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# ABOUT GORDAN’S ALGORITHM FOR BINARY FORMS

MARC OLIVE

ABSTRACT. In this paper, we present a modern version of Gordan’s algorithm on binary forms. Symbolic method is reinterpreted in terms of  $\mathrm{SL}_2(\mathbb{C})$ -equivariant homomorphisms defined upon Cayley operator and polarization process. A graphical approach is thus developed to obtain Gordan’s ideal, a central key to get covariant bases of binary forms. To illustrate the power of the method, we compute a covariant basis of  $S_6 \oplus S_2$  and  $S_8$ .

*Keywords* : Classical invariant theory ; Covariants ; Algorithm

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## 1. INTRODUCTION

*Classical invariant theory* was a very active research field throughout the XIX<sup>e</sup> century. As pointed out by Parshall [44], the birth of this field can be find in the *Disquisitiones arithmeticae* (1801) of Gauss. He studied there linear changes of variables in a quadratic form with integer coefficients. About forty years later, Boole [7] established the main purpose of what will become today *classical invariant theory*. Cayley [16, 17] deeply investigated this field of research and developed important tools still in use nowadays, such as the Cayley Omega process. During about fifteen years (until the 1861 and Cayley’s seventh memoir [14]) the English school of invariant theory, mainly led by Cayley and Sylvester, developed important tools to compute explicit *invariant generators* of binary forms. Thus, the role of calculation deeply influenced this first approach in invariant theory [16].

At that time, a German school mainly conducted by Clebsch, Aronhold and Gordan, developed their own approach, named the *symbolic method*. In 1868, Gordan, who was called the “King of invariant theory”, proved that *covariants* of any binary forms are always finitely generated [26]. As a great part of the mathematic development of that time, such a result was endowed with a constructive proof : the English and the German school were equally preoccupied by calculation and exhibition of invariants and covariants. Despite Gordan’s constructive proof, Cayley was reluctant to make use of the symbolic method to obtain new understanding of invariant theory. In the same spirit, Sylvester claimed that Gordan’s proof was “so long and complicated and so artificial a structure that it requires a very long study to master and there is not one person in Great Britain who has mastered it” [17]. That’s only in 1903, with the work of Grace-Young [28], that the German approach of Gordan and al. became accessible to a wide community of mathematicians. Let also point out that Gordan’s constructive approach led to several explicit results : first, and without no difficulty, Gordan [27] gave the quintic and the sextic

bases for covariants<sup>1</sup>, then he gave the first part of the septic and the octic covariant basis. After that, Von Gall finished the computation for the septic [54] and for the octic [24].

But, in 1890, Hilbert made a critical advance in the field of *invariant theory*. Using a totally new approach [31], which is the cornerstone of all nowadays abstract algebra, he proved the finiteness theorem for all cases dealing with invariants of a *reductive group*. But his first proof [31] was criticized for not being constructive. Facing these critics, Hilbert made another contribution [31] which claimed to be more constructive. This effective approach is nowadays widely used to obtain effective results in the field of invariant theory [46, 22, 11, 12]. As pointed out by Hilbert himself in [31], the main scope of this approach can be summarized into three steps.

The first step is to compute the *Hilbert series* of the graded algebra  $\mathcal{A}$  of invariants<sup>2</sup>. Of course, there exists several methods to compute *a priori* this Hilbert series [5, 38, 47] which is always a rational function by the Hilbert–Serre theorem [15]. The second step is to exhibit what is called a *system of parameters* for the algebra  $\mathcal{A}$  of invariants<sup>3</sup>. Finally, the Hochster–Roberts theorem [32] ensures us that the algebra  $\mathcal{A}$  is *Cohen–Macaulay*<sup>4</sup>. Thanks to that statement, the system of parameters altogether with the Hilbert series give a bound for the degree of invariants still have to be found. We refer the reader to several references [52, 11, 22, 19, 20, 21] to get a general and modern approach to this subject.

But one major lack of this strategy is summed up in the effective computation of a system of parameters. The Noether normalization lemma [36] ensures us that such a system always exists, but as we know, effective algorithms to get such a system [30] are not sufficiently effective because of the extensive use of Grobner bases. In the case of invariants algebra  $\mathbf{Inv}(S_n)$  of a single binary form, one has of course the concept of the *nullcone* and the Mumford–Hilbert criteria [20, 9], to check that a finite family of invariants is a system of parameters of  $\mathbf{Inv}(S_n)$ <sup>5</sup>. But this criteria is not an algorithm to get a system of parameters, and it is no more valid in the case of covariants. Furthermore, in the case of *joint invariants*, that is invariants algebra of  $V := S_{n_1} \oplus \cdots \oplus S_{n_k}$ , such a system of parameters has, in general, a complex shape. Indeed, Brion [10] showed that only in some very few cases, as for instance in the simple case of joint invariants of  $S_4 \oplus S_2$ , there exists a system of parameters which respects the multi-graduation of  $\mathbf{Inv}(V)$ .

Let's point out here that an important motivation for this work was to use an effective approach on invariant theory because we had, for example, to compute joints invariants of  $S_6 \oplus S_2$ . In fact, this motivation is directly taken from the field of *continuum mechanics*, and more precisely from the *theory of elasticity in small deformations* [1]. As an example, to get one part of the invariants basis of the *elasticity tensor* [3], Boehler–Kirilov–Onat used a classical isomorphism between  $\mathcal{SO}(3)$  linear representations over a complex vector space and the one of  $\mathrm{SL}(2, \mathbb{C})$  linear representations on binary forms [50, 6]. Doing so, they directly obtained the part of the invariant bases of the elasticity tensor related to the invariant bases of  $S_8$ , which was first obtained by Von Gall [24] in 1888. Such invariant bases has a direct application to classify orbits space of elasticity tensor, as pointed out by Auffray–Kolev–Petitot [2]. In a forthcoming article, though, we also present a new useful result for continuum mechanics [40], which was a direct consequence of results we obtain in our present paper for the case of joint covariants of  $S_6 \oplus S_2$ .

But we may also observe some other important interests on the subject which come from the field of geometrical arithmetic, illustrated by the work of Lercier–Ritzenthaler [37] on hyperelliptic curves, but also in the field of quantic informatics as illustrated by the work of Luque [39].

<sup>1</sup>The case of a binary quintics presented such a level of difficulty for the English school that Cayley conjectured an infinite number of invariant generators for a binary form of order greater than or equal to five [16].

<sup>2</sup>Writing  $\mathcal{A} = \bigoplus \mathcal{A}_i$  we define the Hilbert series to be the formal series  $H_{\mathcal{A}}(z) := \sum \dim \mathcal{A}_i z^i$ .

<sup>3</sup>The set  $\{\theta_1, \dots, \theta_s\} \subset \mathcal{A}$  is a system of parameters if  $\mathcal{A}$  is finitely generated over its subring  $k[\theta_1, \dots, \theta_s]$ .

<sup>4</sup>Meaning the algebra  $\mathcal{A}$  is a finite and free  $k[\theta_1, \dots, \theta_s]$ -module, where  $\{\theta_1, \dots, \theta_s\}$  is a system of parameters

<sup>5</sup>A set  $\{\theta_1, \dots, \theta_s\} \subset \mathbf{Inv}(S_n)$  is a system of parameters if  $\theta_1(\mathbf{f}) = \cdots = \theta_s(\mathbf{f}) = 0$  implies that  $\mathbf{f} \in S_n$  has a root which multiplicity is of order strictly greater than  $\frac{n}{2}$ .

Of course, the algebraical geometry approach first developed by Hilbert is not the only constructive one. In the case of a single binary form, Olver [42] exhibits another constructive approach, which was generalized for a single  $n$ -ary form and also specified with a “running bound” by Brini–Regonati–Creolis [8]. We also have in Kung–Rota [35] a constructive approach with a combinatorial which became increasingly complex for the cases we had to deal with.

Thus, as it appears to us in the case of joint covariants of  $S_6 \oplus S_2$ , a very simple result stated in Grace–Young (theorem 4.6 of our present paper) gave us a direct algorithm to obtain a covariant basis, although other approaches failed to do so. From this observation, we decided to reformulate Gordan’s theorem<sup>6</sup> on binary forms in the modern language of operators, which was already modernized by Olver [42]. We also decided to represent operators with *directed graphs*, in the spirit of the graphical approach dealt by Olver–Shakiban [41], and to focus on *equivariants morphisms*.

The paper is organized as follows. In section 2 we recall the mathematical background of classical invariant theory, and we introduce classical operators such as the Omega Cayley operator, polarization operators and the transvectant operator. We then introduce *Aronhold molecule* and *molecular covariants* which give graphical representations of equivariants morphisms constructed on the basis of Cayley and polarization operators. We prove Gordan’s theorem for joint covariants in section 4 and for simple covariants in section 5.

Finally, in Appendix A, we illustrate the method<sup>7</sup> by computing explicitly the basis of joint covariants of a sextic and a quadric, and of simple covariants of an octic. This result was already obtained by Von Gall [54], Lercier–Ritzenthaler [37], Cröni [18] and Bedratyuk [4], but the computation is summarized and simplified here.

## 2. COVARIANTS OF BINARY FORMS

Let’s take  $\mathbf{x}$  to be a couple  $(x, y) \in \mathbb{C}^2$ ; we define:

**Definition 2.1.** The  $\mathbb{C}$  vector space of  $n$ th degree binary forms, noted  $S_n$  is the space of homogeneous polynomials

$$\mathbf{f}(\mathbf{x}) = a_0x^n + \binom{n}{1}a_1x^{n-1}y + \cdots + \binom{n}{n-1}a_{n-1}xy^{n-1} + a_ny^n$$

with each  $a_i$  in  $\mathbb{C}$ .

Now we can take  $V$  to be a space of binary forms, that is

$$V := \bigoplus_{i=0}^s S_{n_i}$$

There is a natural  $SL_2(\mathbb{C})$  action on  $\mathbb{C}^2$  and thus on  $V$ , given by

$$(g \cdot \mathbf{f})(\mathbf{x}) := \mathbf{f}(g^{-1} \cdot \mathbf{x}) \text{ for } g \in GL_2(\mathbb{C}) \text{ or } g \in SL_2(\mathbb{C})$$

From this, we naturally define an action<sup>8</sup> on the ring coordinate  $\mathbb{C}[V \oplus \mathbb{C}^2]$ : for  $p \in \mathbb{C}[V \oplus \mathbb{C}^2]$  we define the action to be

$$(g \cdot p)(\mathbf{f}, \mathbf{x}) := p(g^{-1} \cdot \mathbf{f}, g^{-1} \cdot \mathbf{x}) \text{ for } g \in SL_2(\mathbb{C})$$

Thus, all this lead to the classical definition of the covariant ring of binary form

**Definition 2.2.** The covariant algebra of a space  $V$  of binary forms, noted  $C(V)$ , is the algebra of  $SL_2(\mathbb{C})$  polynomial invariant:

$$C(V) := \mathbb{C}[V \oplus \mathbb{C}^2]^{SL_2(\mathbb{C})}$$

A very important result, first due to Gordan [26] and then generalized by Hilbert [31] is:

<sup>6</sup>Remark also that Weynman [56] did an algebra formulation of Gordan’s theorem.

<sup>7</sup>Pasechnik [45] did also an application of this method.

<sup>8</sup>for a general and modern approach of invariants and covariants algebra we refer the reader to the online text of Procesi–Kraft [34]

**Theorem 2.3.** *For every space  $V$  of binary forms, the algebra  $C(V)$  is finitely generated, meaning there exist a finite set  $\mathbf{h}_1, \dots, \mathbf{h}_N$  in  $C(V)$ , called a basis, such that*

$$C(V) = \mathbb{C}[\mathbf{h}_1, \dots, \mathbf{h}_N]$$

We can also attempt to obtain a *minimal* basis [23]. Let's define the subspace  $C_i \subset C(V)$  of  $i$ th degree homogeneous polynomials, and the ideal  $C^+ := \sum_{i>0} C_i$  of the graduated algebra  $C(V)$ . Then we can consider for each  $C_i$  the number  $\delta_i$  to be the cardinal of a supplement to  $(C^+)_i^2 \subset C_i$  in  $C_i$ . Now because of the finiteness, there exist  $k$  such that  $\delta_i = 0$  for  $i \geq k$ ; and we can finally define the invariant number:

$$n(V) = \sum_i \delta_i$$

Now:

**Definition 2.4.** A set  $\mathbf{h}_1, \dots, \mathbf{h}_N$  is a *minimal* basis of  $C(V)$  if their image in the vector space  $C^+/(C^+)^2$  is a basis. In that case we will have  $N = n(V)$

An important observation is that we have a natural bi-graduation on the covariant algebra  $C(V)$ :

- By the **degree**, which is the polynomial degree in the coefficients of the space  $V$  ;
- By the **order** which is the polynomial degree in the variables  $\mathbf{x}$  ;

If then we put  $C_{k,r}(V)$  to be the subspace of  $k$ th degree and  $r$ th order covariants, we thus have:

$$C(V) = \bigoplus_{k \geq 0, r \geq 0} C_{k,r}(V) \quad (2.1)$$

A first way to obtain covariant is to make use of *Cayley's operator* [42], which is a bi-differential operator acting on a tensor product of smooth functions  $\mathbf{f}(\mathbf{x}_\alpha)\mathbf{g}(\mathbf{x}_\beta)$ , and which is given by

$$\Omega_{\alpha\beta}(\mathbf{f}(\mathbf{x}_\alpha)\mathbf{g}(\mathbf{x}_\beta)) := \frac{\partial \mathbf{f}}{\partial x_\alpha} \frac{\partial \mathbf{g}}{\partial y_\beta} - \frac{\partial \mathbf{f}}{\partial y_\alpha} \frac{\partial \mathbf{g}}{\partial x_\beta}$$

We will also make use of the *polarization process*<sup>9</sup>, defined to be

$$\sigma_\alpha := x \frac{\partial}{\partial x_\alpha} + y \frac{\partial}{\partial y_\alpha}$$

Cayley's operator and polarization operator commute with  $SL_2(\mathbb{C})$  action [42]. We then naturally get, with these operators, covariants of a binary form. In fact, as we will see further on, these operators suffice to get all covariants (see theorem 2.10).

Using Cayley's operator, we can now obtain *transvectant* operation, defined to be:

$$\{\mathbf{f}, \mathbf{g}\}_r := \Omega^r \sigma_\alpha^{n-r} \sigma_\beta^{p-r} (\mathbf{f}_\alpha \mathbf{g}_\beta)$$

The classical approach, here, is to give invariants or covariants bases using transvectant operators. For instance, the covariant is of a cubic  $\mathbf{f} \in S_3$  is given by table 1.

Order/degree	1	2	3	4
3	$\mathbf{f}$			
2		$\mathbf{H} := \{\mathbf{f}, \mathbf{f}\}_2$	$\mathbf{T} := \{\mathbf{f}, \mathbf{H}\}_1$	
0				$\{\mathbf{H}, \mathbf{H}\}_2$

TABLE 1. Covariant basis of a binary cubics given in terms of transvectant

*Remark 2.5.* Gordan's proof of finiteness for binary forms, proof that can be find in Grace–Young [28], used what 19th century mathematicians called the symbolic method. As pointed out by Olver [42], differential operators naturally translate into symbolic forms<sup>10</sup>.

<sup>9</sup>This operator was named *scalling process* by Olver [42]

<sup>10</sup>A huge amount of work as been done first by Weyl [55] and afterword by Kung–Rota [35] to get a modern version of this symbolic method, and these leads for example to Umbral calculus.

We now define  $\text{Sym}^k(V)$  to be the space of totally symmetric tensor subspace of  $\otimes^k V$ .

*Remark 2.6.* We have a natural isomorphism between  $C_{k,r}(V)$  and the space  $\text{Hom}_{\text{SL}_2(\mathbb{C})}(\text{Sym}^k(V), S_r)$ .

This isomorphism is a simple trace operation. Indeed, if we take an equivariant morphism  $\varphi \in \text{Hom}_{\text{SL}_2(\mathbb{C})}(\text{Sym}^k(V), S_r)$  we just have to take the covariant  $p(\mathbf{f}, \mathbf{x}) = \varphi(\mathbf{f}(\mathbf{x}), \dots, \mathbf{f}(\mathbf{x}))$ .

Cayley's operator and polarization process carrying us to a natural way to construct  $\text{SL}_2(\mathbb{C})$  equivariant homomorphism from  $S_{n_1} \otimes S_{n_2} \otimes \dots \otimes S_{n_s}$  to  $S_r$ . For instance, we can construct the morphism:

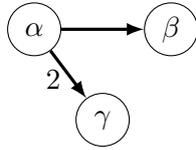
$$\Omega_{\alpha\beta}\Omega_{\alpha\gamma}^2\sigma_{\alpha}^{n-3}\sigma_{\beta}^{p-1}\sigma_{\gamma}^{q-2} : S_n \otimes S_p \otimes S_q \longrightarrow S_r \text{ with } r = n + p + q - 6 \quad (2.2)$$

Such an equivariant morphism will be represented by a digraph [53, 33, 43]. We start with atoms



associated to valences  $\text{val}(\alpha) = n$ ,  $\text{val}(\beta) = p$ ,  $\text{val}(\gamma) = q$ .

Thus we represent the  $\text{SL}_2(\mathbb{C})$  equivariant morphism with the digraph<sup>11</sup>



in which  $\text{val}(\alpha) = n - 3$ ,  $\text{val}(\beta) = p - 1$ ,  $\text{val}(\gamma) = q - 2$

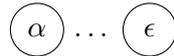
Thus a directed and weighted edge, with weight  $r$ , from two given atoms  $\alpha$  and  $\beta$  will represent the operator  $\Omega_{\alpha\beta}^r$ . Finally we use polarization operator related to atom's valence to get a morphism ; for instance

$$\left( \alpha \xrightarrow{r} \beta \right) \text{ will represent the } \text{SL}_2(\mathbb{C}) \text{ equivariant morphism } \Omega_{\alpha\beta}\sigma_{\alpha}^{n-1}\sigma_{\beta}^{q-1}$$

Is this above example, we have  $\text{val}(\alpha) = n - 1$ .

Following these ideas, we can now construct a more general object on the space  $V = \bigoplus_{i=1}^s S_{n_i}$  of binary forms. When given a digraph  $D$ , its set of vertices will be denoted by  $\mathcal{V}(D)$ , its set of (oriented) edges by  $\mathcal{E}(D)$ . Given an (oriented) edge  $e$  we denote its origin by  $o(e)$  and its termination by  $t(e)$ .

**Definition 2.7.** Let  $\alpha, \beta, \dots, \epsilon$  be symbols associated to orders  $n_{i_\alpha}, \dots, n_{i_\epsilon}$  ; an *Aronhold molecule*  $D$  is a digraph constructed on atoms



which represent a  $\text{SL}_2(\mathbb{C})$  equivariant morphism

$$\tau_D := \prod_{e \in \mathcal{E}(D)} \Omega_{o(e)t(e)}^{w(e)} \prod_{v \in \mathcal{V}(D)} \sigma_v^{\text{val}(v)}$$

from  $S_{n_{i_\alpha}} \otimes \dots \otimes S_{n_{i_\epsilon}}$  to  $S_r$ , with  $r = \text{val}(\alpha) + \dots + \text{val}(\epsilon)$ . The set of all Aronhold molecule will be noted  $\mathfrak{M}(V)$  and the vector space generated by all Aronhold molecules, will be noted  $\mathfrak{A}(V)$ .

Taking  $\mathbf{f}(v) \in V$  for each vertex  $v \in \mathcal{V}(D)$ , we can thus define a covariant in  $C(V)$  taking

$$\tau_D \left( \bigotimes_{v \in \mathcal{V}(D)} \mathbf{f}(v) \right)$$

This define a map  $\Psi$  from  $\mathfrak{A}(V)$  to  $C(V)$ . Now:

**Definition 2.8.** For every space  $V$  of binary forms, we define a *molecular covariant*  $\mathbf{M}$  to be a covariant given by  $\mathbf{M} = \Psi(D)$  where  $D \in \mathfrak{M}(V)$ .

Each molecular covariant will also be represented using a digraph. For instance, the basis of a binary cubic is given in figure 1.

<sup>11</sup>It is very important to note that we represent here a morphism and not a bi-differential operator as did Olver-Shakiban [43]

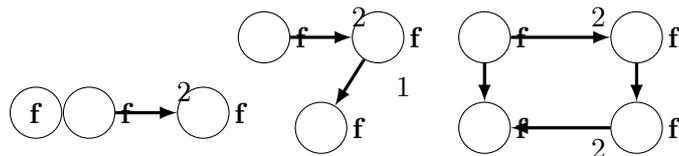


FIGURE 1. Covariant basis of a binary cubic given in molecular form

In fact, we have a relation between covariants given in transvectant form and the ones given in molecular form (see [section 3](#)).

When given an Aronhold molecule  $D \in \mathfrak{A}(V)$ , we define  $w(D)$  to be the *weight* of the weighted digraph  $D$ . We also define the grade  $\text{gr}(D)$  of  $D$  to be the maximal weight of  $D$ .

**Definition 2.9.** For a given integer  $r$ , we define  $\mathfrak{A}_r(V)$  to be the vector subspace of  $\mathfrak{A}(V)$  generated by all Aronhold molecule  $D$  such that  $\text{gr}(D) \geq r$ .

Now, if we take a given space  $V$  of binary forms, we can define  $\mathcal{M}(V)$  to be the algebra generated by all molecular covariants  $\Psi(\mathfrak{M}(V))$ . We then have a very important result, which non trivial proof can be found for example in Olver [\[42\]](#):

**Theorem 2.10.** *Every covariant of a given space of binary forms  $V$  is a polynomial in molecular covariants ; that is:*

$$C(V) = \mathcal{M}(V)$$

For nineteenth century mathematicians, this result was the fact that every covariant may be expressible as a polynomial in symbolic forms.

Nevertheless, this result doesn't assure us that every covariant of a given space  $V$  can be written with transvectants operations. To get this result, one must make use of relations between transvectant covariants and molecular covariants: such a relation is given in Olver [\[42\]](#), but we also give such a result in property [3.5](#).

When we want to express covariants as molecular covariants, we don't have a unique expression. Indeed, (see Olver [\[42\]](#) and Olver–Shakiban [\[43\]](#)) we have fundamentals relations, called syzygies, among operators and thus among Aronhold molecule and also among molecular covariants. Take  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\delta$  be four symbols associated to valence  $n_1$ ,  $n_2$ ,  $n_3$  and  $n_4$ .

- (1) The first syzygie comes from the equality:

$$\Omega_{\alpha\beta}\sigma_{\alpha}^{n_1-1}\sigma_{\beta}^{n_2-1} = -\Omega_{\beta\alpha}\sigma_{\alpha}^{n_1-1}\sigma_{\beta}^{n_2-1}$$

which gives, in graphical forms:

$$\begin{array}{c} \alpha \longrightarrow \beta \\ \hline \end{array} = - \begin{array}{c} \alpha \longleftarrow \beta \\ \hline \end{array} \quad (2.3)$$

- (2) The second one, comes from a determinantal property [\[42\]](#):

$$\Omega_{\alpha\beta}\sigma_{\alpha}^{n_1-1}\sigma_{\beta}^{n_2-1}\sigma_{\gamma}^{n_3} = \Omega_{\alpha\gamma}\sigma_{\alpha}^{n_1-1}\sigma_{\beta}^{n_2-1}\sigma_{\gamma}^{n_3} + \Omega_{\gamma\beta}\sigma_{\alpha}^{n_1}\sigma_{\beta}^{n_2-1}\sigma_{\gamma}^{n_3-1}$$

which gives, in graphical forms:

$$\begin{array}{c} \alpha \longrightarrow \beta \\ \hline \gamma \end{array} = \begin{array}{c} \alpha \\ \searrow \\ \gamma \end{array} \begin{array}{c} \beta \end{array} + \begin{array}{c} \alpha \\ \nearrow \\ \gamma \end{array} \begin{array}{c} \beta \end{array} \quad (2.4)$$

- (3) The last one is a peculiar case of the previous one.

$$\Omega_{\alpha\beta}\Omega_{\gamma\delta}\sigma_{\alpha}^{n_1-1}\sigma_{\beta}^{n_2-1}\sigma_{\gamma}^{n_3-1}\sigma_{\delta}^{n_4-1} = \Omega_{\alpha\delta}\Omega_{\beta\gamma}\sigma_{\alpha}^{n_1-1}\sigma_{\beta}^{n_2-1}\sigma_{\gamma}^{n_3-1}\sigma_{\delta}^{n_4-1} + \Omega_{\alpha\gamma}\Omega_{\delta\beta}\sigma_{\alpha}^{n_1-1}\sigma_{\beta}^{n_2-1}\sigma_{\gamma}^{n_3-1}\sigma_{\delta}^{n_4-1}$$

which gives, in graphical forms:

$$\begin{array}{c}
 \begin{array}{ccc}
 \alpha & \xrightarrow{\quad} & \beta \\
 \delta & \xrightarrow{\quad} & \gamma
 \end{array} \\
 = \\
 \begin{array}{ccc}
 \alpha & & \beta \\
 \downarrow & & \downarrow \\
 \delta & & \gamma
 \end{array}
 + \\
 \begin{array}{ccc}
 \alpha & & \beta \\
 \swarrow & & \searrow \\
 \delta & & \gamma
 \end{array}
 \end{array}
 \tag{2.5}$$

One may observe that these syzygies are in fact rewriting rules for molecular covariants. For example, by 2.3 we will have

$$\begin{array}{c}
 \alpha \xrightarrow{2} \beta \\
 \alpha \xleftarrow{2} \beta \\
 \alpha \xrightarrow{2} \beta
 \end{array}
 = 0$$

thus, for an even number of edges, we will not precise the direction.

Another important observation is that the syzygies 2.4 and 2.5 leads to a huge amount of relations among molecular covariants.

As an example, let's now<sup>12</sup> take the space  $V = S_n$  the syzygies 2.4 and 2.3 will give us

$$\begin{array}{c}
 \begin{array}{ccc}
 \alpha & \xrightarrow{2} & \beta \\
 & \searrow & \swarrow \\
 & \gamma & \\
 \end{array} \\
 = \\
 \begin{array}{ccc}
 \alpha & \xrightarrow{\quad} & \beta \\
 \swarrow & & \searrow \\
 \gamma & & \gamma
 \end{array}
 + \\
 \begin{array}{ccc}
 \alpha & \xrightarrow{\quad} & \beta \\
 \swarrow & & \searrow \\
 \gamma & & \gamma
 \end{array}
 \\
 = - \\
 \begin{array}{ccc}
 \alpha & \xrightarrow{\quad} & \beta \\
 \swarrow & & \searrow \\
 \gamma & & \gamma
 \end{array}
 + \\
 \begin{array}{ccc}
 \alpha & \xrightarrow{\quad} & \beta \\
 \swarrow & & \searrow \\
 \gamma & & \gamma
 \end{array}
 \end{array}$$

and finally, in the case we are in  $\text{Sym}^3(V)$ , all symbols are equivalent, so

$$\begin{array}{ccc}
 \mathbf{f} & \xrightarrow{2} & \mathbf{f} \\
 & \searrow & \swarrow \\
 & \mathbf{f} & \\
 \end{array}
 = 0$$

Because Cayley's operator and polarization operator commutes, we will have other important relations. One of them is simply an application of the binomial formula:

$$\begin{array}{ccc}
 \alpha & \xrightarrow{r} & \beta \\
 & & \gamma
 \end{array}
 = \sum_{i=0}^r \binom{r}{i} \begin{array}{ccc} \alpha & \xrightarrow{i} & \beta \\ & \searrow & \swarrow \\ & \gamma & \end{array} \xrightarrow{r-i}
 \tag{2.6}$$

Now we can get, with fine enough computations [28] the following relation, obtained by Stroh [51], which can be directly applied to operators Cayley's operators and polarization operators which all commute:

**Lemma 2.11.** *Let  $u_1, u_2$  and  $u_3$  be three commutative variables such that*

$$u_1 + u_2 + u_3 = 0$$

*Then we have*

$$\begin{aligned}
 (-1)^{k_2} \sum_{i=0}^{k_1} \binom{g}{i} \binom{k_1 + k_3 - i}{k_3} u_3^{g-i} u_1^i + (-1)^{k_3} \sum_{i=0}^{k_2} \binom{g}{i} \binom{k_2 + k_1 - i}{k_1} u_1^{g-i} u_2^i + \\
 (-1)^{k_1} \sum_{i=0}^{k_3} \binom{g}{i} \binom{k_3 + k_2 - i}{k_2} u_2^{g-i} u_3^i = 0
 \end{aligned}
 \tag{2.7}$$

<sup>12</sup>This example is directly taken from [42, 43]

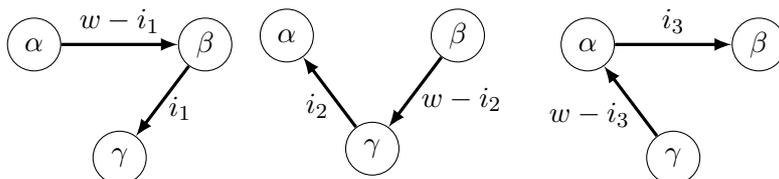
with  $k_1 + k_2 + k_3 = g - 1$ .

Take here a degree three Aronhold molecule with  $V = S_n$ , given by:

$$D(e_0, e_1, e_2) := \begin{array}{c} \alpha \xrightarrow{e_0} \beta \\ \swarrow \quad \searrow \\ \gamma \end{array} \quad \text{with weight } w = e_0 + e_1 + e_2 \quad (2.8)$$

We then have an important lemma, which proof can be found in Grace–Young [28]:

**Corollary 2.12.** *If  $w \leq n$  and  $m_1, m_2, m_3$  are integers such that  $m_1 + m_2 + m_3 = w + 1$  then the Aronhold molecule  $D(e_0, e_1, e_2)$  is a linear combination of the Aronhold molecule.*



with  $i_s = 0 \dots m_s$ .

From this we deduce two very important lemmas:

**Corollary 2.13.** *Let  $D(e_0, e_1, e_2)$  be given by 2.8.*

(1) *If  $w \leq n$  then*

$$D(e_0, e_1, e_2) \in \mathfrak{A}_r(V) \text{ with } r \geq \frac{2}{3}w$$

(2) *If  $w > n$  then*

$$D(e_0, e_1, e_2) \in \mathfrak{A}_r(V)$$

**Corollary 2.14.** *Let  $D(e_0, e_1, e_2)$  be given by 2.8 of grade  $e_0$  and suppose that*

$$e_0 \leq \frac{n}{2} \text{ and } e_1 + e_2 > \frac{e_0}{2}$$

then

$$D(e_0, e_1, e_2) \in \mathfrak{A}_{e_0+1}(V)$$

unless  $e_0 = e_1 = e_2 = \frac{n}{2}$ .

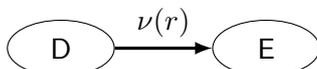
One may remark that these relations are upon morphism, thus these lemma give new syzygies among molecular covariants.

### 3. TRANSVECTANTS AND MOLECULAR COVARIANTS

It's important here to understand the way transvectants and molecular covariants are linked. To get molecular covariants when given a transvectant is the easiest way: it is a direct consequence of Leibnitz formula for derivatives.

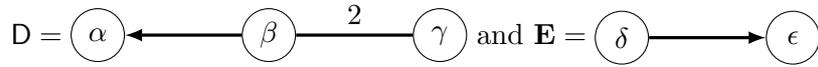
Because molecular covariants come from Aronhold molecule, we will give in fact relations between transvectant and Aronhold molecule. Transvectants can be seen as  $SL_2(\mathbb{C})$  equivariant morphisms ; using composition, we thus can make transvectants of Aronhold molecule.

**Definition 3.1.** If  $D$  and  $E$  are two Aronhold molecules, for a given integer  $r$  and a given symbol  $\nu(r)$ , we define the Aronhold molecule  $\mathcal{L}^{\nu(r)}(D, E)$ , graphically noted

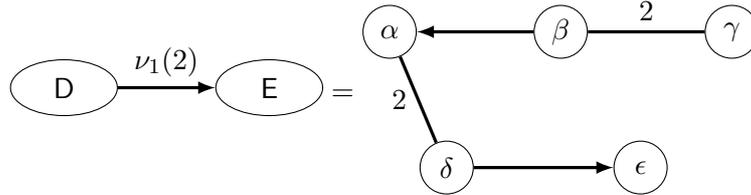


to be a new Aronhold molecule constructed by linking  $D$  and  $E$  with  $r$  edges in a given way.

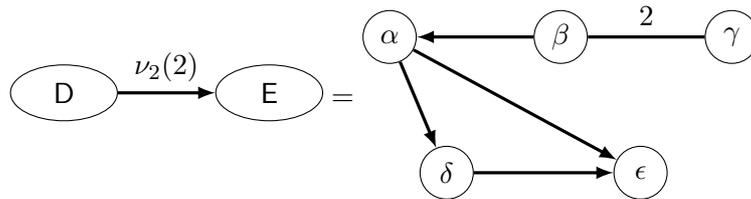
If we take for example



we can define



or



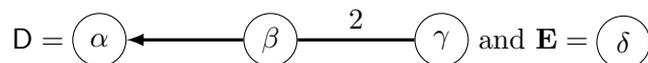
We then get a property which proof can be find in [42]:

**Proposition 3.2.** *If D and E are two Aronhold molecules, for every integer r, the r<sup>th</sup> transvectant {D, E}<sub>r</sub> can be obtain as a linear combination of Aronhold molecules  $\mathcal{L}_r^\nu(D, E)$ , for each possible link  $\nu(r)$  between D and E:*

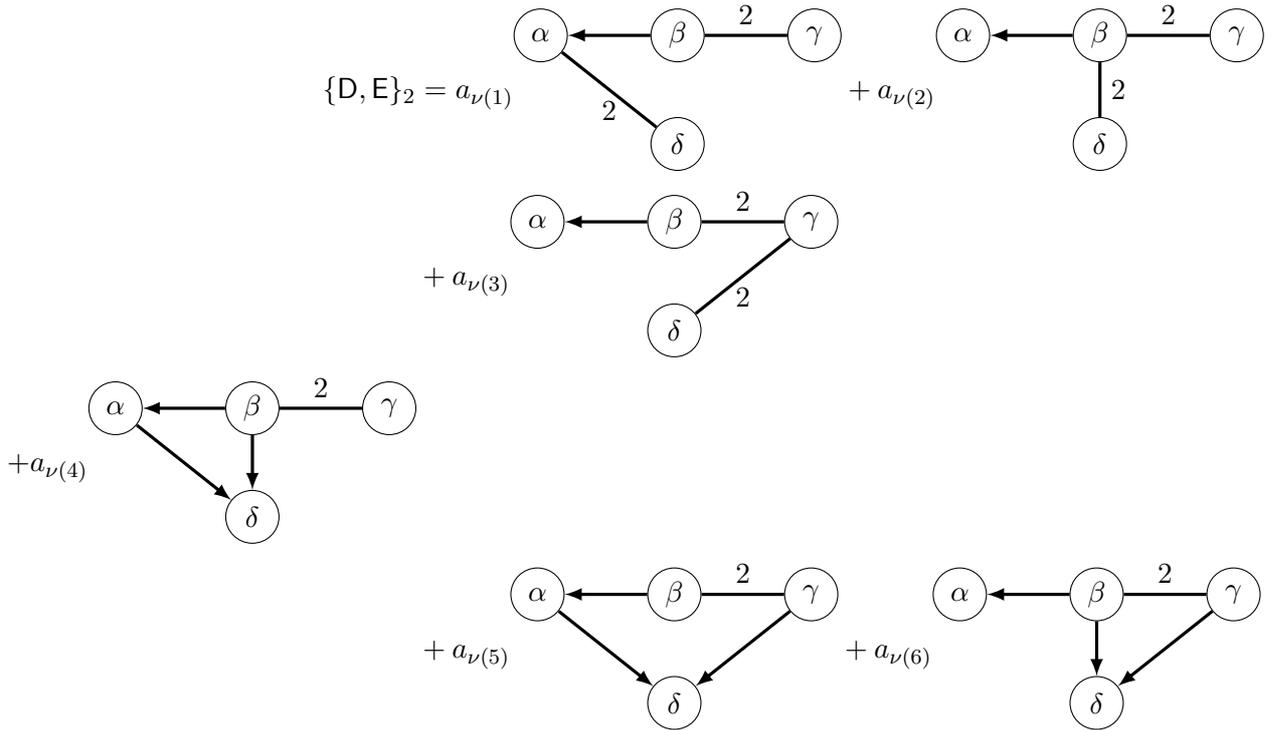
$$\{D, E\}_r = \sum_{\nu(r)} a_{\nu(r)} D \xrightarrow{\nu(r)} E$$

Because of Aronhold molecule's definition, which differ from Olver–Shakiban's molecular definition, the coefficients are not as simple as the ones given in Olver [42]. In fact, we won't have to use exact expression of these coefficients.

As an example, we can take



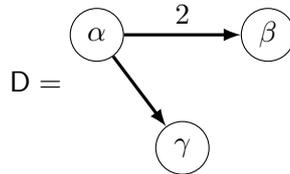
We will thus have:



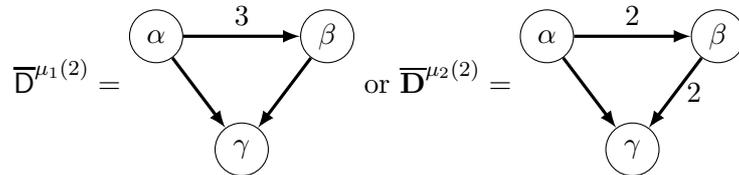
For the opposite link, that is the link between an Aronhold molecule and transvectants, we will make use of another molecular operation<sup>13</sup>:

**Definition 3.3.** Given an Aronhold molecule  $D$ , and an integer  $k$ , we define  $\overline{D}^{\mu(k)}$  as the Aronhold molecule obtained by adding  $k$  edges on  $D$  in a certain way  $\mu(k)$ .

For example, we can take the Aronhold molecule

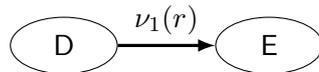


and then consider

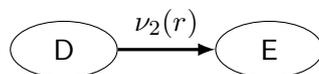


The proofs of the following two propositions will be omitted. They can be found in Olver [42]:

**Proposition 3.4.** Let be given two Aronhold molecules  $D$  and  $E$ , an integer  $r$  and two links  $\nu_1(r)$  and  $\nu_2(r)$  in the transvectant  $\{D, E\}_r$ , then the molecular transvectant



is a linear combination of



<sup>13</sup>This operation was called *convolution* in [28]

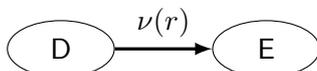
and transvectants

$$\{\bar{D}^{\mu_1(k_1)}, \bar{E}^{\mu_2(k_2)}\}_{r'}$$

with  $k_1 + k_2 + r' = r$  being constant and  $r' < r$ .

Furthermore we also have:

**Proposition 3.5.** *Let be given two Aronhold molecules D and E, an integer r and a link  $\nu(r)$  in the transvectant  $\{D, E\}_r$ , then the Aronhold molecule*

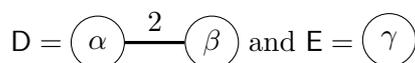


is a linear combination of the transvectants

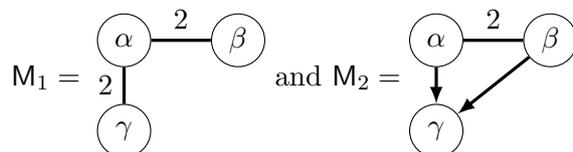
$$\{D, E\}_r \text{ and } \{\bar{D}^{\mu_1(k_1)}, \bar{E}^{\mu_2(k_2)}\}_{r'}$$

with  $k_1 + k_2 + r' = r$  being constant and  $r' < r$ .

If we take for example the Aronhold molecules:



we can consider the transvectant  $\{D, E\}_2$  and the two Aronhold molecules:



Then, property 3.4 assures us that

$$M_1 = \lambda_1 M_2 + \lambda_2 \left\{ \alpha \overset{3}{\text{---}} \beta, \gamma \right\}_1 + \lambda_3 \left\{ \alpha \overset{4}{\text{---}} \beta, \gamma \right\}_0$$

Furthermore property 3.5 assures us that

$$M_1 = \mu_1 \left\{ \alpha \overset{2}{\text{---}} \beta, \gamma \right\}_2 + \mu_2 \left\{ \alpha \overset{3}{\text{---}} \beta, \beta \right\}_1 + \mu_3 \left\{ \alpha \overset{4}{\text{---}} \beta, \gamma \right\}_0$$

all coefficients depending on the valences degrees of the atoms  $\alpha, \beta$  and  $\gamma$ .

#### 4. GORDAN'S ALGORITHM FOR JOINT COVARIANTS

Let recall here that Gordan's proof is constructive and was given in the original paper of Gordan [26] ; but, what follows is directly inspired by the proof given in Grace-Young [28] and, in a very much similar form, in a book by Glenn [25].

The idea is to argue on molecular covariants.

Let's take A to be a **covariant** family taken from a space V of binary forms:

$$A \subset C(V)$$

Now, we define **Cov**(A) to be the covariants algebra taken from A, which can be obtained by doing all possible transvectants<sup>14</sup> from elements of A. We know by theorem 2.10 and proposition 3.5 that **Cov**(V) is nothing else than the covariant algebra C(V).

First of all it is clear that

$$A \subset B \Rightarrow \mathbf{Cov}(A) \subset \mathbf{Cov}(B) \tag{4.1}$$

Then we have a direct lemma, consequence of theorem 2.10:

**Lemma 4.1.** *Let  $V = S_n$  and  $\mathbf{f} \in V$ . If any family  $A \subset \mathbf{Cov}(V)$  contains  $\mathbf{f}$  then  $\mathbf{Cov}(A) = \mathbf{Cov}(V)$ .*

Furthermore, using (4.1) we get the following lemma:

<sup>14</sup>We can also take all possible molecular covariants.

**Lemma 4.2.** *Let  $A_1$  and  $A_2$  be two families of  $\mathbf{Cov}(V)$ . If  $A_1 \subset A_2 \subset \mathbf{Cov}(A_1)$  then  $\mathbf{Cov}(A_1) = \mathbf{Cov}(A_2)$*

Now there is an important definition:

**Definition 4.3.** A covariants family  $A$  of  $V$  is said to be *complete* if it generates its covariant algebra  $\mathbf{Cov}(A)$ ; that is

$$\mathbb{C}[A] = \mathbf{Cov}(A)$$

It is important to notice that the notion of complete family is weaker than the one of a covariant basis<sup>15</sup>. For instance, let us take  $V = S_3$  and  $\mathbf{f} \in V$  to be a cubic. We define

$$\mathbf{H} := \{\mathbf{f}, \mathbf{f}\}_2; \mathbf{T} := \{\mathbf{f}, \mathbf{H}\}_1 \text{ and } \Delta := \{\mathbf{H}, \mathbf{H}\}_2$$

We then know that the family  $A_1 = \{\mathbf{f}, \mathbf{H}, \mathbf{T}, \Delta\}$  is a covariant basis of  $\mathbf{Cov}(A_1) = \mathbb{C}(S_3)$ . Now if we take

$$A_2 = \{\mathbf{H}, \Delta\} \text{ we will have } \mathbf{Cov}(A_2) \subsetneq \mathbf{Cov}(V)$$

But we also observe that  $A_2$  is exactly the covariant basis [28] of the quadratic form  $\mathbf{H} \in S_2$ ; thus  $A_2$  is a complete family but is not a covariant basis of  $\mathbf{Cov}(V)$ .

Let now take two finite covariant families  $A$  and  $B$ :

$$A := \{\mathbf{f}_1, \dots, \mathbf{f}_p\}; B := \{\mathbf{g}_1, \dots, \mathbf{g}_q\}$$

We define  $a_i$  (resp.  $b_j$ ) to be the order of  $\mathbf{f}_i$  (resp.  $\mathbf{g}_j$ ). If we put  $U$  (resp.  $V$ ) to be a monomial in  $k[A]$  (resp.  $k[B]$ ) we will write

$$U := \mathbf{f}_1^{\alpha_1} \dots \mathbf{f}_p^{\alpha_p}; V := \mathbf{g}_1^{\beta_1} \dots \mathbf{g}_q^{\beta_q}$$

We will also write  $\alpha := (\alpha_1, \dots, \alpha_p) \in \mathbb{N}^p$  and  $\beta := (\beta_1, \dots, \beta_q) \in \mathbb{N}^q$ .

To each non nul transvectant

$$\{\mathbf{U}, \mathbf{V}\}_r$$

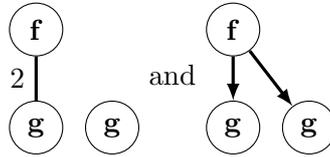
we can associate a non nul integer solution  $\kappa := (\alpha, \beta, u, v, r)$  taking from the system of linear diophantine equation:

$$(S) \quad \begin{cases} a_1\alpha_1 + \dots + a_p\alpha_p = u + r, \\ b_1\beta_1 + \dots + b_q\beta_q = v + r, \end{cases} \quad (4.2)$$

Now, it is clear that reciprocally, to each non null integer solution  $\kappa$  of  $(S)$  we can associate a non null transvectant  $\{\mathbf{U}, \mathbf{V}\}_r$ . For each solution  $\kappa$ , let  $\mathcal{F}(\kappa)$  be the finite family of all molecular covariants occurring in the molecular decomposition of the transvectant  $\{\mathbf{U}, \mathbf{V}\}_r$ , directly taken from proposition 3.2.

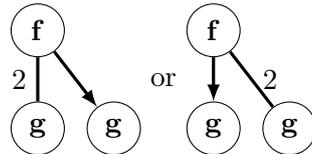
Let's take for example the case when  $A = \{\mathbf{f}\}$ , with  $\mathbf{f} \in S_5$  and  $B = \{\mathbf{g}\}$ , with  $\mathbf{g} \in S_2$ . For  $\mathbf{a} = (1)$  and  $\mathbf{b} = (1, 1)$  we can consider

- The family  $\mathcal{F}(\mathbf{a}, \mathbf{b}, 2)$  which contains the molecular covariants



where the first one is a non connected molecular covariant.

- The family  $\mathcal{F}(\mathbf{a}, \mathbf{b}, 3)$  which contains the molecular covariants



In this case there is no non connected molecular covariant.

If fact we have:

<sup>15</sup>All examples are directly taken from Grace-Young [28]

**Lemma 4.4.** *If  $\kappa$  is a reducible integer solution of (S), then  $\mathcal{F}(\kappa)$  contains a non connected molecular covariant.*

*Proof.* Take the integer solution  $\kappa = (\mathbf{a}, \mathbf{b}, u, v, r)$  to be reducible, that is  $\kappa = \kappa_1 + \kappa_2$  with

$$\kappa_i = (\mathbf{a}^i, \mathbf{b}^i, u^i, v^i, r^i) \text{ solution of (4.2)}$$

Thus we will be able to write  $\mathbf{U} = \mathbf{U}_1 \mathbf{U}_2$  and  $\mathbf{V} = \mathbf{V}_1 \mathbf{V}_2$ . Now there exist  $\nu(r), \nu_1(r^1)$  and  $\nu_2(r^2)$  such that

$$\begin{array}{ccccccc} \textcircled{\mathbf{U}} & \xrightarrow{\nu(r)} & \textcircled{\mathbf{V}} & = & \textcircled{\mathbf{U}_1} & \xrightarrow{\nu_1(r^1)} & \textcircled{\mathbf{V}_1} & \textcircled{\mathbf{U}_2} & \xrightarrow{\nu_2(r^2)} & \textcircled{\mathbf{V}_2} \end{array} \quad (4.3)$$

which is a non connected covariant molecular occurring in  $\mathcal{F}(\kappa)$ .  $\square$

Now we know that there exists a finite family of irreducible integer solutions of (4.2) (see [49, 48, 52] for details). Let then define  $\kappa^1, \dots, \kappa^l$  to be the irreducible integer solutions of (4.2). We also define  $\tau^i$  to be the transvectant associated to the solution  $\kappa^i$ . We thus get a main result [26, 28]:

**Theorem 4.5.** *Let  $V_1$  and  $V_2$  be two spaces of binary forms. Define  $A = \{\mathbf{f}_1, \dots, \mathbf{f}_p\} \subset \mathbf{Cov}(V_1)$  and  $B = \{\mathbf{g}_1, \dots, \mathbf{g}_q\} \subset \mathbf{Cov}(V_2)$  to be two finite and complete families. Then  $\mathbf{Cov}(A \cup B)$  is generated by the finite and complete family  $\tau := \{\tau^1, \dots, \tau^l\}$ .*

*Proof.* Let first remark that each  $\mathbf{f}_i$  (resp. each  $\mathbf{g}_j$ ) correspond to an irreducible solution of (4.2). Thus we know that  $A \subset \tau$  and  $B \subset \tau$ .

From theorem 2.10 we have to prove that each molecular covariant  $\mathbf{M} \in \mathbf{C}(A \cup B)$  is in a finite algebra. But, using definition 3.1 we can write the molecular covariant  $\mathbf{M}$  as

$$\mathbf{M} = \textcircled{\mathbf{D}} \xrightarrow{\nu(r)} \textcircled{\mathbf{E}}$$

with a molecular covariant  $\mathbf{D} \in \mathbf{Cov}(A)$  and  $\mathbf{E} \in \mathbf{Cov}(B)$ ;  $r$  being some integer. Because  $A$  is complete, we can suppose  $\mathbf{D}$  to be a monomial expression  $\mathbf{U}$  on the  $\mathbf{f}_i$ 's; and in the same way we can suppose  $\mathbf{E}$  to be a monomial expression  $\mathbf{V}$  on the  $\mathbf{g}_j$ 's. We then have to consider molecular covariants

$$\mathbf{M} = \textcircled{\mathbf{U}} \xrightarrow{\nu(r)} \textcircled{\mathbf{V}}$$

with

$$\mathbf{U} = \mathbf{f}_1^{a_1} \dots \mathbf{f}_p^{a_n} \text{ and } \mathbf{V} = \mathbf{g}_1^{b_1} \dots \mathbf{g}_q^{b_p}$$

Now we can make a direct induction on the index  $r$  of the transvectant. Put  $\tau^1, \dots, \tau^{i_1}$  to be transvectants from the family  $\tau$  which indexes are lower than  $r$ . If we take a transvectant  $\{\mathbf{U}, \mathbf{V}\}_{r+1}$  which correspond to a reducible integer solution, then by proposition 3.2, we can extend this transvectant as a linear combination of a non connected molecular covariant  $\mathbf{T}$  and transvectants  $\{\mathbf{U}', \mathbf{V}'\}_{r'}$  of lower index  $r' < r+1$ . By induction hypothesis, all these transvectants  $\{\mathbf{U}', \mathbf{V}'\}_{r'}$  are in  $k[\tau]$ .

Let suppose without loss of generality that  $\mathbf{T} = \mathbf{T}_1 \mathbf{T}_2$  where each term correspond to an irreducible integer solution of (4.2). Using proposition 3.5 we can thus write each term as a linear combination of on  $\tau_i \in \tau$  and transvectants of index  $r' < r+1$ . We can thus conclude the first part of the lemma stating that  $\mathbf{Cov}(A \cup B)$  is generated by the finite family  $\tau$ .

To conclude, we have to show that  $\tau$  is a complete family. For that purpose, let just remark that

$$A \cup B \subset \tau \subset \mathbf{Cov}(A \cup B)$$

and then

$$\mathbf{Cov}(\tau) = \mathbf{Cov}(A \cup B) = \mathbf{C}[\tau]$$

$\square$

One direct application of theorem 4.5 is about joint covariants. Indeed, this theorem gives us a constructive approach to get a basis covariant of  $S_n \oplus S_p$ , once we know a basis covariant of each space  $S_n$  and  $S_p$ . Of course, this algorithm depend on the resolution of an integer system.

Nevertheless, there is a simple procedure to get a basis covariant of  $S_n \oplus S_2$ , as detailed in theorem 4.6, which proof is given in [28]. From now on, we define  $\mathbf{u}$  to be a quadratic form.

**Theorem 4.6.** *If  $\{\mathbf{h}_1, \dots, \mathbf{h}_s\}$  is a covariant basis of  $\mathbf{Cov}(S_n)$ , then irreducible covariants of  $\mathbf{Cov}(S_n \oplus S_2)$  are taken from one of this set:*

- $\{\mathbf{h}_i, \mathbf{u}^r\}_{2r-1}$  for  $i = 1 \dots s$  ;
- $\{\mathbf{h}_i, \mathbf{u}^r\}_{2r}$  for  $i = 1 \dots s$  ;
- $\{\mathbf{h}_i \mathbf{h}_j, \mathbf{u}^r\}_{2r}$  where  $\mathbf{h}_i$  is of order  $2p+1$  and  $\mathbf{h}_j$  is of order  $2r-2p-1$ .

We also have another important property:

**Lemma 4.7.** *Let  $\mu := \max(a_i)$  and  $\nu := \max(b_j)$ . If*

$$u + v \geq \mu + \nu, \tag{4.4}$$

*then, the transvectant  $\{U, V\}_r$  is reducible.*

*Proof.* Condition (4.4) implies that  $u \geq \mu$  or  $v \geq \nu$  and thus that the transvectant  $\{U, V\}_r$  contains a reducible term  $T$  (the corresponding integer solution  $(\alpha, \beta, u, v, r)$  is thus not minimal). By virtue of proposition 3.5, the transvectant is a linear combination the term  $T$  and transvectants

$$\{\bar{U}^{c(k_1)}, \bar{V}^{c(k_2)}\}_{r'},$$

where  $r' < r$  and  $k_1 + k_2 = r - r'$ . Note that, because both families  $A$  and  $B$  are supposed to be complete, we have

$$\bar{U}^{c(k_1)} = f_1^{\alpha'_1} \dots f_p^{\alpha'_p}, \quad \bar{V}^{c(k_2)} = g_1^{\beta'_1} \dots g_q^{\beta'_q},$$

where, moreover, the order of the transvectant  $\{\bar{U}^{c(k_1)}, \bar{V}^{c(k_2)}\}_{r'}$  is of order  $u' + v' = u + v$ . Since we have supposed that  $u + v \geq \mu + \nu$ , we get that  $u' + v' \geq \mu + \nu$  and the proof is achieved by a recursive argument on the index of the transvectant  $r$ .  $\square$

*Remark 4.8.* The statement  $u + v \geq \mu + \nu$  can't be replaced by the hypothesis  $u \geq \mu$  or  $v \geq \nu$ . Indeed, taking  $\mathbf{f} \in S_6$  and the covariant bases given in A, we can compute the first covariant  $\mathbf{h}_{3,8} := \{\{\mathbf{f}, \mathbf{f}\}_4, \mathbf{f}\}_1$  from this bases and the second covariant  $\mathbf{h} := \{\mathbf{f}^2, \mathbf{f}\}_5$ . For this last covariant we have  $u = 7 \geq 6$  but

$$\mathbf{h} = \frac{65}{66} \mathbf{h}_{3,8}$$

and then  $\mathbf{h}$  is no reducible.

Note that the lemma 4.7 gives a bound for the order of each element of a minimal basis of joint covariants. More precisely:

**Corollary 4.9.** *If*

$$V = S_{n_1} \oplus \dots \oplus S_{n_s},$$

*and if  $\mu_i$  is the maximal order of a minimal basis for  $S_{n_i}$ , then, for each element  $\mathbf{h}$  of a minimal basis for  $V$ , we get*

$$\text{ord}(\mathbf{h}) \leq \sum_{i=1}^s \mu_i.$$

## 5. GORDAN'S ALGORITHM FOR SIMPLE COVARIANTS

Now, to get the finiteness result when dealing with a space of binary form  $V = S_n$ , we will have to introduce a weaker version of the notion of complete family. Note also that we will always consider homogeneous families.

**Definition 5.1.** Let  $I \subset \mathbf{Cov}(V)$  be an ideal, a family  $A$  is said to be relatively complete modulo  $I$  if every homogeneous covariant  $\mathbf{h} \in \mathbf{Cov}(A)$  can be written

$$\mathbf{h} = p(A) + \mathbf{h}_I \text{ with } \mathbf{h}_I \in I$$

and  $p(A)$  being a polynomial expression in  $A$ , all expression having the same degree.

Now, related to grade's definition 2.9:

**Definition 5.2.** Let  $r$  be an integer ; we define  $\mathcal{G}_r(V) \subset \mathcal{M}(V)$  to be the set of all molecular covariants with grade at least  $r$ :

$$\mathcal{G}_r(V) := \Psi(\mathfrak{A}_r(V))$$

As a first observation, it is clear that for  $V = S_n$ , we have  $\mathcal{G}_r(S_n) = \{0\}$  as soon as  $r > n$ . Furthermore, we will have

$$\mathcal{G}_{i+1}(V) \subset \mathcal{G}_i(V) \text{ for all } i \quad (5.1)$$

We now get the

**Definition 5.3** (Gordan's ideals). Let  $r$  be an integer. We define the Gordan ideal  $I_r(V)$  to be the ideal generated by  $\mathcal{G}_r(V)$  ; we will write

$$I_r(V) := \langle \mathcal{G}_r(V) \rangle$$

We observe directly that:

- $I_r(S_n) = \{0\}$  for all  $r > n$  ;
- By equation 5.1, we have  $I_{r+1}(V) \subset I_r(V)$  for every integer  $r$ .

By the property 3.2, we immediately have:

**Lemma 5.4.** If  $\mathbf{h}_r \in I_r(V)$ , for every covariant  $\mathbf{h} \in \mathbf{Cov}(V)$  and for every integer  $j$ , we have

$$\{\mathbf{h}_r, \mathbf{h}\}_j \in I_r(V)$$

Let's now take the vector space  $S_n$  of  $n^{\text{th}}$  degree binary forms,  $\mathbf{f} \in S_n$ . We will write  $I_r$  to be the associated Gordan's ideal. We also put  $\Delta$  to be an invariant.

One important result, close to theorem 4.5, is:

**Theorem 5.5.** Let  $A$  and  $B$  be two families of  $\mathbf{Cov}(S_n)$ . Let's suppose that

- $\mathbf{f} \in A$  ;
- $A$  is relatively complete modulo  $I_{2k}$  ;
- $B$  is relatively complete modulo  $I_{2k+1}$  (resp. modulo  $I_{2k+1} + \langle \Delta \rangle$ ).
- $B$  contains  $\mathbf{H}_{2k} = \{\mathbf{f}, \mathbf{f}\}_{2k}$

Then there exist a finite family  $C$ , is relatively complete modulo  $I_{2k+1}$  (resp. modulo  $I_{2k+1} + \langle \Delta \rangle$ ) such that

$$\mathbf{Cov}(C) = \mathbf{Cov}(A \cup B) = \mathbf{Cov}(S_n)$$

*Proof.* Using theorem 2.10 and property 3.5, we can consider transvectants

$$\{\mathbf{h}_A, \mathbf{h}_B\}_r \text{ avec } \mathbf{h}_A \in \mathbf{Cov}(A) \text{ and } \mathbf{h}_B \in \mathbf{Cov}(B) \quad (5.2)$$

Now we can write, by hypothesis

$$\mathbf{h}_A = p(A) + \mathbf{h}_{2k} \text{ and } \mathbf{h}_B = q(B) + \mathbf{h}_{2k+2} \quad (5.3)$$

Thus (5.2) can be decomposed as

$$\{p(A), q(B)\}_r \quad (5.4)$$

$$\{\mathbf{h}_{2k}, q(B)\}_r \quad (5.5)$$

$$\{p(A), \mathbf{h}_{2k+2}\}_r \text{ et } \{\mathbf{h}_{2k}, \mathbf{h}_{2k+2}\}_r \quad (5.6)$$

Thus we may directly observe that :

- The case (5.4) had been studied in proof of theorem 4.5 ;
- all transvectant of (5.6) are in  $I_{2k+2}$  by lemma 5.4 ;

Thus we just have to deal with the case (5.5), when  $\mathbf{h}_{2k} \in I_{2k} - I_{2k+2}$ .

For that purpose, we will make here an induction on :

- The order  $r$  of the transvectant in (5.5) ;
- The degree  $d$  in  $\mathbf{f}$  of the covariant  $\mathbf{h}_A$  ; this degree is the same as the one of  $\mathbf{h}_{2k}$  in (5.3).

Suppose indeed that for two given integers  $d$  and  $r$  we have a finite family  $\mathbf{C}_1, \dots, \mathbf{C}_l$  such that, as soon as the degree in  $\mathbf{f}$  of  $\mathbf{h}_A$  is  $d_1 < d$

$$\{\mathbf{h}_A, q(\mathbf{B})\}_m = \phi_2(\mathbf{C}_i) + \mathbf{h}_{2k+2} \text{ for all } m \quad (5.7)$$

and for all  $r_1 < r$

$$\{\mathbf{h}_{2k}, q(\mathbf{B})\}_{r_1} = \phi_1(\mathbf{C}_i) + \mathbf{h}_{2k+2}$$

Let's now consider the transvectant  $\{\mathbf{h}_A, \mathbf{h}_B\}_r$  with  $\mathbf{h}_A$  of degree  $d$  in  $\mathbf{f}$ . If a molecular covariant of this transvectant 3.2 is non connected, then we will have a linear combination of transvectants of order  $r' < r$  ; either we will only consider transvectants  $\{\mathbf{h}_{2k}, q(\mathbf{B})\}_r$  with  $\mathbf{h}_{2k}$  of degree  $d$  in  $\mathbf{f}$ . Thus we can write  $\mathbf{h}_{2k}$  as

$$\textcircled{\mathbf{M}} \xrightarrow{\nu(r)} \textcircled{\mathbf{H}_{2k}}$$

for some integer  $r'$  and some molecular covariant  $\mathbf{M} \in \mathbf{Cov}(V)$  of degree in  $\mathbf{f}$  strictly less than  $d$  ; and thus  $\{\mathbf{h}_{2k}, q(\mathbf{B})\}_r$  will decompose, modulo  $I_{2k+2}$ , into

$$\textcircled{\textcircled{\mathbf{M}} \xrightarrow{\nu(r)} \textcircled{\mathbf{H}_{2k}}} \xrightarrow{\nu(r)} \textcircled{q(\mathbf{B})}$$

thus, modulo  $I_{2k+2}$ , into

$$\textcircled{\mathbf{M}} \xrightarrow{\nu(r'')} \textcircled{q'(\mathbf{B})}$$

because  $\mathbf{H}_{2k} \in \mathbf{B}$  and every molecular covariant which come from  $\mathbf{H}_{2k}$  and  $q(\mathbf{B})$  will be in  $\mathbf{Cov}(\mathbf{B})$ . We can thus make use of (5.7) : we will only have to consider non-connected molecular covariants of  $\{p(A), q(\mathbf{B})\}_r$  : we already saw in proof of theorem 4.5 that we only have finite cases.  $\square$

We now give some important lemmas before getting to the proof of theorem 5.5. Let's first define

$$\mathbf{H}_{2k} := \{\mathbf{f}, \mathbf{f}\}_{2k} \text{ of order } 2n - 4k$$

It is clear that this is the molecular covariant

$$\mathbf{H}_{2k} := \textcircled{\mathbf{f}} \xrightarrow{2k} \textcircled{\mathbf{g}}$$

and thus  $\mathbf{H}_{2k} \in I_{2k}$ . Now, using 2.13:, we get:

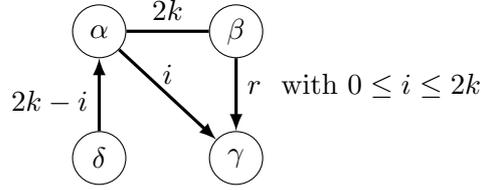
**Lemma 5.6.** *If  $\mathbf{H}_{2k}$  is of order strictly greater than  $n$ , that is if  $2n - 4k > n$ , then the family  $\mathbf{B} = \{\mathbf{H}_{2k}\}$  is relatively complete modulo  $I_{2k+2}$*

*Proof.* We have to consider Aronhold molecule which contain the Aronhold molecule, all symbol being equivalent:

$$\begin{array}{ccc} \textcircled{\alpha} & \xrightarrow{2k} & \textcircled{\beta} \\ & & \downarrow r \\ \textcircled{\delta} & \xrightarrow{2k} & \textcircled{\gamma} \end{array} \text{ with } 1 \leq r \leq 2k$$

When  $r > k$ , we can directly use lemma 2.14 with  $e_0 = 2k$  and  $e_1 = r$ , and conclude that this Aronhold molecule is in  $\mathfrak{A}_{2k+1}$ , and thus in  $\mathfrak{A}_{2k+2}$ .

When  $r < k$ , using syzygie (2.6) we may decompose this Aronhold molecule as a linear combination of



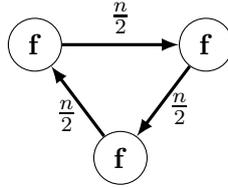
But now; to conclude:

- either  $i \geq k$ , and thus  $2k + r + i \geq 3k$ ; because  $2k + r + i \leq n$  we may use lemma 2.13 and we will have an Aronhold molecule in  $\mathfrak{A}_r$  with  $r \geq \frac{2}{3}w > 2k$ ;
- or  $i < k$ , and thus  $2k - i > k$ : the same argument as above, using lemma 2.14 will be used.

□

And:

**Lemma 5.7.** *If  $\mathbf{H}_{2k}$  is of order  $n$ , that is if  $n = 4k$ , then the family  $\mathbf{B} = \{\mathbf{H}_{2k}\}$  is relatively complete modulo  $I_{2k+2} + \langle \Delta \rangle$  where  $\Delta$  is an invariant given by:*



Furthermore, using property 3.5 we get:

**Lemma 5.8.** *For all integer  $k \geq 1$  we have*

$$I_{2k-1} = I_{2k}$$

And a direct lemma:

**Lemma 5.9.** *The family  $\mathbf{A}_0 := \{\mathbf{f}\}$  is relatively complete modulo  $I_2$*

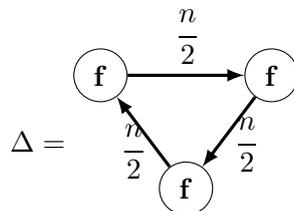
Now, using lemma 4.1, we will have  $\mathbf{Cov}(\mathbf{A}_0) = \mathbf{Cov}(S_n)$ ; this lemma 5.9 just mean that every covariant  $\mathbf{h} \in \mathbf{Cov}(S_n)$  can be written

$$\mathbf{h} = p(\mathbf{f}) + \mathbf{h}_2 \text{ avec } \mathbf{h}_2 \in I_2 \text{ where } p \text{ is a polynomial}$$

We then define  $\mathbf{A}_k$  to be a finite family, relatively complete modulo  $I_{2k+2}$ , and containing  $\mathbf{f}$ : we will show by induction that such a family exist. Let's first observe that, by lemma 4.1, we will have for every integer  $k$ ,  $\mathbf{Cov}(\mathbf{A}_k) = \mathbf{Cov}(S_n)$ . We will also have  $\mathbf{A}_k \subset \mathbf{A}_{k+1}$ ; thus, because for some  $k$  we will have  $I_{2k+2} = \{0\}$ , this induction will give us the desired covariant basis.

The main clue is to construct for every integer  $k$  an auxiliary family  $\mathbf{B}_k$ :

- If  $\mathbf{H}_{2k}$  is of order  $p > n$ , we take  $\mathbf{B}_k := \{\mathbf{H}_{2k}\}$  which, by lemma 5.6, will be relatively complete modulo  $I_{2k+2}$ ; applying theorem 5.5 leads us to the family  $\mathbf{A}_{k+1} := \mathbf{C}$ .
- If  $\mathbf{H}_{2k}$  is of order  $p = n$ , we take  $\mathbf{B}_k := \{\mathbf{H}_{2k}, \Delta\}$  which, by lemma 5.7, will be relatively complete modulo  $I_{2k+2} + \langle \Delta \rangle$ ; where  $\Delta$  is the invariant



In that case a direct induction shows that, applying theorem 5.5, we can take  $A_{k+1}$  to be  $C \cup \{\Delta\}$ .

- If  $\mathbf{H}_{2k}$  is of order  $p < n$ , we suppose already known a covariant basis of  $S_p$ ; we then take  $B_k$  to be this basis, which will be finite and complete, thus finite and relatively complete modulo  $I_{2k+2}$ ; we directly apply theorem 5.5 to get  $A_{k+1} := C$ .

Thus in each case we get the construction of the family  $A_{k+1}$ .

Now, depending on  $n$ 's parity:

- If  $n = 2q$  is even, we know that the family  $A_{q-1}$  is relatively complete modulo  $I_{2q}$ ; furthermore the family  $B_{q-1}$  only contains the invariant  $\Delta_q := \{\mathbf{f}, \mathbf{f}\}_{2q}$ ; finally we observe that  $A_p$  will be given by

$$A_p := A_{p-1} \cup \{\Delta_q\}$$

and it will be relatively complete modulo  $I_{2q+2} = \{0\}$ ; this gives us the wanted basis.

- If  $n = 2q + 1$  is odd, the family  $B_{q-1}$  will contain the quadratic form  $\mathbf{H}_{2q} := \{\mathbf{f}, \mathbf{f}\}_{2q}$ ; we then know that the family  $B_{q-1}$  will be given by the covariant  $\mathbf{H}_{2q}$  and the invariant  $\delta_q := \{\mathbf{H}_{2q}, \mathbf{H}_{2q}\}_2$ . The family  $A_q$  obtained using theorem 5.5 will then be relatively complete modulo  $I_{2q+2} = \{0\}$ ; which gives us the wanted basis.

#### APPENDIX A. JOINT COVARIANTS OF $S_6 \oplus S_2$

We write  $\mathbf{h}_{d,o}$  to be a covariant of degree  $d$  and order  $o$ , taken from the covariant basis of  $S_6$  in table A, issue from Grace–Young [28], and  $\mathbf{u}$  to be a quadratic form in  $S_2$ . By theorem 4.6 we have to consider covariants given by

$$\{\mathbf{h}, \mathbf{u}^r\}_{2r-1} \text{ or } \{\mathbf{h}, \mathbf{u}^r\}_{2r}$$

D/O	0	2	4	6
1				$\mathbf{f}$
2	$\{\mathbf{f}, \mathbf{f}\}_6$		$\mathbf{h}_{2,4} := \{\mathbf{f}, \mathbf{f}\}_4$	
3		$\mathbf{h}_{3,2} := \{\mathbf{h}_{2,4}, \mathbf{f}\}_4$		$\mathbf{h}_{3,6} := \{\mathbf{h}_{2,4}, \mathbf{f}\}_2$
4	$\{\mathbf{h}_{2,4}, \mathbf{h}_{2,4}\}_4$		$\{\mathbf{h}_{3,2}, \mathbf{f}\}_2$	$\mathbf{h}_{4,6} := \{\mathbf{h}_{3,2}, \mathbf{f}\}_1$
5		$\{\mathbf{h}_{2,4}, \mathbf{h}_{3,2}\}_2$	$\{\mathbf{h}_{2,4}, \mathbf{h}_{3,2}\}_1$	
6	$\{\mathbf{h}_{3,2}, \mathbf{h}_{3,2}\}_2$			$\mathbf{h}_{6,6a} := \{\mathbf{h}_{3,8}, \mathbf{h}_{3,2}\}_2$ $\mathbf{h}_{6,6b} := \{\mathbf{h}_{3,6}, \mathbf{h}_{3,2}\}_1$
7		$\{\mathbf{f}, \mathbf{h}_{3,2}^2\}_4$	$\{\mathbf{f}, \mathbf{h}_{3,2}^2\}_3$	
8		$\{\mathbf{h}_{2,4}, \mathbf{h}_{3,2}^2\}_3$		
9			$\{\mathbf{h}_{3,8}, \mathbf{h}_{3,2}^2\}_4$	
10	$\{\mathbf{h}_{3,2}^3, \mathbf{f}\}_6$	$\{\mathbf{h}_{3,2}^3, \mathbf{f}\}_5$		
12		$\{\mathbf{h}_{3,8}, \mathbf{h}_{3,2}^3\}_6$		
15	$\{\mathbf{h}_{3,8}, \mathbf{h}_{3,2}^4\}_8$			

D/O	8	10	12
2	$\mathbf{h}_{2,8} := \{\mathbf{f}, \mathbf{f}\}_2$		
3	$\mathbf{h}_{3,8} := \{\mathbf{h}_{2,4}, \mathbf{f}\}_1$		$\{\mathbf{h}_{2,8}, \mathbf{f}\}_1$
4		$\{\mathbf{h}_{2,8}, \mathbf{h}_{2,4}\}_1$	
5	$\mathbf{h}_{5,8} := \{\mathbf{h}_{2,8}, \mathbf{h}_{3,2}\}_1$		

TABLE 2. Covariant basis of  $S_6$

Recall the covariant algebra  $\mathbf{Cov}(V) := \mathbf{Cov}(S_6 \oplus S_2)$  is a multi-graded algebra. We can write

$$\mathbf{Cov}(V) = \bigoplus_{d_1 \geq 0, d_2 \geq 0, o \geq 0} \mathbf{Cov}(V)_{d_1, d_2, o}$$

where  $d_1$  is the degree in the binary form  $\mathbf{f} \in S_6$ ,  $d_2$  is the degree in the binary form  $\mathbf{u} \in S_2$  and  $o$  the degree in the variable  $\mathbf{x} \in \mathbb{C}^2$ . We can define the Hilbert series:

$$\mathcal{H}(z_1, z_2, t) := \sum_{d_1, d_2, o} \dim(\mathbf{Cov}(V)_{d_1, d_2, o}) z_1^{d_1} z_2^{d_2} t^o$$

Hilbert series of the covariant algebra of  $S_6 \oplus S_2$  has been computed using maple package of Bedratyuk [5].

Thanks to this Hilbert series and theorem 4.6, we finally get a minimal basis of 103 covariants: it's worth noting that, by using this algorithm, we had to check invariant homogeneous space's dimensions up to degree 15.

- Order 0: 27 invariants

Degree 2	Degree 4	Degree 6	Degree 7	Degree 8	Degree 9	Degree 10
$\{\mathbf{f}, \mathbf{f}\}_6$ $\{\mathbf{u}, \mathbf{u}\}_2$	$\{\mathbf{h}_{2,4}, \mathbf{h}_{2,4}\}_4$ $\{\mathbf{h}_{3,2}, \mathbf{u}\}_2$ $\{\mathbf{h}_{2,4}, \mathbf{u}^2\}_4$ $\{\mathbf{f}, \mathbf{u}^3\}_6$	$\{\mathbf{h}_{3,2}, \mathbf{h}_{3,2}\}_2$ $\{\mathbf{h}_{5,2}, \mathbf{u}\}_2$ $\{\mathbf{h}_{4,4}, \mathbf{u}^2\}_4$ $\{\mathbf{h}_{3,6}, \mathbf{u}^3\}_6$ $\{\mathbf{h}_{2,8}, \mathbf{u}^4\}_8$	$\{\mathbf{h}_{5,4}, \mathbf{u}^2\}_4$ $\{\mathbf{h}_{4,6}, \mathbf{u}^3\}_6$ $\{\mathbf{h}_{3,8}, \mathbf{u}^4\}_8$	$\{\mathbf{h}_{7,2}, \mathbf{u}\}_2$	$\{\mathbf{h}_{8,2}, \mathbf{u}\}_2$ $\{\mathbf{h}_{7,4}, \mathbf{u}^2\}_4$ $\{\mathbf{h}_{6,6a}, \mathbf{u}^3\}_6$ $\{\mathbf{h}_{6,6b}, \mathbf{u}^3\}_6$ $\{\mathbf{h}_{5,8}, \mathbf{u}^4\}_8$ $\{\mathbf{h}_{4,10}, \mathbf{u}^5\}_{10}$ $\{\mathbf{h}_{3,12}, \mathbf{u}^6\}_{12}$	$\{\mathbf{h}_{3,2}^3, \mathbf{f}\}_6$

Degree 11	Degree 13	Degree 15
$\{\mathbf{h}_{9,4}, \mathbf{u}^2\}_4$ $\{\mathbf{h}_{10,2}, \mathbf{u}\}_2$	$\{\mathbf{h}_{12,2}, \mathbf{u}\}_2$	$\{\mathbf{h}_{3,8}, \mathbf{h}_{3,2}^4\}_2$

- Order 2: 33 covariants

Degree 1	Degree 3	Degree 4	Degree 5	Degree 6	Degree 7	Degree 8
$\mathbf{u}$	$\mathbf{h}_{3,2}$ $\{\mathbf{f}, \mathbf{u}^2\}_4$ $\{\mathbf{h}_{2,4}, \mathbf{u}\}_2$	$\{\mathbf{h}_{3,2}, \mathbf{u}\}_1$ $\{\mathbf{h}_{2,4}, \mathbf{u}^2\}_3$ $\{\mathbf{f}, \mathbf{u}^3\}_5$	$\mathbf{h}_{5,2}$ $\{\mathbf{h}_{3,4}, \mathbf{u}\}_2$ $\{\mathbf{h}_{3,6}, \mathbf{u}^2\}_4$ $\{\mathbf{h}_{2,8}, \mathbf{u}^3\}_6$	$\{\mathbf{h}_{5,2}, \mathbf{u}\}_1$ $\{\mathbf{h}_{4,4}, \mathbf{u}^2\}_3$ $\{\mathbf{h}_{5,4}, \mathbf{u}\}_2$ $\{\mathbf{h}_{3,6}, \mathbf{u}^3\}_5$ $\{\mathbf{h}_{4,6}, \mathbf{u}^2\}_4$ $\{\mathbf{h}_{2,8}, \mathbf{u}^4\}_7$ $\{\mathbf{h}_{3,8}, \mathbf{u}^3\}_6$	$\mathbf{h}_{7,2}$	$\mathbf{h}_{8,2}$ $\{\mathbf{h}_{7,2}, \mathbf{u}\}_1$ $\{\mathbf{h}_{7,4}, \mathbf{u}\}_2$ $\{\mathbf{h}_{6,6a}, \mathbf{u}^2\}_4$ $\{\mathbf{h}_{6,6b}, \mathbf{u}^2\}_4$ $\{\mathbf{h}_{5,8}, \mathbf{u}^3\}_6$ $\{\mathbf{h}_{4,10}, \mathbf{u}^4\}_8$ $\{\mathbf{h}_{3,12}, \mathbf{u}^5\}_{10}$

Degree 10	Degree 11	Degree 12	Degree 13
$\mathbf{h}_{10,2}$ $\{\mathbf{h}_{9,4}, \mathbf{u}\}_2$	$\{\mathbf{h}_{10,2}, \mathbf{u}\}_1$ $\{\mathbf{h}_{9,4}, \mathbf{u}^2\}_3$	$\mathbf{h}_{12,2}$	$\{\mathbf{h}_{12,2}, \mathbf{u}\}_1$

- Order 4: 21 covariants

Degree 2	Degree 3	Degree 4	Degree 5	Degree 7	Degree 9	Degree 10
$\mathbf{h}_{2,4}$ $\{\mathbf{f}, \mathbf{u}\}_2$	$\{\mathbf{h}_{2,4}, \mathbf{u}\}_1$ $\{\mathbf{f}, \mathbf{u}^2\}_3$	$\mathbf{h}_{4,4}$ $\{\mathbf{h}_{3,6}, \mathbf{u}\}_2$ $\{\mathbf{h}_{2,8}, \mathbf{u}^2\}_4$	$\mathbf{h}_{5,4}$ $\{\mathbf{h}_{4,4}, \mathbf{u}\}_1$ $\{\mathbf{h}_{3,6}, \mathbf{u}^2\}_3$ $\{\mathbf{h}_{4,6}, \mathbf{u}\}_2$ $\{\mathbf{h}_{2,8}, \mathbf{u}^2\}_5$ $\{\mathbf{h}_{3,8}, \mathbf{u}^2\}_4$	$\mathbf{h}_{7,4}$ $\{\mathbf{h}_{6,6b}, \mathbf{u}\}_2$ $\{\mathbf{h}_{5,8}, \mathbf{u}^2\}_4$ $\{\mathbf{h}_{4,10}, \mathbf{u}^3\}_6$ $\{\mathbf{h}_{3,12}, \mathbf{u}^4\}_8$ $\{\mathbf{h}_{6,6a}, \mathbf{u}\}_2$	$\mathbf{h}_{9,4}$	$\{\mathbf{h}_{9,4}, \mathbf{u}\}_1$

- Order 6: 12 covariants

Degree 1	Degree 2	Degree 3	Degree 4	Degree 6
$\mathbf{f}$	$\{\mathbf{f}, \mathbf{u}\}_1$	$\mathbf{h}_{3,6}$ $\{\mathbf{h}_{2,8}, \mathbf{u}\}_2$	$\mathbf{h}_{4,6}$ $\{\mathbf{h}_{3,8}, \mathbf{u}\}_2$ $\{\mathbf{h}_{3,6}, \mathbf{u}\}_1$	$\mathbf{h}_{6,6a}$ $\mathbf{h}_{6,6a}$ $\{\mathbf{h}_{5,8}, \mathbf{u}\}_2$ $\{\mathbf{h}_{4,10}, \mathbf{u}^4\}_4$ $\{\mathbf{h}_{3,12}, \mathbf{u}^3\}_6$

- Order 8: 7 covariants

Degree 2	Degree 3	Degree 4	Degree 5
$\mathbf{h}_{2,8}$	$\mathbf{h}_{3,8}$ $\{\mathbf{h}_{2,8}, \mathbf{u}\}_1$	$\{\mathbf{h}_{2,8}, \mathbf{u}^2\}_3$	$\mathbf{h}_{5,8}$ $\{\mathbf{h}_{4,10}, \mathbf{u}\}_2$ $\{\mathbf{h}_{3,12}, \mathbf{u}^2\}_4$

- Order 10: 2 degree 4 covariants

$$\mathbf{h}_{4,10} \text{ and } \{\mathbf{h}_{3,12}, \mathbf{u}\}_2$$

- Order 12: 1 degree 3 covariant  $\mathbf{h}_{3,12}$

### APPENDIX B. COVARIANT BASES OF $S_8$

Here we have a direct application of constructive theorem 2.10.

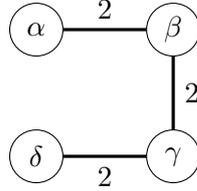
- (1) As a first step the family  $A_0$  is simply the binary form  $\mathbf{f} \in S_8$ ; the set  $B_0$  is simply the form

$$\mathbf{h}_{2,12} := \{\mathbf{f}, \mathbf{f}\}_2 \in S_{12}$$

- (2) To obtain  $A_1$  we have to consider transvectants

$$\{f^a, \mathbf{h}_{2,12}^b\}_r$$

with no reducible molecular covariants modulo  $I_4$ . From lemma 5.8 we deduce that necessarily  $r \leq 2$ . Furthermore, if an Aronhold molecule contain the Aronhold molecule



then we can directly use lemma 2.14 with  $e_0 = 2$  and  $e_1 = 2$ , and conclude that this Aronhold molecule is in  $\mathfrak{A}_3$ , and thus in  $\mathfrak{A}_4$ .

We can deduce from all this that  $A_1$  is the family

$$\mathbf{f}; \mathbf{h}_{2,12}; \mathbf{h}_{3,18} := \{\mathbf{f}, \mathbf{h}_{2,12}\}_1$$

Now the family  $B_1$  is simply the form

$$\mathbf{h}_{2,8} := \{\mathbf{f}, \mathbf{f}\}_4 \in S_8$$

- (3) To get  $A_2$  we have to consider transvectants

$$\{f^{a_1} \mathbf{h}_{2,12}^{a_2} \mathbf{h}_{3,18}^{a_3}, \mathbf{h}_{2,8}^b\}_r$$

The same kind of argument as above, using lemma such as lemma 2.14 leads to [28, 27]:

**Lemma B.1.** *The family  $A_2$  is given by the seven covariants*

$$\mathbf{f}; \mathbf{h}_{2,8} = \{\mathbf{f}, \mathbf{f}\}_4; \mathbf{h}_{2,12} = \{\mathbf{f}, \mathbf{f}\}_2; \mathbf{h}_{3,12} := \{\mathbf{f}, \mathbf{h}_{2,8}\}_2; \mathbf{h}_{3,14} := \{\mathbf{f}, \mathbf{h}_{2,8}\}_1$$

$$\mathbf{h}_{3,18} := \{\mathbf{f}, \mathbf{h}_{2,12}\}_1; \mathbf{h}_{4,18} := \{\mathbf{h}_{2,12}, \mathbf{h}_{2,8}\}_1$$

We also recall that we have to take into account the invariant

$$\{\mathbf{f}, \mathbf{h}_{2,8}\}_8$$

The family  $B_2$  is given by the covariant basis of

$$\mathbf{h}_{2,4} := \{\mathbf{f}, \mathbf{f}\}_6 \in S_4$$

As a classical result [28], such a basis is given by

$$\mathbf{h}_{2,4}; \mathbf{h}_{4,4} := \{\mathbf{h}_{2,4}, \mathbf{h}_{2,4}\}_2; \mathbf{h}_{6,6} := \{\mathbf{h}_{2,4}, \{\mathbf{h}_{2,4}, \mathbf{h}_{2,4}\}_2\}_1$$

and two invariants

$$\mathbf{h}_{4,0} := \{\mathbf{h}_{2,4}, \mathbf{h}_{2,4}\}_4; \mathbf{h}_{6,0} := \{\mathbf{h}_{2,4}, \{\mathbf{h}_{2,4}, \mathbf{h}_{2,4}\}_2\}_4$$

(4) To get family  $B_3$ , we have to consider transvectants

$$\{f^{a_1} \mathbf{h}_{2,8}^{a_2} \mathbf{h}_{2,12}^{a_3} \mathbf{h}_{3,12}^{a_4} \mathbf{h}_{3,14}^{a_5} \mathbf{h}_{3,18}^{a_6} \mathbf{h}_{4,18}^{a_7}, \mathbf{h}_{2,4}^{b_1} \mathbf{h}_{4,4}^{b_2} \mathbf{h}_{6,6}^{b_3}\}_r$$

which is associated to the integer system

$$\begin{cases} 8a_1 + 8a_2 + 12a_3 + 12a_4 + 14a_5 + 18a_6 + 18a_7 & = u + r \\ 4b_1 + 4b_2 + 6b_3 & = v + r \end{cases} \quad (\text{B.1})$$

Using *Normaliz* package [13] of Macaulay2 software [29], we get the integer solutions of B.1. To get a basis reduction, we make use of the fundamental and well known relation between covariants of a binary quartic

$$12\mathbf{h}_{6,6}^2 + 6\mathbf{h}_{4,4}^3 + 2\mathbf{h}_{6,0}\mathbf{h}_{2,4}^3 - 3\mathbf{h}_{2,4}^2\mathbf{h}_{4,4}\mathbf{h}_{4,0} = 0$$

From this, we have a bound on  $b_3$  in the system (B.1), and this remark leads us to important reduction on transvectants. With computations in Macaulay2 [29], we finally get a covariant basis of  $S_8$  given below.

- Degree 1 : the binary form  $\mathbf{f}$  of order 8
- Degree 2 : 4 covariants

Order	0	4	8	12
Covariants	$\mathbf{h}_{2,0} := \{\mathbf{f}, \mathbf{f}\}_8$	$\mathbf{h}_{2,4} := \{\mathbf{f}, \mathbf{f}\}_6$	$\mathbf{h}_{2,8}$	$\mathbf{h}_{2,12}$

- Degree 3 : 8 covariants

Order	0	4	6	8	10	12
Covariants	$\{\mathbf{f}, \mathbf{h}_{2,8}\}_8$	$\{\mathbf{f}, \mathbf{h}_{2,4}\}_4$	$\{\mathbf{f}, \mathbf{h}_{2,4}\}_3$	$\{\mathbf{f}, \mathbf{h}_{2,4}\}_2$	$\{\mathbf{f}, \mathbf{h}_{2,4}\}_1$	$\mathbf{h}_{3,12} := \{\mathbf{f}, \mathbf{h}_{2,8}\}_2$

Order	14	18
Covariants	$\mathbf{h}_{3,14} := \{\mathbf{f}, \mathbf{h}_{2,8}\}_1$	$\mathbf{h}_{3,18} := \{\mathbf{f}, \mathbf{h}_{2,12}\}_1$

- Degree 4 : 12 covariants

Order	0	4	6	8	10
Covariants	$\{\mathbf{h}_{2,4}, \mathbf{h}_{2,4}\}_4$	$\mathbf{h}_{4,4} := \{\mathbf{h}_{2,4}, \mathbf{h}_{2,4}\}_2$ $\{\mathbf{h}_{2,8}, \mathbf{h}_{2,4}\}_4$	$\{\mathbf{h}_{2,8}, \mathbf{h}_{2,4}\}_3$	$\{\mathbf{h}_{2,12}, \mathbf{h}_{2,4}\}_4$	$\{\mathbf{h}_{2,12}, \mathbf{h}_{2,4}\}_3$ $\{\mathbf{h}_{2,8}, \mathbf{h}_{2,4}\}_1$

Order	12	14	18
Covariants	$\{\mathbf{h}_{2,12}, \mathbf{h}_{2,4}\}_2$	$\{\mathbf{h}_{2,12}, \mathbf{h}_{2,4}\}_1$	$\{\mathbf{h}_{2,12}, \mathbf{h}_{2,8}\}_1$

- Degree 5 : 11 covariants

Order	0	2	4	6	8	10	14
Covariants	$\{\mathbf{f}, \mathbf{h}_{2,4}^2\}_8$	$\{\mathbf{f}, \mathbf{h}_{2,4}^2\}_7$	$\{\mathbf{f}, \mathbf{h}_{4,4}\}_4$ $\{\mathbf{f}, \mathbf{h}_{2,4}^2\}_6$	$\{\mathbf{f}, \mathbf{h}_{4,4}\}_3$ $\{\mathbf{f}, \mathbf{h}_{2,4}^2\}_5$	$\{\mathbf{f}, \mathbf{h}_{4,4}\}_2$	$\{\mathbf{h}_{3,14}, \mathbf{h}_{2,4}\}_4$ $\{\mathbf{h}_{3,12}, \mathbf{h}_{2,4}\}_3$ $\{\mathbf{f}, \mathbf{h}_{4,4}\}_1$	$\{\mathbf{h}_{3,12}, \mathbf{h}_{2,4}\}_1$

- Degree 6 : 9 covariants

Order	0	2	4	6
Covariants	$\{\mathbf{h}_{4,4}, \mathbf{h}_{2,4}\}_4$	$\{\mathbf{h}_{2,8}, \mathbf{h}_{2,4}^2\}_7$	$\{\mathbf{h}_{2,12}, \mathbf{h}_{2,4}^2\}_8$ $\{\mathbf{h}_{2,8}, \mathbf{h}_{4,4}\}_4$	$\mathbf{h}_{6,6} := \{\mathbf{h}_{4,4}, \mathbf{h}_{2,4}\}_1$ $\{\mathbf{h}_{2,12}, \mathbf{h}_{2,4}^2\}_7$ $\{\mathbf{h}_{2,8}, \mathbf{h}_{4,4}\}_3$
Order	8	10		
Covariants	$\{\mathbf{h}_{2,12}, \mathbf{h}_{4,4}\}_4$	$\{\mathbf{h}_{2,12}, \mathbf{h}_{4,4}\}_3$		

- Degree 7 : 8 covariants

Order	0	2	4	6
Covariants	$\{\mathbf{f}, \mathbf{h}_{2,4}\mathbf{h}_{4,4}\}_8$	$\{\mathbf{f}, \mathbf{h}_{6,6}\}_6$ $\{\mathbf{f}, \mathbf{h}_{2,4}\mathbf{h}_{4,4}\}_7$	$\{\mathbf{h}_{3,12}, \mathbf{h}_{2,4}^2\}_8$ $\{\mathbf{f}, \mathbf{h}_{6,6}\}_5$	$\{\mathbf{h}_{3,14}, \mathbf{h}_{2,4}^2\}_8$ $\{\mathbf{h}_{3,12}, \mathbf{h}_{2,4}^2\}_7$ $\{\mathbf{f}, \mathbf{h}_{6,6}\}_4$

- Degree 8 : 7 covariants

Order	0	2	4	6
Covariants	$\{\mathbf{h}_{2,12}, \mathbf{h}_{2,4}^3\}_{12}$	$\{\mathbf{h}_{2,12}, \mathbf{h}_{2,4}^3\}_{11}$ $\{\mathbf{h}_{2,8}, \mathbf{h}_{6,6}\}_6$	$\{\mathbf{h}_{2,12}, \mathbf{h}_{2,4}\mathbf{h}_{4,4}\}_8$ $\{\mathbf{h}_{2,12}, \mathbf{h}_{2,4}^3\}_{10}$	$\{\mathbf{h}_{2,12}, \mathbf{h}_{6,6}\}_6$ $\{\mathbf{h}_{2,12}, \mathbf{h}_{2,4}\mathbf{h}_{4,4}\}_7$

- Degree 9 : 5 covariants

Order	0	2	4
Covariants	$\{\mathbf{h}_{3,12}, \mathbf{h}_{2,4}^3\}_{12}$	$\{\mathbf{h}_{3,14}, \mathbf{h}_{2,4}^3\}_{12}$ $\{\mathbf{h}_{3,12}, \mathbf{h}_{2,4}^3\}_{11}$ $\{\mathbf{f}, \mathbf{h}_{4,4}^2\}_7$	$\{\mathbf{h}_{3,14}, \mathbf{h}_{2,4}^3\}_{11}$

- Degree 10 : 3 covariants

Order	0	2
Covariants	$\{\mathbf{h}_{2,12}, \mathbf{h}_{2,4}^2\mathbf{h}_{4,4}\}_{12}$	$\{\mathbf{h}_{2,12}, \mathbf{h}_{2,4}\mathbf{h}_{6,6}\}_{10}$ $\{\mathbf{h}_{2,12}, \mathbf{h}_{2,4}^2\mathbf{h}_{4,4}\}_{11}$

- Degree 11 : 2 covariants of order 2

$$\{\mathbf{h}_{3,18}, \mathbf{h}_{2,4}^4\}_{16} ; \{\mathbf{h}_{3,14}, \mathbf{h}_{2,4}^2\mathbf{h}_{4,4}\}_{12}$$

- Degree 12 : 1 covariants of order 2

$$\{\mathbf{h}_{4,18}, \mathbf{h}_{2,4}^4\}_{16}$$

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