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# A Second-order Total Variation Metric on the Space of Immersed Curves

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## Abstract

This paper studies the space of  $BV^2$  planar curves endowed with the  $BV^2$  Finsler metric over its tangent space of displacement vector fields. Such a space is of interest for applications in image processing and computer vision because it enables piecewise regular curves that undergo piecewise regular deformations, such as articulations. The main contribution of this paper is the proof of the existence of a shortest path between any two  $BV^2$  curves for this Finsler metric. The method of proof relies on the construction of a martingale on a space satisfying the Radon-Nikodym property and on the invariance under reparametrization of the Finsler metric. This method applies more generally to similar cases such as the space of curves with  $H^s$  metrics for  $s > 3/2$ . When  $s \geq 2$  is integer, this space has a strong Riemannian structure and is geodesically complete. Thus, our result shows that the exponential map is surjective, which is complementary to geodesic completeness in infinite dimensions. We propose a finite element discretization of the minimal geodesic problem, and use a gradient descent method to compute a stationary point of a relaxed energy. Numerical illustrations show the qualitative difference between  $BV^2$  and  $H^s$  geodesics.

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*Keywords:* Geodesics ; Martingale ;  $BV^2$ -curves ; shape registration

## 1 Introduction

This paper addresses the problem of the existence of minimal geodesics in spaces of planar curves endowed with several metric over the tangent spaces. Given two initial curves we prove the existence of a minimizing geodesic joining them. Such a result is proved by direct methods of calculus of variations.

We treat the case of  $BV^2$ -curves and  $H^s$ -curves ( $s > 3/2$ ). Although the proof's strategy is the same, the  $BV^2$  and  $H^s$  cases are slightly different and the proof in the  $H^s$  case is simpler. This difference is essentially due to the inherent geometric structures (Riemannian or Finslerian) of each space.

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We also propose a finite element discretization of the minimal geodesic problem. We further relax the problem to obtain a smooth non-convex minimization problem. This enables the use of a gradient descent algorithm to compute a stationary point of the corresponding functional. Although these stationary points are not in general global minimizers of the energy, they can be used to explore numerically the geometry of the corresponding spaces of curves, and to illustrate the differences between the Sobolev and  $BV^2$  metrics.

## 1.1 Previous Works

**Shape spaces as Riemannian spaces.** The mathematical study of spaces of curves has been largely investigated in the last years, see for instance [40, 21]. The set of curves is naturally modeled over a Riemannian manifold [23]. This corresponds to using a Hilbertian metric on each tangent plane of the space of curves, i.e. the set of vector fields which deform infinitesimally a given curve. Several recent works [23, 13, 39, 38] point out that the choice of the metric notably affects the results of gradient descent algorithms for the numerical minimization of functionals. Carefully designing the metric is thus crucial to reach better local minima of the energy and also to compute descent flow with particular behavior. These issues are important for application in image processing (e.g. image segmentation) and computer vision (e.g. shape registration). Typical examples of such a Riemannian metrics are Sobolev-type metrics [32, 30, 34, 33] which lead to smooth curve evolutions.

**Shape spaces as Finslerian spaces.** It is possible to extend this Riemannian framework by considering more general metrics on the tangent planes of the space of curves. Finsler spaces make use of Banach norms instead of Hilbertian norms [5]. A few recent works [21, 39, 14] have studied the theoretical properties of Finslerian spaces of curves.

Finsler metrics are used in [14] to perform curve evolution in the space of  $BV^2$ -curves. The authors make use of a generalized gradient which is the steepest descent direction according to the Finsler metric. The corresponding gradient flow enables piecewise regular evolutions, which is useful for applications such as registration of articulated shapes. The present work naturally follows [14]. Instead of considering gradient flows to minimize smooth functionals, we consider the minimal geodesic problem. The existence of solutions to this problem is important to understand the underlying space of curves. This problem is also useful to perform registration of two shapes in computer vision applications.

**Geodesics in shape spaces.** The computation of geodesics over Riemannian spaces is now routinely used in many imaging applications. Typical examples of applications include shape registration [31, 37, 35], tracking [31] and shape deformation [19]. Geodesic computations also serve as the basis to perform statistics on shape spaces, see for instance [37, 1] and to generalize standard tools from Euclidean geometry such as averages [2], linear regression [28] and cubic splines [35], to name a few. However, due to the infinite dimensional nature of shape spaces, not all Riemannian structures lead to well-posed length-minimizing problems. For instance, a striking result [23, 38, 39] is that the natural  $L^2$ -metric on the space of curves is degenerate, despite its widespread use in computer vision applications. Indeed, the geodesic distance between any pair of curves is equal to zero.

The study of the geodesic distance over shape spaces (modeled as curves, surfaces or diffeomorphisms) has been widely developed in the past ten years [24, 8, 7]. We refer the reader to [6] for

a review of this field of research. These authors typically address the questions of existence of the exponential map, geodesic completeness (the exponential map is defined for all time) and the computation of the curvature. In some situations of interest, the shape space has a strong Riemannian metric so that the exponential map is a local diffeomorphism. In [22] the authors describe geodesic equations for Sobolev metrics. They show in Section 4.3 the local existence and uniqueness of a geodesic with prescribed initial conditions. This result is improved in [11], where the authors prove the existence for all time. Both previous results are proved by techniques from ordinary differential equations. In contrast, local existence (and uniqueness) of minimizing geodesics with prescribed boundary conditions (i.e. between a pair of curves) is typically obtained using the exponential map. In finite dimension, existence of minimizing geodesics between any two points (global existence) is obtained by the Hopf-Rinow theorem [25]. Indeed, if the exponential map is defined for all times (i.e. the space is geodesically complete) then global existence holds. This however is not true in infinite dimensions, and a counter example of non-existence of a geodesic between two points over a manifold is given in see [17]. An even more pathological case is described in [3] where an example is given where the exponential map is not surjective although the manifold is geodesically complete. Some positive results exist for infinite dimensional manifold (see in particular Theorem B in [15] and Theorem 1.3.36 [21]) but the surjectivity of the exponential map still needs to be checked directly on a case-by-case basis.

In the case of a Finsler structure on the shape space, the situation is more complicated since the norm over the tangent plane is often non-differentiable. This non-differentiability is indeed crucial to deal with curves and evolutions that are not smooth. This implies that geodesic equations need to be understood in a weak sense. More precisely, the minimal geodesic problem can be seen as a Bolza problem on the trajectories  $H^1([0, 1], BV^2(\mathbb{S}^1, \mathbb{R}^2))$ . In [26] several necessary conditions for existence of solutions to Bolza problems in Banach spaces are proved within the framework of differential inclusions. Unfortunately, these results require some hypotheses on the Banach space (for instance the Radon-Nikodym property) that are not satisfied by the space of functions with bounded variation we consider in this paper. We thus tackle these issues in the present work and prove local and global existence of minimal geodesics in the space of  $BV^2$  curves. We also show how similar techniques can be applied to the case of Sobolev metrics.

## 1.2 Contributions

Section 2 deals with the Finsler space of  $BV^2$ -curves. Our main contributions are Theorems 2.15 and 2.25, proving respectively the local and global existence of a minimizing geodesic between two  $BV^2$ -curves. We also explain how these results can be generalized to the setting of geometric curves (i.e. up to reparameterizations).

Section 3 extends these results to  $H^s$ -curves with  $s > 3/2$ , which gives rise to Theorems 3.3 and 3.4. These results cover both parametric curves and geometric curves. Our results are complementary to those presented in [22] and [11] where the authors show the geodesic completeness of curves endowed with the  $H^s$ -metrics with  $s \geq 2$ . We indeed show that the exponential map is surjective.

Section 4 proposes a discretized minimal geodesic problem for  $BV^2$  and Sobolev curves. We show numerical simulations for the calculation of stationary points of a relaxed energy, which is smooth and thus allows to use gradient descent schemes.

## 2 Geodesics in the Space of $BV^2$ -Curves

In this section we define immersed parameterized  $BV^2$ -curves and prove useful properties (Section 2.1). We also define a geodesic Finsler distance on that space and prove results of local and global existence (Sections 2.3 and 2.4). Finally, we extend this framework to the case of geometric curves (Section 2.5).

The strategy is as follows: we first prove that the geodesic distance is locally equivalent to the ambient Banach norm and then prove local existence using a convergent martingale to a minimum. Global existence is then obtained by controlling the length of the path (in the  $BV^2(\mathbb{S}^1, \mathbb{R}^2)$  case). Last, local existence of geodesics between geometric curves is obtained thanks to weak compactness.

### 2.1 The Finsler Space of Parameterized $BV^2$ -curves

In the next definition we define the space of immersed parameterized  $BV^2$ -curves. This is the obvious generalization of the space of smooth immersed curves

$$\text{Imm}(\mathbb{S}^1, \mathbb{R}^2) = \{c \in C^\infty(\mathbb{S}^1, \mathbb{R}^2) : c'(x) \neq 0 \quad \forall x \in \mathbb{S}^1\}$$

to the space of  $BV^2(\mathbb{S}^1, \mathbb{R}^2)$  curves, which are not necessarily smooth. We remind that  $f \in BV(\mathbb{S}^1, \mathbb{R}^2)$  (we identify  $\mathbb{S}^1$  with  $[0, 1]$ ) if  $f \in W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)$  and its second variation is finite:

$$|D^2 f|(\mathbb{S}^1) = \sup \left\{ \int_{\mathbb{S}^1} f(s) \cdot g''(s) \, d\gamma(s) : g \in C_c^\infty(\mathbb{S}^1, \mathbb{R}^2), \|g\|_{L^\infty(\mathbb{S}^1, \mathbb{R})} \leq 1 \right\} < \infty.$$

Then, the  $BV$ -norm is defined as

$$\|f\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)} = \|f\|_{W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)} + |D^2 f|(\mathbb{S}^1).$$

We also remind that  $BV^2(\mathbb{S}^1, \mathbb{R}^2) \subset W^{1,\infty}(\mathbb{S}^1, \mathbb{R}^2)$  and, in particular,  $BV^2$ -functions are continuous.

**Definition 2.1** ( *$BV^2$  immersed curves*). We consider the Banach space  $BV^2(\mathbb{S}^1, \mathbb{R}^2)$ , where  $\mathbb{S}^1$  is the unit circle. For every  $\gamma \in BV^2(\mathbb{S}^1, \mathbb{R}^2)$  we consider the following property

$$0 \notin \text{Conv}(\gamma'(s^+), \gamma'(s^-)) \quad \forall s \in \mathbb{S}^1 \tag{2.1}$$

where  $\text{Conv}$  denotes the convex envelope (a line segment) of the right and left limits  $(\gamma'(s^-), \gamma'(s^+))$  of the derivative of  $\gamma$  at  $s$ . The set  $\mathcal{B}$  is a connected component of immersed  $BV^2$  parametrized curves defined as

$$\mathcal{B} = \{\gamma \in BV^2(\mathbb{S}^1, \mathbb{R}^2) : \gamma \text{ satisfies (2.1)}\}. \tag{2.2}$$

Note that due to the existence of both clockwise and counter-clockwise parametrizations, the open set (see Proposition 2.3) defined in (2.2) is not connected. In the following we consider counterclockwise oriented curves.

**Remark 2.2.** By Claim 3 p.218 in [16], we have the continuous embedding of  $BV$  in  $L^\infty(\mathbb{S}^1, \mathbb{R}^2)$ :

$$\forall u \in BV(\mathbb{S}^1, \mathbb{R}^2), \quad \|u\|_{L^\infty(\mathbb{S}^1, \mathbb{R}^2)} \leq \|u\|_{BV(\mathbb{S}^1, \mathbb{R}^2)}. \quad (2.3)$$

We use this continuous embedding to prove in the following proposition that  $\mathcal{B}$  is an open set of  $BV^2(\mathbb{S}^1, \mathbb{R}^2)$ .

**Proposition 2.3.** Let  $\gamma_0 \in \mathcal{B}$  be parameterized with respect to the arc-length parameter. We have

$$\left\{ \gamma \in BV^2(\mathbb{S}^1, \mathbb{R}^2) : \|\gamma - \gamma_0\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)} \leq \frac{\mathcal{L}(\gamma_0)}{2} \right\} \subset \mathcal{B}, \quad (2.4)$$

which proves that  $\mathcal{B}$  is an open set of  $BV^2(\mathbb{S}^1, \mathbb{R}^2)$ .

*Proof.* As  $\gamma_0 \in \mathcal{B}$  is parameterized with respect to the arc-length parameter we have  $|\gamma_0'(s)| = \mathcal{L}'(\gamma_0)$  for every  $s \in \mathbb{S}^1$ , so that we have

$$\mathcal{L}(\gamma_0) = \min_{s \in \mathbb{S}^1} |\gamma_0'(s)|.$$

Now, by (2.3), we have  $\|u\|_{L^\infty(\mathbb{S}^1, \mathbb{R}^2)} \leq \|u\|_{BV(\mathbb{S}^1, \mathbb{R}^2)}$  for every  $u \in BV(\mathbb{S}^1, \mathbb{R}^2)$ , so every curve  $\gamma \in BV^2(\mathbb{S}^1, \mathbb{R}^2)$  such that

$$\|\gamma - \gamma_0\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)} \leq \frac{\mathcal{L}(\gamma_0)}{2}$$

satisfies (2.1). □

Property (2.1) allows to generalize standard properties from the setting of smooth curves to  $BV^2$ -curves. Note also that this condition implies that the support of the curve has no cusp points.

**Remark 2.4 (Arc-length parameter).** Property (2.1) allows one to define the arc-length parameter for  $\gamma$ :

$$s_\gamma : \mathbb{S}^1 \rightarrow \mathbb{S}^1$$

$$s_\gamma(s) = \frac{1}{\mathcal{L}(\gamma)} \int_{s_0}^s |\gamma'(t)| dt, \quad s_0 \in \mathbb{S}^1$$

where  $\mathcal{L}(\gamma)$  denotes the length of  $\gamma$  defined as

$$\mathcal{L}(\gamma) = \int_{\mathbb{S}^1} |\gamma'(s)| ds.$$

Finally we point out that for every curve  $\gamma$  its arc-length parameterization is given by  $\gamma \circ \varphi_\gamma$  where  $\varphi_\gamma = s_\gamma^{-1}$ .

We can now generalize to  $BV^2$  curves the usual properties of smooth curves.

**Proposition 2.5.** For every  $\gamma \in \mathcal{B}$  the set  $[\gamma] = \gamma(\mathbb{S}^1)$  can be locally represented as the graph of a  $BV^2$ -function. For every  $\gamma_1, \gamma_2 \in \mathcal{B}$  such that  $[\gamma_1] = [\gamma_2]$ , there exists a homeomorphism  $\varphi \in BV^2(\mathbb{S}^1, \mathbb{S}^1)$  such that

$$\gamma_1 = \gamma_2 \circ \varphi. \quad (2.5)$$

*Proof.* See [14], Proposition 3.7. □

Note that the diffeomorphism  $\varphi$  appearing in (2.5) is not unique.

**Definition 2.6 (Tangent space).** For any  $\gamma \in \mathcal{B}$ , we set  $T_\gamma \mathcal{B} = BV^2(\gamma)$  the space  $BV^2(\mathbb{S}^1, \mathbb{R}^2)$  equipped with the measure  $d\gamma(s) = |\gamma'(s)| ds$ . We define in a similar way the tangent space for curves in  $W^{2,1}(\gamma)$ . In  $BV^2(\gamma)$  and  $W^{2,1}(\gamma)$ , integration and differentiation is done with respect to the arc-length  $d\gamma(s)$ . Namely, the notation  $\frac{dg}{d\gamma(s)}(s)$  stands for  $\frac{1}{|\gamma'(s)|} \frac{dg}{ds}$ .

Moreover, for every  $f \in BV^2(\mathbb{S}^1, \mathbb{R}^2)$  we denote its second variations with respect to the measure  $d\gamma$  by

$$TV_\gamma^2(f) = \sup \left\{ \int_{\mathbb{S}^1} f(s) \cdot \frac{d^2g}{d\gamma(s)^2}(s) d\gamma(s) : g \in C_c^1(\mathbb{S}^1, \mathbb{R}^2), \|g\|_{L^\infty(\mathbb{S}^1, \mathbb{R}^2)} \leq 1 \right\}. \quad (2.6)$$

The next proposition proves that the tangent space to a given curve in  $\mathcal{B}$  and  $BV^2(\mathbb{S}^1, \mathbb{R}^2)$  represent the same space of functions with equivalent norms.

**Proposition 2.7.** Let  $\gamma \in \mathcal{B}$ . The following properties hold:

1. The sets  $BV^2(\gamma)$  and  $BV^2(\mathbb{S}^1, \mathbb{R}^2)$  coincide and their norm are equivalent. More precisely, there exist two positive constants  $M_\gamma, m_\gamma$  such that, for all  $f \in BV^2(\mathbb{S}^1, \mathbb{R}^2)$

$$m_\gamma \|f\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)} \leq \|f\|_{BV^2(\gamma)} \leq M_\gamma \|f\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)}. \quad (2.7)$$

2. We have

$$\|f\|_{L^\infty(\mathbb{S}^1, \mathbb{R}^2)} = \|f\|_{L^\infty(\gamma)}. \quad (2.8)$$

*Proof. (Proof of (2.7))* We suppose that  $f$  is not equal to zero. In the case of the  $L^1$ -norm on  $f$ , the result follows from (2.1) and the compactness of  $\mathbb{S}^1$ . The constants are given respectively by

$$M_\gamma^0 = \|\gamma'\|_{L^\infty(\mathbb{S}^1, \mathbb{R}^2)} \quad m_\gamma^0 = \min_{s \in \mathbb{S}^1} |\gamma'(s)|.$$

Moreover by a straightforward calculation one can easily see that the  $L^1(\gamma)$  and  $L^1(\mathbb{S}^1, \mathbb{R}^2)$ -norms of the first derivative coincide. So it is sufficient to obtain the result for the second variation of  $f \in BV^2(\gamma)$ .

By integration by part, we have

$$\int_{\mathbb{S}^1} f(s) \cdot \frac{d^2g}{d\gamma^2(s)}(s) d\gamma(s) = \int_{\mathbb{S}^1} \frac{f'}{|\gamma'|} g'(s) ds$$

where we used the fact that  $\frac{dg}{d\gamma(s)} = \frac{g'}{|\gamma'|}$ . This implies in particular that

$$TV_\gamma^2(f) = \left| D \frac{f'}{|\gamma'|} \right|(\mathbb{S}^1).$$

Since  $\frac{1}{|\gamma'|} \in BV(\mathbb{S}^1, \mathbb{R}^2)$  and  $BV(\mathbb{S}^1, \mathbb{R}^2)$  is a Banach algebra, we get

$$\left| D \frac{f'}{|\gamma'|} \right|(\mathbb{S}^1) \leq |Df'|(\mathbb{S}^1) \left| D \frac{1}{|\gamma'|} \right|(\mathbb{S}^1).$$

Now, as  $|Df'|(\mathbb{S}^1) \leq |D^2f|(\mathbb{S}^1)$ , we get the upper bound taking the constant

$$M_\gamma^2 = \left| D \frac{1}{|\gamma'|} \right|(\mathbb{S}^1).$$

On the other hand, we have

$$\int_{\mathbb{S}^1} f'g'(s) ds = \int_{\mathbb{S}^1} \frac{df}{d\gamma(s)}(s) \frac{dg}{d\gamma(s)}(s) |\gamma'| d\gamma(s)$$

so that

$$|Df'|(\mathbb{S}^1) = TV_\gamma \left( \frac{df}{d\gamma(s)} |\gamma'| \right),$$

which implies that

$$|D^2f|(\mathbb{S}^1) \leq TV_\gamma(|\gamma'|) TV_\gamma^2(f) = \|\gamma'\|_{BV(\mathbb{S}^1, \mathbb{R})} TV_\gamma^2(f). \quad (2.9)$$

Therefore, the result is proved by taking the constant

$$m_\gamma^2 = \frac{1}{\|\gamma'\|_{BV(\mathbb{S}^1, \mathbb{R}^2)}}.$$

The lemma ensues setting

$$\begin{aligned} M_\gamma &= \max \{M_\gamma^0, M_\gamma^2\} = \max \left\{ \|\gamma'\|_{L^\infty(\mathbb{S}^1, \mathbb{R}^2)}, \left| D \frac{1}{|\gamma'|} \right|(\mathbb{S}^1) \right\}, \\ m_\gamma &= \min \{m_\gamma^0, m_\gamma^2\} = \min \left\{ \min_{s \in \mathbb{S}^1} |\gamma'(s)|, 1/\|\gamma'(\cdot)\|_{BV(\mathbb{S}^1, \mathbb{R}^2)} \right\}. \end{aligned} \quad (2.10)$$

(*Proof of (2.8)*) It follows straightforwardly from (2.1) that the two measures are equivalent, i.e.

$$\int_A ds = 0 \Leftrightarrow \int_A d\Gamma(s) = 0$$

for every open set  $A$  of the circle. □

## 2.2 Useful Results for the Existence of Geodesics

In this section, we prove that geodesics for the induced Finsler metric exist for any given couple of curves in the same homotopy class.

**Definition 2.8 (Paths in  $\mathcal{B}$ ).** For every  $\gamma_0, \gamma_1 \in \mathcal{B}$ , we define a path in  $\mathcal{B}$  joining  $\gamma_0$  and  $\gamma_1$  as a function

$$\Gamma : t \in [0, 1] \mapsto \Gamma(t) \in \mathcal{B}$$

such that

$$\Gamma(0) = \gamma_0 \quad \Gamma(1) = \gamma_1. \quad (2.11)$$

For every  $\gamma_0, \gamma_1 \in \mathcal{B}$ , we denote  $\mathcal{P}(\gamma_0, \gamma_1)$  the class of all paths joining  $\gamma_0, \gamma_1$  and belonging to  $H^1([0, 1], \mathcal{B})$ . It holds in particular

$$\forall a.e. s \in \mathbb{S}^1, \quad \int_0^1 \Gamma_t(t)(s) dt = \gamma_1(s) - \gamma_0(s) \quad (2.12)$$

where  $\Gamma_t$  denotes the derivative of  $\Gamma$  with respect to  $t$ . In the following  $\Gamma'(t)$  denotes the derivative of the curves  $\Gamma(t) \in BV^2(\mathbb{S}^1, \mathbb{R}^2)$  with respect to  $s$ .



**Definition 2.9 (Geodesic paths in  $\mathcal{B}$ ).** For every path  $\Gamma$  we consider the following energy

$$E(\Gamma) = \int_0^1 \|\Gamma_t(t)\|_{BV^2(\Gamma(t))}^2 dt. \quad (2.13)$$

The geodesic distance between  $\gamma_0$  and  $\gamma_1$  is denoted by  $d(\gamma_0, \gamma_1)$  and defined by

$$d^2(\gamma_0, \gamma_1) = \inf \{E(\Gamma) : \Gamma \in \mathcal{P}(\gamma_0, \gamma_1)\}. \quad (2.14)$$

A geodesic between  $\gamma_0$  and  $\gamma_1$  is a path  $\tilde{\Gamma} \in \mathcal{P}(\gamma_0, \gamma_1)$  such that

$$E(\tilde{\Gamma}) = d^2(\gamma_0, \gamma_1).$$

Note that because of the lack of smoothness of the  $BV^2$  norm over the tangent space, it is not possible to define an exponential map. Geodesics should thus be understood as minimal geodesics, and are thus homotopies defined using the variational problem 2.14. Recall that the existence of (minimizing) geodesics is not granted in infinite dimension. When the metric is Riemannian and even if the manifold is geodesically complete, the existence of minimizers is not guaranteed since the Hopf-Rinow theorem does not hold in infinite dimension, see Section 1.1.

**Remark 2.10 (Time-reparameterization).** We point out that analogously to Remark 2.4 we can reparameterize every non-trivial homotopy  $\Gamma$  (i.e. satisfying  $E(\Gamma) \neq 0$ ) with respect to the time-arc-length defined as the inverse of the following parameter:

$$t_\Gamma : \mathbb{S}^1 \rightarrow \mathbb{S}^1$$

$$\forall t \in [0, 1], \quad t_\Gamma(t) = \frac{1}{E(\Gamma)} \int_0^t \|\Gamma_\tau(\tau)\|_{BV^2(\Gamma(\tau))} d\tau.$$

In particular, if  $\|\Gamma_\tau(\tau)\|_{BV^2(\Gamma(\tau))} \neq 0$  for every  $t$ , by reparameterizing the homotopy with respect to previous parameter we obtain a parameterization with constant velocity

$$\|\Gamma_t(t)\|_{BV^2(\Gamma(t))} = E(\Gamma) \quad \forall t \in [0, 1]. \quad (2.15)$$

In the general case, for every  $\varepsilon > 0$ , we can always prove the existence of a time-parameterization such that

$$\|\Gamma_t(t)\|_{BV^2(\Gamma(t))} \leq E(\Gamma) + \varepsilon \quad \forall t \in [0, 1].$$

We refer to the proof of Theorem 8.18 p.175 in [41] for more details.

The next proposition proves simple properties that are used later.

**Proposition 2.11.** Let  $\Gamma \in \mathcal{P}(\gamma_0, \gamma_1)$ . Then the following properties hold:

1. The function

$$t \mapsto g(t) = \|\Gamma'(t)\|_{L^\infty(\mathbb{S}^1, \mathbb{R}^2)}$$

belongs to  $C([0, 1], \mathcal{B})$ , so in particular it admits a maximum and a positive minimum on  $[0, 1]$ .

Moreover

$$\min_{t \in [0, 1]} \min_{s \in \mathbb{S}^1} |\Gamma'(t)(s)| > 0.$$

2. For every  $t \in [0, 1]$  we have

$$e^{-\mathcal{L}(\gamma_0) - E(\Gamma)} \leq \mathcal{L}(\Gamma(t)) \leq e^{-\mathcal{L}(\gamma_0) + E(\Gamma)} \quad (2.16)$$

and for all  $s \in \mathbb{S}^1$ ,

$$\exp\left(\min_{s \in \mathbb{S}^1} |\gamma'_0(s)| - E(\Gamma)\right) \leq |\Gamma'(t)(s)| \leq \exp\left(\|\gamma'_0\|_{L^\infty(\mathbb{S}^1, \mathbb{R}^2)} + E(\Gamma)\right). \quad (2.17)$$

*Proof.* 1. By Definition 2.9, every  $\Gamma \in \mathcal{P}(\gamma_0, \gamma_1)$  belongs to  $H^1([0, 1], \mathcal{B})$  so in particular to  $C([0, 1], \mathcal{B})$ . Now, as  $BV^2(\mathbb{S}^1, \mathbb{R}^2)$  is embedded in  $L^\infty(\mathbb{S}^1, \mathbb{R}^2)$ , we get the continuity of  $g$ . Moreover, from (2.1) and the compactness of  $\mathbb{S}^1$ , it follows that

$$\min_{s \in \mathbb{S}^1} |\Gamma'(t)(s)| > 0, \quad \forall t \in [0, 1]$$

which implies that  $\min_{t \in [0, 1]} g(t) > 0$ . The second statement is a straightforward consequence of (2.1) and the compactness of  $[0, 1]$ .

2. By Remark 2.10 we can suppose that the time-velocity is constant. We have

$$\frac{\partial \mathcal{L}(\Gamma(t))}{\partial t} = \int_{\mathbb{S}^1} \left\langle \frac{\Gamma'_t(t)}{|\Gamma'(t)|}, \Gamma'(t) \right\rangle ds \leq \left\| \frac{d\Gamma_t(t)}{d\Gamma(t)} \right\|_{L^\infty(\mathbb{S}^1, \mathbb{R}^2)} \mathcal{L}(\Gamma(t)).$$

so, by (2.15) and (2.3), we get

$$\frac{\partial \log(\mathcal{L}(\Gamma(t)))}{\partial t} \leq E(\Gamma)$$

and, by integrating in time between 0 and  $t$ , we get

$$e^{-\mathcal{L}(\gamma_0) - E(\Gamma)} \leq \mathcal{L}(\Gamma(t)) \leq e^{-\mathcal{L}(\gamma_0) + E(\Gamma)}.$$

Similarly we get

$$\frac{\partial |\Gamma'(t)(s)|}{\partial t} = \left\langle \frac{\Gamma'_t(t)}{|\Gamma'(t)|}, \Gamma'(t)(s) \right\rangle \leq \left\| \frac{d\Gamma_t(t)(s)}{d\Gamma(t)} \right\|_{L^\infty(\mathbb{S}^1, \mathbb{R}^2)} |\Gamma'(t)(s)|.$$

so, by (2.15) and (2.3), we get

$$\frac{\partial \log(|\Gamma'(t)(s)|)}{\partial t} \leq E(\Gamma), \quad \forall s \in \mathbb{S}^1$$

and, by integrating in time between 0 and  $t$ , we get

$$e^{|\gamma'_0(s)| - E(\Gamma)} \leq |\Gamma'(t)(s)| \leq e^{|\gamma'_0(s)| + E(\Gamma)}, \quad \forall s \in \mathbb{S}^1$$

and the result ensues.  $\square$

The previous proposition proves in particular that the constants  $m_\gamma$  and  $M_\gamma$  defined in (2.20) are bounded on every path belonging to  $\mathcal{P}(\gamma_0, \gamma_1)$  so that

$$0 < \inf_{t \in [0, 1]} m_{\Gamma(t)} < \sup_{t \in [0, 1]} M_{\Gamma(t)} < +\infty, \quad \forall \Gamma \in \mathcal{P}(\gamma_0, \gamma_1).$$

We now prove a result which gives a condition to obtain a uniform bound of  $m_\gamma$  and  $M_\gamma$  on every minimizing sequence of  $d(\gamma_0, \gamma_1)$ . For that we need the following lemma.

**Lemma 2.12 (Cutting paths).** Let  $\gamma_0, \gamma_1 \in \mathcal{B}$  such that  $\gamma_1 \in B_\varepsilon$  with

$$B_\varepsilon = \{\gamma \in BV^2(\mathbb{S}^1, \mathbb{R}^2) : \|\gamma - \gamma_0\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)} \leq \varepsilon\}$$

where  $\varepsilon > 0$  is chosen so that  $B_\varepsilon \subset \mathcal{B}$ . Let  $\Gamma \in \mathcal{P}(\gamma_0, \gamma_1)$  be a path which goes out  $B_\varepsilon$ . This means that there exists  $(a, b) \subset [0, 1]$  such that for every  $t \in (a, b)$  we have  $\Gamma(t) \notin B_\varepsilon$ . Then

$$E(\Gamma) \geq \varepsilon \inf_{\gamma \in B_\varepsilon} m_\gamma.$$

*Proof.* Remark that, as  $B_\varepsilon$  is sequentially compact with respect to the  $BV^{2,*}$  topology, the terms defining  $m_\gamma$  in (2.10) are bounded which implies that

$$\inf_{\gamma \in B_\varepsilon} m_\gamma > 0.$$

In the following we set  $\partial B_\varepsilon = \{\gamma \in \mathcal{B} : \|\gamma - \gamma_0\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)} = \varepsilon\}$  and

$$\mathcal{P}(\gamma_0, \gamma_1) \cap B_\varepsilon = \{\Gamma \in \mathcal{P}(\gamma_0, \gamma_1) : \Gamma(t) \in B_\varepsilon, \forall t \in [0, 1]\}.$$

Then, for every  $\gamma_1 \in B_\varepsilon$  and for every path  $\Gamma \in \mathcal{P}(\gamma_0, \gamma_1) \cap B_\varepsilon$  we have

$$\|\gamma_1 - \gamma_0\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)} \leq \int_0^1 \frac{\|\Gamma_t(t)\|_{BV^2(\Gamma(t))}}{m_{\Gamma(t)}} dt \leq \frac{E(\Gamma)}{m} \quad (2.18)$$

which implies

$$E(\Gamma) \geq m \|\gamma_1 - \gamma_0\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)}, \quad \forall \gamma_1 \in B_\varepsilon \forall \Gamma \in \mathcal{P}(\gamma_0, \gamma_1) \cap B_\varepsilon. \quad (2.19)$$

Now if  $\Gamma \in \mathcal{P}(\gamma_0, \gamma_1)$  be a path which goes out  $B_\varepsilon$  we can consider

$$\tilde{t} = \min \{t \in [0, 1] : \Gamma(t) \in \partial B_\varepsilon\}$$

and the path  $\Gamma$  on  $[0, \tilde{t}]$  can be seen (up to a time-reparameterization) as a path  $\tilde{\Gamma} \in \mathcal{P}(\gamma_0, \Gamma(\tilde{t}, \cdot))$  joining  $\gamma_0$  and  $\Gamma(\tilde{t}, \cdot)$ . Then we have we have  $E(\Gamma) \geq E(\tilde{\Gamma}) \geq m\varepsilon$ , which proves the result.  $\square$

We can now prove the following important result which is used later to show that for each curve there exists a ball on which the geodesic boundary value problem (for that curve and any other curve in the ball) admits a solution.

**Proposition 2.13.** For every  $\gamma_0 \in \mathcal{B}$  there exists  $r$  and  $R$ ,  $0 < r < R$ , such that if

$$\|\gamma_1 - \gamma_0\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)} \leq r$$

then for every minimizing sequence  $\{\Gamma^h\}_h \subset \mathcal{P}(\gamma_0, \gamma_1)$  for  $d(\gamma_0, \gamma_1)$ , for  $h$  large enough, we have

$$\|\Gamma^h(t) - \gamma_0\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)} \leq R \quad \forall t \in [0, 1]$$

and

$$0 < \inf_h \inf_{t \in [0, 1]} m_{\Gamma^h(t)} < \sup_h \sup_{t \in [0, 1]} M_{\Gamma^h(t)} < +\infty,$$

where  $m_\gamma$  and  $M_\gamma$  are the constants defined in (2.10). Remark that  $r$  and  $R$  depend on  $\gamma_0$  and  $R$  is such that

$$B_R = \{\gamma \in \mathcal{B} : \|\gamma - \gamma_0\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)} \leq R\} \subset \mathcal{B}.$$

*Proof.* We remind that  $\mathcal{B}$  is an open set of  $BV^2(\mathbb{S}^1, \mathbb{R}^2)$  so there exists  $R > 0$  such that

$$B_R = \{\gamma \in \mathcal{B} : \|\gamma - \gamma_0\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)} \leq R\} \subset \mathcal{B}.$$

Moreover,  $B_R$  is compact with respect to the  $BV^2$ -\* topology and the usual embedding theorems for  $BV^2$  imply that the terms defining the constants  $m_\gamma$  and  $M_\gamma$  in (2.10) are uniformly bounded on  $B_R$ , which means

$$0 < m = \inf_{\gamma \in B_R} m_\gamma < M = \sup_{\gamma \in B_R} M_\gamma < +\infty. \quad (2.20)$$

Moreover, by considering the path  $\Gamma(t) = t\gamma_1 + (1-t)\gamma_0$ , we get

$$E(\Gamma) = \int_0^1 \|\gamma_1 - \gamma_0\|_{BV^2(\Gamma(t))} dt \leq M \|\gamma_1 - \gamma_0\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)}, \quad (2.21)$$

which implies

$$d^2(\gamma_0, \gamma_1) \leq M \|\gamma_1 - \gamma_0\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)} \quad \forall \gamma_1 \in B_R. \quad (2.22)$$

Now we consider a smaller ball

$$B_r = \{\gamma \in \mathcal{B} : \|\gamma - \gamma_0\|_{BV(\mathbb{S}^1, \mathbb{R}^2)} \leq r\} \subset \mathcal{B} \quad (2.23)$$

with  $r$  such that

$$Mr < mR \quad (2.24)$$

then for every  $\gamma_1 \in B_r$  by (2.22) we have  $d^2(\gamma_0, \gamma_1) \leq Mr$ . Moreover, by Lemma 2.12, for every  $\gamma_1 \in B_r$  and for every path  $\Gamma \in \mathcal{P}(\gamma_0, \gamma_1)$  going out  $B_R$  we get  $E(\Gamma) \geq mR$ . Now, as  $Mr < mR$ , we get that if  $\gamma_1 \in B_r$ , then every minimizing path for  $d(\gamma_0, \gamma_1)$  belongs to  $B_R$ .

The bounds on  $m_{\Gamma^h(t)}$  and  $M_{\Gamma^h(t)}$  follow from (2.20) which ends the proof.  $\square$

**Remark 2.14.** Concerning the choice of  $R$  in the previous proposition, we point out that if  $\gamma_0$  is parameterized with respect to the arc-length parameter, then we can set

$$R = \frac{\mathcal{L}(\gamma_0)}{2}.$$

Moreover in this case we have

$$m_{\gamma_0} = \min \left\{ \mathcal{L}(\gamma_0), \frac{1}{\mathcal{L}(\gamma_0)} \right\}, \quad M_{\gamma_0} = \mathcal{L}(\gamma_0). \quad (2.25)$$

### 2.3 Local Existence of Geodesics

We can now prove a theorem of local existence for geodesics. This result is based on the martingale technique used in [20] (Proposition 3.2). Such a technique allows one to define a suitable topology in order to get compactness for minimizing sequences of paths.

**Theorem 2.15 (Local existence).** Let  $\gamma_0 \in \mathcal{B}$  and let  $\gamma_1 \in \mathcal{B}$  such that

$$\|\gamma_1 - \gamma_0\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)} \leq r$$

where  $r$  is given by Proposition 2.13. Then, there exists a geodesic between  $\gamma_0$  and  $\gamma_1$ .

*Proof.* Let  $\{\Gamma^h\}_h \subset \mathcal{P}(\gamma_0, \gamma_1)$  be a minimizing sequence for  $E$ . Without loss of generality we can suppose  $\sup_h E(\Gamma_h) < +\infty$ . The proof is divided into two steps.

**Step 1: Definition of a limit path.** For every  $n > 1$  we consider the dyadic decomposition of  $[0, 1]$  given by the intervals

$$I_{n,k} = \left[ \frac{k}{2^n}, \frac{k+1}{2^n} \right[ \quad \text{for } k \in [0, 2^n - 1] \quad (2.26)$$

and, for every  $t \in [0, 1]$  we define

$$f_n^h(t) = 2^n \int_{I_{n,k}} \Gamma_\tau^h(\tau) d\tau,$$

where  $I_{n,k}$  is the interval containing  $t$ . Remark that, for every  $n$  and  $h$ ,  $f_n^h : [0, 1] \rightarrow BV^2(\mathbb{S}^1, \mathbb{R}^2)$  is piecewise-constant and

$$\int_0^1 f_n^h(\tau) d\tau = \int_0^1 \Gamma_t^h(t) dt = \Gamma(1) - \Gamma(0). \quad (2.27)$$

Now, by Proposition 2.13, we have  $m = \inf_h \min_{t \in [0,1]} m_{\Gamma^h(t)} > 0$  and similarly to (2.18) we get

$$\|f_n^h(t)\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)} \leq \frac{2^n}{m} E(\Gamma^h) \quad \forall t \in [0, 1]. \quad (2.28)$$

So, by a diagonal extraction, we can take a subsequence (not relabeled) and a piecewise-constant function  $f_n^\infty : [0, 1] \rightarrow BV^2(\mathbb{S}^1, \mathbb{R}^2)$  such that

$$\forall (n, k), \quad f_n^h(t) \xrightarrow{*BV^2} f_n^\infty(t), \quad \forall t \in [0, 1]$$

and

$$\int_0^1 \|f_n^\infty(t)\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)}^2 dt \leq \liminf_{h \rightarrow \infty} \int_0^1 \|f_n^h(t)\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)}^2 dt.$$

Moreover we can write  $I_{n,k} = [k2^{-n}, (2k+1)2^{-n-1}[ \cup [(2k+1)2^{-n-1}, (k+1)2^{-n}[$  and

$$f_{n+1}^h(t) = 2^{n+1} \int_{I_{n+1,2k}} \Gamma_t^h(t) dt \quad \forall t \in [k2^{-n}, (2k+1)2^{-n-1}[$$

$$f_{n+1}^h(t) = 2^{n+1} \int_{I_{n+1,2k+1}} \Gamma_t^h(t) dt \quad \forall t \in [(2k+1)2^{-n-1}, (k+1)2^{-n}[$$

therefore

$$\begin{aligned} \int_{I_{n,k}} f_{n+1}^h(t) dt &= \int_{I_{n,k}} \Gamma_t^h(t) dt \\ f_n^h(t) &= 2^n \int_{I_{n,k}} f_{n+1}^h(t) dt \quad \forall t \in I_{n,k}. \end{aligned}$$

Then, by the Dominated Convergence Theorem, we get

$$f_n^\infty(t) = 2^n \int_{I_{n,k}} f_{n+1}^\infty(t) dt \quad (2.29)$$

and in addition, by Fatou's Lemma and (2.28), we get

$$\begin{aligned} \int_0^1 \|f_n^\infty(t)\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)}^2 dt &\leq \liminf_{h \rightarrow \infty} \int_0^1 \|f_n^h(t)\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)}^2 dt \\ &\leq \liminf_{h \rightarrow \infty} \frac{E(\Gamma^h)}{m}. \end{aligned}$$

Now, as  $BV^2(\mathbb{S}^1, \mathbb{R}^2)$  is embedded in  $H^1(\mathbb{S}^1, \mathbb{R}^2)$ , this implies that  $\{f_n^\infty\}$  is a bounded martingale in  $L^2([0, 1], H^1(\mathbb{S}^1, \mathbb{R}^2))$  so, by the Convergence Theorem for Martingales,  $f_n^\infty(t) \rightarrow f(t)$  in  $H^1(\mathbb{S}^1, \mathbb{R}^2)$  for almost every  $t$ . Note also that, as  $f_n^\infty \in BV^2(\mathbb{S}^1, \mathbb{R}^2)$  and the second variation is lower semicontinuous with respect to the  $W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)$ -convergence, we actually get  $f \in L^2([0, 1], BV^2(\mathbb{S}^1, \mathbb{R}^2))$ .

We can now define a candidate to be a minimum of  $E$  by setting

$$\Gamma^\infty(t) = \int_0^t f(\tau) d\tau + \Gamma(0), \quad \forall t \in [0, 1]. \quad (2.30)$$

We can easily verify that  $\Gamma^\infty \in H^1([0, 1], BV^2(\mathbb{S}^1, \mathbb{R}^2))$  and that  $\Gamma^\infty$  verifies (2.12). In fact, by the Dominated Convergence Theorem and (2.27), we have

$$\int_0^1 f(\tau) d\tau = \lim_{n \rightarrow \infty} \int_0^1 f_n^\infty(\tau) d\tau = \lim_{n \rightarrow \infty} \lim_{h \rightarrow \infty} \int_0^1 f_n^h(\tau) d\tau = \Gamma(1, \cdot) - \Gamma(0, \cdot).$$

This implies in particular that  $\Gamma^\infty$  verifies (2.11). So we get  $\Gamma^\infty \in \mathcal{P}(\gamma_0, \gamma_1)$ .

**Step 2:  $\Gamma^\infty$  is a geodesic path.** We now prove that  $\Gamma^h(t) \rightarrow \Gamma^\infty(t)$  in  $W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)$  for every  $t$ . For that we denote by  $\Gamma_n^\infty$  and  $\Gamma_n^h$  the paths defined by  $f_n^\infty$  and  $f_n^h$  through (2.30). Now, by definition of  $f_n^h$  and a straightforward calculation, we get that  $\|\Gamma^h(t) - \Gamma_n^h(t)\|_{W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)}$  is small for  $n$  large enough. Moreover, as  $f_n^h(t) \xrightarrow{*BV^2} f_n^\infty(t)$  for every  $t$  and  $f_n^\infty \rightarrow f$  in  $L^2([0, 1], BV^2(\mathbb{S}^1, \mathbb{R}^2))$  we have  $\|\Gamma_n^\infty(t) - \Gamma_n^h(t)\|_{W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)} \rightarrow 0$ , as  $h \rightarrow \infty$  and that  $\|\Gamma_n^\infty(t) - \Gamma^\infty(t)\|_{W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)}$  is small for  $n$  large enough. This implies that

$$\|\Gamma^h(t) - \Gamma^\infty(t)\|_{W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)} \rightarrow 0. \quad (2.31)$$

By the same arguments we can show that

$$\|\Gamma_t^h(t) - \Gamma_t^\infty(t)\|_{W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)} \rightarrow 0, \quad \forall t \in [0, 1], \quad (2.32)$$

which implies in particular that  $\{\Gamma_t^h(t)\}_h$  is bounded in  $BV^2(\mathbb{S}^1, \mathbb{R}^2)$  for almost every  $t$ .

Now, as  $\Gamma^h(t)$  converges in  $W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)$  towards  $\Gamma^\infty(t) \in BV^2(\mathbb{S}^1, \mathbb{R}^2)$ , the arc-length parameterizations at the time  $t$ , denoted respectively by  $\varphi^h(t)$  and  $\varphi^\infty(t)$ , also converge in  $W^{1,1}(\mathbb{S}^1, \mathbb{S}^1)$ . Moreover, for a fixed time  $t$ , we have  $\|\Gamma_t^h(t)\|_{BV^2(\Gamma^h(t))}^2 = \|\Gamma_t^h \circ \varphi^h(t)\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)}^2$ . Since  $\{\Gamma_t^h(t)\}_h$  is bounded in  $BV^2(\mathbb{S}^1, \mathbb{R}^2)$  and the two terms involved in the composition converge in  $W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)$ , we have

$$\|\Gamma_t^h \circ \varphi^h(t) - \Gamma_t^h \circ \varphi^\infty(t)\|_{W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)} \leq \|\Gamma_t^h\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)} \|\varphi^h(t) - \varphi^\infty(t)\|_{W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)}$$

$$\|\Gamma_t^h \circ \varphi^\infty(t) - \Gamma_t^\infty \circ \varphi^\infty(t)\|_{W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)} \leq C \|\Gamma_t^h - \Gamma_t^\infty\|_{W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)}$$

where

$$C = \max \{1, \|(\varphi^{\infty^{-1}})'\|_{L^\infty(\mathbb{S}^1, \mathbb{R}^2)}\}.$$

This implies in particular that

$$\Gamma_t^h \circ \varphi^h(t) \xrightarrow{W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)} \Gamma_t^\infty \circ \varphi^\infty(t). \quad (2.33)$$

Moreover, as the second variation is lower semi-continuous with respect to the  $W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)$  convergence, for every  $t$  we get

$$\begin{aligned} \|\Gamma_t^\infty(t)\|_{BV^2(\Gamma^\infty(t))}^2 &= \|\Gamma_t^\infty \circ \varphi^\infty(t)\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)}^2 \\ &\leq \liminf_{h \rightarrow \infty} \|\Gamma_t^h \circ \varphi^h(t)\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)}^2 = \liminf_{h \rightarrow \infty} \|\Gamma_t^h(t)\|_{BV^2(\Gamma^h(t))}^2. \end{aligned} \quad (2.34)$$

By integrating the previous inequality and using Fatou's Lemma we get that  $\Gamma^\infty$  minimizes  $E$ , which ends the proof.  $\square$

**Remark 2.16.** In order to get the semicontinuity's inequality (2.34), we actually just would need the convergence in (2.33) with respect to the  $BV^2$ -weak topology. We point out that it is very difficult to characterize such a topology, which explains the choice of the weak-\* topology (i.e. we look at  $BV^2(\mathbb{S}^1, \mathbb{R}^2)$  as a dual Banach space) in the previous proof.

Moreover, the martingale approach allows one to get strong convergence in  $W^{1,1}$  without applying any strong-compactness criterion for Sobolev spaces. This is a key point of the proof because the  $BV^2$ -norm is semicontinuous with respect to the strong  $W^{1,1}$ -topology (proved at Step 2).

Inspired by the previous proof we can define the following topology on  $H^1([0, 1], \mathcal{B})$ :

**Definition 2.17 ( $\sigma$ -topology).** Let  $\{\Gamma^h\}_h \subset H^1([0, 1], \mathcal{B})$  and  $\Gamma \in H^1([0, 1], \mathcal{B})$ . We say that  $\Gamma_h$  converges to  $\Gamma$  with respect to the  $\sigma$ -topology (denoted by  $\Gamma^h \xrightarrow{\sigma} \Gamma$ ) if there exists a sequence of piecewise constant functions  $\{f_n^\infty\}$  such that:

- (i) ( **$BV^2$ -\* weak convergence**). Let  $\{I_{n,k}\}_{n,k}$  the collection of the intervals giving the dyadic decomposition of  $[0, 1]$  defined in (2.26), then the sequence

$$f_n^h(t) = 2^n \int_{I_{n,k}} \Gamma_\tau^h(\tau) d\tau$$

verifies

$$\forall (n, k) \quad f_n^h(t) \xrightarrow{*BV^2} f_n^\infty(t) \quad \forall t \in [0, 1]$$

as  $h \rightarrow \infty$ ;

- (ii) (**Martingale convergence**). We have

$$\lim_{n \rightarrow \infty} \|f_n^\infty - \Gamma_t\|_{L^2([0,1], H^1(\mathbb{S}^1, \mathbb{R}^2))} = 0.$$

Then, the proof of Theorem 2.15 proves actually the following result.

**Theorem 2.18.** The following properties hold:

- (i) Every bounded set of  $H^1([0, 1], \mathcal{B})$  is sequentially compact with respect to  $\sigma$ -topology; in particular the  $\sigma$ -convergence implies the strong convergence in  $H^1([0, 1], W^{1,1}(\mathbb{S}^1, \mathbb{R}^2))$ ;
- (ii) The energy  $E$  is lower semicontinuous with respect to the  $\sigma$ -topology.

**Remark 2.19.** The proof of Theorem 2.15 can now be presented as it follows. For  $h$  large enough, we can suppose that  $\{\Gamma^h\} \subset H^1([0, 1], \mathcal{B} \cap B_R(\gamma_0))$  where

$$B_R(\gamma_0) = \{\gamma \in \mathcal{B} : \|\gamma - \gamma_0\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)} \leq R\}$$

for some  $\gamma_0 \in \mathcal{B}$  and  $R$  is given by Proposition 2.13. We also suppose that  $\sup E(\Gamma^h) < \infty$ .

Then  $\{\Gamma^h\}$  is uniformly bounded in  $H^1([0, 1], \mathcal{B})$  and, by points (i) and (ii) of previous theorem, energy  $E$  reaches its minimum on  $H^1([0, 1], \mathcal{B} \cap B_R(\gamma_0))$ .

## 2.4 Global Existence of Geodesics

We are now able to prove global existence of geodesics between an arbitrary pair of curves.

**Theorem 2.20 (Global existence).** Let  $\gamma_0, \gamma_1 \in \mathcal{B}$  with  $d(\gamma_0, \gamma_1) < \infty$ . Then there exists a minimizing geodesic path joining them.

*Proof.* Remark that, since the distance  $d$  is invariant under reparameterization, so that

$$d(\gamma_0, \gamma_1) = d(\gamma_0 \circ \varphi, \gamma_1 \circ \varphi), \quad \forall \varphi \in \text{Diff}(\mathbb{S}^1), \quad (2.35)$$

Then we can suppose (up to a composition with  $\varphi_{\gamma_0}$ ) that  $\gamma_0$  is parameterized with respect to the arc-length parameter. Remark that, as the reparameterizations do not change the length of the curves, for every  $\gamma \in \mathcal{B}$  we have  $\gamma \circ \varphi \in \mathcal{B}$  for every  $\varphi \in \text{Diff}(\mathbb{S}^1)$ .

The proof is based on an iteration argument starting at  $\gamma_0$ . At each step, by the argument used in the proof of Theorem 2.15, we find locally a part of the geodesic path starting at some current curve. We conclude the proof by showing that we can reach  $\gamma_1$  by a finite number of iterations.

**Step 1.** By Remark 2.14 we can take  $R_0 = \frac{\mathcal{L}(\gamma_0)}{2}$  as the constant  $R$  in order to apply Proposition 2.13. Then we get

$$m_0 = \inf_{\gamma \in B_{R_0}} m_\gamma < M_0 = \sup_{\gamma \in B_{R_0}} M_\gamma < +\infty, \quad (2.36)$$

where  $m_\gamma, M_\gamma$  are defined in (2.10) and

$$B_{R_0} = \{\gamma \in BV^2(\mathbb{S}^1, \mathbb{R}^2) : \|\gamma - \gamma_0\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)} \leq R_0\} \subset \mathcal{B}.$$

We consider  $0 < \varepsilon_0 < m_0 R_0$  and we set

$$B_{\varepsilon_0}^d = \{\gamma \in \mathcal{B} : d^2(\gamma, \gamma_0) \leq \varepsilon_0\},$$

$$\partial B_{\varepsilon_0}^d = \{\gamma \in \mathcal{B} : d^2(\gamma, \gamma_0) = \varepsilon_0\}.$$

Remark that, as  $\varepsilon_0 < m_0 R_0$ , by Lemma 2.12, we get  $B_{\varepsilon_0}^d \subset B_{R_0}$ .



Let  $\{\Gamma^h\} \subset \mathcal{P}(\gamma_0, \gamma_1)$  be a minimizing sequence for  $d(\gamma_0, \gamma_1)$ . For every  $h$  we define (up to a time-reparameterization) the path  $\tilde{\Gamma}^h$  obtained as a restriction of  $\Gamma^h$  on  $B_{\varepsilon_0}^d$ .

Such a path belongs to  $\mathcal{P}(\gamma_0, \Gamma^h(\tilde{t}_h, \cdot))$  where

$$\tilde{t}_h = \min \{t \in [0, 1] : \Gamma^h(t) \in \partial B_{\varepsilon_0}\}.$$

Remark that all paths  $\tilde{\Gamma}^h$  verify  $\tilde{\Gamma}^h(0, \cdot) = \gamma_0$  and  $\tilde{\Gamma}^h(1, \cdot)$  belongs to  $\partial B_{\varepsilon_0}$ . Although the end point of these paths is not fixed, the martingale argument used in the Step 1 of the proof of Theorem 2.15 can still be applied.

So, we can find  $\tilde{\gamma}_1 \in \partial B_{\varepsilon_0}$  and a path  $\tilde{\Gamma}^1 \in \mathcal{P}(\gamma_0, \tilde{\gamma}_1)$  such that

$$E(\tilde{\Gamma}^1) = \min_h E(\tilde{\Gamma}^h).$$

In particular we have

$$E(\tilde{\Gamma}^1) = d^2(\gamma_0, \tilde{\gamma}_1) = \varepsilon_0, \quad d^2(\gamma_1, \tilde{\gamma}_1) = d^2(\gamma_0, \gamma_1) - \varepsilon_0.$$

**Step 2: Iteration.** We can now consider a minimizing sequence for  $d(\tilde{\gamma}_1, \gamma_1)$  and iterate the previous argument starting at  $\tilde{\gamma}_1$ . Because of the invariance of  $d$  we can minimize  $d(\tilde{\gamma}_1 \circ \varphi_{\tilde{\gamma}_1}, \gamma_1 \circ \varphi_{\tilde{\gamma}_1})$ . Proposition 2.13 and Remark 2.14 define a constant  $R_1 = \frac{\mathcal{L}(\tilde{\gamma}_1)}{2}$  such that (2.36) is verified on

$$B_{R_1} = \{\gamma \in BV^2(\mathbb{S}^1, \mathbb{R}^2) : \|\gamma - \tilde{\gamma}_1\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)} \leq R_1\} \subset \mathcal{B}.$$

We denote by  $m_1$  and  $M_1$  the constants given by (2.36) at this step.

By setting  $0 < \varepsilon_1 < m_1 R_1$ , we have

$$B_{\varepsilon_1}^d = \{\gamma \in \mathcal{B} : d^2(\gamma, \tilde{\gamma}_1) \leq \varepsilon_1\} \subset B_{R_1}.$$

This allows one to apply the argument used in the proof of Theorem 2.15 and define a second part of the geodesic path  $\tilde{\Gamma}^1$  joining  $\tilde{\gamma}_1 \circ \varphi_{\tilde{\gamma}_1}$  to some curve belonging to  $\tilde{\gamma}_2 \in \partial B_{\varepsilon_1}^d$ .

As the distance  $d$  is invariant under reparameterization we can reparameterize all the curves belonging to  $\tilde{\Gamma}^1$  by  $\varphi_{\tilde{\gamma}_1}^{-1}$  in order to have a path joining the initial curve  $\tilde{\gamma}_1$  and  $\tilde{\gamma}_2 \circ \varphi_{\tilde{\gamma}_1}^{-1}$ .

Remark that as  $\tilde{\gamma}_2 \in B_{R_1}$  and, by Lemma 2.12,  $\tilde{\Gamma}^1(t) \in B_{R_1}$  for every  $t$ , then the reparameterization of the path by  $\varphi_{\tilde{\gamma}_1}$  lives in  $\mathcal{B}$  (i.e.  $\tilde{\Gamma}^1(t) \circ \varphi_{\tilde{\gamma}_1}$  verifies (2.1) for every  $t$ ). More precisely, as the martingale convergence argument used Theorem 2.15 preserves the length of the curves of the paths (see (2.31)), the reparameterized path lives in  $\mathcal{B}$ .

By joining the paths  $\tilde{\Gamma}^0$  and  $\tilde{\Gamma}^1$  (and after a time-reparameterization), we get locally the geodesic path, denoted by  $\tilde{\Gamma}^{0,1}$ , between  $\gamma_0$  and  $\tilde{\gamma}_2$  and we have

$$d^2(\gamma_0, \tilde{\gamma}_2) = \varepsilon_0 + \varepsilon_1, \quad d^2(\gamma_1, \tilde{\gamma}_2) = d^2(\gamma_0, \gamma_1) - \varepsilon_0 - \varepsilon_1.$$

At the last step of the iteration, we are led to find a geodesic  $\tilde{\Gamma}^{k-1}$  between some curve  $\tilde{\gamma}_{k-1}$  and  $\gamma_1$ . Then we join it to  $\tilde{\Gamma}^{0,k-2}$  and, by a time-reparameterization, we find a geodesic  $\tilde{\Gamma}^{0,k-1}$  between  $\gamma_0$  and  $\gamma_1$ .

Remark that we supposed that  $\gamma_0$  is parameterized with respect to the arc-length parameter, but if we start by reparameterizing with respect by  $\varphi_{\gamma_0}$  we should now reparameterize all the curves of the geodesic with respect to  $\varphi_{\gamma_0}^{-1}$ .

**Step 3: Conclusion.** As  $d(\gamma_0, \gamma_1) < \infty$  we can conclude by iterating the previous arguments if

$$\inf_h \varepsilon_h > 0. \quad (2.37)$$

In fact this allows one to use  $B_\varepsilon$  (where  $\varepsilon < \inf_h \varepsilon_h$ ) instead of  $B_{\varepsilon_h}$  at each step so that we can reach  $\gamma_1$  in at most  $k = \lceil d(\gamma_0, \gamma_1)/\varepsilon \rceil$  iterations.

In order to conclude the proof we have to show that (2.37) is verified. Now, at each step  $h = 0, \dots, k$  of the iteration we reparameterize the starting point  $\tilde{\gamma}_h$  with respect to the arc-length parameter, so by Remark 2.14 the constants  $R_h$  are always defined as  $R_h = \frac{\mathcal{L}(\gamma_h)}{2}$ . Then, from (2.16), it follows that

$$l = e^{\mathcal{L}(\Gamma_0) - d^2(\gamma_0, \gamma_1)} \leq \mathcal{L}(\Gamma(t)) \leq e^{\mathcal{L}(\Gamma_0) + d^2(\gamma_0, \gamma_1)} = L, \quad (2.38)$$

and, setting  $R = \frac{l}{2}$  we have  $R_h \geq R \quad \forall h$ . In particular, for every  $\gamma_h \in \mathcal{B}$  parameterized with respect to the arc-length parameter we have

$$B_R(\gamma_h) = \{\gamma \in BV^2(\mathbb{S}^1, \mathbb{R}^2) : \|\gamma - \gamma_h\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)} \leq R\} \subset \mathcal{B}$$

and

$$C_{\gamma_h} = \min_{\gamma \in B_R(\gamma_h)} \min_{s \in \mathbb{S}^1} |\gamma'(t)(s)| > 0.$$

Remark that  $\inf C_{\gamma_h} > 0$  where the infimum is taken on the set of curves belonging to  $\mathcal{B}$  and parameterized with respect to the arc-length. This follows from the fact that  $|\gamma'_h| = \mathcal{L}(\gamma_h)$  on  $\mathbb{S}^1$  and, by the embedding (2.3), we have  $\|\gamma' - \gamma'_h\|_{L^\infty(\mathbb{S}^1, \mathbb{R}^2)} < R$  on  $B_R(\gamma_h)$ .

Moreover, by (2.38), for every  $\gamma \in B_R(\gamma_h)$  we have also

$$\|\gamma'\|_{BV(\mathbb{S}^1, \mathbb{R}^2)} \leq \|\gamma'_h\|_{BV(\mathbb{S}^1, \mathbb{R}^2)} + R = \|\gamma'_h\|_{L^1(\mathbb{S}^1, \mathbb{R}^2)} + R \leq L + R, \quad (2.39)$$

for every  $\gamma_h$  parameterized with respect to the arc-length parameter and belonging to  $\mathcal{B}$ .

Then, by the definition of  $m_\gamma$  in (2.10), we get that the  $m_h$ 's are uniformly bounded from below by a positive constant  $m$ . Then we can choose

$$\varepsilon < mR \leq \inf_h m_h R_h$$

and the proposition ensues.  $\square$

**Remark 2.21 (Length variation and almost local metrics).** We point out that property (2.16) plays a central role in the previous proof. This property gives a control of the length of the curves of a path depending on the length of such a path with respect to our metric. Because of the feature of the metric, such a property allows one to state a uniform bound on constants  $m_{\gamma_h}$ 's. In fact, as  $\gamma_h$  are parameterized with respect to the arc-length parameter,  $|\gamma'_h|$  is constant and the  $BV$ -norm in (2.39) coincides with the length of the curve.

Then, in order to generalize previous proof to other metrics, we have to make sure that the metric verifies Proposition 2.11,2.

For instance, one can verify that a property similar to (2.16) holds for the almost local Finsler metric

$$F(\gamma, v) = \mathcal{L}(\gamma)^\alpha \|v\|_{BV^2(\gamma)}, \quad \alpha \geq 0, \alpha \neq 1.$$

Following the proof of Proposition 2.11,2., it suffices indeed to estimate  $\log(\mathcal{L}(\Gamma(t)^{1-\alpha}))$ . We refer to [22] for an overview of the properties of almost local Riemannian metrics on the space of curves. We remark in particular that the smooth scale invariant metric defined in [22] (Section 3.8) allows one to control the variation of the length similarly to (2.16).

## 2.5 Geodesic Distance Between Geometric Curves

Theorem 2.20 shows the existence of a geodesic between any two parameterized closed curves in  $\mathcal{B}$ . We are interested now in defining a geometric distance between geometric curves (i.e., up to parameterization).

In the following we denote by  $\text{Diff}(\mathbb{S}^1)$  the set of homeomorphisms  $\varphi \in BV^2(\mathbb{S}^1, \mathbb{S}^1)$  such that  $\varphi^{-1} \in BV^2(\mathbb{S}^1, \mathbb{S}^1)$ . The space of geometric curves is defined as  $\mathcal{B}/G_{BV^2}$ . For every  $\gamma \in \mathcal{B}$  its equivalence class (called also geometric curve) is denoted by  $[\gamma]$ .

**Remark 2.22 (Reparameterizations and  $BV^2$ -norms).** Remark that for every  $\gamma \in \mathcal{B}$  and  $\varphi \in \text{Diff}(\mathbb{S}^1)$  we have

$$\|(\gamma \circ \varphi)'\|_{BV(\mathbb{S}^1, \mathbb{R}^2)} = \|\gamma'\|_{BV(\mathbb{S}^1, \mathbb{R}^2)}.$$

Moreover,

$$\|\gamma \circ \varphi\|_{L^1(\mathbb{S}^1, \mathbb{R}^2)} \leq \|\gamma \circ \varphi\|_{L^\infty(\mathbb{S}^1, \mathbb{R}^2)} = \|\gamma\|_{L^\infty(\mathbb{S}^1, \mathbb{R}^2)},$$

so by (2.3), we get

$$\|\gamma \circ \varphi\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)} \leq 2\|\gamma\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)}. \quad (2.40)$$

The next proposition defines a distance on the set of curves belonging to  $\mathcal{B}$  up to reparameterization.

**Proposition 2.23.** The Procrustean dissimilarity measure defined by

$$\mathcal{D}([\gamma_0], [\gamma_1]) = \inf \{d(\gamma_0 \circ \varphi, \gamma_1 \circ \psi) : \varphi, \psi \in \text{Diff}(\mathbb{S}^1)\} \quad (2.41)$$

is a distance on the set of  $\mathcal{B}$ -curves up to reparameterization.

*Proof.* The function  $\mathcal{D}$  is symmetric, non-negative and it is equal to zero if  $[\gamma_0] = [\gamma_1]$ .

Remark also that, the distance  $d$  is invariant under reparameterization, so that

$$d(\gamma_0, \gamma_1) = d(\gamma_0 \circ \varphi, \gamma_1 \circ \varphi), \quad \forall \varphi \in \text{Diff}(\mathbb{S}^1). \quad (2.42)$$

Then from the invariance (2.42) it follows that for every  $\varphi_1, \varphi_2, \varphi_3 \in \text{Diff}(\mathbb{S}^1)$  and  $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{B}$  we have

$$\begin{aligned} d(\gamma_1 \circ \varphi_1, \gamma_2 \circ \varphi_2) &\leq d(\gamma_1 \circ \varphi_1, \gamma_3 \circ \varphi_3) + d(\gamma_3 \circ \varphi_3, \gamma_2 \circ \varphi_2) \\ &= d(\gamma_1 \circ \varphi_1 \circ \varphi_3^{-1}, \gamma_3) + d(\gamma_3, \gamma_2 \circ \varphi_2 \circ \varphi_3^{-1}) \end{aligned}$$

which implies that the triangle inequality is satisfied for  $\mathcal{D}$ .

We prove now that  $\mathcal{D}([\gamma_0], [\gamma_1]) = 0$  implies that  $[\gamma_0] = [\gamma_1]$ . We assume that  $\gamma_0$  is parameterized by the arc-length parameter (this is possible because of the invariance of  $d$  under reparameterization). So there exists a sequence  $\varphi_h$  of reparameterizations such that

$$d(\gamma_0, \gamma_1 \circ \varphi_h) \leq \frac{1}{h}.$$

By Lemma 2.12, for  $h$  large enough the minimal path  $\Gamma^h$  for  $d(\gamma_0, \gamma_1 \circ \varphi_h)$  lives in a neighborhood of  $\gamma_0$  contained in  $\mathcal{B}$ . On such a neighborhood the minimal energies  $E(\Gamma^h)$  are uniformly bounded and  $C = \inf_h \inf_t m_{\Gamma^h(t)} > 0$ .

Then, similarly to (2.18), we get

$$\|\gamma_0 - \gamma_1 \circ \varphi_h\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)} \leq \frac{1}{Ch}$$

for  $h$  large enough. Thus, the support of  $\gamma_0$  and  $\gamma_1$  coincide which implies, by Proposition 2.5, that there exists  $\varphi \in \text{Diff}(\mathbb{S}^1)$  such that  $\gamma_0 = \gamma_1 \circ \varphi$ .  $\square$

Next theorem proves a local result for the existence of geodesics.

**Theorem 2.24 (Geometric local existence).** Let  $\gamma_0 \in \mathcal{B}$  and  $R, r$  the constants given by Proposition 2.13. Let  $\gamma_1 \in \mathcal{B}$  be such that

$$\|\gamma_1 - \gamma_0\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)} \leq r/2 \quad (2.43)$$

and  $\mathcal{D}([\gamma_0], [\gamma_1]) \leq \varepsilon$  where  $\varepsilon > 0$  depends on  $\gamma_0$ . Then there exists a minimizer of  $\mathcal{D}([\gamma_0], [\gamma_1])$ .

*Proof.* In the following we denoted by  $\gamma_0$  and  $\gamma_1$  the parameterizations with respect to the arc-length parameter. Because of the invariance (2.42) we can write

$$\mathcal{D}([\gamma_0], [\gamma_1]) = \inf \{d(\gamma_0, \gamma_1 \circ \psi) : \psi \in \text{Diff}(\mathbb{S}^1)\}.$$

We consider a sequence  $\{\psi_h\}_h \subset \text{Diff}(\mathbb{S}^1)$  such that  $d(\gamma_0, \gamma_1 \circ \psi_h) \rightarrow \mathcal{D}([\gamma_0], [\gamma_1])$ . Remark that, because of (2.40) and (2.43),  $\gamma_0$  and  $\gamma_1 \circ \psi_h$  verify the hypotheses of Theorem 2.15 for every  $h$  and  $\gamma_1 \circ \psi_h$  are uniformly bounded in  $BV^2(\mathbb{S}^1, \mathbb{R}^2)$ .

This implies that there exists a geodesic path denoted by  $\Gamma^h$  joining  $\gamma_0$  and  $\gamma_1 \circ \psi_h$  and we suppose that  $\sup_h d(\gamma_0, \gamma_1 \circ \psi_h) < \infty$ . We have in particular  $d^2(\gamma_0, \gamma_1 \circ \psi_h) = E(\Gamma^h)$  for every  $h$ . We show that there exists a  $\psi_\infty \in \text{Diff}(\mathbb{S}^1)$  such that  $\mathcal{D}([\gamma_0], [\gamma_1]) = d(\gamma_0, \gamma_1 \circ \psi_\infty)$ .

By the same arguments used in the proof of Theorem 2.15, we can define (see Step 1) a path  $\Gamma^\infty \in H^1([0, 1], BV^2(\mathbb{S}^1, \mathbb{R}^2))$  such that

$$\int_0^1 \Gamma_t^h(t)(s) dt \rightarrow \int_0^1 \Gamma_t^\infty(t)(s) dt, \quad (2.44)$$

and (see Step 2)

$$\Gamma^h(t) \xrightarrow{W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)} \Gamma^\infty(t), \quad \forall t \in \mathbb{S}^1 \quad (2.45)$$

$$\Gamma_t^h(t) \xrightarrow{W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)} \Gamma_t^\infty(t), \quad \forall t \in \mathbb{S}^1. \quad (2.46)$$

Now, because of (2.12), we have

$$\int_0^1 \Gamma_t^h(t)(s) dt = \gamma_0(s) - \gamma_1 \circ \psi_h(s). \quad (2.47)$$

As  $\{\gamma_1 \circ \psi_h\}$  is bounded in  $BV^2(\mathbb{S}^1, \mathbb{R}^2)$  there exists a subsequence (not relabeled) converging  $BV^{2,*}$  weakly towards some curve having the same support that  $\gamma_1$ . So, by Proposition 2.5, there exists  $\psi_\infty \in \text{Diff}(\mathbb{S}^1)$  such that

$$\gamma_1 \circ \psi_h \xrightarrow{*BV^2(\mathbb{S}^1, \mathbb{R}^2)} \gamma_1 \circ \psi_\infty.$$

Because of (2.11), this implies that

$$\Gamma^\infty(0) = \gamma_0 \quad \Gamma^\infty(1) = \gamma_1 \circ \psi_\infty$$

and, from (2.44) and (2.47), it follows that

$$\int_0^1 \Gamma_t^\infty(t)(s) dt = \gamma_0 - \gamma_1 \circ \psi_\infty(s)$$

which implies that  $\Gamma^\infty \in \mathcal{P}(\gamma_0, \gamma_1 \circ \psi_\infty)$ . Moreover, denoting by  $\varphi^h, \varphi^\infty$  the arc-length parameter of  $\Gamma^h(t)$  and  $\Gamma^\infty(t)$ , by (2.45) we have  $\varphi^h \rightarrow \varphi^\infty$  in  $W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)$ . So, by (2.46) and the lower semi-continuity of the second variation with respect to the  $W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)$  strong topology, we get

$$\begin{aligned} \|\Gamma_t^\infty(t)\|_{BV^2(\Gamma^\infty(t))}^2 &= \|(\Gamma^\infty \circ \varphi^\infty)_t(t)\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)}^2 \\ &\leq \liminf_{h \rightarrow \infty} \|(\Gamma^h \circ \varphi^h)_t(t)\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)}^2 = \liminf_{h \rightarrow \infty} \|\Gamma_t^h(t)\|_{BV^2(\Gamma^h(t))}^2. \end{aligned}$$

By integrating the previous inequality and using Fatou's Lemma we get

$$E(\Gamma^\infty) \leq \lim_{h \rightarrow \infty} E(\Gamma^h) = \lim_{h \rightarrow \infty} d(\gamma_0, \gamma_1 \circ \psi_h) = \mathcal{D}([\gamma_0], [\gamma_1]),$$

which implies that  $\mathcal{D}^2([\gamma_0], [\gamma_1]) = d^2(\gamma_0, \gamma_1 \circ \psi_\infty) = E(\Gamma^\infty)$ . □

We finally prove the following result for the global existence between geometric curves.

**Theorem 2.25 (Geometric global existence).** Let  $\gamma_0, \gamma_1 \in \mathcal{B}$ , then there exists a geodesic path joining  $[\gamma_0]$  and  $[\gamma_1]$ .

*Proof.* Because of the invariance (2.42) we can write

$$\mathcal{D}([\gamma_0], [\gamma_1]) = \inf \{d(\gamma_0, \gamma_1 \circ \psi) : \psi \in \text{Diff}(\mathbb{S}^1)\}.$$

We consider a sequence  $\{\psi_h\}_h \subset \text{Diff}(\mathbb{S}^1)$  such that  $d(\gamma_0, \gamma_1 \circ \psi_h) \rightarrow \mathcal{D}([\gamma_0], [\gamma_1])$ .

Now, for every  $h$  we consider a minimizing path for  $d(\gamma_0, \gamma_1 \circ \psi_h)$  such that

$$E(\Gamma^h) \leq \mathcal{D}^2([\gamma_0], [\gamma_1]) + \frac{1}{h},$$

which implies that  $E(\Gamma^h) \rightarrow \mathcal{D}([\gamma_0], [\gamma_1])$ . We can then apply to the sequence  $\{\Gamma^h\}_h$  the same argument used to prove Theorem 2.20.

Remark that at each step we can use the constant  $R$  and  $\varepsilon$  defined at the Step 3 of the proof of Theorem 2.20. Without loss of generality, we can consider  $\gamma_0$  parameterized with respect to the arc-length parameter. Then we can define a curve  $\tilde{\gamma}_1 \in \mathcal{B}$  and a path  $\tilde{\Gamma}^1$  joining  $\gamma_0$  and  $\tilde{\gamma}_1$  such that

$$E(\tilde{\Gamma}^1) = d^2(\gamma_0, \tilde{\gamma}_1) = \varepsilon, \quad d^2(\tilde{\gamma}_1, \gamma_1) = \mathcal{D}^2([\gamma_0], [\gamma_1]) - \varepsilon.$$

We can now iterate the argument on a sequence of minimizing paths  $\{\Gamma^h\}$  such that

$$E(\Gamma^h) \leq \mathcal{D}^2([\tilde{\gamma}_1], [\gamma_1]) + \frac{1}{h}.$$

As in the proof of Theorem 2.20 we must reparameterize all the minimizing paths with respect to  $\varphi_{\tilde{\gamma}_1}$  and afterwards we have to reparameterize the limit local path by  $\varphi_{\tilde{\gamma}_1}^{-1}$ .

By conveniently choosing  $\varepsilon$ , at the last step of the iteration, we are led to find a geodesic  $\tilde{\Gamma}^{k-1}$  between some curve  $[\tilde{\gamma}_{k-1}]$  and  $[\gamma_1]$  where  $\tilde{\gamma}_{k-1}$  and  $\gamma_1$  verify the hypothesis of Theorem 2.24. Then by Theorem 2.24 we find a path joining  $\tilde{\gamma}_{k-1}$  and  $\gamma_1 \circ \varphi_\infty$  ( $\varphi_\infty \in \text{Diff}(\mathbb{S}^1)$ ) and, by joining it to the others local paths, we find a path  $\Gamma^\infty$  between  $\gamma_0$  and  $\gamma_1 \circ \varphi_\infty$  such that  $E(\Gamma^\infty) = \mathcal{D}^2([\gamma_0], [\gamma_1])$ . Such a path is a geodesic between  $[\gamma_0]$  and  $[\gamma_1]$  and this ends the proof. □

### 3 Geodesics in the Space of Sobolev Curves

In this section, we study the geodesic boundary value problem in the class of curves belonging to  $H^s(\mathbb{S}^1, \mathbb{R}^2)$  with  $s > 3/2$ . We remind that, by identifying  $\mathbb{S}^1$  with  $[0, 1]$ ,  $u \in H^s(\mathbb{S}^1, \mathbb{R})$  if  $u$  is 1-periodic and

$$\|u\|_{H^s(\mathbb{S}^1, \mathbb{R})}^2 = \sum_{n \in \mathbb{Z}} (1 + n^{2s}) |\hat{f}(n)|^2 < \infty$$

with

$$\hat{f}(n) = \int_{\mathbb{S}^1} f(s) e^{-ins} ds.$$

Moreover,  $\|u\|_{H^s(\mathbb{S}^1, \mathbb{R})}$  defines a norm and  $H^s(\mathbb{S}^1, \mathbb{R})$  equipped with such a norm is a Hilbert space.

We also remind that

$$H^s(\mathbb{S}^1, \mathbb{R}^2) \subset C^{0, s - \frac{1}{2}}(\mathbb{S}^1, \mathbb{R}^2), \quad \forall s > 1/2. \quad (3.1)$$

In our framework, as  $s > 3/2$ , every  $H^s$ -curve is  $C^1$ . This implies in particular that the derivative is bounded which is the main hypothesis needed to prove (analogously to Proposition 2.11) a uniform bound on the length of the curves. We refer to [27] for more details about fractional order Sobolev spaces. Let us remark that (see [29])  $H^s(\mathbb{S}^1, \mathbb{R}^2)$  is not embedded in  $BV(\mathbb{S}^1, \mathbb{R}^2)$  for  $s \in (0, 1)$  so if  $s > 3/2$  then  $H^s(\mathbb{S}^1, \mathbb{R}^2)$  is embedded in  $BV(\mathbb{S}^1, \mathbb{R}^2)$  but not in  $BV^2(\mathbb{S}^1, \mathbb{R}^2)$ . This implies that the Sobolev case can not be seen as a special case of the  $BV^2$  framework. The proof of (local) existence of geodesics is simpler, since  $H^s(\mathbb{S}^1, \mathbb{R}^2)$  being a Hilbert space, the weak topology can be used instead of the  $BV^2$ -\*weak one.

In the following, we apply the strategy presented in Section 2 to  $H^s$ -curves in order to prove the existence of a geodesics. Since the proofs are very similar to those presented in the previous section, we do not give every detail, but we develop the key arguments particular to  $H^s$  curves in order to retrieve the same results.

We define the class of the parameterized  $H^s$ -curves as follows.

**Definition 3.1 ( $H^s$ -curves).** We define  $\mathcal{H}_s$  as the class of counterclockwise oriented curves belonging to  $H^s(\mathbb{S}^1, \mathbb{R}^2)$  (for  $s > 3/2$ ) and verifying (2.1). We consider  $\mathcal{H}_s$  equipped with the  $H^s$ -norm. For every  $\gamma \in \mathcal{H}_s$  the arc-length parameterization  $\varphi_\gamma$  can be as in Remark 2.4. Moreover, in this section we denote by  $\text{Diff}(\mathbb{S}^1)$  the set of homeomorphisms  $\varphi \in H^s(\mathbb{S}^1, \mathbb{S}^1)$  such that  $\varphi^{-1} \in H^s(\mathbb{S}^1, \mathbb{S}^1)$ .

From (3.1) it follows that there exists a constant  $C(s)$  such that

$$\|\gamma'\|_{L^\infty(\mathbb{S}^1, \mathbb{R}^2)} \leq C(s) \|\gamma\|_{H^s(\mathbb{S}^1, \mathbb{R}^2)}, \quad \forall \gamma \in \mathcal{H}_s. \quad (3.2)$$

Moreover, it is easy to verify that  $\mathcal{H}_s$  is an open set of  $H^s(\mathbb{S}^1, \mathbb{R}^2)$ . In fact, similarly to the  $BV^2$ -case, for every  $\gamma_0 \in \mathcal{H}_s$ , which is parameterized with respect to the arc-length parameter, (3.2) implies that

$$\left\{ \gamma \in \mathcal{H}_s : \|\gamma - \gamma_0\|_{H^s(\mathbb{S}^1, \mathbb{R}^2)} \leq \frac{\mathcal{L}(\gamma_0)}{2C(s)} \right\} \subset \mathcal{H}_s. \quad (3.3)$$

We point out that local properties of reparameterization proved in Proposition 2.5 can be easily generalized to  $H^s$ -curves. The arc-length is thus defined for any  $H^s$ -curve. As in the previous section we define the space  $H^s(\gamma)$  as the space  $H^s(\mathbb{S}^1, \mathbb{R}^2)$ , where integration is performed with respect to the measure  $d\Gamma = |\Gamma'| ds$ . Concerning the tangent space, for any  $\gamma \in \mathcal{H}_s$ , we set  $T_\gamma \mathcal{H}_s = H^s(\gamma)$ .

As in the previous section, we prove the equivalence of the norms of  $H^s(\mathbb{S}^1, \mathbb{R}^2)$  and  $H^s(\gamma)$ .

**Proposition 3.2.** Let  $\gamma \in H^s(\mathbb{S}^1, \mathbb{R}^2)$ . The following properties hold.

1. The sets  $H^s(\gamma)$  and  $H^s(\mathbb{S}^1, \mathbb{R}^2)$  coincide and their norm are equivalent. More precisely, there exist two positive constants  $M_\gamma, m_\gamma$  such that, for all  $f \in H^s(\mathbb{S}^1, \mathbb{R}^2)$

$$m_\gamma \|f\|_{H^s(\mathbb{S}^1, \mathbb{R}^2)} \leq \|f\|_{H^s(\gamma)} \leq M_\gamma \|f\|_{H^s(\mathbb{S}^1, \mathbb{R}^2)}.$$

2. Constants  $m_\gamma$  and  $M_\gamma$  admit some positive uniform upper and lower bounds on each ball of  $H^s(\mathbb{S}^1, \mathbb{R}^2)$  contained in  $\mathcal{H}_s$ .

*Proof.* We use the following property for  $s > 3/2$  proved in [18]: given  $M, C > 0$  there exists  $C_s = C_s(M, C)$ , such that for every  $f \in H^s(\mathbb{S}^1, \mathbb{R}^2)$ ,

$$\|f \circ \varphi\|_{H^s(\mathbb{S}^1, \mathbb{R}^2)} \leq C_s \|f\|_{H^s(\mathbb{S}^1, \mathbb{R}^2)}.$$

for all  $\varphi \in \text{Diff}(\mathbb{S}^1)$  such that

$$\min_{x \in \mathbb{S}^1} \det D\varphi(x) > M \quad \text{and} \quad \|\varphi - \text{Id}\|_{H^s(\mathbb{S}^1, \mathbb{R}^2)} < C. \quad (3.4)$$

The map  $R : H^s(\mathbb{S}^1, \mathbb{R}^2) \mapsto H^s(\mathbb{S}^1, \mathbb{S}^1)$  defined by  $R(\gamma) = \varphi_\gamma$  (the arc-length parametrization is defined up to the choice of a basepoint in  $\mathbb{S}^1$ ) is continuous. Indeed,  $R$  is the composition of the inverse mapping (continuous) on  $\text{Diff}(\mathbb{S}^1)$  with the map  $S : \gamma \in H^s(\mathbb{S}^1, \mathbb{R}^2) \mapsto s_\gamma \in \text{Diff}(\mathbb{S}^1)$  (see Remark 2.4). Using the Hilbert algebra property of  $H^{s-1}(\mathbb{S}^1, \mathbb{R})$ , we get that the map  $S$  is smooth. Thus,  $R$  is continuous.

More precisely, if  $\gamma$  lies in a bounded ball  $B \subset H^s(\mathbb{S}^1, \mathbb{R}^2)$  then  $s'_\gamma(s) = |\gamma'(s)| > C(B)$  and, by using the injection of  $H^{s-1}(\mathbb{S}^1, \mathbb{R}^2)$  into  $L^\infty(\mathbb{S}^1, \mathbb{R})$ , we have that  $R(\gamma)$  verifies the first condition in (3.4).

The second condition can be checked on  $s_\gamma$  instead of  $\varphi_\gamma$  by using symmetric neighborhoods at identity in  $\text{Diff}(\mathbb{S}^1)$ . In addition the map  $S$  is uniformly Lipschitz on every ball  $B \subset H^s(\mathbb{S}^1, \mathbb{R}^2)$ .

Indeed, we have

$$DS(\gamma)(X) = \left[ s \rightarrow \int_0^s \frac{1}{|\gamma'(t)|} \langle \gamma'(t), X(t) \rangle dt \right] \quad \forall X \in C^\infty(\mathbb{S}^1, \mathbb{R}^2).$$

In addition the map  $h \in H^{s-1}(\mathbb{S}^1, \mathbb{R}) \mapsto 1/h \in H^{s-1}(\mathbb{S}^1, \mathbb{R})$  is bounded on every bounded ball such that  $\min_{x \in \mathbb{S}^1} h'(x) \geq \eta > 0$ . It can be checked by using the formula

$$\|h\|_{H^{s'}(\mathbb{S}^1, \mathbb{R})}^2 = \int_{\mathbb{S}^1} |h(x)|^2 dx + \iint_{\mathbb{S}^1 \times \mathbb{S}^1} \frac{|h(x) - h(y)|^2}{|x - y|^{1+2s'}} dx dy,$$

so that

$$\left\| \frac{1}{h} - \frac{1}{g} \right\|_{H^{s'}(\mathbb{S}^1, \mathbb{R})}^2 \leq \eta^4 \|h - g\|_{H^{s'}(\mathbb{S}^1, \mathbb{R})}^2.$$

It implies that the map  $DS(\gamma)$  is bounded on a bounded ball  $B$  included in  $H^s(\mathbb{S}^1, \mathbb{R}^2)$ . Therefore,  $S(B)$  is bounded in  $H^s$ . This means that the second condition in (3.4) is satisfied.

On that bounded ball we thus have, for  $f \in H^s(\mathbb{S}^1, \mathbb{R}^2)$

$$\|f\|_{H^s(\gamma)} = \|f \circ R(\gamma)\|_{H^s(\mathbb{S}^1, \mathbb{R}^2)} \leq C(B) \|f\|_{H^s(\mathbb{S}^1, \mathbb{R}^2)}, \quad (3.5)$$

and also

$$\|f\|_{H^s(\mathbb{S}^1, \mathbb{R}^2)} = \|f \circ R(\gamma) \circ R(\gamma)^{-1}\|_{H^s(\mathbb{S}^1, \mathbb{R}^2)} \leq C(B) \|f \circ R(\gamma)\|_{H^s(\mathbb{S}^1, \mathbb{R}^2)} = \|f\|_{H^s(\gamma)}. \quad (3.6)$$

Note that the neighborhood of is chosen invariant by the inverse map so that the constant  $C(B)$  applies to the composition with  $R(\gamma)^{-1}$  as well.  $\square$

The energy of a path is defined as

$$E(\Gamma) = \int_0^1 \|\Gamma_t(t)\|_{H^s(\Gamma(t))}^2 dt$$

and the geodesic distance is defined accordingly (see Definition 2.9).

We point out that Lemma 2.12 and Proposition 2.13 are directly generalized to the Sobolev case by changing the  $BV^2$ -norms by the  $H^s$ -norms and use the  $H^s$ -weak topology instead of the  $BV^2$ -\* weak one.

Analogously to Proposition 2.11, we can prove that the length of every curve of a path is bounded, which is a fundamental property to prove global existence of geodesics:

$$e^{\mathcal{L}(\Gamma_0) - E(\Gamma)} \leq \mathcal{L}(\Gamma(t)) \leq e^{\mathcal{L}(\Gamma_0) + E(\Gamma)}, \quad \forall \Gamma \in \mathcal{P}(\gamma_0, \gamma_1), \quad \forall t \in [0, 1]. \quad (3.7)$$

We also point out that the proofs of Lemma 2.12 and Proposition 2.13 are based on the fact that constants  $m_\gamma$  and  $M_\gamma$  are bounded on balls. This is actually true in the Sobolev framework because of Proposition 3.2. Then, Proposition 2.13 also applies to the Sobolev case.

**Theorem 3.3 (Local existence).** Let  $\gamma_0 \in \mathcal{H}_s$ . Let  $\gamma_1 \in \mathcal{H}_s$  such that

$$\|\gamma_1 - \gamma_0\|_{BV^2(\mathbb{S}^1, \mathbb{R}^2)} \leq r$$

where  $r$  is given by Proposition 2.13. Then, there exists a geodesic between  $\gamma_0$  and  $\gamma_1$ .

*Proof.* Because of Propositions 3.2 and 2.13 we have

$$\int_0^1 \|\Gamma_t^h(t)\|_{H^s(\mathbb{S}^1, \mathbb{R}^2)} dt \leq \int_0^1 \frac{\|\Gamma_t^h(t)\|_{H^s(\Gamma(t))}}{m_{\Gamma^h(t)}} dt \leq \frac{E(\Gamma)}{m} \quad (3.8)$$

which implies that  $\Gamma_t^h$  is uniformly bounded in  $L^2([0, 1], H^s(\mathbb{S}^1, \mathbb{R}^2))$  and, because of the boundary conditions, that  $\Gamma^h$  is uniformly bounded in  $H^1([0, 1], H^s(\mathbb{S}^1, \mathbb{R}^2))$ . Therefore there exists a subsequence of  $\Gamma^h$  that weakly converges in  $H^1([0, 1], H^s(\mathbb{S}^1, \mathbb{R}^2))$ . Since the embedding

$$H^1([0, 1], H^s(\mathbb{S}^1, \mathbb{R}^2)) \subset C([0, 1], H^{s-1}(\mathbb{S}^1, \mathbb{R}^2))$$

is compact, this subsequence (not relabeled) converges to a path  $\Gamma^\infty \in L^2([0, 1], C(\mathbb{S}^1, \mathbb{R}^2))$ ,

$$\Gamma^h \rightarrow \Gamma^\infty \quad \text{in} \quad C([0, 1], H^{s-1}(\mathbb{S}^1, \mathbb{R}^2)).$$

This proves in particular that

$$\Gamma^h(t) \xrightarrow{W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)} \Gamma^\infty(t), \quad \forall t \in [0, 1] \quad \text{and} \quad \Gamma_t^h \xrightarrow{H^s(\mathbb{S}^1, \mathbb{R}^2)} \Gamma_t^\infty(t), \quad \forall t \in [0, 1].$$



Now, as  $\Gamma^h(t)$  converges in  $W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)$  towards  $\Gamma^\infty(t)$ , the arc-length parameterizations at the time  $t$ , denoted respectively by  $\varphi^h(t)$  and  $\varphi^\infty(t)$ , also converge in  $W^{1,1}(\mathbb{S}^1, \mathbb{S}^1)$ .

By the same arguments used for (2.33), we have

$$\Gamma_t^h \circ \varphi^\infty(t) \xrightarrow{H^s(\mathbb{S}^1, \mathbb{R}^2)} \Gamma_t^\infty \circ \varphi^\infty(t), \quad \forall t \in [0, 1].$$

By the lower semi-continuity of the  $H^s(\mathbb{S}^1, \mathbb{R}^2)$ -norm, for a.e.  $t$  we get

$$\begin{aligned} \|\Gamma_t^\infty(t)\|_{H^s(\Gamma(t))}^2 &= \|\Gamma_t^\infty \circ \varphi^\infty(t)\|_{H^s(\mathbb{S}^1, \mathbb{R}^2)}^2 \\ &\leq \liminf_{h \rightarrow \infty} \|\Gamma_t^h \circ \varphi^h(t)\|_{H^s(\mathbb{S}^1, \mathbb{R}^2)}^2 = \liminf_{h \rightarrow \infty} \|\Gamma_t^h(t)\|_{H^s(\Gamma(t))}^2. \end{aligned}$$

By integrating the previous inequality and using Fatou's Lemma we get that  $\Gamma^\infty$  minimizes  $E$  and the theorem ensues.  $\square$

As in the case of  $BV^2$ -curves we can prove the following result of global existence for geodesics.

**Theorem 3.4 (Global existence).** Let  $\gamma_0, \gamma_1 \in \mathcal{H}_s$  be two given curves. There exists a geodesic path joining them and there exists a minimizing geodesic joining  $[\gamma_0]$  and  $[\gamma_1]$ .

*Sketch of the proof.* The proof is the same than that one given for Theorem 2.20. We just point out that, as in the  $BV^2$  framework, we can reach  $\gamma_1$  by a finite number of iterations because of two main arguments:

1. at each step we reparameterize the current curve with respect to the arc-length parameter which allows one to choose a ball  $B_\varepsilon$  with radius equal to the half of the length of the curve. Moreover, because of (3.7), we can find a uniform minimum radius  $R$  for the balls  $B_\varepsilon$ . Then we can always use  $B_R$  at each step of the iteration;
2. thanks to property (3.7) we can find an uniform upper and lower bound on  $B_R$  for the constants  $m_\gamma$  and  $M_\gamma$  given by Proposition (2.13).

By the same arguments used in the previous section we can prove existence of geodesics between any two elements in  $\mathcal{H}_s$ .  $\square$

In [11], the authors prove (Theorem 1.1) that the space of immersed curves is geodesically complete with respect to the  $H^s$ -metrics ( $s$  integer and  $s \geq 2$ ). This means in particular the the exponential map is defined for all time.

Moreover, from Theorem 3.4, we have the existence of minimal geodesics between any two points. These two results prove the following corollary.

**Corollary 3.5 (Surjectivity of the exponential map).** The exponential map on  $\mathcal{H}_s$  for  $s \geq 2$  integer is defined for all time and is surjective.

The so-called Fréchet or Kärcher means are a particular case of minimizers of the distance to a closed subset. The surjectivity of the exponential map enables the use of [4, Theorem 3.5] which proves that the projection onto a closed subset is unique on a dense subset. A direct theoretical consequence of this surjectivity result is the following result.

**Proposition 3.6.** Let  $s \geq 2$  be an integer. For any integer  $n \geq 1$ , there exists a dense subset  $D \subset \mathcal{H}_s^n$ , such that the Kärcher mean associated with any  $(\gamma_1, \dots, \gamma_n) \in D$ , defined as a minimizer on

$$\min_{\Gamma \in \mathcal{H}_s} \mathcal{K}(\Gamma) \sum_{i=1}^n d(\gamma, \gamma_i)^2, \quad (3.9)$$

is unique.

*Proof.* Let  $S$  be the diagonal in  $\mathcal{H}_s^n$ . The set  $S$  is a closed subset of  $\underbrace{\mathcal{H}_s \times \dots \times \mathcal{H}_s}_{n \text{ times}}$ . In [4, Theorem 3.5] the authors prove that the set of minimizers of  $\arg \min_{y \in S} d(x, y)$  is a singleton for a dense subset in  $\mathcal{H}_s^n$ .  $\square$

Obviously, the function  $\mathcal{K}$  may have several local minima but the mean is uniquely defined on a dense subset.

## 4 Numerical computations of Geodesics

In this section we discretize and relax Problem (2.14) in order to approximate numerically geodesic paths. Note that since we use a gradient descent to minimize a discretized and relaxed energy, the resulting optimal discrete homotopy aims at approximating stationary points of the geodesic energy, and that these homotopies cannot be guaranteed to be globally minimizing geodesics.

### 4.1 Penalized Boundary Condition at $t = 1$

To obtain a simple numerical scheme, we first relax the constraint  $\Gamma(1) = \gamma_1$  by adding at the energy  $E$  a data fidelity term  $H(\Gamma(1), \gamma_1)$  taking into account the deviation between  $\Gamma(1)$  and  $\gamma_1$ . In the following we make use of the  $H$  functional defined in [14] (equation 5.4).

Such a functional is defined as the following distance between two curves

$$H(\gamma, \lambda)^2 = T(\gamma, \gamma) + T(\lambda, \lambda) - 2T(\gamma, \lambda), \quad \forall \gamma, \lambda \in \mathcal{B}$$

where

$$T(\gamma, \lambda) = \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \mathbf{n}_\gamma(s) \cdot \mathbf{n}_\lambda(t) k(\gamma(s), \lambda(t)) d\gamma(s) d\lambda(t)$$

with

$$k(v, w) = e^{-\frac{\|v-w\|^2}{2\sigma^2}} + e^{-\frac{\|v-w\|^2}{2\delta^2}}, \quad \forall v, w \in \mathbb{R}^2.$$

This functional  $H$  was initially proposed in [36] as a norm on a reproducing Hilbert space of currents. It can be shown to be a metric on the space of geometric curves, which explains why it is a good candidate to enforce approximately the boundary constraint at time  $t = 1$ . We remind that  $H$  is continuous with respect to strong topology of  $W^{1,1}(\mathbb{S}^1, \mathbb{R}^2)$ . We refer to [14] for its properties and its discretization using finite elements.

Then, given two curves  $\gamma_0, \gamma_1 \in \mathcal{B}$ , we consider the following problem

$$\begin{aligned} \min \{ F(\Gamma) : \Gamma \in H^1([0, 1], \mathcal{B}), \Gamma(0) = \gamma_0 \} \\ F(\Gamma) = H(\Gamma(1), \gamma_1) + E(\Gamma). \end{aligned} \quad (4.1)$$

To allow for more flexibility in the numerical experiments, we introduce a weighted  $BV^2$  norm in the definition (2.13) of the energy  $E$ . Given some positive weights  $(\lambda_0, \lambda_1, \lambda_2) \in (\mathbb{R}^+)^3$ , we consider in this section

$$E(\Gamma) = \int_0^1 \|\Gamma_t\|_{BV^2(\Gamma(t))} dt, \quad (4.2)$$

where, for all  $\gamma \in \mathcal{B}$  and  $v \in T_\gamma \mathcal{B}$ ,

$$\|v\|_{BV^2(\gamma)} = \int_{\mathbb{S}^1} \left( \lambda_0 |v(s)| + \lambda_1 \left| \frac{dv}{d\gamma}(s) \right| + \lambda_2 \left| \frac{d^2v}{d\gamma^2}(s) \right| \right) d\gamma(s).$$

## 4.2 Regularized Problem and $\Gamma$ -convergence

The energy minimized in (4.1) is both non-smooth and non-convex. In order to compute stationary points using a gradient descent scheme, we further relax this problem by smoothing the  $\mathbb{R}^2$ -norm used to calculate the  $BV^2$ -norm. This approach is justified by a  $\Gamma$ -convergence result in Theorem 4.1.

The energy  $E$  is regularized as

$$E_\varepsilon(\Gamma) = \int_0^1 \|\Gamma_t\|_{BV^2(\Gamma(t))}^\varepsilon dt. \quad (4.3)$$

where  $\varepsilon > 0$  controls the amount of smoothing, and the smoothed  $BV^2$ -norm is defined, for  $\gamma \in \mathcal{B}$  and  $v \in T_\gamma \mathcal{B}$ , as

$$\|v\|_{BV^2(\gamma)}^\varepsilon = \int_{\mathbb{S}^1} \left( \lambda_0 |v(s)|_\varepsilon + \lambda_1 \left| \frac{dv}{d\gamma}(s) \right|_\varepsilon \right) |\gamma'(s)|_\varepsilon ds + \lambda_2 TV_\gamma^2(v),$$

where

$$\forall x \in \mathbb{R}^2, \quad |x|_\varepsilon = \sqrt{|x|^2 + \varepsilon^2}.$$

The initial problem (4.1) is then replaced by

$$\begin{aligned} \min \{ F_\varepsilon(\Gamma) : \Gamma \in H^1([0, 1], \mathcal{B}), \Gamma(0) = \gamma_0 \} \\ F_\varepsilon(\Gamma) = H(\Gamma(1), \gamma_1) + E_\varepsilon(\Gamma) \end{aligned} \quad (4.4)$$

This smoothing approach is justified by the following theorem

**Theorem 4.1.** Let  $\gamma_0 \in \mathcal{B}$  and  $X = \{\Gamma \in H^1([0, 1], \mathcal{B} \cap B_r(\gamma_0)) : \Gamma(0) = \gamma_0\}$ , where  $B_r(\gamma_0)$  is defined as in the point (iii) of Theorem 2.18. Then

$$\lim_{\varepsilon \rightarrow 0} \min_{\Gamma \in X} F_\varepsilon(\Gamma) = \min_{\Gamma \in X} F(\Gamma).$$

Moreover if  $\{\Gamma_\varepsilon\}_\varepsilon$  is a sequence of minimizers of  $F_\varepsilon$  then there exists a subsequence (not relabelled) such that  $\Gamma_\varepsilon \xrightarrow{\sigma} \Gamma$  as  $\varepsilon \rightarrow 0$  (see Definition 2.17) and  $\Gamma$  is a minimizer of  $F$ .

*Proof.* Remark that  $F$  and  $F_\varepsilon$  are not equal to infinity. Then, by Theorem 2.18 (iii),  $F$  and  $F_\varepsilon$  reach their minima on  $X$ .

As  $\{F_\varepsilon\}_\varepsilon$  is a decreasing sequence converging to  $F$  pointwise, we get

$$\lim_{\varepsilon \rightarrow 0} \min_{\Gamma \in X} F_\varepsilon(\Gamma) = \inf_{\varepsilon > 0} \min_{\Gamma \in X} F_\varepsilon(\Gamma) = \min_{\Gamma \in X} \inf_{\varepsilon > 0} F_\varepsilon(\Gamma) = \min_{\Gamma \in X} F(\Gamma). \quad (4.5)$$

Thus, if  $\{\Gamma_\varepsilon\}$  is a sequence of minimizers of  $F_\varepsilon$  we have  $F(\Gamma_\varepsilon) \leq F_\varepsilon(\Gamma_\varepsilon)$  so that  $\{\Gamma_\varepsilon\}$  is a minimizing sequence for  $F$ . Then, by Theorem 2.18, there exists  $\Gamma \in X$  and a subsequence (not relabelled) such that  $\Gamma_\varepsilon \xrightarrow{\sigma} \Gamma$  as  $\varepsilon \rightarrow 0$ . As  $F$  is lower semicontinuous with respect to the  $\sigma$  convergence, from (4.5), it follows that  $\Gamma$  is a minimizer of  $F$ .  $\square$

### 4.3 Finite Element Space

In the following, to ease the notation, we identify  $\mathbb{R}^2$  with  $\mathbb{C}$  and  $\mathbb{S}^1$  with  $[0, 1]$  using periodic boundary conditions.

To approximate numerically stationary points of (4.4), we discretize this problem by using finite element approximations of the homotopies, which are piecewise linear along the  $s$  variable and piecewise constant along the  $t$  variable. This choice of elements is justified by the fact that the evaluation of the energy requires the use of two derivatives along the  $s$  variable, and a single derivative along the  $t$  variable.

**Finite elements curves.** A piecewise affine curve with  $n$  nodes is defined as

$$\forall s \in [0, 1], \quad \gamma(s) = \sum_{j=1}^n \tilde{\gamma}_j \xi_j(s)$$

where we used piecewise affine finite elements

$$\begin{aligned} \xi_j(s) &= \max \left\{ 0, 1 - n \left| s - \frac{j}{n} \right| \right\} \quad s \in [0, 1], \quad \forall j = 1, \dots, n-1 \\ \xi_n(s) &= \max \{0, 1 - n |s|\} + \max \{0, 1 - n |s - 1|\}, \quad s \in [0, 1]. \end{aligned}$$

Here,  $\tilde{\gamma} \in \mathbb{C}^n$  denotes the coordinates of  $\gamma$  and we denote  $\gamma = P_1(\tilde{\gamma})$  the corresponding bijection.

**Finite elements homotopies.** We consider the finite dimensional space of homotopies of the form

$$\forall (t, s) \in [0, 1]^2, \quad \Gamma(t)(s) = \sum_{i=1}^N \sum_{j=1}^n \tilde{\Gamma}_{i,j} \zeta_i(t) \xi_j(s) \quad (4.6)$$

where we used piecewise constant finite elements

$$\zeta_i(t) = \mathbb{I}_{\left[\frac{i}{N}, \frac{i+1}{N}\right]}(t) \quad \forall i = 1, \dots, N-1, \quad \zeta_N(t) = \mathbb{I}_{\left[0, \frac{1}{N}\right]}(t),$$

Here,  $\tilde{\Gamma} \in \mathbb{C}^{N \times n}$  denotes the coordinates of  $\Gamma$  and we denote  $\Gamma = P_{0,1}(\tilde{\Gamma})$  the corresponding bijection.

## 4.4 Discretized Energies

The initial infinite dimensional optimization problem (4.4) is discretized by restricting the minimization to the finite element space described by (4.6) as follow

$$\min \left\{ \mathcal{F}_\varepsilon(\tilde{\Gamma}) : \tilde{\Gamma} \in \mathbb{C}^{N \times n}, \tilde{\Gamma}_{1,\cdot} = \tilde{\gamma}_0 \right\} \quad \text{where} \quad \mathcal{F}_\varepsilon(\tilde{\Gamma}) = F_\varepsilon(\Gamma) \quad (4.7)$$

where  $\Gamma = P_{0,1}(\tilde{\Gamma})$  and where the input boundary curves are  $\gamma_0 = P_1(\tilde{\gamma}_0), \gamma_1 = P_1(\tilde{\gamma}_1)$ , which are assumed to be piecewise affine finite elements. We have denoted here  $\Gamma_{i,\cdot} = (\Gamma_{i,j})_{j=1}^n \in \mathbb{R}^n$ .

In order to ease the computation of gradients, we note that the energy  $\mathcal{F}_\varepsilon$  can be decomposed as

$$\mathcal{F}_\varepsilon(\tilde{\Gamma}) = H(P_1(\tilde{\Gamma}_{N,\cdot}), \gamma_1) + \mathcal{E}_\varepsilon(\tilde{\Gamma}) \quad \text{where} \quad \mathcal{E}_\varepsilon(\tilde{\Gamma}) = \frac{1}{N-1} \sum_{i=1}^{N-1} J(\tilde{\Gamma}_{i,\cdot}, \tilde{v}_i) \quad (4.8)$$

where we denoted the discrete time derivative vector field as

$$\tilde{v}_i = \frac{\tilde{\Gamma}_{i+1,\cdot} - \tilde{\Gamma}_{i,\cdot}}{N-1} \in \mathbb{C}^n.$$

For  $\tilde{\gamma} \in \mathbb{C}^n$  and  $\tilde{v} \in \mathbb{C}^n$ , we used the notation

$$J(\tilde{\gamma}, \tilde{v}) = \sum_{\ell=0}^2 \lambda_\ell J_\ell(\tilde{\gamma}, \tilde{v}).$$

and we define bellow the explicit computation of the terms  $J_\ell(\tilde{\gamma}, \tilde{v})$  for  $\ell = 0, 1, 2$ .

**Zero order energy term ( $\ell = 0$ ).** The  $L^1$  norm of a piecewise affine field  $v = P_1(\tilde{v})$  tangent to a piecewise affine curve  $\gamma = P_1(\tilde{\gamma})$  can be computed as

$$\int_{\mathbb{S}^1} |v(s)|_\varepsilon |\gamma'(s)|_\varepsilon ds = \sum_{i=1}^n n |\Delta^+(\tilde{\gamma})_i|_{\frac{\varepsilon}{n}} \int_{\frac{i}{n}}^{\frac{i+1}{n}} |\tilde{v}_i \xi_i(s) + \tilde{v}_{i+1} \xi_{i+1}(s)|_\varepsilon ds.$$

This quantity cannot be evaluated in closed form. For numerical simplicity, we thus approximate the integral by the trapezoidal rule. With a slight abuse of notation (this is only an approximate equality), we define the discrete  $L^1$ -norm as

$$J_0(\tilde{\gamma}, \tilde{v}) = \frac{1}{2} \sum_{i=1}^n |\Delta^+(\tilde{\gamma})_i|_{\frac{\varepsilon}{n}} \left( |\tilde{v}_i|_\varepsilon + |\tilde{v}_{i+1}|_\varepsilon \right),$$

where we used the following forward finite difference operator

$$\Delta^+ : \mathbb{C}^n \rightarrow \mathbb{C}^n, \quad \Delta^+(\tilde{\gamma})_i = \tilde{\gamma}_{i+1} - \tilde{\gamma}_i.$$

**First order energy term** ( $\ell = 1$ ). We point out that

$$\frac{dv}{d\gamma} = \sum_{i=1}^n \frac{\Delta^+(\tilde{v})_i}{|\Delta^+(\tilde{\gamma})_i|_{\frac{\varepsilon}{n}}} \zeta_i \quad (4.9)$$

which implies that

$$\int_{\mathbb{S}^1} \left| \frac{dv}{d\gamma(s)} \right| |\gamma'(s)|_{\varepsilon} ds = \sum_{i=1}^n \int_{\frac{i}{n}}^{\frac{i+1}{n}} n |\Delta^+(\tilde{v})_i|_{\frac{\varepsilon}{n}} ds.$$

Then the discretized  $L^1$ -norm of the first derivative is defined by

$$J_1(\tilde{\gamma}, \tilde{v}) = \sum_{i=1}^n |\Delta^+(\tilde{v})_i|_{\frac{\varepsilon}{n}}.$$

**Second order energy term** ( $\ell = 2$ ). As the first derivative is piecewise constant the second variation coincides with the sum of the jumps of the first derivative. In fact, for every  $g \in C_c^1(\mathbb{S}^1, \mathbb{R}^2)$ , we have

$$\begin{aligned} \int_{\mathbb{S}^1} \left\langle \frac{dv}{d\gamma(s)}, \frac{dg}{d\gamma(s)} \right\rangle d\gamma(s) &= \sum_{i=1}^n \int_{\frac{i}{n}}^{\frac{i+1}{n}} \left\langle \frac{dv}{d\gamma(s)}, g'(s) \right\rangle ds = \\ &= \sum_{i=1}^n \left\langle \frac{dv}{d\gamma} \left( \frac{i-1}{n} \right) - \frac{dv}{d\gamma} \left( \frac{i}{n} \right), g \left( \frac{i}{n} \right) \right\rangle. \end{aligned}$$

Then, by (4.9), the second variation  $TV_{\gamma}^2(v)$  can be defined as

$$J_2(\tilde{\gamma}, \tilde{v}) = \sum_{i=1}^n \left| \frac{\Delta^+(\tilde{v})_{i+1}}{|\Delta^+(\tilde{\gamma})_{i+1}|_{\frac{\varepsilon}{n}}} - \frac{\Delta^+(\tilde{v})_i}{|\Delta^+(\tilde{\gamma})_i|_{\frac{\varepsilon}{n}}} \right|_{\varepsilon}.$$

## 4.5 Minimization with Gradient Descent

The finite problem (4.7) is an unconstrained optimization on the variable  $(\tilde{\Gamma}_{2,\cdot}, \dots, \tilde{\Gamma}_{N,\cdot})$ , since  $\tilde{\Gamma}_{1,\cdot} = \tilde{\gamma}_0$  is fixed. The function  $\mathcal{F}_{\varepsilon}$  being minimized is  $C^1$  with a Lipschitz gradient, and we thus make use of a gradient descent method. In the following, we compute gradient for the canonical inner product in  $\mathbb{C}^{N \times n}$ .

Starting from some  $\tilde{\Gamma}^{(0)} \in \mathbb{C}^{N \times n}$ , we iterate

$$\tilde{\Gamma}^{(k+1)} = \tilde{\Gamma}^{(k)} - \tau_k \nabla \mathcal{F}_{\varepsilon}(\tilde{\Gamma}^{(k)}) \quad (4.10)$$

where  $\tau_k > 0$  is the descent step. A small enough gradient step size (or an adaptive line search strategy) ensures that the iterates converge toward a stationary point  $\Gamma^{(\infty)}$  of  $\mathcal{F}_{\varepsilon}$ .

The gradient  $\nabla \mathcal{F}_{\varepsilon}(\tilde{\Gamma})$  is given by its partial derivatives as, for  $i = 2, \dots, N-1$ ,

$$\partial_{\tilde{\Gamma}_i} \mathcal{F}_{\varepsilon}(\tilde{\Gamma}) = \frac{1}{N-1} \left( \partial_1 J(\tilde{\Gamma}_i, \tilde{v}_i) - \frac{1}{N-1} \partial_2 J(\tilde{\Gamma}_{i+1}, \tilde{v}_{i+1}) + \frac{1}{N-1} \partial_2 J(\tilde{\Gamma}_{i-1}, \tilde{v}_{i-1}) \right)$$

where  $\partial_1 J$  (resp.  $\partial_2 J$ ) is the derivative of  $J$  with respect to the first (resp. second) variable and

$$\partial_{\tilde{\Gamma}_N} \mathcal{F}_{\varepsilon} = \delta + \frac{1}{(N-1)^2} \partial_1 J(\tilde{\Gamma}_{N-1}, \tilde{v}_{N-1}).$$

where  $\delta$  is the gradient of the map  $\tilde{\gamma} \mapsto H(P_1(\tilde{\gamma}), \gamma_1)$  at  $\tilde{\gamma} = \tilde{\Gamma}_{N,\cdot}$ . This gradient can be computed as detailed in [14].

## 4.6 Numerical Results

In this section we show some numerical examples of computations of a stationary points  $\tilde{\Gamma}^{(\infty)}$  of the problem (4.7) that is intended to approximate geodesics for the  $BV^2$  metric. We use a similar approach to approximate geodesic for the  $H^s$  metric, for  $s = 2$ , by replacing  $E$  in (4.2) by

$$E(\Gamma) = \int_0^1 \|\Gamma_t\|_{H^s(\Gamma(t))} dt. \quad (4.11)$$

where, for all  $\gamma \in H^s(\mathbb{S}^1, \mathbb{R}^2)$  and  $v \in H^s(\gamma)$ ,

$$\|v\|_{H^s(\gamma)} = \int_{\mathbb{S}^1} \left( \mu_0 |v(s)|^2 + \mu_1 \left| \frac{dv}{d\gamma}(s) \right|^2 + \mu_2 \left| \frac{d^2v}{d\gamma^2}(s) \right|^2 \right) d\gamma(s),$$

where the parameter  $(\mu_0, \mu_1, \mu_2) \in (\mathbb{R}^+)^3$  can be tuned for each particular application. Note that, on contrast to the  $BV^2$  case, this Sobolev energy is a smooth functional, and one thus does not need to perform a regularization (4.3), or equivalently, one can use  $\varepsilon = 0$  in this case. We do not detail the computation of the gradient of the discretized version of the functional (4.11), since these computations are very similar to the  $BV^2$  case.

In the following experiments, we use a discretization grid of size  $(N, n) = (10, 256)$ . The weights are set to  $(\lambda_0, \lambda_1, \lambda_2) = (1, 0, 1)$  and  $(\mu_0, \mu_1, \mu_2) = (1, 0, 1)$  (the curves are normalized to fit in  $[0, 1]^2$ ). These experiments can be seen as toy models illustrations for the shape registration problem, where one seeks for a meaningful bijection between two geometric curves parameterized by  $\gamma_0$  and  $\gamma_1$ . Note that the energies being minimized are highly non-convex, so that the initialization  $\Gamma^{(0)}$  of the gradient descent (4.10) plays a crucial role.

Figure 1, top row, shows a simple test case, for which using a trivial constant initialization  $\tilde{\Gamma}_i^{(0)} = \tilde{\gamma}_0$ , for both  $BV^2$  and  $H^2$  metric, produces a valid homotopy  $\Gamma^{(\infty)}$  between  $\gamma_0$  and  $\gamma_1$ . One can observe that while both homotopies are similar, the Sobolev metric produces a slightly smoother regular evolution of curves. This is to be expected, since piecewise affine curves are not in the Sobolev space  $H^2(\mathbb{S}^1, \mathbb{R}^2)$ .

Figure 1, bottom row, shows a more complicated case, where using a trivial initialization  $\tilde{\Gamma}^{(0)}$  fails to give a correct result  $\tilde{\Gamma}^{(\infty)}$ , because the gradient descent is trapped in a poor local minimum. We thus use as initialization the natural bijection  $\tilde{\Gamma}^{(0)}$  which is piecewise affine and correctly links the singular points of the curves  $\gamma_0$  and  $\gamma_1$ . It turns out that this homotopy is a stationary point of the energy (4.7), which can be checked on the numerical results obtained by the gradient descent. On the contrary, the Sobolev metric finds a slightly different homotopy, which leads to smoother intermediate curves.

## 5 Conclusion

The variational approach defined in this work represents a general strategy to prove existence of minimal geodesics with respect to Finslerian metrics. The uniqueness (local and global) is at the moment an open question.

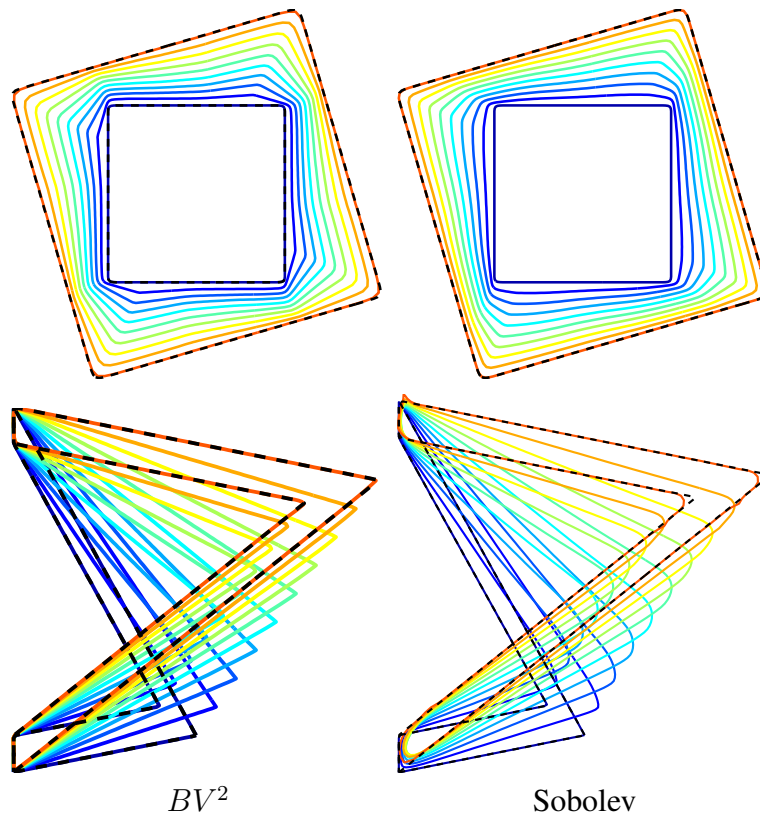


Figure 1: Homotopies  $\Gamma^{(\infty)}$  obtained for  $BV^2$ -Finsler energy (left) and Sobolev metric (right). Each image displays the initial curve  $\gamma_0$  (black one) and  $\gamma_1$  (dash line) and the optimal  $\{\tilde{\Gamma}_{i,\cdot}\}_i$  where the index  $1 \leq i \leq N$  is indicated by color variations between blue ( $i = 1$ ) and red ( $i = N$ ).

In order to generalize previous results to more general Banach spaces, we point out the main properties which must be satisfied by the Banach topology:

1. In order to prove local existence, two facts are needed:
  - (i) the two constants  $m_\gamma, M_\gamma$  appearing in Proposition 2.7 must be bounded on the balls of the space;
  - (ii) the topology of the space must imply a suitable convergence of the reparameterizations in order to get semi-continuity of the norm of  $\{\Gamma_t^h \circ \varphi^h(t)\}_h$ ; in the  $BV^2$ -case such a topology is the  $W^{1,1}$ -strong topology.
2. In order to prove global existence, the main hypothesis for the iteration argument is the uniform bound of the length of the curves; this actually requires that the first derivative of  $\Gamma_t$  is  $L^\infty$ .

For the  $BV^2$  metric, the major difficulty concerns the characterization of the weak topology of the space of the paths. The usual characterization of the dual of Boncher spaces  $H^1([0, 1], B)$  ( $B$  is a Banach space) requires that the dual of  $B$  verifies the Radon-Nikodym property ([9], [12]). We point out that the martingale argument used to prove Theorem 2.15 avoids this problem and allows one



to define a suitable topology in such a space guaranteeing the lower semi-continuity of the geodesic energy.

Moreover, as pointed out in the introduction, the necessary conditions proved in [26] are not valid in our case. This represents an interesting direction of research because optimality conditions allow one to study regularity properties of minimal geodesics. It is an open question if the generalized Euler-Lagrange equations in [26] can be generalized to our case and give the Hamiltonian geodesic equations. Strongly linked to this question is the issue of convergence of the numerical method. Indeed, the convergence of the sequence of discretized problems would imply the existence of geodesic equations.

From a numerical point of view, as we pointed out, the minimal geodesic energy suffers from many poor local minima. To avoid some of these poor local minima, it is possible to modify the metric to take into account some prior on the set of deformations. For instance, in the spirit of [14], a Finsler metric can be designed to favor piecewise rigid motions.

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