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## Dynamical modeling of spindle with active magnetic bearing for milling process

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**Keywords**: Dynamical; Milling; Spindle; Active Magnetic Bearing.

**Abstract.** A dynamical modeling of spindle with Active Magnetic Bearing (AMB) is presented. All the required parameters are included in the model for stability analysis. The original map of stability is generated by Time Domain Simulation. The major importance of forced vibrations is highlighted for a spindle with AMB. Milling test are used to quickly evaluate the stability. Finally, the simulation results are then validated by cutting tests on a 5 axis machining center with AMB.

#### Introduction

The productivity of machining process is still severely limited by machining vibrations, so called chatter. The work of Tobias [1] have presented the surface regeneration as the major cause of chatter.

One first method is to evaluate the asymptotic stability of the process, and it is possible to plot the stability chart. This description introduced for turning has been extended by the work of Altintas for milling [2]. Newly, improved methods have been developed with a more detailed stability analysis [3-4]. For high-speed milling with highly interrupted cut and low helix angle, a new unstable area has been detected, called period-doubling, or flip bifurcation [5].

Another approach is the Time Domain Simulation (TDS), based on numerical computation. In this case, the equation of motion is numerically integrated. All the information is available like the displacement, the tool position, or the cutting forces [6-8]. In some simple case, it's even possible to simulate the surface finish [8,9]. This technique is very powerful and it's possible to model all the nonlinear parameters of machining: ploughing or the fact that the tool leaves the cut under large vibrations [6].

The development of spindles with active magnetic bearing became very fast during the last ten years. A quite large bibliography is available, see for example [10-13]. The life span and their robustness to force impact are major advantages of AMB, much larger than the conventional roller bearing spindle [13]. In addition to these advantages, it is also very easy to use the included spindle displacement sensors and feedback currents, for position and force measurement. For example, Auchet et al. [14] developed an original method for indirect cutting force measurement by analyzing the command voltage of AMB. Chen and Knospe developed approaches to maximize damping, by using an additional AMB on the spindle, and in some case to it's possible to control chatter on dedicated simplified test bench [13, 15]. Kyung and Lee [16] have presented a study on the stability of a spindle with AMB, but only for low spindle speeds corresponding to conventional machining.

In this work, the dynamical modeling of a spindle with AMB is presented in the high-speed domain (40000 rpm), in the area of the first Hopf lobe. The original AMB machining modeling is presented, and the results are validated by experimental test. In section 2 the dynamical modeling is presented. The stability results are studied in Section 3. Experimental tests are collected in Section 4. Finally, conclusions are gathered in Section 5.

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#### **Dynamical modeling**

**Modeling of the spindle.** The point M is located at the center of the rotor at the height z, this displacement is defined by u(z, t) (Figure 1).

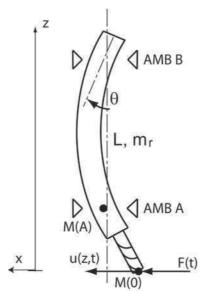


Figure 1. Modeling of the spindle.

The point M(0) is located at the end of the tool, the points M(A) and M(B) are at the height of the A and B active magnetic bearings. The modal base is defined to represent the displacement vibrations of the spindle:

$$u(z,t) = \sum_{i=1}^{n} \varphi_i(z)q_i(t)$$
(1)

 $\varphi_i(z)$  is the  $i^{th}$  modal shape, associated to the natural pulsation  $\omega_i$  and the modal displacement coefficient  $q_i(t)$ , is obtained solving the motion equation:

$$\left\{ m_i \ddot{q}_i(t) + c_i \dot{q}_i(t) + k_i q_i(t) = f_i(t) \right\}_i \tag{2}$$

 $m_i$ ,  $c_i$  and  $k_i$  are the modal mass, damping and stiffness of the  $i^{th}$  mode.  $f_i(t)$  is the projection of the external forces F(t). This projection considers the fact that the displacement is different along the height of the spindle:

$$f_i(t) = \varphi_i(0)F(t) \tag{3}$$

The system of equations is defined by:

$$\left\{ m_{i}\ddot{q}_{i}(t) + c_{i}\dot{q}_{i}(t) + k_{i}q_{i}(t) = \varphi_{i}(0)F(t) \right\}_{i} \tag{4}$$

The analysis of the machining stability at high spindle speed requires taking into account the spindle modal behavior. An analytical model of the spindle and measurements has shown that gyroscopic effect on modal frequencies is not important. For example (Figure 2), the first modal frequency (approximately 1 kHz) varies less 10% during spindle speed variation. We note a doubling of this frequency due to the gyroscopic effect (Campbell diagram) and at 40000 rpm; two close frequencies appear at 885 and 990 Hz. Thus this light variation can be disregarded and gyroscopic effect won't be taken into account in this study.

Also, the unbalance is not taken into account because a part of the control algorithm, not defined here, is designed to center the rotation axis of the spindle at his inertia axis and thus avoid unbalance force compensation.

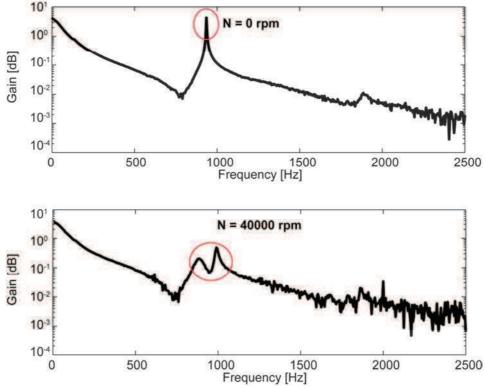


Figure 2. Gyroscopic effect for the first modal frequency.

The rotor is free, and it is necessary to consider the rigid body mode of the rotor. In fact there is a static component of the excitation force, which is balanced by the feedback control loop. As an assumption, only the displacement at the bearing A is model, we consider that the B bearing is a hinge. The mass of the rotor is  $m_r$  and its length L. The rotor is model as a uniform bar. The  $\theta$  angle is considered very small (L  $\gg$ x), the motion equation is:

$$\frac{m_r}{3}\ddot{q}_0 \approx F(t) \tag{5}$$

The spindle dynamics is modeled in the xz plane, defined by its mass and the two first flexible modes. The modal parameters were identified by hammer impact at the end of the tool, and the mass rotor was measured.

**Excitation by the cutting forces.** The relationship between the chip thickness and the tool's vibration may create self-generated vibrations, which are different from forced or transitional vibrations (Figure 3).

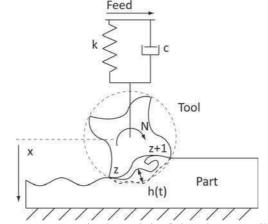


Figure 3. Dynamical model of the chatter milling.

A linear cutting law was used [5], the force is proportional to the chip thickness h(t):

$$F_c(t) = [K_1 h(t)]g(t)r(h(t))$$
(6)

g(t) is an step function that is equal to 1 when the tooth is cutting, and equal to 0 otherwise.  $K_1$  is a cutting coefficient defined by:

$$K_1 = \frac{1}{2} A_p K_t \alpha_x \tag{7}$$

 $K_t$  is the specific tangential cutting coefficient,  $A_p$  is the axial depth of cut,  $\alpha_x$  is the x directional milling force coefficient, is defined by:

$$\alpha_{x} = \frac{1}{2} \left[ -\cos(2\theta) - 2\theta k_{r} - k_{r} \sin(2\theta) \right]_{\phi_{st}}^{\phi_{ex}}$$
(8)

 $k_r$  is the reduced radial cutting coefficient,  $\phi_{st}$  and  $\phi_{ex}$  are the entry and exit cutting angle of the tool, see [3]. The instantaneous chip thickness involved in Equation 6, is defined by:

$$h(t) = f_z + x(t - \tau) - x(t) \tag{9}$$

Without vibrations, the chip thickness is equal to the feed per tooth  $f_z$ . When vibration occurs, the static chip thickness is modulated by the regenerative effect. The delay between two tooth pass is a fundamental parameter for modeling self-excited vibrations, it is defined by:

$$\tau = \frac{60}{Nz} \tag{10}$$

With N the spindle speed, in rpm, and z the number of teeth of the tool. In real machining, the tool vibrations still have finite amplitude. When the vibrations are too strong, the tool leaves the cut, and no effort is applied during this time. In order to model this, a unit function r(h(t)) was introduced. It can be described as follows:

- if h(t) < 0, i.e. chip thickness is zero, then r(h(t)) = 0, the cutting forces are nil, and no effort is applied.
- if h(t) > 0, i.e. the tool is cutting, then r(h(t)) = 1, the cutting forces are calculated by a linear cutting law (Equation 6).

Active Magnetic Bearing. The AMB generate a force, by the feedback intensity current i, in order to maintain the rotor axis position x. The magnetic bearings are controlled in differential mode, i.e. the current in the coils is the sum of the bias current  $i_0$ , and the control current i (Figure 4). The bias current  $i_0>0$  allows the double control force compared to no bias current, because the magnetic forces on the spindle can only be attractive. The reactivity of the control is increased, but a heating effect is possible only when i = 0.

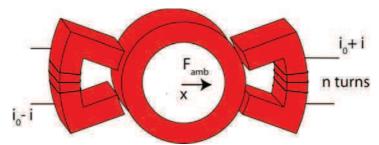


Figure 4. Active Magnetic Bearings.

The nonlinear law is defined in the following equation:

$$F_{amb}(i,x) = k \left[ \frac{(i_0 + i)^2}{(x_0 - x)^2} - \frac{(i_0 - i)^2}{(x_0 + x)^2} \right]$$
(11)

 $i_0$  is the bias current,  $x_0$  is the nominal air gap, and k is the global magnetic permeability calculated by:

$$k = \frac{1}{4} \,\mu_0 A_g \, n^2 \tag{12}$$

 $\mu_0$  is the vacuum magnetic permeability, also called magnetic constant,  $A_g$  is the air gap area an n is the number of turns. The knowledge of the control loop is an important aspect, because it generates the forces applied to the spindle in reaction to the cutting forces. In this work, the magnetic bearing manufacturer (MECOS) has given all the information about the control loop. First, a semi-static control loop compensates the static forces. Secondly, there is a dynamic control loop, similar to a proportional derivate regulator with band-cut filters. Third, there is the bias current generation.

#### Time Domain Simulation with AMB

**Numerical integration.** The Time Domain Simulation of milling process requires to choice a suitable resolution method, because there are many strong non-linearity: delay term, screen function, cutting forces, maximum current limitations... The explicit schema of Runge-Kutta (2,3) type [17] was used. In order to control the stability and the error the time step was adapted. This algorithm is improved for strongly nonlinear problem. The modeling was developed in Matlab-Simulink, and all the parameters are collected in Table 1. These parameters come for measures on the spindle by various tests of magnetic bearings excitation.

Table 11 drameters used for the simulation					
Dynamical	Values	Cutting	Values	AMB	Values
$m_1$	2.510 kg	$K_t$	900 MPa	$i_0$	3 A
$c_{I}$	305.56 Ns/m	$k_r$	0.2	$x_0$	0.4 mm
$k_{I}$	$9.3*10^7 \text{N/m}$	$f_z$	0.1 mm/tooth		
$\omega_I$	970 Hz	R	6 mm		
$\phi_1$	0.01	$A_e$	4 mm		
$m_2$	0.782 kg	Z	3		
$c_2$	170.52 Ns/m	$\alpha_{\scriptscriptstyle X}$	-1.198		
$k_2$	1.2*10 <sup>8</sup> N/m				
$\omega_2$	1970 Hz				
$\phi_2$	0.08				
$m_r$	21 kg				

Table 1 Parameters used for the simulation

**Results of simulation.** The computation of the stability or instability by Time Domain Simulation it's not direct. The peak to peak criterion was used because is easy to compute and more reliable [6]. The machining is considered as unstable if the peak to peak amplitude exceeds a given level. The simulation chart obtained with a 0.1 mm criterion at tool's end is presented Figure 5. The time of computation is 6 hours on a desktop computer (Intel Core Duo 1.73 GHz – 1 Go RAM). It may be noted that the dichotomy method may reduce the computing time [9], but full simulation allows the absence of closed areas.

The stability map is presented on figure 5. It's possible to divide the contribution of the stability and the contribution of the forced vibrations. By removing the nonlinearity when the tool leave the cut under large vibrations, i.e. assuming r(h(t)) = h(t), only the asymptotic stability is detected (thick line). Great forced vibrations (dotted line) are present at 10000, 14000, 20000 and 40000 rpm. These areas are corresponding to simple forced vibrations but they affect the behavior of the spindle by a safety mode (the spindle is stopped).

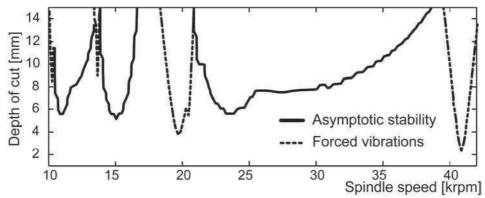
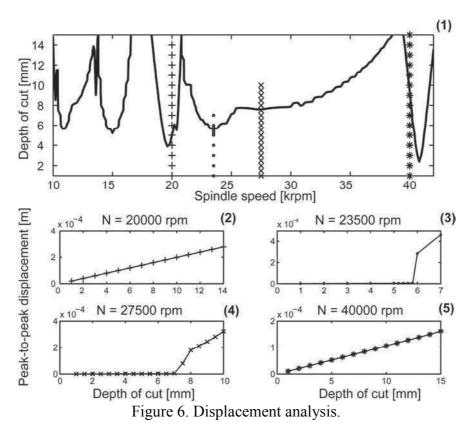


Figure 5. Map of stability for a spindle with active magnetic bearing.

The evolution of peak to peak displacement function of the axial depth of cut is presented Figure 6. The (3) and (4) graphs show that after a linear part, an abrupt increase of the amplitudes is present. This behavior corresponds to chatter at the critical depth of cut, linked to the asymptotic instability. In graphs (2) and (3), the vibration amplitudes increase proportionally with the axial depth of cut, like forced vibrations.



A frequency analysis is used to determine the nature of these forced vibrations during machining. Figure 7 show the frequency analysis of the displacement for 40000 rpm with an axial depth of cut of 15 mm. Only the tooth passing frequency is present, with a frequency of 2000 Hz, corresponding to the tooth passing frequency of a 3 tooth tool at 40000 rpm. Classically, for roller bearing spindle, these cutting cases are associated with stable machining but with a high level of vibrations. For milling with AMB, these forced vibrations are unsafe and the spindle is automatically stopped by the control loop.

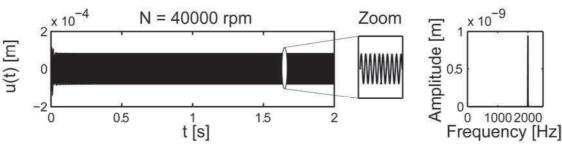


Figure 7. Frequency analysis for 40000 rpm.

In this paper, the concept of stability, widely used for roller bearing spindle, is developed by including the forced vibrations. This new aspect would be explained by the superposition of the asymptotic stability and the forced vibrations.

#### **Experimental part**

Cutting tests were conducted on a 5 axis high-speed milling center (Mikron UCP 600 Vario) with a spindle with Active Magnetic Bearing (Ibag HF400M). The tool is a monolithic carbide end mill, three teeth, 12 mm diameter, helix angle 30°, mounted on a HSK50E. Milling tests were carried out on a solid block (80×80 mm) of aluminum 2017A. A schematic experimental diagram is presented Figure 8. The part was down-milled with a feed per tooth of 0.1 mm, and a radial depth of cut of 4 mm. The axial depth of cut is increased during the milling test from 0 mm to 12 mm. The vibrations of the spindle were measured directly using the control signals of the magnetic bearings. These sensors are included on the spindle but the bandwidth is limited to 2.5 kHz.

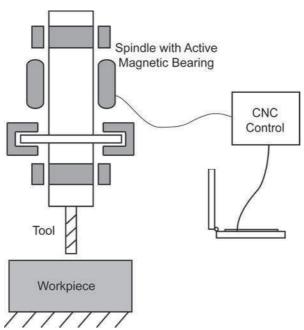


Figure 8. Experiment schematic diagram.

**Vibrations analysis.** Much information is available from the control signals, for example: the position of the center of the rotor, the drive currents signals... In practice, it's more reliable to use the drive currents signals, because they naturally amplify the cutting force variations. In contrast, the extracted information is indirectly related to the level of vibration.

The raw signal and the sampled signal once per tool revolution are presented in Figure 9 for N = 20000 rpm. For this case, it is difficult to identify the critical depth of cut only by the observation of the measured signals. By sampling the signal at the tooth passing frequency, a loss of regularity of the cut is observed at t = 0.5 s. Moreover, this machining with N = 20000 rpm is problematic,

because the vibrations are so strong and the spindle is automatically stopped (safety mode). Indirect measurement of machining vibrations with the control signals of magnetic bearings is a reliable means to control the milling stability.

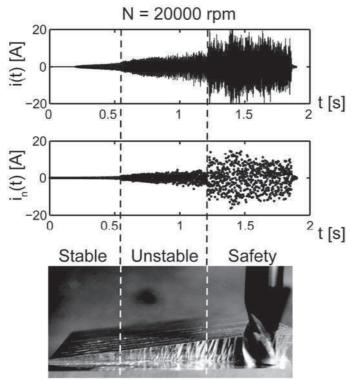


Figure 9. Continuous time histories and 1/rev sampled signals for 20000 rpm.

**Discussion simulations-experiments.** The experimental results are collected on the map of stability (Figure 10). A good correlation is observed between numerical simulations and milling tests. The classical Hopf instability is found at 15000 rpm, 22000 rpm and 23000 rpm. Moreover, high forced vibrations, predicted by the simulation, at 20000 rpm have been found by experiment. In this case the spindle was automatically stopped by the control loop (safety). The test at 40000 rpm does not show so much vibration than the 20000 rpm one, but the test at 37000 rpm shows high forced vibrations much sooner than predicted. The discrepancies between the experiment and the modeling are observed for some spindle speed. It is presently not clear what causes these discrepancies. Possible causes could be: the too simple linear cutting law or the simplified AMB modeling.

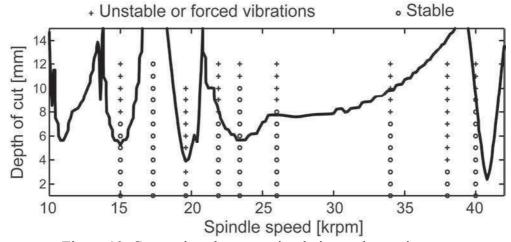


Figure 10. Comparison between simulation and experiments.

#### **Conclusions**

In this paper, a dynamical modeling of spindle with Active Magnetic Bearing (AMB) is presented. A complete but simple numerical model for machining with AMB is proposed. This model uses: the AMB and the servo, the flexible mode of the rotor, the regenerative effect and the nonlinearity when the tool leaves the cut under large vibrations. All this elements are naturally linked to the electro-mechanical system and must absolutely be included in the model in order to obtain acceptable results. For a spindle with AMB, it is indispensable to consider the strong forced vibrations, because they can induce safety mode (stop of the spindle). The stability map is modified, because new areas of large forced vibration appear for high spindle speed. Experimentally, the side milling ramp test was used to easily find the stability limit. There is a good correlation between simulation and experiment, which confirms the assumptions, made for the modeling. This original model can be used, modified and improved easily. All the modeling equations are presented in the paper, all the parameters are obtained by the machining properties and the resolution is made with Matlab-Simulink.

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