quad (\mathbf{f}, \mathbf{h}_{2,4}\mathbf{h}_{4,4})_7$
Degree 8	$(\mathbf{h}_{2,12}, \mathbf{h}_{2,4}^3)_{11} \quad (\mathbf{h}_{2,8}, \mathbf{h}_{6,6})_6$
Degree 9	$(\mathbf{h}_{3,14}, \mathbf{h}_{2,4}^3)_{12} \quad (\mathbf{h}_{3,12}, \mathbf{h}_{2,4}^3)_{11} \quad (\mathbf{f}, \mathbf{h}_{4,4}^2)_7$
Degree 10	$(\mathbf{h}_{2,12}, \mathbf{h}_{2,4}\mathbf{h}_{6,6})_{10} \quad (\mathbf{h}_{2,12}, \mathbf{h}_{2,4}^2\mathbf{h}_{4,4})_{11}$
Degree 11	$(\mathbf{h}_{3,18}, \mathbf{h}_{2,4}^4)_{16} \quad (\mathbf{h}_{3,14}, \mathbf{h}_{2,4}^2\mathbf{h}_{4,4})_{12}$
Degree 12	$(\mathbf{h}_{4,18}, \mathbf{h}_{2,4}^4)_{16}$
13 covariants of order 4.	
Degree 2	$\mathbf{h}_{2,4} := (\mathbf{f}, \mathbf{f})_6$
Degree 3	$(\mathbf{f}, \mathbf{h}_{2,4})_4$
Degree 4	$\mathbf{h}_{4,4} := (\mathbf{h}_{2,4}, \mathbf{h}_{2,4})_2$
	$(\mathbf{h}_{2,8}, \mathbf{h}_{2,4})_4$
Degree 5	$(\mathbf{f}, \mathbf{h}_{4,4})_4 \quad (\mathbf{f}, \mathbf{h}_{2,4}^2)_6$
Degree 6	$(\mathbf{h}_{2,12}, \mathbf{h}_{2,4}^2)_8 \quad (\mathbf{h}_{2,8}, \mathbf{h}_{4,4})_4$
Degree 7	$(\mathbf{h}_{3,12}, \mathbf{h}_{2,4}^2)_8 \quad (\mathbf{f}, \mathbf{h}_{6,6})_5$
Degree 8	$(\mathbf{h}_{2,12}, \mathbf{h}_{2,4}\mathbf{h}_{4,4})_8 \quad (\mathbf{h}_{2,12}, \mathbf{h}_{2,4}^3)_{10}$
Degree 9	$(\mathbf{h}_{3,14}, \mathbf{h}_{2,4}^3)_{11}$
12 covariants of order 6.	
Degree 3	$(\mathbf{f}, \mathbf{h}_{2,4})_3$
Degree 4	$(\mathbf{h}_{2,8}, \mathbf{h}_{2,4})_3$
Degree 5	$(\mathbf{f}, \mathbf{h}_{4,4})_3 \quad (\mathbf{f}, \mathbf{h}_{2,4}^2)_5$
Degree 6	$\mathbf{h}_{6,6} := (\mathbf{h}_{4,4}, \mathbf{h}_{2,4})_1 \quad (\mathbf{h}_{2,12}, \mathbf{h}_{2,4}^2)_7 \quad (\mathbf{h}_{2,8}, \mathbf{h}_{4,4})_3$

*continued on next page*

continued from previous page

Degree 7	$(\mathbf{h}_{3,14}, \mathbf{h}_{2,4}^2)_8$	$(\mathbf{h}_{3,12}, \mathbf{h}_{2,4}^2)_7$	$(\mathbf{f}, \mathbf{h}_{6,6})_4$
Degree 8	$(\mathbf{h}_{2,12}, \mathbf{h}_{6,6})_6$	$(\mathbf{h}_{2,12}, \mathbf{h}_{2,4}\mathbf{h}_{4,4})_7$	
6 covariants of order 8.			
Degree 1	$\mathbf{f}$		
Degree 2	$\mathbf{h}_{2,8}$		
Degree 3	$(\mathbf{f}, \mathbf{h}_{2,4})_2$		
Degree 4	$(\mathbf{h}_{2,12}, \mathbf{h}_{2,4})_4$		
Degree 5	$(\mathbf{f}, \mathbf{h}_{4,4})_2$		
Degree 6	$(\mathbf{h}_{2,12}, \mathbf{h}_{4,4})_4$		
7 covariants of order 10.			
Degree 3	$(\mathbf{f}, \mathbf{h}_{2,4})_1$		
Degree 4	$(\mathbf{h}_{2,12}, \mathbf{h}_{2,4})_3$	$(\mathbf{h}_{2,8}, \mathbf{h}_{2,4})_1$	
Degree 5	$(\mathbf{h}_{3,14}, \mathbf{h}_{2,4})_4$	$(\mathbf{h}_{3,12}, \mathbf{h}_{2,4})_3$	
	$(\mathbf{f}, \mathbf{h}_{4,4})_1$		
Degree 6	$(\mathbf{h}_{2,12}, \mathbf{h}_{4,4})_3$		
3 covariants of order 12.			
Degree 2	$\mathbf{h}_{2,12}$		
Degree 3	$\mathbf{h}_{3,12} := (\mathbf{f}, \mathbf{h}_{2,8})_2$		
Degree 4	$(\mathbf{h}_{2,12}, \mathbf{h}_{2,4})_2$		
3 covariants of order 14.			
Degree 3	$\mathbf{h}_{3,14} := (\mathbf{f}, \mathbf{h}_{2,8})_1$		
Degree 4	$(\mathbf{h}_{2,12}, \mathbf{h}_{2,4})_1$		
Degree 5	$(\mathbf{h}_{3,12}, \mathbf{h}_{2,4})_1$		
2 covariants of order 18.			
Degree 3	$\mathbf{h}_{3,18} := (\mathbf{f}, \mathbf{h}_{2,12})_1$		
Degree 4	$(\mathbf{h}_{2,12}, \mathbf{h}_{2,8})_1$		

## APPENDIX A. THE STROH FORMULA AND SOME COROLLARIES

The following general algebraic relation was obtained by Stroh [56] (see also [34]).

**Lemma A.1.** *Let  $u_1, u_2$  and  $u_3$  be three commutative variables such that*

$$u_1 + u_2 + u_3 = 0.$$

*Then we have*

$$\begin{aligned}
 (-1)^{k_2} \sum_{i=0}^{k_1} \binom{g}{i} \binom{k_1 + k_3 - i}{k_3} u_3^{g-i} u_1^i + (-1)^{k_3} \sum_{i=0}^{k_2} \binom{g}{i} \binom{k_2 + k_1 - i}{k_1} u_1^{g-i} u_2^i + \\
 (-1)^{k_1} \sum_{i=0}^{k_3} \binom{g}{i} \binom{k_3 + k_2 - i}{k_2} u_2^{g-i} u_3^i = 0, \quad (\text{A.1})
 \end{aligned}$$

*with  $k_1 + k_2 + k_3 = g - 1$ .*

This formula leads to new degree three relations. Let  $V = S_n$  and  $(e_0, e_1, e_2)$  be three integers such that  $e_i + e_j \leq n$  ( $i \neq j$ ). Define:

$$D(e_0, e_1, e_2) := \begin{array}{ccc} & \xrightarrow{e_0} & \\ \circlearrowleft \alpha & & \circlearrowright \beta \\ & \xleftarrow{e_2} & \xrightarrow{e_1} \\ & \circlearrowright \gamma & \end{array} \quad \text{with weight } w = e_0 + e_1 + e_2, \quad (\text{A.2})$$

Note that  $D(e_0, e_1, e_2) \in \text{Hom}_{\text{SL}(2, \mathbb{C})}(S_n \otimes S_n \otimes S_n, S_{3n-2w})$ .

**Lemma A.2.** *Let  $w \leq n$  and  $m_1, m_2, m_3 \geq 1$  be integers such that  $m_1 + m_2 + m_3 = w + 1$ , then the molecule  $D(e_0, e_1, e_2)$  is a linear combination of*

$$D(w - i_1, i_1, 0), \quad D(0, w - i_2, i_2), \quad D(i_3, 0, w - i_3),$$

with  $i_s = 0 \dots m_s - 1$ ,

*Sketch of proof.* Using Clebsch–Gordan decomposition, first observe that

$$\dim \text{Hom}_{\text{SL}(2, \mathbb{C})}(S_n \otimes S_n \otimes S_n, S_{3n-2w}) = w + 1$$

Suppose we have a linear relation

$$\sum_{i=0}^w \lambda_i D(w - i, i, 0) = 0.$$

Taking  $\mathbf{f}_\alpha = x_\alpha^n$ ,  $\mathbf{f}_\beta = y_\beta^n$  and  $\mathbf{f}_\gamma = y_\gamma^n$  leads to  $\lambda_0 = 0$ ; and by induction we get  $\lambda_i = 0$  for all  $i$ . Thus  $\mathcal{F}_1 := \{D(w - i, i, 0), i = 0 \dots w\}$  is a basis of  $\text{Hom}_{\text{SL}(2, \mathbb{C})}(S_n \otimes S_n \otimes S_n, S_{3n-2w})$ . There is the same statement for  $\mathcal{F}_2 := \{D(0, w - i, i), i = 0 \dots w\}$  and  $\mathcal{F}_3 := \{D(i, 0, w - i), i = 0 \dots w\}$ .

Let

$$u_1 = \Omega_{\alpha\beta}\sigma_\gamma, \quad u_2 = \Omega_{\beta\gamma}\sigma_\alpha, \quad u_3 = \Omega_{\gamma\alpha}\sigma_\beta$$

These are commutative variables verifying  $u_1 + u_2 + u_3 = 0$ . Now, taking the family

$$\mathcal{F} := \{D(w - i_1, i_1, 0), D(0, w - i_2, i_2), D(i_3, 0, w - i_3), \quad i_s = 0 \dots m_s - 1\}$$

lemma A.1 with  $k_1 = m_1, k_2 = m_2, k_3 = m_3 + 1$  (for  $m_3 < w$ ) and  $g = w + 3$  induces that  $D(m_3 + 1, 0, w - m_3 - 1) \in \mathcal{F}_3$  is generated by the family  $\mathcal{F}$ . By induction,  $\mathcal{F}_3$  and thus all molecules are generated by  $\mathcal{F}$ .  $\square$

**Lemma A.3.** *Let  $D(e_0, e_1, e_2)$  be given by A.2.*

(1) *If  $w \leq n$  then*

$$D(e_0, e_1, e_2) \text{ is of grade } r \geq \frac{2}{3}w.$$

(2) *If  $w > n$  then*

$$D(e_0, e_1, e_2) \text{ is of grade } r \geq n - \frac{w}{3}.$$

*Sketch of proof.* The detailed proof is in [34]. Just consider here the case when  $w \leq n$  with  $w = 3k - 1$ . Taking  $m_1 = m_2 = m_3 = m$  in lemma A.2 leads to a family  $\mathcal{F}$  whose molecules are of grade at least  $2k$ . We use the same kind of arguments for  $w = 3k + 2$  and  $w = 3k$ .  $\square$

A special case of A.3 is:

**Lemma A.4.** *Let  $D(e_0, e_1, e_2)$  be given by A.2 with  $e_i + e_j \leq n$  ( $i \neq j$ ). Suppose that*

$$e_0 \leq \frac{n}{2} \text{ and } e_1 + e_2 > \frac{e_0}{2},$$

then

$$D(e_0, e_1, e_2) \text{ is of grade } e_0 + 1,$$

unless  $e_0 = e_1 = e_2 = \frac{n}{2}$ .

## APPENDIX B. RELATIVELY COMPLETE FAMILIES OF A SINGLE BINARY FORM

We give her results about reduction of some families modulo an ideal. First of all:

**Lemma B.1.** *Let  $k$  be an integer such that  $2k \leq n$ ; then*

$$I_{2k-1} = I_{2k}.$$

*Proof.* On has to consider molecular covariants of grade  $2k - 1$ , that is molecular covariants containing

$$\mathbf{E} := \left( \mathbf{f}_\alpha \right) \xrightarrow{2k-1} \left( \mathbf{f}_\beta \right)$$

Then, such a molecular covariant is a molecular covariant in a transvectant  $(\mathbf{E}, \mathbf{E}')_r$  for some integer  $r$  and some molecular covariant  $\mathbf{E}'$ . By proposition 4.7,  $\mathbf{D}$  is a linear combination of

$$(\mathbf{E}, \mathbf{E}')_r \text{ and } (\overline{\mathbf{E}}^{\mu_1}, \overline{\mathbf{E}'}^{\mu_1})_r.$$

But, all symbols being equivalent, we know that  $\mathbf{E} = 0$  and each transvectant  $(\overline{\mathbf{E}}^{\mu_1}, \overline{\mathbf{E}'}^{\mu_1})_r$  are in  $I_{2k}$  by lemma 6.4.  $\square$

Every molecular covariant of grade 1 is thus in  $I_2$ , and then:

**Corollary B.2.** *The family  $A_0 := \{\mathbf{f}\}$  is relatively complete modulo  $I_2$*

The following lemma is about degree three molecular covariants, and is used in the following:

**Lemma B.3.** *Let  $V$  be a space of binary forms,  $\alpha, \beta$  and  $\gamma$  be three atoms of respective valence  $n, p, q$ . Let  $r$  be an integer such that  $r \leq \min(n, p, q)$ ; then*

$$\begin{array}{c} \left( \alpha \right) \xrightarrow{r} \left( \beta \right) \\ \left( \gamma \right) \end{array} = \sum_{i=0}^r \binom{r}{i} \begin{array}{c} \left( \alpha \right) \xrightarrow{i} \left( \gamma \right) \\ \left( \beta \right) \xrightarrow{r-i} \left( \gamma \right) \end{array} \quad (\text{B.1})$$

*Proof.* Starting with relation (3.2):

$$\Omega_{\alpha\beta}\sigma_\gamma = \Omega_{\alpha\gamma}\sigma_\beta + \Omega_{\gamma\beta}\sigma_\alpha,$$

we get

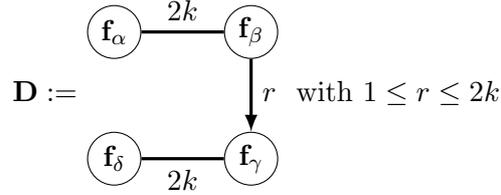
$$\Omega_{\alpha\beta}^r \sigma_\gamma^r = \sum_{i=0}^r \binom{r}{i} \Omega_{\alpha\gamma}^i \Omega_{\gamma\beta}^{r-i} \sigma_\beta^i \sigma_\gamma^{r-i},$$

and we just have to multiply each side of the equation by  $\sigma_\alpha^{n-r} \sigma_\beta^{p-r} \sigma_\gamma^{q-r}$ .  $\square$

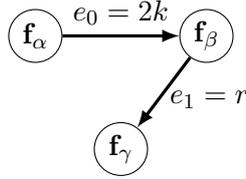
Recall here that, for  $\mathbf{f} \in S_n$ , for a given integer  $k \geq 0$ , we define  $\mathbf{H}_{2k} := (\mathbf{f}, \mathbf{f})_{2k}$ .

**Lemma B.4.** *If  $2n - 4k > n$ , then the  $\mathbf{H}_{2k}$  strictly greater than  $n$  and the family  $B = \{\mathbf{H}_{2k}\}$  is relatively complete modulo  $I_{2k+2}$*

*Proof.* We have to consider molecular covariants containing

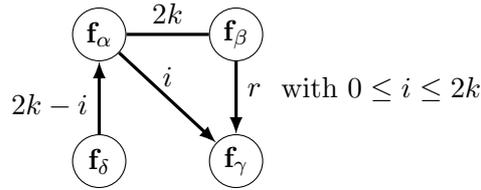


all symbol being equivalent. When  $r > k$ , the molecular covariant



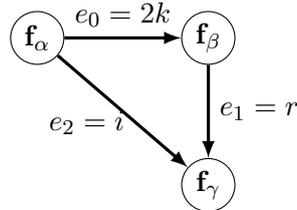
is of grade  $2k + 1$  by lemma A.4. Thus the molecular covariant associated to  $\mathbf{D}$  is in  $I_{2k+1} = I_{2k+2}$  (lemma B.1).

When  $r < k$ , by relation (B.1),  $\mathbf{D}$  decomposes as a linear combination of



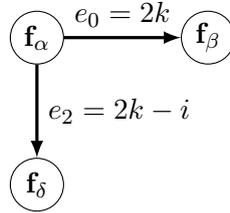
Now:

- If  $i \geq k$ , we consider the molecular covariant



of weight  $w = 2k + r + i \geq 3k + r > 3k$ . Since  $2k + r + i \leq n$ , this molecular covariant is of grade  $rgeq \frac{2}{3}w > 2k$  by lemma A.3;

- If  $i < k$ , we consider the molecular covariant

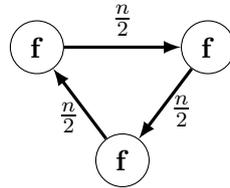


and we conclude by lemma A.4.

□

In the same way:

**Lemma B.5.** *If  $n = 4k$ , then  $\mathbf{H}_{2k}$  is of order  $n$  and the family  $\mathbf{B} = \{\mathbf{H}_{2k}\}$  is relatively complete modulo  $I_{2k+2} + \langle \mathbf{f}_\delta \rangle$  where  $\mathbf{f}_\delta$  is an invariant given by:*



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